



Simultaneous approximations to p -adic numbers and algebraic dependence via multidimensional continued fractions

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Abstract

Unlike the real case, there are not many studies and general techniques for providing simultaneous approximations in the field of p -adic numbers \mathbb{Q}_p . Here, we study the use of multidimensional continued fractions (MCFs) in this context. MCFs were introduced in \mathbb{R} by Jacobi and Perron as a generalization of continued fractions and they have been recently defined also in \mathbb{Q}_p . We focus on the dimension two and study the quality of the simultaneous approximation to two p -adic numbers provided by p -adic MCFs, where p is an odd prime. Moreover, given algebraically dependent p -adic numbers, we see when infinitely many simultaneous approximations satisfy the same algebraic relation. This also allows to give a condition that ensures the finiteness of the p -adic Jacobi–Perron algorithm when it processes some kinds of \mathbb{Q} -linearly dependent inputs.

Keywords Jacobi–Perron algorithm · Multidimensional continued fractions · p -Adic numbers · Simultaneous approximations

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1 Introduction

Continued fractions give a representation for any real number by means of a sequence of integers, providing along the way rational approximations. In particular, they pro-

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vide best approximations, i.e., the n th convergent of the continued fraction of a real number is closer to it than any other rational number with a smaller or equal denominator. Multidimensional continued fractions (MCFs) are a generalization of classical continued fractions introduced by Jacobi [22] and Perron [34] in an attempt to answer a question posed by Hermite about a possible generalization of the Lagrange theorem for continued fractions to other algebraic irrationalities. A MCF is a representation of a m -tuple of real numbers $(\alpha_0^{(1)}, \dots, \alpha_0^{(m)})$ by means of m sequences of integers $((a_n^{(1)})_{n \geq 0}, \dots, (a_n^{(m)})_{n \geq 0})$ (finite or infinite) obtained by the Jacobi–Perron algorithm:

$$\begin{cases} a_n^{(i)} = [\alpha_n^{(i)}], & i = 1, \dots, m, \\ \alpha_{n+1}^{(1)} = \frac{1}{\alpha_n^{(m)} - a_n^{(m)}}, & n = 0, 1, 2, \dots, \\ \alpha_{n+1}^{(i)} = \frac{\alpha_n^{(i-1)} - a_n^{(i-1)}}{\alpha_n^{(m)} - a_n^{(m)}}, & i = 2, \dots, m, \end{cases} \quad (1)$$

We shall write

$$(\alpha_0^{(1)}, \dots, \alpha_0^{(m)}) = [(a_0^{(1)}, a_1^{(1)}, \dots), \dots, (a_0^{(m)}, a_1^{(m)}, \dots)].$$

The Jacobi–Perron algorithm has been widely studied concerning its periodicity and approximation properties. For instance, in [8,9,26,35] the authors provided some classes of algebraic irrationalities whose expansion by the Jacobi–Perron algorithm becomes eventually periodic. In [30], a criterion of periodicity, involving linear recurrence sequences, is given. The periodicity of the Jacobi–Perron algorithm is also related to the study of Pisot numbers [19,20]. Further studies on MCFs can be found in [2,16,29,41].

Continued fractions for p -adic numbers were introduced in three different ways [10, 36,37] and subsequently studied by several authors like [7,11,13,21]. More recently they have been generalized to higher dimensions. In [31], the authors studied the fundamental properties of MCFs in \mathbb{Q}_p , focusing on convergence properties and finite expansions, whereas in [32] further properties regarding finiteness and periodicity of the p -adic Jacobi–Perron algorithm have been proved.

The study of simultaneous approximations of real numbers is a very important topic in Diophantine approximation; classical and fundamental results can be found in [3,15,17,18,23]. Also in the p -adic setting simultaneous approximation has been investigated, e.g., in [1,27,39]. Some results can also be found regarding simultaneous approximations in \mathbb{Q}_p , involving a p -adic number and its integral powers [12,28]. Specific results regarding the case of a p -adic number and its square are investigated in [6,40]. However, there are no applications of MCFs for providing simultaneous approximations of p -adic numbers and for studying the quality of such approximations.

MCFs have been deeply studied in this context for the real case, since they provide simultaneous rational approximations to real numbers. The quality of these simultaneous approximations has been studied in several works, such as [5,14,24,33,38],

thus, it seems natural to exploit MCFs in \mathbb{Q}_p for approaching the problem of constructing simultaneous approximations to p -adic numbers. In this paper, we give a first study in this direction and we also investigate the relation between simultaneous approximations and algebraic dependence.

The paper is structured as follows. In Sect. 2, we introduce the notation and we give some basic definitions and properties. Section 3 is devoted to the study of the quality of the simultaneous approximations provided by p -adic MCFs. Finally, in Sect. 4, we focus on algebraically dependent pairs of p -adic numbers; firstly we find a condition on the quality of approximation under which a sequence of simultaneous rational approximations satisfies the same algebraic relation. Secondly, we apply this result to MCFs and deduce a condition that ensures the finiteness of the p -adic Jacobi–Perron algorithm when it processes some kinds of \mathbb{Q} -linearly dependent inputs.

2 Definitions and useful properties

In the following, we will focus on p -adic MCFs of dimension 2, i.e., using the notation of the previous section, we set $m = 2$. Most of the results obtained in this paper can be adapted to any dimension $m \geq 2$, but in the general case the notation is very annoying and possibly confusing. Hence, we now recall the p -adic Jacobi–Perron algorithm for the case $m = 2$; for more details see [31]. From now on, p will be an odd prime number.

Definition 1 The *Browkin s -function* $s : \mathbb{Q}_p \longrightarrow \mathcal{Y} = \mathbb{Z} \left[\frac{1}{p} \right] \cap \left(-\frac{p}{2}, \frac{p}{2} \right)$ is defined by

$$s(\alpha) = \sum_{j=k}^0 x_j p^j,$$

with $\alpha \in \mathbb{Q}_p$ written as $\alpha = \sum_{j=k}^{\infty} x_j p^j, k \in \mathbb{Z}$, and $x_j \in \mathbb{Z} \cap \left(-\frac{p}{2}, \frac{p}{2} \right)$.

