Multiplication operator and average characteristic polynomial associated with exceptional Jacobi polynomials

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Abstract

Studying the multiplication operator associated with exceptional Jacobi polynomials, the zero distribution of the corresponding average characteristic polynomials is determined. Applying this result, the location of zeros of certain self-inversive polynomials is examined.

Keywords Exceptional Jacobi polynomials · Average characteristic polynomial · Multiplication operator · Self-inversive polynomials

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1 Introduction

The three term recurrence relation fulfilled by standard orthogonal polynomials implies that the multiplication operator $M : f(x) \rightarrow xf(x)$, acting on the weighted L_w^2 space can be represented by a tridiagonal (Jacobi) matrix. Denoting by π_n the projection operator to the n -dimensional subspace, the eigenvalues of $\pi_n M \pi_n$ are just the zeros of the orthogonal polynomials in question.

Considering the probability density

$$Q_n(x_1, \dots, x_n) = c(n) \det |K_n(x_i, x_j)|_{i,j=1}^n \prod_{i=1}^n W(x_i),$$

where $c(n)$ is a normalization factor, and the average characteristic polynomial is defined as follows:

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$$\chi_n(z) := \mathbb{E} \left(\prod_{i=1}^n (z - x_i) \right),$$

where the expectation $\mathbb{E}$ refers to ϱ_n . According to [28, (2.2.10)], the zeros of the average characteristic polynomial are just the zeros of q_n again.

Exceptional orthogonal polynomials are complete systems of polynomials with respect to a positive measure. They are different from the standard or from the classical orthogonal polynomials, since exceptional families have finite codimension in the space of polynomials. Similarly to the classical ones, exceptional polynomials are eigenfunctions of Sturm–Liouville-type differential operators but unlike the classical cases, the coefficients of these operators are rational functions. Exceptional orthogonal polynomials also possess a Bochner-type characterization as each family can be derived from one of the classical families applying finitely many Darboux transformations, see [5].

Exceptional orthogonal polynomials were introduced recently by Gómez-Ullate, Kamran and Milson, cf. e.g. [7,8] and the references therein. These families of polynomials play a fundamental role for instance in the construction of bound-state solutions to exactly solvable potentials in quantum mechanics. In the last few years, they have seen a great deal of activity in this area both by mathematicians and physicists, cf. e.g. [4,6,13,25], etc. The location of zeros of exceptional orthogonal polynomials is also examined, cf. e.g. [2,3,9,15,16,19].

Exceptional orthogonal polynomials fulfil $2L + 1$ recurrence formulae with $L \geq 2$. Thus the matrices that represent the multiplication operators associated with exceptional polynomials are $2L + 1$ diagonal. Unlike the standard case, the n th average characteristic polynomial does not coincide with the n th orthogonal polynomial; furthermore, these two polynomials are of different degrees.

Below we deal with exceptional Jacobi polynomials. In this case, it is pointed out that the normalized counting measure associated with the modified characteristic polynomial tends to the equilibrium measure of the interval of orthogonality in weak-star sense, see Theorem 1 below, i.e. it coincides asymptotically with the normalized zero-counting measure of the corresponding exceptional Jacobi polynomials, see [2, Theorem 6.5].

Finally, this multiplication operator method is applied to determine the zeros on the unite circle of certain self-inversive polynomials.

2 Preliminaries, notation

2.1 General construction of exceptional orthogonal polynomials

There are two different approaches of derivation exceptional orthogonal polynomials from classical ones. Both methods are based on the quasi-rational eigenfunctions of the classical (Hermite, Laguerre, Jacobi) differential operators. The method, developed first for exceptional Hermite polynomials (see [6]), defines exceptional polynomials by the Wronskian of certain partition of quasi-rational eigenfunctions mentioned above.

The other method defines exceptional polynomials recursively, which is by application of finitely many Crum–Darboux transformations to the classical differential operators. A Bochner-type characterization theorem (see [5]) ensures that all exceptional classes can be derived by this method.

Subsequently we follow the construction by Crum–Darboux transformation. According to [5, Propositions 3.5 and 3.6], it is as follows.

Classical orthogonal polynomials $\{P_n^{[0]}\}_{n=0}^{\infty}$ are eigenfunctions of the second-order linear differential operator with polynomial coefficients

$$T[y] = py'' + qy' + ry,$$

and its eigenvalues are denoted by λ_n . T can be decomposed as follows:

$$T = BA + \tilde{\lambda}, \text{ with } A[y] = b(y' - wy), \quad B[y] = \hat{b}(y' - \hat{w}y), \quad (1)$$

where w is the logarithmic derivative of a quasi-rational eigenfunction of T with eigenvalue $\tilde{\lambda}$, which is w is rational and fulfils the Riccati equation

$$p(w' + w^2) + qw + r = \tilde{\lambda}. \quad (2)$$

b is a suitable rational function. The coefficients of B can be expressed as follows:

$$\hat{b} = \frac{p}{b}, \quad \hat{w} = -w - \frac{q}{p} + \frac{b'}{b}. \quad (3)$$

Then the exceptional polynomials are the eigenfunctions of $\hat{T}$ that is the partner operator of T , which is

$$\hat{T}[y] = (AB + \tilde{\lambda})[y] = py'' + \hat{q}y' + \hat{r}y, \quad (4)$$

where

$$\begin{aligned} \hat{q} &= q + p' - 2\frac{b'}{b}p, \\ \hat{r} &= r + q' + wp' - \frac{b'}{b}(q + p') + \left(2\left(\frac{b'}{b}\right)^2 - \frac{b''}{b} + 2w'\right)p, \end{aligned} \quad (5)$$

(2) and (4) imply that

$$\hat{T}AP_n^{[0]} = \lambda_n AP_n^{[0]}, \quad (6)$$

so exceptional polynomials can be obtained from the classical ones by application of (finitely many) appropriate first-order differential operator(s) to the classical polynomials. This observation motivates the notation below:

$$AP_n^{[0]} = b \left(P_n^{[0]} \right)' - bw P_n^{[0]} =: P_n^{[1]}, \quad (7)$$

(and recursively $A_s P_n^{[s-1]} =: P_n^{[s]}$ in case of s Darboux transformations.) The degree of $P_n^{[1]}$ is usually greater than n . $\left\{ P_n^{[1]} \right\}_{n=0}^{\infty}$ is an orthogonal system on I with respect to the weight

$$W := \frac{pw_0}{b^2}, \quad (8)$$

where w_0 is one of the classical weights.

Remark Since at the endpoints of I (if there is any) p may possess zeros, b can be zero here as well, but b does not have zeros inside I , otherwise the moments of W would not be finite.

- As each $P_n^{[1]}$ is a polynomial ($n = 0, 1, \dots$), applying the operator A to $P_0^{[0]}$ it can be seen that bw must be a polynomial and to $P_1^{[0]}$ shows that b itself is also a polynomial.
- Let us recall that $r = 0$ in the classical differential operators. Considering (2) and comparing degrees, w itself cannot be a polynomial. By the same reasons, pw cannot be a polynomial unless it is of degree one. w is a rational function, and it has no poles in $(-1, 1)$.
- Expressing (2) as

$$bBP_n^{[1]} = p \left(P_n^{[1]} \right)' + \left(pw + q - p \frac{b'}{b} \right) P_n^{[1]} = (\lambda_n - \tilde{\lambda}) b P_0^{[0]}, \quad (9)$$

it can be easily seen that if $P_n^{[1]}$ had got a double zero at $x_0 \in \text{int} I$, then $P_0^{[0]}(x_0) = 0$, and by (7) $\left(P_n^{[0]} \right)'(x_0) = 0$, which is impossible, i.e. that zeros of $P_n^{[1]}$ which are in the interior of the interval of orthogonality are simple.

- Let a be (one of) the finite endpoint(s) of the interval of orthogonality. Again by (9), one can derive that if $b(a) \neq 0$, then $P_n^{[1]}(a) \neq 0$ as well.
- If there was an $n \in \mathbb{N}$ such that $P_n^{[1]}(a) = 0$, then $b(a) = 0$ and by (7) $(bw)(a) = 0$ and so $P_n^{[1]}(a) = 0$ for all n . That is, taking $\tilde{P}_n^{[1]} := b_1 \left(P_n^{[0]} \right)' - b_1 w P_0^{[0]}$, where $b_1(x) = \frac{b(x)}{x-a}$, we arrive to the exceptional system orthogonal with respect to $W = \frac{pw_0}{b_1^2}$. Thus, we can assume that $\forall n \ P_n^{[1]}(a) \neq 0$, and if $b(a) = 0$, then $(bw)(a) \neq 0$.

