Bounds for the generalized Marcum function of the second kind

Árpád Baricz^{1,2} · Nitin Bisht³ · Sanjeev Singh³ · V. Antony Vijesh³

Dedicated to the memory of Richard Askey

Received: 29 July 2020 / Accepted: 25 March 2021 / Published online: 5 June 2021

© The Author(s) 2021

Abstract

In this paper, we focus on the generalized Marcum function of the second kind of order $\nu > 0$, defined by

$$R_\nu(a, b) = \frac{c_{a,\nu}}{a^{\nu-1}} \int_b^\infty t^\nu e^{-\frac{t^2+a^2}{2}} K_{\nu-1}(at) dt,$$

where $a > 0$, $b \geq 0$, K_ν stands for the modified Bessel function of the second kind, and $c_{a,\nu}$ is a constant depending on a and ν such that $R_\nu(a, 0) = 1$. Our aim is to find some new tight bounds for the generalized Marcum function of the second kind and compare them with the existing bounds. In order to deduce these bounds, we include the monotonicity properties of various functions containing modified Bessel functions of the second kind as our main tools. Moreover, we demonstrate that our bounds in some sense are the best possible ones.

Keywords Modified Bessel function of the second kind · Bounds · Survival function · Complementary error function · Generalized Marcum Q -function

Mathematics Subject Classification 33E20 · 33C10 · 94A13 · 94A05 · 26A48

The authors are very thankful to the reviewers for a careful evaluation of the manuscript and their several helpful comments which improved the quality of the paper. N. Bisht is grateful to the Council of Scientific and Industrial Research India (Grant No. 09/1022(0014)/2013-EMR-I) for the financial support, and S. Singh is thankful to the Science and Engineering Research Board (SERB), Department of Science and Technology, Government of India for the financial support through Project CRG/2020/002875..

✉ Árpád Baricz
bariczoci@yahoo.com

1 Introduction and preliminary results

Let $R_\nu(a, b)$ be the so-called generalized Marcum function of the second kind [4], defined by

$$R_\nu(a, b) = \frac{2}{\Gamma(\nu)\Gamma\left(1 - \nu, \frac{a^2}{2}\right)} \frac{1}{a^{\nu-1}} \int_b^\infty t^\nu e^{-\frac{t^2+a^2}{2}} K_{\nu-1}(at) dt, \quad (1.1)$$

where $a > 0$, $b \geq 0$, $\nu > 0$, and K_ν stands for the modified Bessel function of the second kind. As $R_\nu(a, 0) = 1$, we can write $R_\nu(a, b)$ as follows:

$$R_\nu(a, b) = 1 - \frac{2}{\Gamma(\nu)\Gamma\left(1 - \nu, \frac{a^2}{2}\right)} \frac{1}{a^{\nu-1}} \int_0^b t^\nu e^{-\frac{t^2+a^2}{2}} K_{\nu-1}(at) dt, \quad (1.2)$$

where $a > 0$, $b \geq 0$, and $\nu > 0$. In [4], Baricz et al. proved some monotonicity properties of the generalized Marcum function of the second kind with respect to a , b and ν , and log-concavity and convexity properties with respect to the parameter b . They also deduced several bounds for the Marcum function of the second kind (that is, when $\nu = 1$) and for the generalized Marcum function of the second kind.

The study of the generalized Marcum function of the second kind is motivated by the importance of the generalized Marcum Q -function in the electrical engineering literature, which was studied by several authors in the last few decades, see for example [1,2,9] and the references therein. In particular, Baricz and Sun [2] obtained several tight bounds for the generalized Marcum Q -function which are based on the monotonicity properties of the functions of the form $x \mapsto x^{\alpha\nu+\beta} I_\nu(x)/f(e^x, \sinh x, \cosh x)$, where α and β are real numbers and I_ν stands for the modified Bessel function of the first kind. These bounds are sharp in nature in the sense that they cannot be further improved. Motivated by the work of Baricz and Sun [2], in this paper our aim is to find some new bounds for the generalized Marcum function of the second kind. These bounds are obtained from the monotonicity properties of the functions of the form $x \mapsto x^{\alpha\nu+\beta} K_\nu(x)/f(e^x, \sinh x, \cosh x)$, where α and β are real numbers. The monotonicity properties of these functions depend upon some monotonicity properties of the ratio of modified Bessel functions of the second kind, which were investigated in [5] and [11]. The bounds which we deduce in this paper are tight enough and cannot be further improved as we prove that the power $\alpha\nu + \beta$ is the smallest or largest constant so that the corresponding monotonicity property for the function $x \mapsto x^{\alpha\nu+\beta} K_\nu(x)/f(e^x, \sinh x, \cosh x)$ holds.

It is worth mentioning that during the preparation of our manuscript we found out that the distribution whose survival function we study in this paper resembles a distribution of Nadarajah [6]. More precisely, motivated by a Bayesian inference of an inverse Gaussian sample, Nadarajah [6] introduced a new modified Bessel distribution of the second kind supported on the real line. In [4], it has been shown that the generalized Marcum function of the second kind is in fact the survival function of the truncated distribution of a special case (when $p = 1/2$) of the modified Bessel distribution of the second kind considered by Nadarajah. Taking into account the vast

application field in engineering and physical problems of the generalized Marcum Q -function and of the Bessel distributions, we believe that our paper may be useful for people who meet in their study such kind of exponential integrals involving the modified Bessel function of the second kind.

This paper is organized as follows: in this section we prove the preliminary results which contain the monotonicity of the functions involving the modified Bessel function of the second kind. Using the preliminary results in Sect. 2, we present some new lower and upper bounds for the generalized Marcum function of the second kind for two cases: when $b \geq a > 0$ and $a > b > 0$. In Sect. 3, we provide some comparison between our new bounds which are obtained in Sect. 2 and the bounds given in [4] for both cases when $b \geq a > 0$ and $a > b > 0$. The discussion is concluded in Sect. 4, by illustrating the theoretical results numerically.

It is interesting to note that the functions $x \mapsto x^\nu K_\nu(x)e^x$ and $x \mapsto x^{-\nu} K_\nu(x)e^x$ have different monotonic behavior, see [4]. More specifically

- (1) for all $\nu \geq 1/2$, the function $x \mapsto x^\nu K_\nu(x)e^x$ is increasing on $(0, \infty)$.
- (2) for all $\nu \geq -1/2$, the function $x \mapsto x^{-\nu} K_\nu(x)e^x$ is decreasing on $(0, \infty)$.

Moreover, Yang and Zheng [11] showed that for fixed $\nu \geq 1/2$ the function $x \mapsto \sqrt{x} K_\nu(x)e^x$ is decreasing on $(0, \infty)$. Then the natural question which we can ask is that for a fixed ν what is the monotonic behavior of the function $x \mapsto x^{\alpha_\nu} K_\nu(x)e^x$ on $(0, \infty)$? Does there exist the smallest α_ν such that the function $x \mapsto x^{\alpha_\nu} K_\nu(x)e^x$ is increasing on $(0, \infty)$? Does there exist the largest α_ν such that the function $x \mapsto x^{\alpha_\nu} K_\nu(x)e^x$ is decreasing on $(0, \infty)$? These questions will be answered in this section. Moreover, this study improves one of the results of Yang and Zheng [11] and provides an alternative proof for [10, Corollary 1]. Similarly, the behavior of the functions $x \mapsto x^{\beta_\nu} K_\nu(x)/\sinh x$ and $x \mapsto x^{\beta_\nu} K_\nu(x)/\cosh x$ are studied and we find the largest and the smallest powers of x such that these functions possess the monotonic decreasing/increasing property. This section also discusses the monotonic behavior of the functions $x \mapsto x^{\delta_\nu} K_\nu(x) \sinh x$ and $x \mapsto x^{\beta_\nu} K_\nu(x) \cosh x$.