Given $\alpha, \beta \in \mathbb{Q}_p$, we get the corresponding MCF $(\alpha, \beta) = [(a_0, a_1, \dots), (b_0, b_1, \dots)]$ by the following iterative equations:

$$\begin{cases} a_k = s(\alpha_k), \\ b_k = s(\beta_k), \\ \alpha_{k+1} = \frac{1}{\beta_k - b_k}, \\ \beta_{k+1} = \frac{\alpha_k - a_k}{\beta_k - b_k}, \end{cases}$$

for $k = 0, 1, \dots$, with $\alpha_0 = \alpha$ and $\beta_0 = \beta$. Thus, the complete quotients satisfy the following relations:

$$\alpha_k = a_k + \frac{\beta_{k+1}}{\alpha_{k+1}}, \quad \beta_k = b_k + \frac{1}{\alpha_{k+1}}$$

and, if the algorithm does not stop, then the initial values are represented by the following MCF:

$$\alpha = a_0 + \frac{1}{b_1 + \frac{1}{a_2 + \frac{1}{b_3 + \frac{1}{a_3 + \frac{1}{\ddots}}}}} \quad \text{and} \quad \beta = b_0 + \frac{1}{a_1 + \frac{1}{b_2 + \frac{1}{a_3 + \frac{1}{b_3 + \frac{1}{a_2 + \frac{1}{b_3 + \frac{1}{\ddots}}}}}}}$$

We define the sequences of integers $(A_k)_{k \geq -2}, (B_k)_{k \geq -2}, (C_k)_{k \geq -2}$ of the numerators and denominators of the convergents, i.e.,

$$[(a_0, \dots, a_n), (b_0, \dots, b_n)] = \left(\frac{A_n}{C_n}, \frac{B_n}{C_n} \right) = (Q_n^\alpha, Q_n^\beta) \tag{2}$$

as follows:

$$\begin{cases} A_{-2} = 0, & A_{-1} = 1, & A_0 = a_0, \\ B_{-2} = 1, & B_{-1} = 0, & B_0 = b_0, \\ C_{-2} = 0, & C_{-1} = 0, & C_0 = 1, \end{cases} \quad \begin{cases} A_n = a_n A_{n-1} + b_n A_{n-2} + A_{n-3}, \\ B_n = a_n B_{n-1} + b_n B_{n-2} + B_{n-3}, \\ C_n = a_n C_{n-1} + b_n C_{n-2} + C_{n-3}, \end{cases} \tag{3}$$

for any $n \geq 1$. Then

$$\prod_{k=0}^n \begin{pmatrix} a_k & 1 & 0 \\ b_k & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} A_n & A_{n-1} & A_{n-2} \\ B_n & B_{n-1} & B_{n-2} \\ C_n & C_{n-1} & C_{n-2} \end{pmatrix} \tag{4}$$

for any $n \geq 0$.

We define the sequences $(\tilde{A}_k)_{k \geq -1} = (A_k C_{k-1} - A_{k-1} C_k)$ and $(\tilde{B}_k)_{k \geq -1} = (B_k C_{k-1} - B_{k-1} C_k)$, arising from the difference between two consecutive conver-

gents:

$$Q_n^\alpha - Q_{n-1}^\alpha = \frac{\tilde{A}_n}{C_n C_{n-1}}, \quad Q_n^\beta - Q_{n-1}^\beta = \frac{\tilde{B}_n}{C_n C_{n-1}}.$$

The following relations hold true:

$$\begin{cases} \tilde{A}_{-1} = 0, & \tilde{A}_0 = -1, & \tilde{A}_1 = b_1, & \begin{cases} \tilde{A}_n = -b_n \tilde{A}_{n-1} - a_{n-1} \tilde{A}_{n-2} + \tilde{A}_{n-3}, \\ \tilde{B}_n = -b_n \tilde{B}_{n-1} - a_{n-1} \tilde{B}_{n-2} + \tilde{B}_{n-3}, \end{cases} \\ \tilde{B}_{-1} = 0, & \tilde{B}_0 = 0, & \tilde{B}_1 = 1, & \end{cases} \tag{5}$$

for any $n \geq 2$. Indeed,

$$\begin{aligned} \tilde{A}_n &= A_n C_{n-1} - A_{n-1} C_n \\ &= (a_n A_{n-1} + b_n A_{n-2} + A_{n-3}) C_{n-1} - A_{n-1} (a_n C_{n-1} + b_n C_{n-2} + C_{n-3}) \\ &= -b_n (A_{n-1} C_{n-2} - A_{n-2} C_{n-1}) + A_{n-3} (a_{n-1} C_{n-2} + b_{n-1} C_{n-3} + C_{n-4}) \\ &\quad - C_{n-3} (a_{n-1} A_{n-2} + b_{n-1} A_{n-3} + A_{n-4}) \\ &= -b_n \tilde{A}_{n-1} - a_{n-1} \tilde{A}_{n-2} + \tilde{A}_{n-3}; \end{aligned}$$

similarly for the \tilde{B}_k s. From (4), we have

$$\begin{aligned} 1 &= \det \begin{pmatrix} A_n & A_{n-1} & A_{n-2} \\ B_n & B_{n-1} & B_{n-2} \\ C_n & C_{n-1} & C_{n-2} \end{pmatrix} \\ &= A_n B_{n-1} C_{n-2} - A_{n-1} B_n C_{n-2} - A_n B_{n-2} C_{n-1} + A_{n-2} B_n C_{n-1} \\ &\quad + A_{n-1} B_{n-2} C_n - A_{n-2} B_{n-1} C_n, \end{aligned}$$

from which

$$\frac{1}{C_n C_{n-1} C_{n-2}} = (Q_n^\alpha - Q_{n-1}^\alpha)(Q_n^\beta - Q_{n-2}^\beta) - (Q_n^\beta - Q_{n-1}^\beta)(Q_n^\alpha - Q_{n-2}^\alpha).$$

Moreover, since

$$\alpha = \frac{\alpha_n A_{n-1} + \beta_n A_{n-2} + A_{n-3}}{\alpha_n C_{n-1} + \beta_n C_{n-2} + C_{n-3}} \quad \text{and} \quad \beta = \frac{\alpha_n B_{n-1} + \beta_n B_{n-2} + B_{n-3}}{\alpha_n C_{n-1} + \beta_n C_{n-2} + C_{n-3}},$$

we have

$$\begin{aligned} &(\alpha - Q_{n-1}^\alpha)(\beta - Q_{n-2}^\beta) - (\beta - Q_{n-1}^\beta)(\alpha - Q_{n-2}^\alpha) \\ &= \frac{1}{C_{n-1} C_{n-2} (\alpha_n C_{n-1} + \beta_n C_{n-2} + C_{n-3})}. \end{aligned} \tag{6}$$

Let us observe that the previous properties hold for general MCFs, while we now give some specific results regarding only p -adic MCFs. In the following we will use