2.2 Exceptional Jacobi polynomials

Let the n^{th} Jacobi polynomial defined by Rodrigues' formula be:

$$(1-x)^\alpha (1+x)^\beta P_n^{\alpha, \beta}(x) = \frac{(-1)^n}{2^n n!} \left((1-x)^{\alpha+n} (1+x)^{\beta+n} \right)^{(n)},$$

where $\alpha, \beta > -1$.

$$p_k := p_k^{\alpha, \beta} = \frac{P_k^{\alpha, \beta}}{Q_k^{\alpha, \beta}},$$

$$Q_k^2 := \left(Q_k^{\alpha, \beta}\right)^2 = \frac{2^{\alpha+\beta+1} \Gamma(k+\alpha+1) \Gamma(k+\beta+1)}{(2k+\alpha+\beta+1) \Gamma(k+1) \Gamma(k+\alpha+\beta+1)}. \quad (10)$$

The orthogonal Jacobi polynomials which fulfil the following differential equation (cf. [28, (4.2.1)])

$$(1-x^2)y'' + (\beta - \alpha - (\alpha + \beta + 2)x)y' + n(n + \alpha + \beta + 1)y = 0. \quad (11)$$

With

$$P_n^{[0]} = p_n^{(\alpha, \beta)} = p_n \quad (12)$$

and

$$P_n^{[1]} = AP_n^{[0]} = b(P_n^{[0]})' - bwP_n^{[0]}. \quad (13)$$

In the following examples X_m stands for a family of exceptional Jacobi polynomials of codimension m which are given by a certain Darboux transformation. Examples of X_m can be found in [9]. These are as follows:

$$w_0 = w^{(\alpha, \beta)} = (1-x)^\alpha (1+x)^\beta, \quad (14)$$

where α and β are defined appropriately, see [9, Proposition 5.1].

$$T[y] = (1-x^2)y'' + (\beta - \alpha - (\alpha + \beta + 2)x)y' = BA - (m - \alpha)(m + \beta + 1),$$

where A and B are defined in (2), and

$$b(x) = (1-x)P_m^{(-\alpha, \beta)}(x), \quad w(x) = (\alpha - m) \frac{P_m^{(-\alpha-1, \beta-1)}(x)}{(1-x)P_m^{(-\alpha, \beta)}(x)}.$$

So the defined exceptional Jacobi polynomials, $P_n^{[1]} := AP_n^{[0]}$, are orthogonal with respect to

$$W(x) = \frac{(1-x^2)w^{(\alpha, \beta)}(x)}{(1-x)^2 \left(P_m^{(-\alpha, \beta)}(x)\right)^2} = \frac{w^{(\alpha-1, \beta+1)}(x)}{\left(P_m^{(-\alpha, \beta)}(x)\right)^2},$$

and the space spanned by these classes is m -codimensional in the space of polynomials.

We restrict our investigations to the one-step Darboux transformation case. As it is mentioned above, b can be zero at ± 1 . Indeed, by [2, Table 1] the quasi-rational eigenfunctions of

$$T[y] = (1 - x^2)y'' + (\beta - \alpha - (\alpha + \beta + 2)x)y'$$

cf. (11), are $p_n^{(\alpha, \beta)}(x)$, $(1+x)^{-\beta} p_n^{(\alpha, -\beta)}(x)$, $(1-x)^{-\alpha} p_n^{(-\alpha, \beta)}(x)$ and $(1+x)^{-\beta} (1-x)^{-\alpha} p_n^{(-\alpha, -\beta)}(x)$. It gives three kinds of $b(x)$. According to the three types of quasi-rational eigenfunctions, let

$$b_0(x) = \begin{cases} (1-x)p_n^{(-\alpha, \beta)}(x) & \text{or} \\ (1+x)p_n^{(\alpha, -\beta)}(x) & \text{or} \\ (1-x^2)p_n^{(-\alpha, -\beta)}(x). \end{cases}$$

Thus, $b(x)$ can be taken as

$$b(x) = s(x)b_0(x), \quad (15)$$

where s is a polynomial such that $s(x) \neq 0$ on $[-1, 1]$.

Of course, $\{P_n^{[1]}\}$ is an (exceptional) closed orthogonal system with $s \equiv 1$ in L^2_W if and only if $\{s P_n^{[1]}\}$ is orthogonal and closed in $L^2_{\frac{W}{s^2}}$. If s had a zero in $x = \pm 1$, then closedness could fail. So the examples beyond b_0 do not give really new classes and can be handled as simple consequences of the b_0 -type families. Because it does not cause any difficulty, below we use b instead of using b_0 first and than taking extension.

Subsequently we denote by $\tilde{b}$ that part of b which appears in the weight function,

$$\frac{p}{b} = \frac{\tilde{p}}{\tilde{b}}, \quad (16)$$

which is bounded on $[-1, 1]$. Introducing the notation

$$b(x) = (1-x)^{\frac{1-\varepsilon_1}{2}} (1+x)^{\frac{1-\varepsilon_2}{2}} \tilde{b}(x), \quad (17)$$

where $\varepsilon_i = \pm 1$, $i = 1, 2$, the exceptional Jacobi polynomial system, $\{P_n^{[1]}\}_{n=0}^\infty$, is orthogonal on $(-1, 1)$ with respect to

$$W = \frac{w^{(\alpha+\varepsilon_1, \beta+\varepsilon_2)}}{\tilde{b}^2}. \quad (18)$$

The orthonormal system is denoted by $\{\hat{P}_n\}_{n=0}^\infty$. The codimension is given by the degree of $\tilde{b}$ cf. [5].

3 Multiplication operator

Below we investigate the location of zeros of the modified characteristic polynomial introduced in [14]. In our case, it is as follows.

Let us denote by $\{\hat{P}_n\}_{n=0}^\infty$ the orthonormal system of exceptional Jacobi polynomials, i.e.

$$\hat{P}_n := \frac{P_n^{[1]}}{\sigma_n}, \quad \text{where } \sigma_n := \|P_n^{[1]}\|_{W,2}. \quad (19)$$

The Christoffel–Darboux kernel is

$$K_n(x, y) := \sum_{k=0}^{n-1} \hat{P}_k(x) \hat{P}_k(y). \quad (20)$$

The joint probability density generated by an n -point ensemble has the form:

$$\rho_N(x_1, \dots, x_N) = c(N) \det |K_N(x_i, x_j)|_{i,j=1}^N \prod_{i=1}^N W(x_i). \quad (21)$$

Thus the correlation functions are

$$\rho_{N,n}(x_1, \dots, x_n) = c(n, N) \det |K_N(x_i, x_j)|_{i,j=1}^n \prod_{i=1}^n W(x_i),$$

where $c(N)$ and $c(n, N)$ are normalization factors. The expectation $\mathbb{E}$ refers to (21). The eigenvalue measure becomes a point process with determinantal correlation kernel, cf. [17, Example 2.12], i.e. it is a determinantal point process.

Now we define the modified characteristic polynomial. Let b and $\tilde{b}$ be given by (13) and (16), respectively. Recalling that b is a polynomial, let

$$Q(x) := \int^x \tilde{b}. \quad (22)$$

The constant term of Q is chosen to be zero.

Considering Q , the modified average characteristic polynomial is defined as follows:

$$\chi_n(z) := \chi_n^Q(z) = \mathbb{E} \left(\prod_{i=1}^n (z - Q(x_i)) \right), \quad (23)$$

cf. [14]. Denote by z_i the zeros, of $\chi_n(z)$ and define the not necessarily unique pre-images of z_i by y_i , i.e. $z_i = Q(y_i)$, $i = 1, \dots, n$. The normalized counting measure based on y_i is

$$\tilde{v}_n = \frac{1}{n} \sum_{i=1}^n \delta_{y_i} \quad (24)$$

that is $\int Q^l(y) d\tilde{v}_n = \int z^l d\nu_n(z)$. Note, that since $\tilde{b} > 0$ on $[-1, 1]$, Q is increasing here and so if $z_i \in Q([-1, 1])$, y_i can be chosen from $[-1, 1]$ uniquely.