The following two lemmas are used to prove the main Lemma in this section. For Lemma 1 we refer [11, Theorem 2.6], while for Lemma 2 we refer to [3, Theorem 2] and [11, Lemma 2.4].

Lemma 1 *The function $\nu \mapsto K'_\nu(x)/K_\nu(x)$ is strictly decreasing on $(0, \infty)$ for $x > 0$.*

Lemma 2 *The function $x \mapsto x K'_\nu(x)/K_\nu(x)$ is decreasing on $(0, \infty)$ for all $\nu \in \mathbb{R}$.*

The main Lemma in this section is the following result.

Lemma 3 *Let $\nu \in \mathbb{R}$. Then the following assertions are true:*

- a. *For $\nu \geq 1/2$ the function $x \mapsto \sqrt{x} K_\nu(x)e^x$ is strictly decreasing on $(0, \infty)$.*
- b. *For $\nu > 1/2$ the smallest constant α_ν for which the function $x \mapsto x^{\alpha_\nu} K_\nu(x)e^x$ is increasing on $(0, \infty)$ is $\alpha_\nu = \nu$. For $\nu \geq 1/2$ the largest constant α_ν for which the function $x \mapsto x^{\alpha_\nu} K_\nu(x)e^x$ is decreasing on $(0, \infty)$ is $\alpha_\nu = 1/2$.*
- c. *The largest constant β_ν for which the function $x \mapsto x^{\beta_\nu} K_\nu(x)/\sinh x$ is strictly decreasing on $(0, \infty)$ is $\beta_\nu = \nu + 1$.*

- d. The function $x \mapsto x^\nu K_\nu(x)/\cosh x$ is strictly decreasing on $(0, \infty)$. Moreover, the largest constant γ_ν for which the function $x \mapsto x^{\gamma_\nu} K_\nu(x)/\cosh x$ is strictly decreasing on $(0, \infty)$ is $\gamma_\nu = \nu$.
- e. For $\nu > 1/2$ the function $x \mapsto x^{\delta_\nu} K_\nu(x) \sinh x$ is increasing on $(0, \infty)$ when $\delta_\nu \geq \nu$.
- f. For $\nu > 1/2$ the function $x \mapsto x^{\epsilon_\nu} K_\nu(x) \cosh x$ is increasing on $(0, \infty)$ when $\epsilon_\nu \geq \nu + \theta$, where

$$\theta = \sup_{x>0} \left(\frac{2x}{e^{2x} + 1} \right) = 0.278 \dots$$

For $\nu \geq 1/2$ the largest constant for which the function $x \mapsto x^{\epsilon_\nu} K_\nu(x) \cosh x$ is decreasing on $(0, \infty)$ is $\epsilon_\nu = 1/2$.

Proof a. This follows from [11, Corollary 3.2].

b. Consider the function $f_\nu : (0, \infty) \rightarrow \mathbb{R}$, defined by $f_\nu(x) = x^{\alpha_\nu} K_\nu(x) e^x$. Then

$$f'_\nu(x) = x^{\alpha_\nu-1} K_\nu(x) e^x \left[x + \alpha_\nu + \frac{x K'_\nu(x)}{K_\nu(x)} \right].$$

In view of [7, 10.29.2]

$$K'_\nu(x) = -\frac{\nu}{x} K_\nu(x) - K_{\nu-1}(x) \quad (1.3)$$

we have

$$\begin{aligned} f'_\nu(x) &= x^{\alpha_\nu-1} K_\nu(x) e^x \left[x + \alpha_\nu - \nu - \frac{x K_{\nu-1}(x)}{K_\nu(x)} \right] \\ &= x^{\alpha_\nu-1} K_\nu(x) e^x (\alpha_\nu - \nu + \phi_\nu(x)), \end{aligned} \quad (1.4)$$

where $\phi_\nu : (0, \infty) \rightarrow \mathbb{R}$ is defined by $\phi_\nu(x) = x - x K_{\nu-1}(x)/K_\nu(x)$.

To find the smallest value of α_ν for which the function f_ν is increasing on $(0, \infty)$, it is necessary to find the minimum of the function ϕ_ν . By Soni inequality [8, Eq. 5] and the fact $K_\nu(x) = K_{-\nu}(x)$, we have

$$K_\nu(x) \geq K_{\nu-1} \text{ for all } \nu \geq \frac{1}{2} \text{ and } x > 0. \quad (1.5)$$

This gives

$$\phi_\nu(x) = x - \frac{x K_{\nu-1}(x)}{K_\nu(x)} \geq 0 \quad \text{for all } x > 0. \quad (1.6)$$

Now, in view of the asymptotic formula [7, 10.30.2]

$$K_\nu(x) \sim \frac{1}{2} \left(\frac{x}{2} \right)^{-\nu} \Gamma(\nu), \quad (1.7)$$

where $\nu > 0$ and $x \rightarrow 0$, we get

$$\lim_{x \rightarrow 0} \phi_\nu(x) = 0. \quad (1.8)$$

By using (1.6) and (1.8) we obtain $f'_\nu(x) \geq 0$ for $\alpha_\nu \geq \nu$. Thus, $\alpha_\nu = \nu$ is the smallest α_ν for which $x \mapsto x^{\alpha_\nu} K_\nu(x)e^x$ is increasing on $(0, \infty)$.

Now, our aim is to find the largest constant α_ν for which the function $x \mapsto x^{\alpha_\nu} K_\nu(x)e^x$ is decreasing on $(0, \infty)$ for all $\nu \geq 1/2$. We again rewrite the derivative of $f_\nu(x)$ as follows:

$$\begin{aligned} f'_\nu(x) &= x^{\alpha_\nu-1} K_\nu(x) e^x \left[\alpha_\nu - \frac{1}{2} + \left(\frac{1}{2} + x + \frac{x K'_\nu(x)}{K_\nu(x)} \right) \right] \\ &= x^{\alpha_\nu-1} K_\nu(x) e^x \left(\alpha_\nu - \frac{1}{2} + \psi_\nu(x) \right), \end{aligned} \quad (1.9)$$

where $\psi_\nu : (0, \infty) \rightarrow \mathbb{R}$ is defined by $\psi_\nu(x) = 1/2 + x + x K'_\nu(x)/K_\nu(x)$. Thus, to find the largest constant for which the function f_ν is decreasing it is necessary to find the maximum value of the function ψ_ν . Due to Lemma 1, we have that $x K'_\nu(x)/K_\nu(x) \leq x K'_{1/2}(x)/K_{1/2}(x)$ for any $x > 0$ and $\nu \geq 1/2$. Now, by the recurrence relation (1.3) and using the fact that $K_\nu(x) = K_{-\nu}(x)$ we get

$$x \frac{K'_\nu(x)}{K_\nu(x)} \leq -\frac{1}{2} - x. \quad (1.10)$$

Thus for all $x > 0$ we get

$$\psi_\nu(x) = \frac{1}{2} + x + x \frac{K'_\nu(x)}{K_\nu(x)} \leq 0. \quad (1.11)$$