$v_p(\cdot)$ for the p -adic valuation, $|\cdot|_p$ for the p -adic norm and $|\cdot|_\infty$ for the Euclidean norm. Moreover, we define

$$h_n = v_p\left(\frac{b_n}{a_n}\right), \quad k_n = v_p\left(\frac{1}{a_n}\right), \quad K_n = k_1 + \dots + k_n$$

for any $n \geq 1$ and the sequences $(V_k^\alpha)_{k \geq -2} = (C_k\alpha - A_k)_{k \geq -2}$, $(V_k^\beta)_{k \geq -2} = (C_k\beta - B_k)_{k \geq -2}$. We recall from [31] the following properties:

- $|a_n|_p > 1$ and $|b_n|_p < |a_n|_p$, for any $n \geq 1$;
- $|a_n|_p = |\alpha_n|_p$, $|b_n|_p = \begin{cases} |\beta_n|_p & \text{if } |\beta_n|_p \geq 1 \\ 0 & \text{if } |\beta_n|_p < 1 \end{cases}$, for any $n \geq 1$;
- $v_p(C_n) = -K_n$, for any $n \geq 1$;
- $\lim_{n \rightarrow +\infty} |V_n^\alpha|_p = \lim_{n \rightarrow +\infty} |V_n^\beta|_p = 0$.

3 The quality of the approximations of p -adic MCFs

In this section, we investigate how well the convergents of a bidimensional continued fraction approach their limit in \mathbb{Q}_p .

3.1 The rate of convergence

In the first instance, we give some results about the rate of convergence of the real sequences $|V_n^\alpha|_p$ and $|V_n^\beta|_p$.

Theorem 1 *Let $[(a_0, a_1, \dots), (b_0, b_1, \dots)]$ be the p -adic MCF expansion of $(\alpha, \beta) \in \mathbb{Q}_p^2$, then*

$$v_p(\alpha - Q_n^\alpha) \geq K_n + \left\lfloor \frac{n+2}{2} \right\rfloor, \quad v_p(\beta - Q_n^\beta) \geq K_n + \left\lfloor \frac{n+3}{2} \right\rfloor,$$

for $n \geq -2$

Proof We will prove by induction that

$$v_p(Q_{n+1}^\alpha - Q_n^\alpha) = v_p\left(\frac{\tilde{A}_{n+1}}{C_{n+1}C_n}\right) \geq -v_p(C_n) + \left\lfloor \frac{n+2}{2} \right\rfloor,$$

i.e., we have to prove that

$$v_p(\tilde{A}_{n+1}) \geq v_p(C_{n+1}) + \left\lfloor \frac{n+2}{2} \right\rfloor$$

for any $n \geq -2$. We can observe that

$$v_p(\tilde{A}_{-1}) = v_p(C_{-1}) = \infty,$$

for $n = -2$, and

$$v_p(\tilde{A}_0) = v_p(\tilde{C}_0) = 0,$$

for $n = -1$. Moreover,

$$v_p(\tilde{A}_1) = v_p(b_1), \quad v_p(C_1) = v_p(a_1),$$

for $n = 0$, and we know that $v_p(b_1) > v_p(a_1)$. Now, we proceed by induction, supposing that it is true $v_p(\tilde{A}_n) \geq v_p(C_n) + \lfloor \frac{n+1}{2} \rfloor$. Consider

$$\begin{aligned} v_p(\tilde{A}_{n+1}) &= v_p(-b_{n+1}\tilde{A}_n - a_n\tilde{A}_{n-1} + \tilde{A}_{n-2}) \\ &\geq \inf\{v_p(b_{n+1}\tilde{A}_n), v_p(a_n\tilde{A}_{n-1}), v_p(\tilde{A}_{n-2})\}, \end{aligned}$$

by the inductive hypothesis we have

$$\begin{aligned} v_p(b_{n+1}\tilde{A}_n) &\geq v_p(b_{n+1}) + v_p(C_n) + \left\lfloor \frac{n+1}{2} \right\rfloor \\ &\geq v_p(a_{n+1}) + 1 + v_p(C_n) + \left\lfloor \frac{n+1}{2} \right\rfloor \geq v_p(C_{n+1}) + \left\lfloor \frac{n+2}{2} \right\rfloor. \end{aligned}$$

Similarly,

$$\begin{aligned} v_p(a_n\tilde{A}_{n-1}) &\geq v_p(a_n) + v_p(C_{n-1}) + \left\lfloor \frac{n}{2} \right\rfloor \geq v_p(C_{n+1}) + \left\lfloor \frac{n}{2} \right\rfloor + 1 \\ &= v_p(C_{n+1}) + \left\lfloor \frac{n+2}{2} \right\rfloor \end{aligned}$$

and

$$\begin{aligned} v_p(\tilde{A}_{n-2}) &\geq v_p(C_{n-2}) + \left\lfloor \frac{n-1}{2} \right\rfloor \geq v_p(C_{n+1}) + \left\lfloor \frac{n-1}{2} \right\rfloor + 3 \\ &\geq v_p(C_{n+1}) + \left\lfloor \frac{n+2}{2} \right\rfloor. \end{aligned}$$

Thus, we also have $v_p(Q_{n+k}^\alpha - Q_n^\alpha) \geq -v_p(C_n) + \lfloor \frac{n+2}{2} \rfloor$ and for $k \rightarrow \infty$, we have $v_p(\alpha - Q_n^\alpha) \geq -v_p(C_n) + \lfloor \frac{n+2}{2} \rfloor$.

Similar arguments hold for proving $v_p(Q_{n+1}^\beta - Q_n^\beta) \geq -v_p(C_n) + \lfloor \frac{n+3}{2} \rfloor$, i.e., for proving $v_p(\tilde{B}_{n+1}) \geq v_p(C_{n+1}) + \left\lfloor \frac{n+3}{2} \right\rfloor$. We just check the basis of the induction:

$$v_p(\tilde{B}_{-1}) = \infty, \quad v_p(\tilde{B}_0) = \infty, \quad v_p(\tilde{B}_1) = 0$$

and

$$v_p(C_{-1}) = \infty, \quad v_p(C_0) = 0, \quad v_p(C_1) = v_p(a_1) < 0.$$

□

Corollary 1 *Let $[(a_0, a_1, \dots), (b_0, b_1, \dots)]$ be the p -adic MCF expansion of $(\alpha, \beta) \in \mathbb{Q}_p^2$, then*

$$v_p(V_n^\alpha) \geq \left\lfloor \frac{n+2}{2} \right\rfloor, \quad v_p(V_n^\beta) \geq \left\lfloor \frac{n+3}{2} \right\rfloor,$$

so that

$$\min\{v_p(V_n^\alpha), v_p(V_n^\beta)\} \geq \left\lfloor \frac{n+2}{2} \right\rfloor = \lfloor \frac{n}{2} \rfloor + 1.$$

Remark 1 In the real case, given $(\alpha, \beta) = [(a_0, a_1, \dots), (b_0, b_1, \dots)]$ it is well known that

$$\left| \alpha - \frac{A_n}{C_n} \right|_\infty < \frac{1}{|C_n|_\infty}, \quad \left| \beta - \frac{B_n}{C_n} \right|_\infty < \frac{1}{|C_n|_\infty}.$$

In the p -adic case, a stronger result holds, indeed from the previous theorem we have

$$\left| \alpha - \frac{A_n}{C_n} \right|_p < \frac{1}{k|C_n|_p}, \quad \left| \beta - \frac{B_n}{C_n} \right|_p < \frac{1}{k|C_n|_p}$$

where $k = k(n)$ tends to infinity.