The next theorem describes zero distribution of the modified average characteristic polynomial. X_m stands for exceptional orthogonal polynomials generated by one Darboux transformation and of codimension m .

Theorem 1 *In the exceptional Jacobi case X_m , where $m \in \mathbb{N}$ is arbitrary, if $\alpha + \varepsilon_1, \beta + \varepsilon_2 \geq -\frac{1}{2}$, then*

$$\tilde{v}_n \rightarrow \mu_e$$

in weak-star sense, where μ_e is the equilibrium measure of $[-1, 1]$.

For this study, our main tool is the multiplication operator $M : f \rightarrow Qf$, which was introduced in [14]. It is defined as follows.

Exceptional orthogonal polynomials fulfil the next recurrence formula with constant coefficient: $Q P_n^{[1]} = \sum_{k=-L}^L \tilde{u}_{n,k} P_{n+k}^{[1]}$ (see [24] and [14, (3.4)]). After normalization the recurrence relation above is modified as follows:

$$Q \hat{P}_n = \sum_{k=-L}^L u_{n,k} \hat{P}_{n+k}, \quad (25)$$

where $u_{n,k} = \frac{\sigma_{n+k}}{\sigma_n} \tilde{u}_{n,k}$, for σ_n see (19). That is M can be represented by an infinite matrix, M_e , in the orthonormal basis $\{\hat{P}_n\}_{n=0}^\infty$. This operator, acting on $L_W^2[-1, 1]$ and simultaneously on l_2 , is denoted by M_e . By (25), the matrix of M_e is $2L+1$ -diagonal:

$$M_e = \begin{bmatrix} u_{0,0} & u_{0,1} & \dots & \dots & u_{0,L} & 0 & 0 & \dots \\ u_{1,-1} & u_{1,0} & \dots & \dots & u_{1,L-1} & u_{1,L} & 0 & \dots \\ \vdots & \vdots & \dots & \ddots & \vdots & \dots & \vdots & \dots \\ u_{L,-L} & u_{L,-L+1} & \dots & u_{L,0} & \dots & \dots & u_{L,L} & 0 \\ 0 & u_{L+1,-L} & \dots & \vdots & \dots & \vdots & \dots & u_{L+1,L} \\ \vdots & 0 & \dots & u_{L+j,-L} & \dots & \vdots & \dots & \dots \end{bmatrix}. \quad (26)$$

It can be easily seen that M_e is symmetric since

$$u_{k,j} = \int_{-1}^1 Q \hat{P}_k \hat{P}_{k+j} W^2 = \int_{-1}^1 Q \hat{P}_{k+j} \hat{P}_{(k+j)-j} W^2 = u_{k+j,-j}. \quad (27)$$

Let us recall that $\{\hat{P}_n\}_{n=0}^\infty$ is an orthonormal system on $[-1, 1]$ with respect to $W = \frac{p w_0}{b^2}$, where w_0 is a classical Jacobi weight function. Besides this exceptional

orthonormal polynomial system, there is the standard orthonormal polynomial system, $\{q_n\}_{n=0}^\infty$, on $[-1, 1]$ with respect to W . These standard orthonormal polynomials fulfil the three-term recurrence relation:

$$xq_n = a_{n+1}q_{n+1} + b_nq_n + a_nq_{n-1} \quad (28)$$

(see e.g. [28, (3.2.1)]). Since $W > 0$ on $(-1, 1)$, by [23, Theorem 4.5.7] (see also [26]) in formula (28) the recurrence coefficients fulfil the asymptotics

$$\lim_{n \rightarrow \infty} a_n = \frac{1}{2} \quad \text{and} \quad \lim_{n \rightarrow \infty} b_n = 0. \quad (29)$$

According to (28) multiplication operator on $L_W^2[-1, 1]$, $A : f \mapsto xf$ can be represented in the (Schauder) basis $\{q_n\}$ as an infinite tridiagonal matrix (denoted by A again)

$$A = \begin{bmatrix} b_0 & a_1 & 0 & 0 & \dots \\ a_1 & b_1 & a_2 & 0 & \dots \\ 0 & a_2 & b_2 & a_3 & \dots \\ 0 & 0 & a_3 & b_3 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix},$$

and so A acts on $L_W^2[-1, 1]$ and on l^2 . Let $Q(x)$ be as above. Then the multiplication operator, $M : f \mapsto Qf$ can be represented as $M = Q(A)$, and in the basis $\{q_n\}$, it has a matrix $M_q = Q(A)$. Let Π_n^q and Π_n^e be the projections to $\text{span}\{q_0, \dots, q_{n-1}\}$, and to $\text{span}\{\hat{P}_0, \dots, \hat{P}_{n-1}\}$, respectively. Let

$$A_{n \times n} := \Pi_n^q A \Pi_n^q, \quad M_{e,n \times n} := \Pi_n^e M_e \Pi_n^e, \quad M_{q,n \times n} := \Pi_n^q M_q \Pi_n^q.$$

To prove Theorem 1, we compare the trace of the truncated operators in the different bases above.

It is known (see [14] and the references therein) that the characteristic polynomial of the truncated multiplication matrix $M_{e,n \times n}$, coincides with the modified average characteristic polynomial, that is

$$\det(zI_n - \pi_n^e M_e \pi_n^e) = \mathbb{E} \left(\prod_{i=1}^n (z - Q(x_i)) \right), \quad (30)$$

where I_n stands for the n -dimensional identity operator. In the standard case, the characteristic polynomial of the truncated multiplication matrix $A_{n \times n}$ coincides with the standard average characteristic polynomial (see [10, 11]) and the eigenvalues are just the zeros of the standard orthonormal polynomials, q_n , see [28, (2.2.10)]. Thus, we need the next statement.

Proposition 1 *With the notation (18), let us assume that $\alpha + \varepsilon_1, \beta + \varepsilon_2 \geq -\frac{1}{2}$. Then for each $l \in \mathbb{N}$*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left(\text{Tr}((Q(A_{n \times n}))^l) - \text{Tr}((M_{e, n \times n})^l) \right) = 0.$$

Proof of Proposition 1 is postponed to the next section.

Proof (of Theorem 1) Let $\xi_i = \xi_{i,n}$, $i = 1, \dots, n$ the zeros of the standard orthogonal polynomial, q_n . Let us recall that $\{\xi_i\}_{i=1}^n$ are the eigenvalues of $A_{n \times n}$. Since $\frac{1}{n} \text{Tr}((Q(A_{n \times n}))^l) = \frac{1}{n} \sum_{i=1}^n Q^l(\xi_i)$ and by (30) $\frac{1}{n} \text{Tr}((M_{e, n \times n})^l) = \frac{1}{n} \sum_{i=1}^n Q^l(y_i)$, according to Proposition 1

$$\lim_{n \rightarrow \infty} \left| \frac{1}{n} \sum_{i=1}^n Q^l(\xi_i) - \int Q^l d\tilde{\nu}_n \right| = 0.$$

On the other hand, (29) implies that the normalized counting measure based on the zeros of the orthogonal polynomials, q_n , tends to the equilibrium measure of the interval of orthogonality in weak-star sense (see e.g. [18, Proposition 1.1] and the references therein). That is

$$\lim_{n \rightarrow \infty} \left| \frac{1}{n} \sum_{i=1}^n Q^l(\xi_i) - \int_{-1}^1 Q^l d\mu_e \right| = 0.$$

Comparing the two limits and taking into consideration that the spectrum of the multiplication operator is the closure the range of Q on $[-1, 1]$ (see e.g. [27, Sect. 150]) we have that for all $l \in \mathbb{N}$

$$\lim_{n \rightarrow \infty} \left(\int_{-1}^1 Q^l d\tilde{\nu}_n - \int_{-1}^1 Q^l d\mu_e \right) = 0.$$

As it is pointed out in the proof of [14, Theorem 4.1], $\text{span}\{Q^l : l \in \mathbb{N}\}$ is dense in $C[-1, 1]$, which implies the result. $\square$

4 Proof of Proposition 1

First, we need some technical lemmas. Let $C = [c_{ij}]_{i,j=0}^\infty$ be an infinite matrix. $c_{ij} = {}_i C_j$ stands for the elements of C . Its n^{th} principal minor matrix is $C_{n \times n} := [c_{ij}]_{i,j=0}^{n-1}$. Its i th row is ${}_i C$ and j th column is C_j . With this notation, we can state the next lemma.