By using once again the recurrence relation (1.3) and in view of the asymptotic expansion [7, 10.40.2]

$$K_\nu(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x} \sum_{k \geq 0} \frac{a_k(\nu)}{x^k} \quad \text{as } x \rightarrow \infty,$$

where

$$\begin{aligned} a_k(\nu) &= \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2) \dots (4\nu^2 - (2k - 1)^2)}{k! 8^k} \\ &\text{for all } k \geq 1 \quad \text{and} \quad a_0(\nu) = 1, \end{aligned}$$

we conclude that

$$\lim_{x \rightarrow \infty} \psi_\nu(x) = 0. \quad (1.12)$$

By using (1.11) and (1.12), it follows that $f'(x) \leq 0$ for all $x \in (0, \infty)$ when $\alpha_\nu \leq 1/2$. Thus $\alpha_\nu = 1/2$ is the largest value for which the function $x \mapsto e^{\alpha_\nu} e^x K_\nu(x)$ is decreasing on $(0, \infty)$.

c. Consider the function $g_\nu : (0, \infty) \rightarrow \mathbb{R}$, defined by $g_\nu(x) = x^{\beta_\nu} K_\nu(x)/\sinh x$. Then

$$g'_\nu(x) = \frac{x^{\beta_\nu-1} K_\nu(x)}{\sinh x} \left[\frac{x K'_\nu(x)}{K_\nu(x)} + \beta_\nu - x \coth x \right]. \quad (1.13)$$

From Lemma 2 and the monotone decreasing property of $x \mapsto -x \coth x$, we see that for all $\nu \in \mathbb{R}$, the function $x \mapsto x K'_\nu(x)/K_\nu(x) - x \coth x$ is a decreasing function on $(0, \infty)$. Consequently, the maximum value of $x K'_\nu(x)/K_\nu(x) - x \coth x$ is nothing but $\lim_{x \rightarrow 0} (x K'_\nu(x)/K_\nu(x) - x \coth x)$. Using the asymptotic formula (1.7) we get

$$\lim_{x \rightarrow 0} \left[\frac{x K'_\nu(x)}{K_\nu(x)} - x \coth x \right] = \lim_{x \rightarrow 0} \left[-\nu - \frac{x K_{\nu-1}(x)}{K_\nu(x)} - x \coth x \right] = -\nu - 1.$$

Thus from (1.13) we obtain

$$g'_\nu(x) \leq \frac{x^{\beta_\nu-1} K_\nu(x)}{\sinh x} (\beta_\nu - \nu - 1)$$

and consequently for all $\beta_\nu \leq \nu + 1$ we have that $g'_\nu(x) \leq 0$ for all $x > 0$. Hence $x \mapsto x^{\beta_\nu} K_\nu(x)/\sinh x$ is a decreasing function for all $\beta_\nu \leq \nu + 1$. Consequently $\beta_\nu = \nu + 1$ is the largest value for which the function $x \mapsto x^{\beta_\nu} K_\nu(x)/\sinh x$ is decreasing on $(0, \infty)$.

d. Observe that

$$\left[\frac{x^{\gamma_\nu} K_\nu(x)}{\cosh x} \right]' = \frac{x^{\gamma_\nu-1} K_\nu(x)}{\cosh x} \left[\frac{x K'_\nu(x)}{K_\nu(x)} + \gamma_\nu - x \tanh x \right]. \quad (1.14)$$

For all $\nu \in \mathbb{R}$ proceeding like in part c, the function $x \mapsto x K'_\nu(x)/K_\nu(x) - x \tanh x$ is decreasing on $(0, \infty)$. Hence

$$\frac{x K'_\nu(x)}{K_\nu(x)} - x \tanh x \leq \lim_{x \rightarrow 0} \left[\frac{x K'_\nu(x)}{K_\nu(x)} - x \tanh x \right] = -\nu. \quad (1.15)$$

Thus (1.14) becomes

$$\left[\frac{x^{\gamma_\nu} K_\nu(x)}{\cosh x} \right]' \leq \frac{x^{\gamma_\nu-1} K_\nu(x)}{\cosh x} (\gamma_\nu - \nu).$$

Hence for all $\gamma_\nu \leq \nu$ the function $x \mapsto x^{\gamma_\nu} K_\nu(x)/\cosh x$ is decreasing on $(0, \infty)$. Consequently $\gamma_\nu = \nu$ is the largest value for which the function $x \mapsto x^{\gamma_\nu} K_\nu(x)/\cosh x$ is decreasing on $(0, \infty)$.

e. First note that

$$\begin{aligned}(x^{\delta_\nu} K_\nu(x) \sinh x)' &= x^{\delta_\nu-1} K_\nu(x) \sinh x \left[\frac{x K'_\nu(x)}{K_\nu(x)} + \delta_\nu + x \coth x \right] \\ &= x^{\delta_\nu-1} K_\nu(x) \sinh x \left[x \left(\coth x - \frac{K_{\nu-1}(x)}{K_\nu(x)} \right) + \delta_\nu - \nu \right].\end{aligned}\tag{1.16}$$

From (1.5) and $\coth x > 1$, we have

$$\coth x - \frac{K_{\nu-1}(x)}{K_\nu(x)} \geq 0 \quad \text{for all } x > 0.$$

Consequently for $\nu > 1/2$ and $x > 0$ we have

$$x \left[\coth x - \frac{K_{\nu-1}(x)}{K_\nu(x)} \right] \geq 0.$$

Hence for $x > 0$ we get $(x^{\delta_\nu} K_\nu(x) \sinh x)' \geq 0$ when $\delta_\nu \geq \nu$. Thus $x \mapsto x^{\delta_\nu} K_\nu(x) \sinh x$ is an increasing function on $(0, \infty)$ for all $\delta_\nu \geq \nu$.

f. Note that

$$\begin{aligned}(x^{\epsilon_\nu} K_\nu(x) \cosh x)' &= x^{\epsilon_\nu-1} K_\nu(x) \cosh x \left[\frac{x K'_\nu(x)}{K_\nu(x)} + \epsilon_\nu + x \tanh x \right] \\ &= x^{\epsilon_\nu-1} K_\nu(x) \cosh x \\ &\quad \times \left[-\nu - \frac{x K_{\nu-1}(x)}{K_\nu(x)} + \epsilon_\nu + x \tanh x \right] \quad \text{by (1.3)} \\ &\geq x^{\epsilon_\nu-1} K_\nu(x) \cosh x \, (-\nu + \epsilon_\nu - \phi(x)),\end{aligned}\tag{1.17}$$

where

$$\phi(x) = x - x \tanh x = \frac{2x}{e^{2x} + 1}\tag{1.18}$$

and the inequality (1.17) follows from (1.5). It is easy to check that the function ϕ has maximum value at $x_0 = 0.63923 \dots$ which equals $\sup_{x>0}(\phi(x)) = \phi(x_0) = 0.278 \dots$. It then follows

$$(x^{\epsilon_\nu} K_\nu(x) \cosh x)' > 0 \quad \text{if } \epsilon_\nu > \nu + 0.278 \dots$$