On the other hand formula (6) implies

$$v_p(V_{n-1}^\alpha V_{n-2}^\beta - V_{n-1}^\beta V_{n-2}^\alpha) = K_n,$$

which provides an upper bound for the p -adic valuation of the V_n 's, namely

$$\min\{v_p(V_n^\alpha), v_p(V_n^\beta)\} + \min\{v_p(V_{n-1}^\alpha), v_p(V_{n-1}^\beta)\} \leq K_{n+1}. \tag{7}$$

This shows that the lower bound for $\min\{v_p(V_n^\alpha), v_p(V_n^\beta)\}$ provided by Corollary 1 is optimal, in the sense that it is reached in some cases:

Example 1 Consider an infinite MCF such that $v_p(a_n) = -1$ for every $n \geq 1$. Then $K_n = n$ for every n , so that by Corollary 1 and formula (7) we get

$$\min\{v_p(V_n^\alpha), v_p(V_n^\beta)\} + \min\{v_p(V_{n-1}^\alpha), v_p(V_{n-1}^\beta)\} = n + 1$$

so that $\min\{v_p(V_n^\alpha), v_p(V_n^\beta)\} = \left\lfloor \frac{n}{2} \right\rfloor + 1$ for every $n \geq 1$.

However, in many other cases the bound provided by Corollary 1 can be improved, as stated by the following propositions:

Proposition 1 Let $(\ell_n)_{n \geq 0}$ be a sequence of natural numbers > 0 ; put $\ell_{-1} = \ell_{-2} = 0$ and define $f(n) = \sum_{j=0}^n \ell_j$. Let $[(a_0, a_1, \dots), (b_0, b_1, \dots)]$ be an infinite p -adic MCF satisfying $h_{n+1} \geq \ell_n, k_{n+1} \geq \ell_n + \ell_{n-1}$ for $n \geq 0$. Then for every $n \in \mathbb{N}$

$$\min\{v_p(V_n^\alpha), v_p(V_n^\beta)\} \geq f(n).$$

Proof For $n \geq 1$, either $v_p(\beta_n) > 0, b_n = 0, v_p\left(\frac{\beta_n}{\alpha_n}\right) > k_n$ or $v_p(\beta_n) = v_p(b_n) \leq 0, v_p\left(\frac{\beta_n}{\alpha_n}\right) = v_p\left(\frac{b_n}{a_n}\right) = h_n$. In any case $v_p\left(\frac{\beta_{n+1}}{\alpha_{n+1}}\right) \geq \ell_n$, for $n \geq 0$. Let V_n be either V_n^α or V_n^β . From the formula

$$V_n = -\frac{\beta_{n+1}}{\alpha_{n+1}}V_{n-1} - \frac{1}{\alpha_{n+1}}V_{n-2}, \tag{8}$$

we get for $n \geq 0$

$$\frac{V_n}{p^{f(n)}} = \mu_n \frac{V_{n-1}}{p^{f(n-1)}} + \nu_n \frac{V_{n-2}}{p^{f(n-2)}}$$

where

$$\mu_n = -\frac{\beta_{n+1}}{\alpha_{n+1}} \cdot \frac{1}{p^{\ell_n}}, \quad \nu_n = -\frac{1}{\alpha_{n+1}} \cdot \frac{1}{p^{\ell_n + \ell_{n-1}}} \in \mathbb{Z}_p.$$

Since $V_{-1}, V_{-2} \in \mathbb{Z}_p$ we obtain by induction $\frac{V_n}{p^{f(n)}} \in \mathbb{Z}_p$. □

Corollary 2 Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be any function. There are infinitely many $(\alpha, \beta) \in \mathbb{Q}_p^2$ satisfying

$$\min\{v_p(V_n^\alpha), v_p(V_n^\beta)\} \geq f(n).$$

Proof Of course we can assume $f(n)$ strictly increasing, so that $f(n) = \sum_{j=0}^n \ell_j$ with $\ell_n \in \mathbb{N}, \ell_n > 0$; the proof follows from Proposition 1 by observing that there are infinitely many p -adic MCF satisfying $h_{n+1} \geq \ell_n, k_{n+1} \geq \ell_n + \ell_{n-1}$ for $n \geq 0$. □

We would like to investigate in which sense and to which extent the approximations given by p -adic convergents may be considered “good approximations.” Observe that the Browking s -function is locally constant, hence so is the function $\mathbb{Q}_p^2 \rightarrow \mathbb{Q}^2$ associating to a pair (α, β) of its n th convergents (Q_n^α, Q_n^β) (where this function is defined). Therefore every $(\alpha, \beta) \in \mathbb{Q}_p^2$ having a MCF of length $\geq n$ has a neighborhood U such that every $(\alpha', \beta') \in U$ has the same k -convergents as (α, β) for $k \leq n$. The following proposition will provide an explicit radius for this neighborhood.

Proposition 2 *Let $(\alpha, \beta) \in \mathbb{Q}_p^2$ be such that the associated MCF $[(a_0, a_1, \dots), (b_0, b_1, \dots)]$ has length $\geq n$. Let $(\alpha', \beta') \in \mathbb{Q}_p^2$. If $\max\{|\alpha - \alpha'|_p, |\beta - \beta'|_p\} < \frac{1}{p^{2K_n}}$, then the MCF $[(a'_0, a'_1, \dots), (b'_0, b'_1, \dots)]$ associated to (α', β') has length $\geq n$ and $a_i = a'_i, b_i = b'_i$, for $i = 0, \dots, n$.*