Lemma 1 *Let $C = [c_{ij}]_{i,j=0}^\infty$ be a $2k+1$ -diagonal, infinite matrix. Let $P = \sum_{l=0}^M p_l x^l$ be a fixed polynomial. If the entries of C are bounded, i.e. there is a K such that $|c_{ij}| < K$ for all $0 \leq i, j < \infty$, then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} (\text{Tr}(P(C_{n \times n})) - \text{Tr}((P(C))_{n \times n})) = 0. \quad (31)$$

Proof It is obvious that for each $l \in \mathbb{N}$ C^l is $2(kl) + 1$ diagonal:

Indeed, $c_{ij} = 0$ if $|i - j| > k$. Assuming that $(C^l)_{ij} =: C_{ij}^l = 0$ if $|i - j| > kl$ we compute $C_{ij}^{l+1} = \langle C^l, C_j \rangle = \sum_{|i-p| \leq kl} C_{ip}^l C_{pj}$. That is $C_{ij}^{l+1} = 0$ if $|i - j| > (l + 1)k$.

By induction on l , we show that the elements of the principal diagonal are coincided in $C_{n \times n}^l$ and $(C^l)_{n \times n}$ except at most the last $k(l - 1)$ ones.

Let us assume that $(C_{n \times n}^l)_{ij} = ((C^l)_{n \times n})_{ij}$ if $i \leq n - (l - 1)k$ or $j \leq n - (l - 1)k$. (For $l = 1$, it is obviously fulfilled.) Let $i \leq n - kl$. By the assumption and by $2kl + 1$ -diagonality $((C_{n \times n})^l)_i = (C^l)_i$, and similarly if $j \leq n - k$, then $(C_{n \times n})_j = C_j$. Thus, $((C_{n \times n})^{l+1})_{ij} = C_{ij}^{l+1}$ if $i \leq n - kl$ and $j \leq n - k$. If $i \leq n - kl$, then $C_{ip}^l = 0$ if $p > n$ and if $j > n - k$ then $C_j - (C_{n \times n})_j$ starts with n zeros; thus, if $i \leq n - kl$ and $j > n - k$, $((C_{n \times n})^{l+1})_j = C^{l+1}_j$ again. For $j \leq n - kl$, we have the same by symmetry.

Finally considering that the rows and columns contain finitely many non-zero elements, these imply that $\frac{1}{n} (\text{Tr}((C_{n \times n})^l) - \text{Tr}((C^l)_{n \times n})) \leq \frac{1}{n} k(l - 1)(2kl + 1)^l K^l$, that is

$$\lim_{n \rightarrow \infty} \frac{1}{n} (\text{Tr}((C_{n \times n})^l) - \text{Tr}((C^l)_{n \times n})) = 0. \quad (32)$$

Since $\text{Tr}(P(C_{n \times n})) = \sum_{l=0}^M p_l \text{Tr}((C_{n \times n})^l)$ and $\text{Tr}((P(C))_{n \times n}) = \sum_{l=0}^M p_l \text{Tr}((C^l)_{n \times n})$, (32) implies (31). $\square$

To ensure the boundedness of the entries of the matrices in question, we recall the asymptotic behaviour of recurrence coefficients. First, we compute the norm of $P_n^{[1]}$.

Lemma 2 With the notation (19), if $\alpha + \frac{\varepsilon_1}{2}, \beta + \frac{\varepsilon_2}{2} > -\frac{1}{2}$,

$$\sigma_k = \sqrt{k(k + \alpha + \beta + 1) + \tilde{\lambda}}. \quad (33)$$

Proof With the notation $p_k^{(\alpha, \beta)} = p_k$, cf. (12)

$$\begin{aligned} \sigma_k^2 \hat{P}_k^2 W &= (bp'_k - bw p_k)^2 \frac{pw^{(\alpha, \beta)}}{b^2} \\ &= (p'_k)^2 w^{(\alpha+1, \beta+1)} + (bw)^2 p_k^2 \frac{pw^{(\alpha, \beta)}}{b^2} - 2bw p'_k p_k \frac{pw^{(\alpha, \beta)}}{b^2}. \end{aligned}$$

By (2) $pw^2 = \tilde{\lambda} - qw - pw'$ and considering (11) $(w^{(\alpha+1, \beta+1)})' = qw^{(\alpha, \beta)}$

$$\sigma_k^2 \hat{P}_k^2 W = (p'_k)^2 w^{(\alpha+1, \beta+1)} + \tilde{\lambda} p_k^2 w^{(\alpha, \beta)} - (p_k^2 w w^{(\alpha+1, \beta+1)})'.$$

Thus,

$$\sigma_k^2 = \int_{-1}^1 (p'_k)^2 w^{(\alpha+1, \beta+1)} + \tilde{\lambda} p_k^2 w^{(\alpha, \beta)} - (p_k^2 w w^{(\alpha+1, \beta+1)})' = I_1 + I_2 + I_3.$$

According to [28, (4.21.7)]

$$p'_k = \frac{k + \alpha + \beta + 1}{2} p_{k-1}^{(\alpha+1, \beta+1)} \frac{\varrho_{k-1}^{\alpha+1, \beta+1}}{\varrho_k^{\alpha, \beta}}$$

$$I_1 = \left(\frac{k + \alpha + \beta + 1}{2} \frac{\varrho_{k-1}^{\alpha+1, \beta+1}}{\varrho_k^{\alpha, \beta}} \right)^2.$$

$I_2 = \tilde{\lambda}$. By the assumption on α and β , one can see that $I_3 = 0$. Substituting the values of the corresponding $\varrho_k^{\alpha, \beta}$, (33) is proved. $\square$

Analogous to (29), the coefficients in (25) fulfil a symmetric limit relation

$$\lim_{n \rightarrow \infty} u_{n,j} =: U_{|j|}, \quad (34)$$

where $U_{|j|}$ depends on the polynomial $\tilde{b}$ (cf. (2), (16)) as follows. Let

$$\tilde{b}(x) = \sum_{k=0}^{L-1} d_k x^k. \quad (35)$$

Since Lemma 2 ensures that the asymptotics of $\tilde{u}_{n,j}$ and $u_{n,j}$ coincide, according to [14, Proposition 1]

$$U_{|j|} = \begin{cases} \sum_{p=\max\{l, 1\}}^{\lfloor \frac{L}{2} \rfloor} \frac{d_{2p-1}}{2p} \binom{2p}{p-l} \frac{1}{2^{2p}}, & \text{if } |j| = 2l \\ \sum_{p=l}^{\lfloor \frac{L-1}{2} \rfloor} \frac{d_{2p}}{2p+1} \binom{2p+1}{p-l} \frac{1}{2^{2p+1}}, & \text{if } |j| = 2l + 1. \end{cases} \quad (36)$$

Let us recall that $\frac{p}{b}$ is bounded on $[-1, 1]$ and $W(x) = \frac{(1-x)^{\alpha+\varepsilon_1}(1+x)^{\beta+\varepsilon_2}}{b^2}$ ($\varepsilon_i = \pm 1$), where $b(x) = \tilde{b}(x)(1-x)^{\frac{1-\varepsilon_1}{2}}(1+x)^{\frac{1-\varepsilon_2}{2}}$ and $0 < c < \tilde{b} < C$ on $[-1, 1]$.

Subsequently, the next lemma proved by Badkov (see [1]) is useful. Here, we cite the formulation given in [22].

Lemma A [22, Lemma 2.E] *Let $\{q_k\}_{k=0}^\infty$ be the standard orthonormal system with respect to W . For each $j \geq 0$ integer*

$$\left| q_k^{(j)}(x) \right| \leq c \left(\frac{k}{\sqrt{1-x^2} + \frac{1}{k}} \right)^j \frac{1}{(\sqrt{1-x} + \frac{1}{k})^{\alpha+\varepsilon_1} (\sqrt{1+x} + \frac{1}{k})^{\beta+\varepsilon_2} \sqrt{\sqrt{1-x^2} + \frac{1}{k}}}. \quad (37)$$

We need the following estimations on norms of exceptional Jacobi polynomials.