On the other hand, by using the inequality (1.10), we have

$$\begin{aligned}(x^{\epsilon_\nu} K_\nu(x) \cosh x)' &= x^{\epsilon_\nu-1} K_\nu(x) \cosh x \left[\frac{x K'_\nu(x)}{K_\nu(x)} + \epsilon_\nu + x \tanh x \right] \\ &\leq x^{\epsilon_\nu-1} K_\nu(x) \cosh x \left(\epsilon_\nu - \frac{1}{2} - \phi(x) \right),\end{aligned}\tag{1.19}$$

where $\phi(x)$ is defined in (1.18). Since $\inf_{x>0} \phi(x) = 0$, it gives

$$(x^{\epsilon_v} K_v(x) \cosh x)' \leq 0 \quad \text{if } \epsilon_v \leq \frac{1}{2}.$$

In view of

$$\frac{x K'_v(x)}{K_v(x)} + x \tanh x \leq 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} \left(\frac{x K'_v(x)}{K_v(x)} + x \tanh x \right) = -\frac{1}{2},$$

one can conclude that for $v \geq 1/2$ the largest constant for which the function $x \mapsto x^{\epsilon_v} K_v(x) \cosh x$ is decreasing on $(0, \infty)$ is $\epsilon_v = 1/2$. $\square$

Remark 1 Recall that the functions given in part c of [4, Lemma 5] and part a of Lemma 3 are the sharpest in terms of their monotonicity. From part b we can say that these functions cannot be further improved by raising or diminishing the power of x in the functions. As a consequence of part b of Lemma 3, we can conclude that if $1/2 < \alpha < v$, then the function $x \mapsto x^{\alpha_v} e^x K_v(x)$ is neither monotonic increasing nor monotonic decreasing in $(0, \infty)$. Equivalently the function $x \mapsto \alpha_v + x + x K'_v(x)/K_v(x)$ has at least one real positive zero. For example, using the expression $K_{3/2}(x) = \sqrt{(\pi/2)}(e^{-x}/\sqrt{x})(1 + 1/x)$, it is easy to verify that the function $x \mapsto x e^x K_{3/2}(x)$ is decreasing on $(0, 1)$ and increasing on $[1, \infty)$. In fact, by [10, Corollary 1], there is an $x_0 > 0$ such that the function $x \mapsto x^{\alpha_v} e^x K_v(x)$ is decreasing on $(0, x_0)$ and increasing on (x_0, ∞) .

Remark 2 It is worth mentioning that recently Yang and Tian [10, Corollary 1] obtained the result of part b of Lemma 3 by studying the properties of ratios of two Laplace transforms. In this paper an independent proof is provided.

Remark 3 As $x K'_v(x)/K_v(x) - x \coth x \rightarrow -\infty$ whenever $x \rightarrow \infty$, by using (1.13), we conclude that the function $x \mapsto x^{\beta_v} K_v(x)/\sinh x$ is not an increasing function on the whole interval $(0, \infty)$ for any β_v . Similarly, $x \mapsto x^{\gamma_v} K_v(x)/\cosh x$ is not an increasing function on the whole interval $(0, \infty)$ for any γ_v .

2 Lower and upper bounds for the generalized Marcum function of the second kind

In this section some new tight lower and upper bounds for the generalized Marcum function of the second kind are obtained by using the monotonicity property of the functions discussed in Lemma 3. The bounds are expressed in terms of the elementary special function named as the complementary error function $\operatorname{erfc}(t)$. It is defined as follows [7, 7.2.2]

$$\operatorname{erfc}(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-y^2} dy.$$

Recall also that for $a, \nu > 0$, $c_{a,\nu}$ denotes the constant defined by

$$c_{a,\nu} = \frac{2}{\Gamma(\nu)\Gamma\left(1-\nu, \frac{a^2}{2}\right)}.$$

2.1 Case 1 : $b \geq a > 0$

Theorem 1 For $b \geq a > 0$, the following inequalities hold:

$$R_\nu(a, b) \leq \frac{c_{a,\nu}}{a^{\nu-1}} \sqrt{b} K_{\nu-1}(ab) e^{ab} \int_{b+a}^{\infty} (y-a)^{\nu-\frac{1}{2}} e^{-\frac{y^2}{2}} dy, \quad (2.1)$$

$$R_\nu(a, b) \leq \frac{c_{a,\nu}}{a^{\nu-1}} \frac{b^\nu K_{\nu-1}(ab)}{2 \sinh(ab)} \sqrt{\frac{\pi}{2}} \left[\operatorname{erfc}\left(\frac{b-a}{\sqrt{2}}\right) - \operatorname{erfc}\left(\frac{b+a}{\sqrt{2}}\right) \right], \quad (2.2)$$

$$R_\nu(a, b) \leq \frac{c_{a,\nu}}{a^{\nu-1}} \frac{b^{\nu-1} K_{\nu-1}(ab)}{2 \cosh(ab)} \left[e^{\frac{-(b-a)^2}{2}} + e^{\frac{-(b+a)^2}{2}} + a \sqrt{\frac{\pi}{2}} \left(\operatorname{erfc}\left(\frac{b-a}{\sqrt{2}}\right) - \operatorname{erfc}\left(\frac{b+a}{\sqrt{2}}\right) \right) \right], \quad (2.3)$$

where $\nu \geq 3/2$ in (2.1), and $\nu > 0$ in (2.2) and (2.3).

Proof For $\nu \geq 1/2$ and $t \geq b$ part a of Lemma 3 leads to

$$K_\nu(t) \leq \sqrt{\frac{b}{t}} K_\nu(b) \frac{e^b}{e^t}, \quad (2.4)$$

which in view of (1.1) implies that

$$\begin{aligned} R_\nu(a, b) &\leq \frac{c_{a,\nu}}{a^{\nu-1}} \int_b^\infty t^\nu e^{-\frac{a^2+t^2}{2}} \sqrt{\frac{b}{t}} K_{\nu-1}(ab) \frac{e^{ab}}{e^{at}} dt, \\ &= \frac{c_{a,\nu}}{a^{\nu-1}} e^{ab} K_{\nu-1}(ab) \sqrt{b} \int_b^\infty t^{\nu-\frac{1}{2}} e^{-\frac{(t+a)^2}{2}} dt, \\ &= \frac{c_{a,\nu}}{a^{\nu-1}} e^{ab} K_{\nu-1}(ab) \sqrt{b} \int_{b+a}^\infty (y-a)^{\nu-\frac{1}{2}} e^{-\frac{y^2}{2}} dy. \end{aligned}$$