Proof Notice that $\frac{1}{p^{2K_n}} = \frac{1}{|C_n|_p^2}$. We prove the thesis by induction on n . The claim is certainly true for $n = 0$, since in general

$$|x - y|_p < 1 \Leftrightarrow s(x) = s(y).$$

Suppose now $n \geq 1$, and $\max\{|\alpha - \alpha'|_p, |\beta - \beta'|_p\} < \frac{1}{|C_{n+1}|_p^2}$. By the case $n = 0$ we have $a'_0 = a_0, b'_0 = b_0$. Moreover, we observe that our hypothesis implies $|\beta - \beta'| < \frac{1}{|a_1|_p} = |\beta - b_0|_p$. By the properties of the non-Archimedean norm, we have

$$\frac{1}{|a'_1|_p} = |\beta' - b_0|_p = \max\{|\beta' - \beta|_p, |\beta - b_0|_p\} = |\beta - b_0|_p = \frac{1}{|a_1|_p},$$

so that $|a_1|_p = |a'_1|_p$. We have

$$|\alpha_1 - \alpha'_1| = \left| \frac{1}{\beta - b_0} - \frac{1}{\beta' - b_0} \right|_p = |a_1|_p^2 |\beta - \beta'|_p < \prod_{j=2}^{n+1} \frac{1}{|a_j|_p^2} = \frac{1}{|C_n^{(1)}|_p^2}, \tag{9}$$

where $C_n^{(1)}$ is the n th denominator of the convergents of the MCF expansion of (α_1, β_1) . Moreover,

$$\begin{aligned} |\beta_1 - \beta'_1|_p &= |\alpha_1(\alpha - a_0) - \alpha'_1(\alpha' - a_0)|_p = |(\alpha - a_0)(\alpha_1 - \alpha'_1) + \alpha'_1(\alpha - \alpha')|_p \\ &\leq \max\{ |(\alpha - a_0)(\alpha_1 - \alpha'_1)|, |a_1|_p |(\alpha - \alpha')|_p \} < \frac{1}{|C_n^{(1)}|_p^2}. \end{aligned} \tag{10}$$

Thus, by inductive hypothesis we have $a_i = a'_i, b_i = b'_i$ for $i = 1, \dots, n + 1$. □

Unfortunately, in general the pair (Q_n^α, Q_n^β) does not lie in the p -adic ball centered in (α, β) and having radius $\frac{1}{p^{2K_n}}$, as Example 1 shows. The next proposition gives a constructive sufficient condition ensuring this property.

Proposition 3 Consider an infinite MCF such that $k_{n+1} > k_n + k_{n-1}$ and $h_n > k_{n-1}$ for $n \geq 2$. Then for every $n \in \mathbb{N}$, $\max\{|\alpha - Q_n^\alpha|_p, |\beta - Q_n^\beta|_p\} < \frac{1}{p^{2K_n}}$.

Proof It is a consequence of Proposition 1. □

3.2 Diophantine study

In this section, we want to relate the rate of approximation of the convergents of a p -adic MCF to the Euclidean size of its numerators and denominators. First, we give a bound on this size.

Lemma 1 Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of real numbers, such that there exists $m \in \mathbb{N}$, c_0, \dots, c_m positive real numbers such that $c_m > 0$ and

$$|a_{n+m+1}|_\infty < c_m |a_{n+m}|_\infty + c_{m-1} |a_{n+m-1}|_\infty + \dots + c_0 |a_n|_\infty.$$

Let \tilde{x} be the (unique, by the Cartesian rule of signs) positive real root of the polynomial

$$f(X) = X^{m+1} - c_m X^m - \dots - c_1 X - c_0 \tag{11}$$

and let $M \geq \max\{|a_0|_\infty, \frac{|a_1|_\infty}{\tilde{x}}, \dots, \frac{|a_m|_\infty}{\tilde{x}^m}\}$. Then $|a_n|_\infty \leq M \tilde{x}^n$ for every $n \in \mathbb{N}$.

Proof The proof is straightforward by induction on n . □

Notice that $f(0) = -c_0 < 0$, so that $\tilde{x} > 0$, more precisely

$$\tilde{x} = c_m + \frac{c_{m-1}}{\tilde{x}} + \frac{c_{m-2}}{\tilde{x}^2} + \dots + \frac{c_0}{\tilde{x}^m},$$

which implies $c_m < \tilde{x}$. Put $C = \sum_{i=0}^m |c_i|_\infty$, if $C < 1$, then $f(1) = 1 - C > 0$, so that $0 < \tilde{x} < 1$, and we can conclude that $c_m < \tilde{x} < 1$. In the following, \tilde{x}_p will be the real root of the polynomial

$$X^3 - \frac{1}{2}X^2 - \frac{1}{2p}X - \frac{1}{p^3}$$

so that $\frac{1}{2} < \tilde{x}_p < 1$ and $\lim_{p \rightarrow \infty} \tilde{x}_p = \frac{1}{2}$ in \mathbb{R} . By specializing to the case of p -adic MCF we can apply the previous Lemma to the sequences $(A_n), (B_n), (C_n)$ as in (3) with $m = 2$. Considering that $|a_n|_\infty, |b_n|_\infty < \frac{p}{2}$, for every $n \geq 0$, in this special case the role of the polynomial $f(X)$ of the Lemma is played by

$$X^3 - \frac{p}{2}X^2 - \frac{p}{2}X - 1$$

whose real root is $p\tilde{x}_p$. Thus, we obtain the following proposition.

Proposition 4 *Given the sequences (A_n) , (B_n) , (C_n) as in (3), there exists $H > 0$ such that*

$$\max\{|A_n|_\infty, |B_n|_\infty, |C_n|_\infty\} \leq H(p\tilde{x}_p)^n,$$

for every $n \in \mathbb{N}$ and in particular

$$\max\{|A_n|_\infty, |B_n|_\infty, |C_n|_\infty\} = o(p^n).$$

Proposition 5 *Let $\alpha = (\alpha, \beta) \in \mathbb{Q}^2$, and write*

$$\alpha = \frac{x_0}{z_0}, \quad \beta = \frac{y_0}{z_0}$$

with $z_0 \in \mathbb{Z}$ and $x_0, y_0 \in \mathbb{Z} \left[\frac{1}{p} \right]$.