Lemma 3 If $\alpha + \varepsilon_1, \beta + \varepsilon_2 \geq -\frac{1}{2}$, and $0 \leq \delta \leq \min \left\{ \frac{1}{4}, \frac{\alpha + \varepsilon_1}{2} + \frac{1}{4}, \frac{\beta + \varepsilon_2}{2} + \frac{1}{4} \right\}$, then

$$\left\| \hat{P}_k(x) \sqrt{W(x)} (1 - x^2)^{\frac{1}{4} - \delta} \right\|_{\infty} \leq ck^{\max\{-\varepsilon_1, -\varepsilon_2\} - 1 + 2\delta}, \quad (38)$$

where c is a constant (independent of k).

Proof We have

$$\begin{aligned} & \left| \hat{P}_k(x) \sqrt{W(x)} (1 - x^2)^{\frac{1}{4}} \right| \\ & \leq \frac{c}{\sigma_k} (|b(x)p'_k(x)| + |(bw)(x)p_k(x)|) (1 - x)^{\frac{\alpha + \varepsilon_1}{2} + \frac{1}{4}} (1 + x)^{\frac{\beta + \varepsilon_2}{2} + \frac{1}{4}} \\ & = K_1 + K_2. \\ K_1 & \leq \left| p_{k-1}^{(\alpha+1, \beta+1)} \right| (1 - x)^{\frac{\alpha}{2} + \frac{3}{4}} (1 + x)^{\frac{\beta}{2} + \frac{3}{4}}, \end{aligned}$$

which is bounded, see e.g. [28, (8.21.10)], or (37). Since bw is bounded on $[-1, 1]$ (it is a polynomial), by (37)

$$\begin{aligned} K_2 & \leq \frac{c}{k} |p_k(x)| (1 - x)^{\frac{\alpha + \varepsilon_1}{2} + \frac{1}{4} - \delta} (1 + x)^{\frac{\beta + \varepsilon_2}{2} + \frac{1}{4} - \delta} \\ & \leq \frac{c}{k} \frac{1}{(\sqrt{1-x} + \frac{1}{k})^{\alpha} (\sqrt{1+x} + \frac{1}{k})^{\beta} \sqrt{\sqrt{1-x^2} + \frac{1}{k}}} \\ & \leq \frac{c\sqrt{k}}{k} \frac{1}{(\sqrt{1-x} + \frac{1}{k})^{-\varepsilon_1 - \frac{1}{2} + 2\delta} (\sqrt{1+x} + \frac{1}{k})^{-\varepsilon_2 - \frac{1}{2} + 2\delta}} \leq ck^{\max\{-\varepsilon_1, -\varepsilon_2\} - 1 + 2\delta}. \end{aligned}$$

To prove Proposition 1, we introduce the operator O which changes the basis in the Hilbert space in question:

$$O^{-1} M_q O = M_e, \quad (39)$$

where

$$O = [o_{ij}]_{i,j=0}^{\infty},$$

and

$$\hat{P}_j = \sum_{i=0}^{j+m} o_{ij} q_i. \quad (40)$$

□

Proof (of Proposition 1) Notice that Lemma 1 can be applied to A and M_e . Indeed, A is tridiagonal, and M_e is $2L + 1$ -diagonal. Furthermore, the entries of A and M_e are bounded (see (29) and (34) respectively). Thus it is enough to show that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left(\text{Tr}(\pi_n^q Q^l(A) \pi_n^q) - \text{Tr}(\pi_n^e (M_e)^l \pi_n^e) \right) = 0. \quad (41)$$

Considering (39), it is enough to prove that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left(\text{Tr}(\pi_n^q Q^l(A) \pi_n^q) - \text{Tr}(\pi_n^e O^{-1} Q^l(A) O \pi_n^e) \right) = 0. \quad (42)$$

To prove (42), we compute the diagonal elements of $\pi_n^e O^{-1} Q^l(A) O \pi_n^e$. According to (40), for $0 \leq i \leq n - 1$

$$i(O^{-1} Q^l(A) O)_i = \sum_{k=\max\{0, i-L\}}^{\infty} i(O^{-1} Q^l(A))_k O_{ki}$$

and considering the $2L + 1$ -diagonality of $Q^l(A)$

$$\begin{aligned} i(O^{-1} Q^l(A))_k &= \sum_{j=0}^{i+L} i O_j^{-1} j(Q^l(A))_k \\ &= \sum_{\max\{0, k-IL\} \leq j \leq \min\{i+L, k+IL\}} j O_{ji} j(Q^l(A))_k. \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{i=0}^{n-1} i(O^{-1} Q^l(A) O)_i &= \sum_{i=0}^{n-1} \sum_{k=0}^{i+L} \sum_{\max\{0, k-IL\} \leq j \leq \min\{i+L, k+IL\}} j O_{ji} j(Q^l(A))_k \\ &= \sum_{i=0}^{n-1} \sum_{k=0}^{i+L} o_{ki}^2 j(Q^l(A))_k + \sum_{i=0}^{n-1} \sum_{k=0}^{i+L} \sum_{\substack{\max\{0, k-IL\} \leq j \leq \min\{i+L, k+IL\} \\ j \neq k}} j O_{ji} j(Q^l(A))_k \\ &= \tilde{M}_n + H_n. \end{aligned} \quad (43)$$

Reversing the order of summation, we get

$$\begin{aligned} \tilde{M}_n &= \sum_{k=0}^L \sum_{i=0}^{n-1} o_{ki}^2 j(Q^l(A))_k + \sum_{k=L+1}^{n-1} j(Q^l(A))_k \sum_{i=k-L}^{n-1} o_{ki}^2 \\ &\quad + \sum_{k=n}^{n-1+L} j(Q^l(A))_k \sum_{i=k-L}^{n-1} o_{ki}^2 \\ &= E_{1,n} + M_n + E_{2,n}. \end{aligned}$$

Thus, the expression after the limit in (42) becomes

$$\begin{aligned} & \frac{1}{n} \sum_{k=0}^{n-1} \left({}_k(Q^l(A))_k - {}_k(O^{-1}Q^l(A)O)_k \right) \\ &= \frac{1}{n} \left(\sum_{k=0}^L {}_k(Q^l(A))_k + E_{1,n} + E_{2,n} \right) \\ &+ \frac{1}{n} \sum_{k=L+1}^{n-1} {}_k(Q^l(A))_k \left(1 - \sum_{i=k-L}^{n-1} o_{ki}^2 \right) + \frac{1}{n} H_n. \end{aligned} \quad (44)$$

According to (36), the entries ${}_i(Q^l(A))_k$ are bounded, and by orthonormality, we have

$$\sum_{i=k-L}^{n-1} o_{ki}^2 = 1. \quad (45)$$

Thus,

$$\frac{1}{n} \left(\sum_{k=0}^L {}_k(Q^l(A))_k + E_{1,n} + E_{2,n} \right) \leq \frac{K}{n},$$

where K depends on L and on the supremum of $|{}_i(Q^l(A))_k|$, but it is independent of n , that is the first term tends to zero, when n tends to infinity.

We can start the estimation of the second term of (44) in the same way. By the boundedness of the entries of $Q^l(A)$, and by (45), we have

$$\frac{1}{n} \sum_{k=L+1}^{n-1} {}_k(Q^l(A))_k \left(1 - \sum_{i=k-L}^{n-1} o_{ki}^2 \right) \leq C \frac{1}{n} \sum_{k=0}^{n-1} \sum_{i=n}^{\infty} o_{ki}^2. \quad (46)$$

Let $i > k$. Referring to (40), we have

$$o_{ki} = \int_{-1}^1 \hat{P}_i q_k W = \frac{1}{\lambda_i} \int_{-1}^1 \left(p \hat{P}_i'' + \hat{q} \hat{P}_i' + \hat{r} \hat{P}_i \right) q_k W, \quad (47)$$

where the differential equation of $\hat{P}_i$ was taken into consideration, cf. (4) and (6). By the assumption on α , β and (16), integrating by parts the first term, we have

$$\int_{-1}^1 p \hat{P}_i'' q_k W = - \int_{-1}^1 \hat{P}_i' (p q_k W)' = - \int_{-1}^1 \hat{P}_i' (\hat{q} q_k + p q_k') W, \quad (48)$$

because $p W' = (\hat{q} - p') W$, cf. (5).