To prove (2.2), the monotone property of $x \mapsto x^{\nu+1} K_\nu(x) / \sinh x$ is used. Note that part c of Lemma 3 implies

$$K_\nu(t) \leq \left(\frac{b}{t}\right)^{\nu+1} \frac{\sinh t}{\sinh b} K_\nu(b) \quad (2.5)$$

for all $t \geq b$ and $\nu \in \mathbb{R}$, and this implies that

$$\begin{aligned} R_\nu(a, b) &\leq \frac{c_{a,\nu}}{a^{\nu-1}} \int_b^\infty t^\nu e^{-\frac{a^2+t^2}{2}} \left(\frac{b}{t}\right)^\nu \frac{\sinh(at)}{\sinh(ab)} K_{\nu-1}(ab) dt \\ &= \frac{c_{a,\nu}}{a^{\nu-1}} \frac{b^\nu K_{\nu-1}(ab)}{\sinh(ab)} \int_b^\infty e^{-\frac{a^2+t^2}{2}} \left(\frac{e^{at} - e^{-at}}{2}\right) dt \\ &= \frac{c_{a,\nu}}{a^{\nu-1}} \frac{b^\nu K_{\nu-1}(ab)}{2 \sinh(ab)} \int_b^\infty \left(e^{-\frac{(t-a)^2}{2}} - e^{-\frac{(t+a)^2}{2}}\right) dt \\ &= \frac{c_{a,\nu}}{a^{\nu-1}} \frac{b^\nu K_{\nu-1}(ab)}{2 \sinh(ab)} \sqrt{\frac{\pi}{2}} \left[\operatorname{erfc}\left(\frac{b-a}{\sqrt{2}}\right) - \operatorname{erfc}\left(\frac{b+a}{\sqrt{2}}\right) \right]. \end{aligned}$$

Similarly, from part d of Lemma (3) we get

$$K_\nu(t) \leq \left(\frac{b}{t}\right)^\nu \frac{\cosh t}{\cosh b} K_\nu(b) \quad (2.6)$$

for all $t \geq b$ and $\nu \in \mathbb{R}$. By using the above inequality in view of (1.1) we get

$$\begin{aligned} R_\nu(a, b) &\leq \frac{c_{a,\nu}}{a^{\nu-1}} \int_b^\infty t^\nu e^{-\frac{a^2+t^2}{2}} \left(\frac{b}{t}\right)^{\nu-1} \frac{\cosh(at)}{\cosh(ab)} K_{\nu-1}(ab) dt \\ &= \frac{c_{a,\nu}}{a^{\nu-1}} \frac{b^{\nu-1} K_{\nu-1}(ab)}{2 \cosh(ab)} \int_b^\infty t \left(e^{-\frac{(t-a)^2}{2}} + e^{-\frac{(t+a)^2}{2}}\right) dt \\ &= \frac{c_{a,\nu}}{a^{\nu-1}} \frac{b^{\nu-1} K_{\nu-1}(ab)}{2 \cosh(ab)} \left[e^{-\frac{(b-a)^2}{2}} + e^{-\frac{(b+a)^2}{2}} \right. \\ &\quad \left. + a \sqrt{\frac{\pi}{2}} \left(\operatorname{erfc}\left(\frac{b-a}{\sqrt{2}}\right) - \operatorname{erfc}\left(\frac{b+a}{\sqrt{2}}\right) \right) \right]. \end{aligned}$$

□

Theorem 2 For $\nu > 3/2$ and $b \geq a > 0$, the following inequalities hold true:

$$R_\nu(a, b) \geq \frac{c_{a,\nu}}{a^{\nu-1}} b^{\nu-1} K_{\nu-1}(ab) \sinh(ab) \int_b^\infty \frac{t e^{-\frac{t^2+a^2}{2}}}{\sinh(at)} dt, \quad (2.7)$$

$$R_\nu(a, b) \geq \frac{c_{a,\nu}}{a^{\nu-1}} b^\nu K_{\nu-1}(ab) \cosh(ab) \int_b^\infty \frac{e^{-\frac{t^2+a^2}{2}}}{\cosh(at)} dt. \quad (2.8)$$

Proof Using part e of Lemma 3 with constant $\delta_\nu = \nu$ and proceeding like in the previous theorem we can deduce inequality (2.7). Similarly, part f of Lemma 3 with $\epsilon_\nu = \nu + 1$ yields the inequality (2.8) since part f of Lemma 3 is true for all $\epsilon_\nu \geq \nu + 0.278 \dots$. The choice $\epsilon_\nu = \nu + 1$ is just sake of convenience. □

Remark 4 Using the well-known inequality $\frac{1}{1-e^{-2at}} > 1$ for all $a, t > 0$, we can obtain from the lower bound (2.7) a weaker lower bound, which can be expressed in terms of the complementary error function as follows:

$$\begin{aligned} R_\nu(a, b) &\geq \frac{c_{a,\nu}}{a^{\nu-1}} b^{\nu-1} K_{\nu-1}(ab) \sinh(ab) \int_b^\infty \frac{t e^{-\frac{t^2+a^2}{2}}}{\sinh(at)} dt \\ &\geq 2 \frac{c_{a,\nu}}{a^{\nu-1}} b^{\nu-1} K_{\nu-1}(ab) \sinh(ab) \int_b^\infty t e^{-\frac{t^2+a^2}{2}} e^{-at} dt \\ &= 2 \frac{c_{a,\nu}}{a^{\nu-1}} b^{\nu-1} K_{\nu-1}(ab) \sinh(ab) \left[e^{-\frac{(b+a)^2}{2}} - a \sqrt{\frac{\pi}{2}} \operatorname{erfc} \left(-\frac{(b+a)}{2} \right) \right]. \end{aligned}$$

The best choice for ϵ_ν in part f of Lemma 3 is $\epsilon_\nu = \nu + \theta$ where $\theta = 0.278 \dots$ which gives a tighter lower bound for $R_\nu(a, b)$ than the lower bound given in (2.8), that is precisely

$$R_\nu(a, b) \geq \frac{c_{a,\nu}}{a^{\nu-1}} b^{\nu+\theta-1} K_{\nu-1}(ab) \cosh(ab) \int_b^\infty \frac{t^{1-\theta} e^{-\frac{t^2+a^2}{2}}}{\cosh(at)} dt. \quad (2.9)$$

2.2 Case 2 : $a > b > 0$

Theorem 3 For $a > b > 0$, the following inequalities hold:

$$\begin{aligned} R_\nu(a, b) &\leq 1 - \frac{c_{a,\nu}}{a^{\nu-1}} \sqrt{b} K_{\nu-1}(ab) e^{ab} \\ &\quad \int_a^{b+a} (y-a)^{\nu-\frac{1}{2}} e^{-\frac{y^2}{2}} dy, \end{aligned} \quad (2.10)$$

$$\begin{aligned} R_\nu(a, b) &\leq 1 - \frac{c_{a,\nu}}{a^{\nu-1}} \frac{b^\nu K_{\nu-1}(ab)}{2 \sinh(ab)} \sqrt{\frac{\pi}{2}} \\ &\quad \left[\operatorname{erfc} \left(-\frac{a}{\sqrt{2}} \right) - \operatorname{erfc} \left(\frac{b-a}{\sqrt{2}} \right) - \operatorname{erfc} \left(\frac{a}{\sqrt{2}} \right) + \operatorname{erfc} \left(\frac{b+a}{\sqrt{2}} \right) \right], \end{aligned} \quad (2.11)$$

$$\begin{aligned} R_\nu(a, b) &\leq 1 - \frac{c_{a,\nu}}{a^{\nu-1}} \frac{b^{\nu-1} K_{\nu-1}(ab)}{2 \cosh(ab)} \left(2e^{-\frac{a^2}{2}} - e^{-\frac{(b+a)^2}{2}} - e^{-\frac{(b-a)^2}{2}} \right) \\ &\quad - \frac{c_{a,\nu} b^{\nu-1} K_{\nu-1}(ab)}{2 \cosh(ab) a^{\nu-2}} \sqrt{\frac{\pi}{2}} \\ &\quad \left[-\operatorname{erfc} \left(\frac{b-a}{\sqrt{2}} \right) - 2 \operatorname{erfc} \left(\frac{a}{\sqrt{2}} \right) + \operatorname{erfc} \left(\frac{b+a}{\sqrt{2}} \right) \right], \end{aligned} \quad (2.12)$$

where $\nu \geq 3/2$ in (2.10) and $\nu > 0$ in (2.11) and (2.12).