The p -adic Jacobi–Perron algorithm applied to α stops in a number of steps bounded by $-\frac{\log(M)}{\log(\tilde{x}_p)}$ where

$$M = \max \left\{ |z_0|_\infty, \frac{1}{p}|y_0|_\infty + \frac{1}{2}|z_0|_\infty, \frac{1}{p^2}|x_0|_\infty + \frac{1}{2p}|y_0|_\infty + \left(\frac{1}{2p} + \frac{1}{4} \right) |z_0|_\infty \right\}$$

$$\leq \max \left\{ |z_0|_\infty, \frac{1}{2}(|y_0|_\infty + |z_0|_\infty), \frac{1}{4}(|x_0|_\infty + |y_0|_\infty + |z_0|_\infty) \right\} \tag{12}$$

$$\leq |x_0|_\infty + |y_0|_\infty + |z_0|_\infty \tag{13}$$

$$\leq 3 \max\{|x_0|_\infty, |y_0|_\infty, |z_0|_\infty\}. \tag{14}$$

Proof The proof is the same as [31, Theorem 5], but we take into account the number of steps. The p -adic JP algorithm produces the sequence of complete quotients $(\alpha_n)_{n \geq 0}$, where

$$\alpha_n = (\alpha_n, \beta_n) \in \mathbb{Q}^m, \quad \alpha_n = \frac{x_n}{z_n}, \quad \beta_n = \frac{y_n}{z_n},$$

and x_n, y_n, z_n are generated by the following rules:

$$\begin{cases} x_n = a_n z_n + y_{n+1}, \\ y_n = b_n z_n + z_{n+1}, \\ z_n = x_{n+1} \end{cases}$$

with $a_n, b_n \in \mathcal{Y}$, $|y_{n+1}|_p, |z_{n+1}|_p < |z_n|_p$. Then $\frac{z_n}{p^n} \in \mathbb{Z}$ and from the formula

$$z_{n+1} = z_{n-2} - a_{n-1}z_{n-1} - b_n z_n$$

we get by Lemma 1

$$\begin{aligned} \frac{|z_{n+1}|_\infty}{p^{n+1}} &< \frac{1}{2} \frac{|z_n|_\infty}{p^n} + \frac{1}{2p} \frac{|z_{n-1}|_\infty}{p^{n-1}} + \frac{1}{p^3} \frac{|z_{n-2}|_\infty}{p^{n-2}} \\ &< M' \tilde{x}_p^n, \end{aligned}$$

where

$$M' = \max \left\{ \frac{|z_2|_\infty}{p^2}, \frac{|z_1|_\infty}{p}, |z_0|_\infty \right\}.$$

Then $z_{n+1} = 0$ when $\tilde{x}_p^n \leq \frac{1}{M'}$. We have

$$\begin{aligned} \frac{|z_1|_\infty}{p} &= \frac{1}{p} |y_0 - b_0 z_0|_\infty < \frac{1}{p} |y_0|_\infty + \frac{1}{2} |z_0|_\infty \\ \frac{|z_2|_\infty}{p^2} &= \frac{1}{p^2} |y_1 - b_1 z_1|_\infty \\ &= \frac{1}{p^2} |(x_0 - a_0 z_0) - b_1 (y_0 - b_0 z_0)|_\infty \\ &< \frac{1}{p^2} |x_0|_\infty + \frac{1}{2p} |y_0|_\infty + \left(\frac{1}{2p} + \frac{1}{4} \right) |z_0|_\infty. \end{aligned}$$

Therefore $M' < M$, so that

$$\begin{aligned} z_{n+1} = 0 \quad \text{for } \tilde{x}_p^n \leq \frac{1}{M} \\ \text{that is for } n \geq -\frac{\log(M)}{\log(\tilde{x}_p)}. \end{aligned} \tag{15}$$

Inequalities (12) and (14) are straightforward. □

Corollary 3 *Let $t, u \in \mathbb{Z}[\frac{1}{p}]$ and $v \in \mathbb{Z}$ such that the p -adic MCF for $(\frac{t}{v}, \frac{u}{v})$ has length $\geq n + 1$. Then*

$$\max\{|t|_\infty, |u|_\infty, |v|_\infty\} \geq \frac{1}{3\tilde{x}_p^n}.$$

Proof With the notation of the proof of Proposition 5, we have $z_{n+1} \neq 0$, then the claim follows from by (14) and (15). □

Corollary 4 *Let $(\alpha, \beta) \in \mathbb{Q}_p^2$ be a pair having a p -adic MCF expansion of length $\geq n + 1$. Then*

$$\max\{|A_n|_\infty, |B_n|_\infty, |C_n|_\infty\} \geq \frac{1}{3p^{K_n} \tilde{x}_p^n}.$$

Proof If we set $t = p^{K_n} A_n, u = p^{K_n} B_n, v = p^{K_n} C_n$, then the hypothesis of Corollary 3 is fulfilled. \square

The following theorem establishes an explicit lower bound for the Euclidean length of a pair of rational numbers which is a “good approximation” of a p -adic pair w.r.t the corresponding K_n .

Theorem 2 Let $(\alpha, \beta) \in \mathbb{Q}_p^2$ be a pair having a p -adic MCF expansion of length $\geq n + 1$. Let $(\frac{t}{v}, \frac{u}{v}) \in \mathbb{Q}^2$ with $t, u \in \mathbb{Z}[\frac{1}{p}]$, $v \in \mathbb{Z}$, and assume $\max \left\{ \left| \alpha - \frac{t}{v} \right|_p, \left| \beta - \frac{u}{v} \right|_p \right\} < \frac{1}{p^{2K_{n+1}}}$; then $\max\{|t|_\infty, |u|_\infty, |v|_\infty\} \geq \frac{1}{3\bar{x}_p^n}$.

Proof By Proposition 2 the pair $(\frac{t}{v}, \frac{u}{v})$ has the same MCF expansion as (α, β) up to $n + 1$. The claim follows from Corollary 3. \square

4 Results related to algebraic dependence

4.1 A p -adic Liouville-type theorem on algebraic dependence

The quality of rational approximations to real numbers is related to their algebraic dependence. Indeed, if it is possible to find infinitely many good approximations to a m -tuple of real numbers, then they are algebraically independent, see, e.g., [4]. Similar results also hold for the p -adic numbers [25]. In the following theorem, we prove a new result of this kind and then we apply it to p -adic MCFs.

Lemma 2 Let C be a non-zero integer number, then

$$|C|_p \geq \frac{1}{|C|_\infty}.$$

Proof The result follows from $p^{v_p(C)} \leq |C|_\infty$ and $|C|_p = \frac{1}{p^{v_p(C)}}$. \square

The following result is a variant of [25, Theorem 3].

Theorem 3 Given $\alpha, \beta \in \mathbb{Q}_p \setminus \mathbb{Q}$ such that $F(\alpha, \beta) = 0$, for $F(X, Y) \in \mathbb{Z}[X, Y]$ non-zero polynomial with minimal total degree D , let $(t_n)_{n \geq 0}, (u_n)_{n \geq 0}, (v_n)_{n \geq 0}$ be sequences of integers such that $v_n \neq 0$ for every $n \in \mathbb{N}$ and

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \alpha, \quad \lim_{n \rightarrow \infty} \frac{t_n}{v_n} = \beta \tag{16}$$

in \mathbb{Q}_p . Consider $M_n = \max\{|t_n|_\infty, |u_n|_\infty, |v_n|_\infty\}$ and

$$U_n = \max \left\{ \left| \alpha - \frac{t_n}{v_n} \right|_p, \left| \beta - \frac{u_n}{v_n} \right|_p \right\};$$

if

$$\lim_{n \rightarrow \infty} U_n \cdot M_n^D = 0 \tag{17}$$

in \mathbb{R} , then $F\left(\frac{t_n}{v_n}, \frac{u_n}{v_n}\right) = 0$ for $n \gg 0$.