So (47) and (48) imply that

$$o_{ki} = \frac{1}{\lambda_i} \int_{-1}^1 -p \hat{P}'_i q'_k W + \hat{r} \hat{P}_i q_k W = I_{ki} + I I_{ki}. \quad (49)$$

Considering (9), we get

$$p \hat{P}'_i = \frac{\lambda_i - \tilde{\lambda}}{\sigma_i} b p_i - \left(p w + q - p \frac{b'}{b} \right) \hat{P}_i. \quad (50)$$

By (50), we have

$$\begin{aligned} I_{ki} &= -\frac{\lambda_i - \tilde{\lambda}}{\sigma_i \lambda_i} \int_{-1}^1 b p_i q'_k W + \frac{1}{\lambda_i} \int_{-1}^1 \left(p w + q - p \frac{b'}{b} \right) \hat{P}_i q'_k W \\ &= I_{ki}^{(1)} + I_{ki}^{(2)}. \end{aligned} \quad (51)$$

By (16) $\left| p w + q - p \frac{b'}{b} \right|$ is bounded on $[-1, 1]$. Indeed, only the first term needs some investigation; recalling that $b w$ is a polynomial, notice that $p w = \frac{p}{b} b w$.

Thus, according to (37) and (38)

$$\begin{aligned} |I_{ki}^{(2)}| &\leq \frac{ck}{i^2} \left\| \hat{P}_k(x) \sqrt{W(x)} (1-x^2)^{\frac{1}{4}} \right\|_{\infty} \\ &\quad \times \int_{-1}^1 \frac{\sqrt{1-x}^{\alpha+\varepsilon_1} \sqrt{1+x}^{\beta+\varepsilon_2}}{(1-x^2)^{\frac{1}{4}} \left(\sqrt{1-x^2} + \frac{1}{k} \right)^{\frac{3}{2}} (\sqrt{1-x} + \frac{1}{k})^{\alpha+\varepsilon_1} (\sqrt{1+x} + \frac{1}{k})^{\beta+\varepsilon_2}} dx \\ &\leq c \frac{k}{i^2} J_k. \\ J_k &\leq c \int_{-1}^0 \frac{(\sqrt{1+x})^{\beta+\varepsilon_2+\frac{3}{2}-2\delta}}{(\sqrt{1+x} + \frac{1}{k})^{\beta+\varepsilon_2+\frac{3}{2}-2\delta+2\delta} (1+x)^{1-\delta}} \\ &\quad + c \int_0^1 \frac{(\sqrt{1-x})^{\alpha+\varepsilon_1+\frac{3}{2}-2\delta}}{(\sqrt{1-x} + \frac{1}{k})^{\alpha+\varepsilon_1+\frac{3}{2}-2\delta+2\delta} (1-x)^{1-\delta}} \leq c \frac{k^{2\delta}}{\delta}, \end{aligned}$$

where the last inequality fulfils if $\alpha + \varepsilon_1 + \frac{3}{2} - 2\delta \geq 0$ and $\beta + \varepsilon_2 + \frac{3}{2} - 2\delta \geq 0$. Let $\delta = ck^{-\frac{1}{4}}$. Then,

$$|I_{ki}^{(2)}| \leq c \frac{k^{\frac{5}{4}}}{i^2}, \quad (52)$$

provided that $\alpha + \varepsilon_1, \beta + \varepsilon_2 > -\frac{3}{2}$. For sake of simplicity, let us denote by $c(i) := -\frac{\lambda_i - \tilde{\lambda}}{\sigma_i \lambda_i}$. After simplification, we arrive at

$$I_{ki}^{(1)} = c(i) \int_{-1}^1 \frac{\tilde{p}}{b} q'_k p_i w^{(\alpha, \beta)}.$$

Recalling that $\frac{1}{b}$ is bounded on $[-1, 1]$, let S_{i-k-2} be its uniformly best approximating polynomial on $[-1, 1]$ of degree $i - k - 2$. Then, as its degree is less than i , it follows that $S_{i-k-2} \tilde{p} q'_k$ is orthogonal to p_i with respect to $w^{(\alpha, \beta)}$. Thus,

$$I_{ki}^{(1)} = c(i) \int_{-1}^1 \left(\frac{1}{b} - S_{i-k-2} \right) \tilde{p} q'_k p_i w^{(\alpha, \beta)}.$$

Taking into account, (11), (33) and (2), we get $c(i) \leq c \frac{1}{i}$. Since $\left(\frac{1}{b}\right)'$ is bounded on $[-1, 1]$ too, according to the classical Jackson's theorem, we have

$$\begin{aligned} |I_{ki}^{(1)}| &\leq \frac{c}{i(i-k)} \int_{-1}^1 \tilde{p} |q'_k| |p_i| w^{(\alpha, \beta)} \\ &\leq \frac{c}{i(i-k)} \left\| q'_k(x) \sqrt{W(x)} (1-x^2)^{\frac{3}{4}} \right\|_{\infty} \int_{-1}^1 |p_i(x)| \sqrt{W(x)} \frac{b(x)}{(1-x^2)^{\frac{3}{4}}} dx \\ &\leq c \frac{k}{i(i-k)} \int_{-1}^1 |p_i(x)| (1-x)^{\frac{\alpha}{2}-\frac{1}{4}} (1+x)^{\frac{\beta}{2}-\frac{1}{4}}, \end{aligned}$$

where the norm of q'_k is estimated by (37). Finally by [28, (7.34.1)], the last integral can be estimated by a constant independent of i , that is

$$|I_{ki}^{(1)}| \leq c \frac{k}{i(i-k)}. \quad (53)$$

To estimate II_{ki} , first we remark that $\hat{r}$ is bounded on $[-1, 1]$. Indeed, it is clear that we have to deal with only the endpoints. Recalling that $\frac{p}{b}$ is bounded on $[-1, 1]$ and considering (5), we see that $\hat{q}$ is bounded there too. As it is mentioned in the Remark of section 2, $\hat{P}_n(1) \neq 0$ that is due to (4) and (6) $\hat{r}$ must be bounded at 1. By the same chain of ideas, we can show that $\hat{r}$ is bounded at -1 . Thus, by the Cauchy-Schwarz inequality

$$|II_{ki}| \leq \frac{c}{i^2} \int_{-1}^1 |\hat{p}_i| \sqrt{W} |q_k| \sqrt{W} \leq \frac{c}{i^2}. \quad (54)$$

Since $\sum_{i=n}^{\infty} o_{ki}^2 \leq 1$, according to (52), (53) and (54), we have

$$\begin{aligned} \frac{1}{n} \sum_{k=0}^{n-1} \sum_{i=n}^{\infty} o_{ki}^2 &\leq c \frac{1}{n} \sum_{k=n^{\frac{3}{4}}}^{n-n^{\frac{3}{4}}} \sum_{i=n}^{\infty} o_{ki}^2 + O\left(\frac{1}{n^{\frac{1}{4}}}\right) \\ &\leq \frac{c}{n} \sum_{k=n^{\frac{3}{4}}}^{n-n^{\frac{3}{4}}} \sum_{i=n}^{\infty} \left(\frac{k^{\frac{5}{2}}}{i^4} + \frac{k^2}{i^2 n^{\frac{3}{2}}} \right) + O\left(\frac{1}{n^{\frac{1}{4}}}\right) = O\left(\frac{1}{\sqrt{n}}\right) + O\left(\frac{1}{n^{\frac{1}{4}}}\right), \quad (55) \end{aligned}$$

so the expression in (46) tends to zero.