Proof For $\nu > 1/2$ and $0 < t \leq b$, part a of Lemma 3 implies

$$K_\nu(t) \geq \sqrt{\frac{b}{t}} K_\nu(b) \frac{e^b}{e^t}, \quad (2.13)$$

which in view of (1.2) implies that

$$\begin{aligned} R_\nu(a, b) &\leq 1 - \frac{c_{a,\nu}}{a^{\nu-1}} \int_0^b t^\nu e^{-\frac{a^2+t^2}{2}} \sqrt{\frac{b}{t}} K_{\nu-1}(ab) \frac{e^{ab}}{e^{at}} dt \\ &= 1 - \frac{c_{a,\nu}}{a^{\nu-1}} e^{ab} K_{\nu-1}(ab) \sqrt{b} \int_0^b t^{\nu-\frac{1}{2}} e^{-\frac{(t+a)^2}{2}} dt \\ &= \frac{c_{a,\nu}}{a^{\nu-1}} e^{ab} K_{\nu-1}(ab) \sqrt{b} \int_a^{b+a} (y-a)^{\nu-\frac{1}{2}} e^{-\frac{y^2}{2}} dy. \end{aligned}$$

The inequality (2.11) is obtained by using the monotonicity of the function $t \mapsto t^{\nu+1} K_\nu(t)/\sinh t$. For $0 < t \leq b$ and $\nu \in \mathbb{R}$, part c of Lemma 3 leads to

$$K_\nu(t) \geq \left(\frac{b}{t}\right)^{\nu+1} \frac{\sinh t}{\sinh b} K_\nu(b). \quad (2.14)$$

Using the above inequality in view of (1.2), for $\nu > 0$ we get

$$\begin{aligned} R_\nu(a, b) &\leq 1 - \frac{c_{a,\nu}}{a^{\nu-1}} \int_0^b t^\nu e^{-\frac{a^2+t^2}{2}} \left(\frac{b}{t}\right)^\nu \frac{\sinh(at)}{\sinh(ab)} K_{\nu-1}(ab) dt \\ &= 1 - \frac{c_{a,\nu}}{a^{\nu-1}} \frac{b^\nu K_{\nu-1}(ab)}{\sinh(ab)} \int_0^b e^{-\frac{a^2+t^2}{2}} \left(\frac{e^{at} - e^{-at}}{2}\right) dt. \end{aligned}$$

Hence

$$\begin{aligned} R_\nu(a, b) &\leq 1 - \frac{c_{a,\nu}}{a^{\nu-1}} \frac{b^\nu K_{\nu-1}(ab)}{2 \sinh(ab)} \sqrt{\frac{\pi}{2}} \left[\operatorname{erfc}\left(-\frac{a}{\sqrt{2}}\right) - \operatorname{erfc}\left(\frac{b-a}{\sqrt{2}}\right) \right. \\ &\quad \left. - \operatorname{erfc}\left(\frac{a}{\sqrt{2}}\right) + \operatorname{erfc}\left(\frac{b+a}{\sqrt{2}}\right) \right]. \end{aligned}$$

Similarly, by using part d of Lemma (3) we obtain the upper bound for $R_\nu(a, b)$ given in (2.12). $\square$

Theorem 4 For $\nu > 3/2$ and $a > b > 0$, the following inequalities hold:

$$R_\nu(a, b) \geq 1 - \frac{c_{a,\nu}}{a^{\nu-1}} b^{\nu-1} K_{\nu-1}(ab) \sinh(ab) \int_0^b \frac{t e^{-\frac{t^2+a^2}{2}}}{\sinh at} dt, \quad (2.15)$$

$$R_\nu(a, b) \geq 1 - \frac{c_{a,\nu}}{a^{\nu-1}} b^\nu K_{\nu-1}(ab) \cosh(ab) \int_0^b \frac{e^{-\frac{t^2+a^2}{2}}}{\cosh at} dt. \quad (2.16)$$

Proof Using part e of Lemma 3 with constant $\delta_\nu = \nu$ and proceeding similarly as in the previous theorem we can obtain inequality (2.15). Similarly, part f of Lemma 3 with $\epsilon_\nu = \nu + 1$ yields the inequality (2.16) since part f of Lemma 3 is true for all $\epsilon_\nu \geq \nu + 0.278 \dots$. The choice $\epsilon_\nu = \nu + 1$ is just sake of convenience. $\square$

Remark 5 The best choice for ϵ_v in part f of Lemma 3 is $\epsilon_v = v + \theta$ where $\theta = 0.278 \dots$ which gives a tighter lower bound for $R_v(a, b)$ than the lower bound given in (2.16), that is precisely

$$R_v(a, b) \geq 1 - \frac{c_{a,v}}{a^{v-1}} b^{v+\theta-1} K_{v-1}(ab) \cosh(ab) \int_0^b \frac{t^{1-\theta} e^{-\frac{t^2+a^2}{2}}}{\cosh at} dt. \quad (2.17)$$

For $v > 3/2$, we can get from the lower bound (2.16) for $R_v(a, b)$ a weaker lower bound, which can be expressed in terms of complementary error function as follows:

$$\begin{aligned} R_v(a, b) &\geq 1 - \frac{c_{a,v}}{a^{v-1}} b^v K_{v-1}(ab) \cosh(ab) \int_0^b \frac{e^{-\frac{t^2+a^2}{2}}}{\cosh at} dt \\ &\geq 1 - \frac{c_{a,v}}{a^{v-1}} b^v K_{v-1}(ab) \cosh(ab) \int_0^b e^{-\frac{(t+a)^2}{2}} dt \\ &= 1 - \frac{c_{a,v}}{a^{v-1}} b^v K_{v-1}(ab) \cosh(ab) \sqrt{\frac{\pi}{2}} \left[\operatorname{erfc} \left(\frac{a}{\sqrt{2}} \right) - \operatorname{erfc} \left(\frac{b+a}{\sqrt{2}} \right) \right]. \end{aligned}$$

3 Sharpness of the bounds and comparison with other existing bounds

In this section, we discuss the tightness of the bounds obtained in Sect. 2 and compare these bounds with the bounds given in [4]. All the bounds for $R_v(a, b)$ stated in the previous section and the section 2 of [4] are obtained by using the bounds for $K_v(t)$. Thus we compare the bounds for $K_v(t)$, which give immediately the comparison of the bounds for $R_v(a, b)$.