Proof We observe that $v_n^D \cdot F\left(\frac{t_n}{v_n}, \frac{u_n}{v_n}\right) \in \mathbb{Z}$ and

$$\left|v_n^D \cdot F\left(\frac{t_n}{v_n}, \frac{u_n}{v_n}\right)\right|_{\infty} \leq K M_n^D, \tag{18}$$

where K is the sum of the Euclidean absolute values of the coefficients of $F(X, Y)$. Therefore, if $F\left(\frac{t_n}{v_n}, \frac{u_n}{v_n}\right)$ is not zero, we have

$$\left|F\left(\frac{t_n}{v_n}, \frac{u_n}{v_n}\right)\right|_p \geq \left|v_n^D \cdot F\left(\frac{t_n}{v_n}, \frac{u_n}{v_n}\right)\right|_p \geq \frac{1}{\left|v_n^D \cdot F\left(\frac{t_n}{v_n}, \frac{u_n}{v_n}\right)\right|_{\infty}} \geq \frac{1}{K M_n^D} \tag{19}$$

by Lemma 2 and (18). On the other hand, we can write

$$F(X, Y) = \sum_{i,j} A_{ij}(X - \alpha)^i(Y - \beta)^j$$

with $A_{ij} \in \mathbb{Q}_p$. We have

$$A_{10} = \frac{\partial F}{\partial X}(\alpha, \beta), \quad A_{01} = \frac{\partial F}{\partial Y}(\alpha, \beta),$$

if $A_{10} = 0$ and $\frac{\partial F}{\partial X}(X, Y) \neq 0$ then the latter polynomial would give an algebraic dependence relation between α and β of total degree $\leq D - 1$, therefore $\frac{\partial F}{\partial X}(X, Y) = 0$, that is X does not appear in $F(X, Y)$. Analogously $A_{01} = 0$ implies that $\frac{\partial F}{\partial Y}(X, Y) = 0$, that is Y does not appear in $F(X, Y)$. It follows that if $A_{ij} \neq 0$ for some $i > 0$ then $A_{10} \neq 0$; and if $A_{ij} \neq 0$ for some $j > 0$ then $A_{01} \neq 0$. Hence, it is easy to see that for every i, j such that $i + j > 1$ and $n \gg 0$

$$\begin{aligned} &v_p \left(A_{ij} \left(\frac{t_n}{v_n} - \alpha \right)^i \left(\frac{u_n}{v_n} - \beta \right)^j \right) \\ &> \min \left\{ v_p \left(A_{10} \left(\frac{t_n}{v_n} - \alpha \right) \right), v_p \left(A_{01} \left(\frac{u_n}{v_n} - \beta \right) \right) \right\}. \end{aligned}$$

Therefore for $n \gg 0$, we obtain

$$\begin{aligned} \left| F \left(\frac{t_n}{v_n}, \frac{u_n}{v_n} \right) \right|_p &\leq \max_{ij} \left\{ \left| A_{ij} \left(\frac{t_n}{v_n} - \alpha \right)^i \left(\frac{u_n}{v_n} - \beta \right)^j \right| \right\} \\ &= \max \left\{ \left| A_{01} \left(\frac{t_n}{v_n} - \alpha \right) \right|_p, \left| A_{10} \left(\frac{u_n}{v_n} - \beta \right) \right|_p \right\} \leq H \cdot U_n, \end{aligned} \tag{20}$$

for $H = \max\{|A_{01}|_p, |A_{10}|_p\}$. Putting together equations (19) and (20), we get

$$\text{frac1KM}_n^D \leq \left| F \left(\frac{t_n}{v_n}, \frac{u_n}{v_n} \right) \right|_p \leq H \cdot U_n,$$

for every n such that $F \left(\frac{t_n}{v_n}, \frac{u_n}{v_n} \right) \neq 0$. This implies that there exists $C > 0$ such that if $F \left(\frac{t_n}{v_n}, \frac{u_n}{v_n} \right) \neq 0$ then $U_n \cdot M_n > C$. Then hypothesis (17) proves the claim. \square

Remark 2 We shall apply Theorem 3 with

$$\frac{u_n}{v_n} = Q_n^\alpha, \quad \frac{t_n}{v_n} = Q_n^\beta,$$

with $u_n, t_n, v_n \in \mathbb{Z}$ coprime. Set $\delta = \max\{0, -v(\alpha), -v(\beta)\}$, then for $n \gg 0$, we have

$$(t_n, u_n, v_n) = p^{K_n + \delta}(A_n, B_n, C_n).$$

Consequently, if $M_n = \max\{|t_n|_\infty, |u_n|_\infty, |v_n|_\infty\}$ then by Proposition 4 there exists $H > 0$ such that

$$M_n \leq Hp^{K_n + n} \tilde{x}_p^n = o(p^{n + K_n}). \tag{21}$$

4.2 Some consequences on linear dependence

We specialize Theorem 3 to the case $D = 1$, i.e., when we have linear dependence. In [31], the authors proved that if the p -adic Jacobi–Perron algorithm stops in a finite number of steps, then the initial values are \mathbb{Q} -linearly dependent. Further results about linear dependence and p -adic MCFs can be found in [32], where it is conjectured that if we start the p -adic Jacobi–Perron algorithm with a m -tuple of \mathbb{Q} -linearly dependent numbers, then the algorithm is finite or periodic. Here, exploiting the previous results, we can give a condition that ensures the finiteness of the p -adic Jacobi–Perron algorithm when it processes certain \mathbb{Q} -linearly dependent inputs.

Theorem 4 Given $\alpha, \beta \in \mathbb{Q}_p$, consider

$$M_n = \max\{|A_n|_\infty, |B_n|_\infty, |C_n|_\infty\}, \quad U_n = \max \left\{ \left| \alpha - \frac{A_n}{C_n} \right|_p, \left| \beta - \frac{B_n}{C_n} \right|_p \right\},$$

where $(A_n), (B_n), (C_n)$ are the sequences of numerators and denominators of convergents of the MCF representing (α, β) . If

$$\lim_{n \rightarrow \infty} U_n \cdot M_n = 0,$$

then either $\alpha, \beta, 1$ are linearly independent over \mathbb{Q} or the p -adic MCF expansion of (α, β) is finite.