Finally, we estimate the error term $\frac{1}{n} H_n$, cf. (43) and (44). Using the properties of $|_j(Q^l(A))_k|$ and o_{ki} again, it can be estimated as follows:

$$\begin{aligned} \frac{1}{n} |H_n| &\leq c \frac{1}{n} \sum_{i=0}^{(l-1)L} \sum_{k=0}^{i+L} \sum_{\substack{\max\{0, k-lL\} \leq j \leq i+L \\ j \neq k}} |o_{ji}| |o_{ki}| \\ &+ \frac{1}{n} \left| \sum_{i=(l-1)L+1}^{n-1} \sum_{k=0}^{i+L} \sum_{\substack{\max\{0, k-lL\} \leq j \leq k+lL \\ j \neq k}} o_{ji} o_{ki} \, {}_j(Q^l(A))_k \right| = \Sigma_1(n) + \Sigma_2(n). \end{aligned}$$

Certainly, $\lim_{n \rightarrow \infty} \Sigma_1(n) = 0$. Splitting $\Sigma_2(n)$ to two parts and changing the ordering of summation, we get

$$\begin{aligned} \Sigma_2(n) &\leq \frac{1}{n} \left| \sum_{i=(l-1)L+1}^{n-1} \sum_{k=0}^{lL-1} \sum_{\substack{0 \leq j \leq k+lL \\ j \neq k}} o_{ji} o_{ki} \, {}_j(Q^l(A))_k \right| \\ &+ \frac{1}{n} \left| \sum_{i=(l-1)L+1}^{n-1} \sum_{k=lL}^{i+L} \sum_{\substack{k-lL \leq j \leq k+lL \\ j \neq k}} o_{ji} o_{ki} \, {}_j(Q^l(A))_k \right| \\ &\leq \frac{1}{n} \sum_{k=0}^{lL-1} \sum_{\substack{0 \leq j \leq k+lL \\ j \neq k}} \left| {}_j(Q^l(A))_k \right| \sum_{i=(l-1)L+1}^{n-1} |o_{ji} o_{ki}| \\ &+ \left| \frac{1}{n} \sum_{k=lL}^{n-1+L} \sum_{\substack{k-L \leq j \leq k+lL \\ j \neq k}} {}_j(Q^l(A))_k \sum_{i=(l-1)L+1}^{n-1} o_{ji} o_{ki} \right| \\ &= \Sigma_{21}(n) + \Sigma_{22}(n). \end{aligned}$$

Again, $\lim_{n \rightarrow \infty} \Sigma_{21}(n) = 0$. According to (40) $o_{ki} = 0$ if $i \leq k - L$. Thus, by orthogonality $\sum_{i=(l-1)L+1}^{n-1} o_{ji} o_{ki} = \sum_{i=n}^{\infty} o_{ji} o_{ki}$. That is

$$\Sigma_{22}(n) \leq c \frac{1}{n} \sum_{k=lL}^{n-1+L} \sum_{i=n}^{\infty} |o_{ji} o_{ki}| \leq c \frac{1}{n} \sum_{k=lL}^{n-1+L} \max_{k-L \leq j \leq k+L} \sum_{i=n}^{\infty} o_{ji}^2.$$

We can proceed as in (55) again, and so $\lim_{n \rightarrow \infty} \Sigma_{22}(n) = 0$, which finishes the estimation of $\frac{1}{n} |H_n|$ and the proof of Proposition 1 as well. $\square$

5 Certain self-inversive polynomials

In this section, we use the multiplication operator and the infinite matrix M_e introduced in Sect. 3 to investigate certain self-inversive polynomials.

Let b be defined by (15), and $\tilde{b}$ by (16). Investigation of Sect. 3 was independent of the constant term of the polynomial $Q = \int^x \tilde{b}$. At first, as above, take $Q_0(x) := \int^x \tilde{b}$, with zero constant term. Recalling (25), we have $Q_0 \hat{P}_n = \sum_{k=-L}^L u_{n,k} \hat{P}_{n+k}$, and by (34) we get $\lim_{n \rightarrow \infty} u_{n,0} = U_0$, where U_0 depends on $\tilde{b}$, cf. (35). Rearranging the equation above, we have $(Q_0 - U_0) \hat{P}_n = \sum_{\substack{-L \leq k \leq L \\ k \neq 0}} u_{n,k} \hat{P}_{n+k} + (u_{n,0} - U_0) \hat{P}_n$; thus, in this section, we define

$$Q(x) := \int^x \tilde{b} - U_0,$$

where $\int^x \tilde{b}$ means the primitive function without any constant term. The operator M_e refers to this Q , that is, denoting by M_e^0 the operator defined by Q_0 , $M_e = M_e^0 - U_0 I$. Since the asymptotics of the elements of M_e can be described by (36) M_e can be decomposed to a bounded symmetric and a compact symmetric part again (cf. (27)), that is

$$M_e = M_{e,s} + M_{e,c}, \quad (56)$$

where

$$M_e = \begin{bmatrix} u_{0,0} - U_0 & u_{0,1} & \dots & \dots & u_{0,L} & 0 & 0 & \dots \\ u_{1,-1} & u_{1,0} - U_0 & \dots & \dots & u_{1,L-1} & u_{1,L} & 0 & \dots \\ \vdots & \vdots & \dots & \ddots & \vdots & \dots & \vdots & \dots \\ u_{L,-L} & u_{L,-L+1} & \dots & u_{L,0} - U_0 & \dots & \dots & u_{L,L} & 0 \\ 0 & u_{L+1,-L} & \dots & \vdots & \dots & \vdots & \dots & u_{L+1,L} \\ \vdots & 0 & \dots & u_{L+j,-L} & \dots & \vdots & \dots & \dots \end{bmatrix}$$

and

$$M_{e,s} = \begin{bmatrix} 0 & U_1 & \dots & U_L & 0 & 0 & \dots \\ U_1 & 0 & \dots & U_{L-1} & U_L & \dots & \\ \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \dots \\ U_L & \dots & 0 & \dots & \dots & U_L & \dots \\ 0 & U_L & \vdots & \dots & \vdots & \dots & U_L \\ \vdots & \dots & U_L & \dots & \vdots & \dots & \dots \end{bmatrix}. \quad (57)$$

Investigation of $M_{e,s}$ leads to the so-called self-inversive or palindrome polynomials, cf. [21, Lemma 12]. A polynomial P of degree n with real coefficients is self-inversive if $z^n P\left(\frac{1}{z}\right) = P(z)$. The location of zeros of self-inversive polynomials has been extensively studied, see e.g. [20,29], etc. Of course, the zeros of a self-inversive polynomial are symmetric with respect to the unit circle. One of the statements on the location of zeros is the next one (see eg. [29]): if $P_{2m}(z) = \sum_{k=0}^{2m} a_k z^k$ is self-inversive and $|a_m| > \sum_{\substack{0 \leq k \leq m \\ k \neq m}} |a_k|$, then P_{2m} has no zeros on the unit circle. In our special case we get something similar.

Let us recall that $\tilde{b}(x) = \sum_{k=0}^m d_k x^k$, cf. (35). Now define

$$P_{2L,\lambda}(z) = \sum_{k=1}^L U_k \left(z^{L+k} + z^{L-k} \right) - \lambda z^L,$$

where U_k depends on b see (36), and

$$\tilde{P}_{2L}(z) = P_{2L,\lambda}(z) + \lambda z^L.$$

Theorem 2 $P_{2L,\lambda}(z)$ has no zeros on the unit circle if and only if

$$\lambda \notin \left[2 \sum_{k=1}^L (-1)^k U_k, 2 \sum_{k=1}^L U_k \right] = \left[(-1)^L \tilde{P}_{2L}(-1), (-1)^L \tilde{P}_{2L}(1) \right].$$

Let us consider $M_{e,s}$ as an operator on l^2 and on the Hardy space

$$H^2 := \left\{ f(z) = \sum_{k=0}^{\infty} c_k z^k : \text{is holomorphic on } |z| < 1, \right. \\ \left. \lim_{r \rightarrow 1} f(re^{i\varphi}) = f(e^{i\varphi}) \text{ a.e. } \varphi \in (0, 2\pi] \right\}.$$

It is a Hilbert space under the norm $\|f\|^2 = \sum_{k=0}^{\infty} |c_k|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\varphi})|^2 d\varphi$.

Lemma 4 $M_{e,s} - \lambda I$ has a bounded inverse if and only if $P_{2L,\lambda}(z)$ has no zeros on the unit circle.