3.1 Comparison with other existing bounds

Let $b \geq a > 0$. For $v \geq 1/2$, Baricz et al. [4] obtained the following upper bound for $R_v(a, b)$

$$R_v(a, b) \leq \frac{c_{a,v}}{(ab)^{v-1}} K_{v-1}(ab) e^{ab} \int_{b+a}^{\infty} (y-a)^{2v-1} e^{-\frac{y^2}{2}} dy \quad (3.1)$$

by using the inequality

$$K_v(t) \leq \left(\frac{t}{b} \right)^v \frac{e^b}{e^t} K_v(b).$$

In this paper, by using the inequality

$$K_v(t) \leq \sqrt{\frac{b}{t}} K_v(b) \frac{e^b}{e^t}$$

for $\nu \geq 1/2$, a new upper bound is obtained for $R_\nu(a, b)$ in the case when $\nu \geq 3/2$

$$R_\nu(a, b) \leq \frac{c_{a,\nu}}{a^{\nu-1}} \sqrt{b} K_{\nu-1}(ab) e^{ab} \int_{b+a}^{\infty} (y-a)^{\nu-\frac{1}{2}} e^{-\frac{y^2}{2}} dy. \quad (3.2)$$

On the other hand, for $\nu > 3/2$ we have

$$K_\nu(t) \leq \sqrt{\frac{b}{t}} K_\nu(b) \frac{e^b}{e^t} \leq \left(\frac{t}{b}\right)^\nu \frac{e^b}{e^t} K_\nu(b).$$

Hence the new bound (3.2) is sharper than the upper bound (3.1) obtained by Baricz et al. [4] for $\nu > 3/2$. For $\nu > 0$, Baricz et al. [4] obtained the following upper bound for $R_\nu(a, b)$

$$R_\nu(a, b) \leq \frac{c_{a,\nu}}{(ab)^{\nu-1}} \frac{K_{\nu-1}(ab)}{e^{ab}} \int_{b-a}^{\infty} (y+a)^{2\nu-1} e^{-\frac{y^2}{2}} dy \quad (3.3)$$

by using the monotone decreasing property of the function $t \mapsto t^{-\nu} e^{-t} K_\nu(t)$ on $(0, \infty)$ and for $t \geq b$ the inequality

$$K_\nu(t) \leq \left(\frac{t}{b}\right)^\nu \frac{e^t}{e^b} K_\nu(b). \quad (3.4)$$

Note that $t \mapsto e^{-t} \cosh t$ and $t \mapsto t \coth t$ are decreasing and increasing functions, respectively, on $(0, \infty)$. Thus for $t \geq b > 0$

$$\frac{b}{t} \frac{\sinh t}{\sinh b} \leq \frac{\cosh t}{\cosh b} \leq \frac{e^t}{e^b}.$$

Consequently for $t \geq b$ and $\nu > 0$

$$\begin{aligned} K_\nu(t) &\leq \left(\frac{b}{t}\right)^{\nu+1} \frac{\sinh t}{\sinh b} K_\nu(b) \leq \left(\frac{b}{t}\right)^\nu \frac{\cosh t}{\cosh b} K_\nu(b) \\ &\leq \left(\frac{b}{t}\right)^\nu \frac{e^t}{e^b} K_\nu(b) \leq \left(\frac{t}{b}\right)^\nu \frac{e^t}{e^b} K_\nu(b). \end{aligned} \quad (3.5)$$

In view of (3.5), (2.5), and (2.6), we conclude that the upper bound in (2.2) is sharper than the upper bound in (2.3). Moreover, in view of (3.4) and (3.5), we conclude that the new bound (2.3) is sharper than the upper bound in (3.3) obtained by Baricz et al. [4].

Remark 6 In view of (3.5) for $\nu > 0$ and $t \geq b > 0$, we get

$$K_\nu(a, b) \leq \left(\frac{b}{t}\right)^\nu \frac{e^t}{e^b} K_\nu(b).$$

By using this we get the following new upper bound for $R_v(a, b)$

$$R_v(a, b) \leq c_{a,v} \left(\frac{b}{a}\right)^{v-1} \frac{K_{v-1}(ab)}{e^{ab}} \left[e^{-\frac{(b-a)^2}{2}} - a \sqrt{\frac{\pi}{2}} \operatorname{erfc} \left(\frac{b-a}{\sqrt{2}} \right) \right]. \quad (3.6)$$

In view of (3.5), we conclude that the new upper bound (3.6) is sharper than the upper bound (3.3) obtained by Baricz et al. [4].

For $v \geq 3/2$, Baricz et al. [4] obtained the following lower bound for $R_v(a, b)$

$$R_v(a, b) \geq c_{a,v} \left(\frac{b}{a}\right)^{v-1} e^{ab} K_{v-1}(ab) \left[e^{-\frac{(b+a)^2}{2}} - a \sqrt{\frac{\pi}{2}} \operatorname{erfc} \left(\frac{b+a}{\sqrt{2}} \right) \right] \quad (3.7)$$

by using the monotone increasing property of the function $t \mapsto e^t t^v K_v(t)$ on $(0, \infty)$ for all $v \geq 1/2$, that is, for $t \geq b > 0$ and $v \geq 1/2$ the inequality

$$K_v(t) \geq \left(\frac{b}{t}\right)^v \frac{e^b}{e^t} K_v(b). \quad (3.8)$$

Note that $t \mapsto e^{-t} \sinh t$ and $t \mapsto t \coth t$ both are increasing functions on $(0, \infty)$. Thus for $t \geq b > 0$

$$\frac{e^b}{e^t} \geq \frac{\sinh b}{\sinh t} \geq \left(\frac{b}{t}\right) \frac{\cosh b}{\cosh t}.$$

Consequently,

$$K_v(t) \geq \left(\frac{b}{t}\right)^v \frac{e^b}{e^t} K_v(b) \geq \left(\frac{b}{t}\right)^v \frac{\sinh b}{\sinh t} K_v(b) \geq \left(\frac{b}{t}\right)^{v+1} \frac{\cosh b}{\cosh t} K_v(b). \quad (3.9)$$

In view of (3.9), the new lower bound in (2.7) is sharper than the new lower bound in (2.8). Moreover, in view of (3.9) and (3.8), we conclude that the lower bound in (3.7) obtained by Baricz et al. [4] is sharper than the new lower bound in (2.7). It is easy to see that the lower bound in (2.9) is sharper than the lower bound in (2.8) but weaker than the lower bound in (3.7) which is obtained by Baricz et al. [4].

Similarly to the previous discussion, for the case when $a > b > 0$ we can verify the followings:

- i. The upper bound in (2.11) is sharper than the upper bound in (2.12), and the upper bound in (2.12) is sharper than the upper bound in (3.10) obtained by Baricz et al. [4]

$$R_v(a, b) \leq 1 - \frac{c_{a,v}}{(ab)^{v-1}} \frac{K_{v-1}(ab)}{e^{ab}} \int_{-a}^{b-a} (y+a)^{2v-1} e^{-\frac{y^2}{2}} dy \text{ for } v > 0 \quad (3.10)$$

ii. The upper bound in (2.10) is sharper than the upper bound in (3.11) given in [4]

$$R_v(a, b) \leq 1 - \frac{c_{a,v}}{(ab)^{v-1}} K_{v-1}(ab) e^{ab} \times \int_a^{b+a} (y-a)^{2v-1} e^{-\frac{y^2}{2}} dy \quad \text{for } v > \frac{1}{2}. \quad (3.11)$$

iii. The lower bound in (2.15) is sharper than the lower bound in (2.16), but weaker than the lower bound in (3.12) obtained by Baricz et al. [4]

$$R_v(a, b) \geq 1 - c_{a,v} \left(\frac{b}{a}\right)^{v-1} e^{ab} K_{v-1}(ab) \times \left[e^{-\frac{a^2}{2}} - e^{-\frac{(b+a)^2}{2}} - a \sqrt{\frac{\pi}{2}} \left(\operatorname{erfc}\left(\frac{a}{\sqrt{2}}\right) - \operatorname{erfc}\left(\frac{b+a}{\sqrt{2}}\right) \right) \right]. \quad (3.12)$$

iv. The lower bound in (2.17) is sharper than the lower bound in (2.16) but weaker than the lower bound in (3.12).