Proof Assume that the p -adic MCF for (α, β) is not finite, then the sequence (Q_n^α, Q_n^β) p -adically converges to (α, β) by [31, Proposition 3] and $(\alpha, \beta) \notin \mathbb{Q}^2$, by Proposition 5. Suppose that $A\alpha + B\beta + C = 0$ for some $A, B, C \in \mathbb{Q}$ not all zero. We define the sequence $S_n = AA_{n-1} + BB_{n-1} + CC_{n-1}$; Theorem 3 implies that $S_n = 0$ for n sufficiently large. Furthermore, it is straightforward to see that $S_n = AV_{n-1}^\alpha + BV_{n-1}^\beta$ (see also [32]) and by Corollary 1 we should have $V_n^\alpha = V_n^\beta = 0$, which is a contradiction, as $(\alpha, \beta) \notin \mathbb{Q}^2$, since we have supposed the MCF for (α, β) is not finite. \square

Remark 3 Theorem 4 is an improvement of a result implicitly contained in [32, Proposition 10], namely that if $(1, \alpha, \beta)$ are linearly dependent over \mathbb{Q} and there is a constant $K > 0$ such that

$$\max\{|V_n^\alpha|_p, |V_n^\beta|_p\} \leq \frac{K}{p^n}, \tag{22}$$

then the p -adic Jacobi–Perron algorithm stops in finitely many steps when applied to (α, β) . In fact (22) implies

$$U_n \cdot p^{K_n} \leq \frac{K}{p^n},$$

so that, by (21),

$$U_n \cdot M_n \leq H \tilde{x}_p^n U_n p^{K_n+n} \leq KH \tilde{x}_p^n \xrightarrow{\infty} 0 \text{ for } n \rightarrow \infty.$$

4.3 A class of fast convergent p -adic MCFs

Finally, we see some conditions on the partial quotients that produce MCFs converging to algebraically independent numbers or having convergents that satisfy an algebraic relation.

Lemma 3 Given a MCF $[(a_0, a_1, \dots), (b_0, b_1, \dots)]$ such that

$$k_{n+1} \geq (D - 1)(k_n + k_{n-1}) + 2D; \text{ and} \tag{23}$$

$$h_{n+1} \geq (D - 1)k_n + D \tag{24}$$

for $n \gg 0$ and $D \geq 1$, then there exists $C \in \mathbb{R}$ such that

$$\min\{v_p(V_n^\alpha), v_p(V_n^\beta)\} \geq (D - 1)K_n + Dn + C \text{ for } n \in \mathbb{N}.$$

Proof The argument is the same as in the proof of Proposition 1. In fact, if conditions (23) and (24) hold for every $n \geq 0$ then the claim directly follows from Proposition 1 by putting $\ell_n = (D - 1)k_n + D$. In any case hypotheses (23) and (24) imply that

$$v_p \left(\frac{\beta_{n+1}}{\alpha_{n+1}} \right) \geq (D - 1)k_n + D \quad \text{for } n \gg 0. \tag{25}$$

Let V_n be one of V_n^α, V_n^β . From the formula (8) we get for $n \gg 0$

$$\frac{V_n}{p^{(D-1)K_n+Dn}} = \mu_n \frac{V_{n-1}}{p^{(D-1)K_{n-1}+D(n-1)}} + v_n \frac{V_{n-2}}{p^{(D-2)K_{n-1}+D(n-2)}},$$

$$\mu_n = -\frac{\beta_{n+1}}{\alpha_{n+1}} \frac{1}{p^{(D-1)k_n+D}}, \quad v_n = -\frac{1}{\alpha_{n+1}} \frac{1}{p^{(D-1)(k_n+k_{n-1})+2D}}.$$

By (23) and (25) there exists n_0 such that $\mu_n, v_n \in \mathbb{Z}_p$ for $n > n_0$. Then

$$\left| \frac{V_n}{p^{(D-1)K_n+Dn}} \right|_p \leq \max \left\{ \left| \frac{V_i}{p^{(D-1)K_i+Di}} \right|_p, i = 0, \dots, n_0 \right\} \text{ for } n > n_0$$

so that the claim follows by setting

$$C = \min \left\{ v_p \left(\frac{V_i^\alpha}{p^{(D-1)K_i+Di}} \right), v_p \left(\frac{V_i^\beta}{p^{(D-1)K_i+Di}} \right), i = 0, \dots, n_0 \right\}.$$

□

Theorem 5 Assume that α, β are algebraically dependent and let $F(X, Y) \in \mathbb{Q}[X, Y]$ be a non-zero polynomial of minimum total degree D such that $F(\alpha, \beta) = 0$. If the MCF expansion of (α, β) satisfies conditions (23) and (24), then $F(Q_n^\alpha, Q_n^\beta) = 0$ for $n \gg 0$.

Proof Let M_n, U_n be as in Theorem 3. For $n \gg 0$ and a suitable constant $C > 0$ we have

$$\begin{aligned} U_n &= \frac{1}{p^{K_n}} \max \{ |V_n^\alpha|_p, |V_n^\beta|_p \} \\ &\leq \frac{C}{p^{D(K_n+n)}} \text{ by Lemma 3} \end{aligned}$$

so that $\lim_{n \rightarrow \infty} U_n \cdot M_n = 0$ by (21). Then the claim follows from Theorem 3. □

Theorem 6 Given $(\alpha, \beta) = [(a_0, a_1, \dots), (b_0, b_1, \dots)]$ such that

$$\lim_{n \rightarrow \infty} \frac{k_n}{k_{n-1} + k_{n-2}} = \infty, \quad \lim_{n \rightarrow \infty} \frac{h_n}{k_{n-1}} = \infty$$

in \mathbb{R} , then either α, β are algebraically independent or there exists a non-zero polynomial $F(X, Y) \in \mathbb{Q}[X, Y]$ such that $F(Q_n^\alpha, Q_n^\beta) = 0$ for $n \gg 0$.

Proof Let $D > 0$. For $n \gg_D 0$ we have

$$\begin{aligned} k_n &\geq D(k_{n-1} + k_{n-2}) \geq (D - 1)D(k_{n-1} + k_{n-2}) + 2D, \text{ and} \\ h_n &\geq Dk_{n-1} \geq (D - 1)k_{n-1} + D. \end{aligned}$$

Then the claim follows from Theorem 5. \square

Remark 4 By Faltings theorem, an algebraic curve having infinitely many rational points must have genus 0 or 1. This is a strong condition on polynomials $F(X, Y) \in \mathbb{Q}[X, Y]$ such that $F(Q_n^\alpha, Q_n^\beta) = 0$, for $n \gg 0$.

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