Proof With this interpretation if $f \in H^2$,

$$(M_{e,s}f)(z) = \sum_{k=1}^L U_k(z^k + z^{-k})f(z) - \sum_{k=1}^L U_k \sum_{j=0}^{k-1} \frac{f^{(j)}(0)}{j!} z^{j-k}.$$

Let $g \in H^2$ be arbitrary. Then considering the equation

$$(M_{e,s} - \lambda I)f = g$$

we have

$$\begin{aligned} f(z) &= \frac{z^L g(z) + \sum_{k=1}^L U_k \sum_{j=0}^{k-1} \frac{f^{(j)}(0)}{j!} z^{L+j-k}}{P_{2L,\lambda}(z)} \\ &= \frac{z^L g(z) + \sum_{j=0}^{L-1} \frac{f^{(j)}(0)}{j!} \sum_{k=j}^{L-1} U_{L-k+j} z^k}{P_{2L,\lambda}(z)}. \end{aligned} \quad (58)$$

By symmetry, counting with multiplicity, $P_{2L,\lambda}$ has l zeros in the unit disc and $2(L-l)$ on the unit circle. We can determine $f \in H^2$ if and only if these zeros can be compensated. That is if ξ_m are the zeros of the denominator (in the closed unit disc) of multiplicity k_m with $\sum_m k_m = 2L-l$, then it means a system of linear equations in the numerator. According to (58) this system looks like

$$\begin{bmatrix} C_0(\xi_1) & C_1(\xi_1) & \dots & C_{L-1}(\xi_1) \\ \vdots & \vdots & \dots & \vdots \\ C_0(\xi_2) & C_1(\xi_2) & \dots & C_{L-1}(\xi_2) \\ C'_0(\xi_2) & C'_1(\xi_2) & \dots & C'_{L-1}(\xi_2) \\ \vdots & \vdots & \dots & \vdots \\ C_0^{(k_2-1)}(\xi_2) & C_1^{(k_2-1)}(\xi_2) & \dots & C_{L-1}^{(k_2-1)}(\xi_2) \\ \vdots & \vdots & \dots & \vdots \\ c_{2L-l,0} & \dots & \dots & c_{2L-l,L-1} \end{bmatrix} \begin{bmatrix} f(0) \\ f'(0) \\ f''(0) \\ \vdots \\ f^{(L-1)}(0) \end{bmatrix} = \begin{bmatrix} h(\xi_1) \\ \vdots \\ h(\xi_2) \\ h'(\xi_2) \\ \vdots \\ h^{(k_2-1)}(\xi_2) \\ \vdots \end{bmatrix},$$

where $C_j(x) = \frac{1}{j!} \sum_{k=j}^{L-1} U_{L-k+j} x^k$, $j = 0, \dots, L-1$; $h(x) = -x^L g(x)$, the coefficient matrix is of $2L-l \times L$. We deal with the rank of the coefficient matrix. Let us denote by O_j , the j th column of the coefficient matrix ($j = 0; 1; \dots; L-1$). The first step is

$$O_j - \frac{(L-1)!}{j!} \frac{U_{j+1}}{U_L} O_{L-1}, \quad \text{for } j = 0, 1, \dots, L-2.$$

After the first step, the sums in the entries of the new columns $O_j^{(1)}$, are shorter than they were in O_j for $j = 0, 1, \dots, L-2$. Repeating this procedure we finally arrive

at a coefficient matrix which inherits the structure of the original one. Its entries can be calculated by the new functions $\tilde{C}_j(x) = a_j x^j$.

Thus, the rank of the coefficient matrix coincides with its less size. That is to get a unique solution, it must be of $L \times L$ which means that $2L - l = L$, i.e. $l = L$ which means that $P_{2L,\lambda}$ has L zeros inside the unite disc. The unique solution of this linear system is equivalent to the existence of a (well-defined) bounded inverse of the operator. Indeed, after determination of the first L coefficients of f , the remainder ones can be uniquely determined. $\square$

Lemma 5 *The spectrum of $M_{e,s}$ is $Q([-1, 1])$.*

Proof Let us recall the information on $M_{e,s}$. According to Weyl's theorem (see e.g. [27, Sect. 134]), the essential spectrum of $M_{e,s}$ agrees with the essential spectrum of M_e . Recalling that the spectrum of M_e is the closure of $Q([-1, 1])$, the essential spectrum and the spectrum of M_e are the same. As the spectrum of $M_{e,s}$ does not contains any isolated points as well, see [12], it also coincides with $Q([-1, 1])$.

The fact that the essential spectrum and the spectrum of $M_{e,s}$ coincide in this setup can be proved as follows.

Let us consider $P_{2L,\lambda}(r, \varphi)$ ($z = (r \cos \varphi, r \sin \varphi)$) as a function from $\mathbb{R}^{1+2}$ to $\mathbb{R}^2$. If there was an isolated point of $\sigma(M_{e,s})$, then there would be a $(\lambda_0, r_0, \varphi_0)$, $r_0 = 1$, such that $P_{2L,\lambda_0}(r_0, \varphi_0) = 0$ and there would be a neighbourhood of λ_0 such that for any λ from this neighbourhood, the zeros of $P_{2L,\lambda}$ are not on the unit circle.

So $P_{2L,\lambda}(r, \varphi) = (p_1(\lambda, r, \varphi), p_2(\lambda, r, \varphi))$. Let us consider $\partial_2 P_{2L,\lambda} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, where $a_{i1} = \frac{\partial}{\partial r} p_i$, $a_{i2} = \frac{\partial}{\partial \varphi} p_i$, $i = 1, 2$. As $\det \partial_2 P_{2L,\lambda} = \frac{1}{r}(a_{22}^2 + a_{12}^2)$, we can apply the implicit function theorem at $(\lambda_0, r_0, \varphi_0)$ that is there is a neighbourhood of λ_0 denoted by U and an arc g in U such that if $\lambda \in U$ then $P_{2L}(\lambda, g(\lambda)) = P_{2L,\lambda}(r(\lambda), \varphi(\lambda)) = 0$. If $\lambda \in U$ $g'(\lambda) = [r'(\lambda)\varphi'(\lambda)]$, where $r'(\lambda) = \frac{r^{2L+1}}{a_{22}^2 + a_{12}^2} \sum_{k=1}^L k U_k \cos k\varphi(r^k - r^{-k})$ (that is $r'(\lambda_0) = 0$), and $\varphi'(\lambda) = \frac{-r^{2L}}{a_{22}^2 + a_{12}^2} \sum_{k=1}^L k U_k \sin k\varphi(r^k + r^{-k})$. Thus, the slope of the tangent line at λ_0 to g is $-\cot(\varphi(\lambda_0))$, which is just the the slope of the tangent line to the unit circle at the same point, and the curvature of g at λ_0 is 1. Considering the symmetry of the zeros with respect to the unite circle g has to coincide locally with the unit circle, which means that λ_0 cannot be isolated. $\square$

Proof (of Theorem 2) Notice, that $b \geq 0$ on $[-1, 1]$. Thus Q is increasing here and $Q(-1) < Q(1)$. That is to prove the statement it is enough to compute these two values. In view of (36)

$$\begin{aligned} & 2 \sum_{k=1}^L (\pm 1)^k U_k \\ &= 2 \sum_{l=1}^{\lfloor \frac{L}{2} \rfloor} \sum_{p=l}^{\lfloor \frac{L}{2} \rfloor} \frac{d_{2p-1}}{2p} \binom{2p}{p-l} \frac{1}{2^{2p}} \pm 2 \sum_{l=0}^{\lfloor \frac{L-1}{2} \rfloor} \sum_{p=l}^{\lfloor \frac{L-1}{2} \rfloor} \frac{d_{2p}}{2p+1} \binom{2p+1}{p-l} \frac{1}{2^{2p+1}} =: S. \end{aligned}$$

Changing the order of summation and by the definition of Q

$$S = \sum_{p=l}^{\left[\frac{L}{2}\right]} \frac{d_{2p-1}}{2p} \left(1 - \frac{1}{2^{2p}} \binom{2p}{p}\right) \pm \sum_{p=l}^{\left[\frac{L-1}{2}\right]} \frac{d_{2p}}{2p+1} = \sum_{k=1}^L (\pm 1)^k \frac{d_{k-1}}{k} - U_0 = Q(\pm 1).$$

Thus, the spectrum of the operator $[M]_{e,s}$ is $\left[\sum_{k=1}^L (-1)^k U_k, 2 \sum_{k=1}^L U_k\right]$, which together with Lemma 4 proves the statement. $\square$

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