Remark 7 By using the monotone decreasing property of the functions $t \mapsto t^{-v} e^{-t} K_v(t)$, $t \mapsto e^{-t} \cosh t$ and the monotone increasing property of the function $t \mapsto t \cosh t$ we get

$$\begin{aligned} K_v(t) &\geq \left(\frac{b}{t}\right)^{v+1} \frac{\sinh t}{\sinh b} K_v(b) \geq \left(\frac{b}{t}\right)^v \frac{\cosh t}{\cosh b} K_v(b) \\ &\geq \left(\frac{b}{t}\right)^v \frac{e^t}{e^b} K_v(b) \geq \left(\frac{t}{b}\right)^v \frac{e^t}{e^b} K_v(b). \end{aligned} \quad (3.13)$$

By comparing the first and the last one, we get the following new upper bound for $v > 0$,

$$R_v(a, b) \leq 1 - c_{a,v} \left(\frac{b}{a}\right)^{v-1} \frac{K_{v-1}(ab)}{e^{ab}} \times \left[e^{-\frac{a^2}{2}} - e^{-\frac{(b-a)^2}{2}} - a \sqrt{\frac{\pi}{2}} \left(\operatorname{erfc}\left(\frac{-a}{\sqrt{2}}\right) - \operatorname{erfc}\left(\frac{b-a}{\sqrt{2}}\right) \right) \right]. \quad (3.14)$$

Clearly, the new upper bound in (3.14) is sharper than the upper bound in (3.10) obtained by Baricz et al. [4].

3.2 Tightness of the bounds for $b \rightarrow 0$ and $b \rightarrow \infty$

In this subsection, the tightness of the new upper and lower bounds are discussed. Section 3.1 shows that all the new upper bounds obtained in this study are sharper

than the upper bounds obtained by Baricz et al. [4]. Moreover, the upper bounds obtained by Baricz et al. [4] are tight bounds. Consequently, all the new upper bounds obtained in this study are also tight. It is also possible to verify that the relative error for the upper bounds tend to zero as b approaches infinity. In this subsection, proofs are discussed to show that all the new lower bounds are tight as well as the relative error in the lower bounds tends to zero as $b \rightarrow \infty$.

The following limits show the tightness of the relative error for the lower bounds:

$$\begin{aligned} \lim_{b \rightarrow \infty} \left[b^{\nu-\frac{1}{2}} \int_b^{\infty} \frac{e^{-\frac{t^2+a^2}{2}}}{\cosh(at)} dt \right] \\ = \lim_{b \rightarrow \infty} \frac{(e^{-\frac{b^2+a^2}{2}} / \cosh(ab))}{(-\nu + 1/2)b^{-\nu-1/2}} \\ = \lim_{b \rightarrow \infty} \frac{e^{-\frac{(b+a)^2}{2}}}{(-\nu + 1/2)b^{-\nu-1/2}} = 0. \end{aligned} \quad (3.15)$$

and

$$\lim_{b \rightarrow \infty} \left[\frac{\int_b^{\infty} (e^{-\frac{a^2+t^2}{2}} / \cosh(at)) dt}{\int_{b+a}^{\infty} e^{-\frac{t^2}{2}} dt} \right] = 2. \quad (3.16)$$

Using the limit (3.15), we conclude that the lower bound L_2 in (2.8) tends to zero as $b \rightarrow \infty$. Moreover, by using (3.16) and the following asymptotic relation

$$R_\nu(a, b) \sim c_{a,\nu} \sqrt{\frac{\pi}{2}} \left(\frac{b}{a} \right)^{\nu-\frac{1}{2}} \int_{b+a}^{\infty} e^{-\frac{y^2}{2}} dy \quad (3.17)$$

we can verify that the relative error $(L_2 - R_\nu(a, b))/R_\nu(a, b)$ tends to zero as $b \rightarrow \infty$.

As the lower bound L_1 in (2.7) is sharper than the lower bound L_2 in (2.8), L_1 is also tight as $b \rightarrow \infty$. Similarly, we can verify that the relative error in L_1 approaches to zero as $b \rightarrow \infty$. Similarly, for the case $a > b > 0$ all the new lower bounds (2.15) and (2.16) are also tight as well as the relative error tends to zero as $b \rightarrow 0$.

4 Numerical results

In this section, we present numerical results in order to compare the various upper and lower bounds of $R_\nu(a, b)$. To draw the figures for the bounds of $R_\nu(a, b)$ we have used Mathematica 8.0.

Fig. 1 Numerical results for $R_\nu(a, b)$ for $b \geq a = 1$ and $\nu = 3$

Fig. 2 Numerical results for $R_\nu(a, b)$ for $b \geq a = 1$ and $\nu = 4$

4.1 Case 1 : $b \geq a > 0$

Let the upper bounds given in (2.1), (2.2), (2.3), (3.3), and (3.1) be denoted by U_1 , U_2 , U_3 , U_4 , and U_5 , respectively. Let L_1 , L_2 , L_3 , and L_4 denote the lower bounds (2.7), (2.8), (3.7) and [4, Eq. 2.22].

For the choice $\nu = 3$ and $a = 1$, Fig. 1 illustrates the comparison between all the upper bounds with $R_3(1, b)$ in the interval $(1, 2.5)$. Let R_3 denote $R_3(1, b)$. Figure 1 supports the theoretical results obtained in Sect. 3. Note that the new upper bound U_1 is sharper than all the other upper bounds. For large b , the upper bound U_5 is sharper than the upper bound U_3 .

For the choice $\nu = 4$ and $a = 1$, Fig. 2 illustrates the comparison between all the upper bounds with $R_4(1, b)$ in the interval $(1, 2.5)$. Let R_4 denote $R_4(1, b)$. Figure 2

Fig. 3 Numerical results for $R_\nu(a, b)$ for $b \leq a = 3$ and $\nu = 2$

Fig. 4 Numerical results for $R_\nu(a, b)$ for $b \leq a = 1$ and $\nu = 3$

supports the theoretical results obtained in Sect. 3. Clearly, both lower bounds L_1 and L_2 are sharper than the lower bound L_4 .

4.2 Case 2 : $a > b > 0$

Let the upper bounds given in (2.10), (2.11), (2.12), (3.10), and (3.11) be denoted by U_6 , U_7 , U_8 , U_9 , and U_{10} , respectively. Let L_1 , L_2 , L_3 , and L_4 denote the lower bounds (2.15), (2.16), (3.12), and [4, Eq. 2.23].

For the choice $\nu = 2$ and $a = 3$, Fig. 3 illustrates the comparison between all the upper bounds for $R_2(3, b)$ in the interval $(0, 3)$. Let R_2 denote $R_2(3, b)$. Figure 3 supports the theoretical results obtained in Sect. 3. Clearly, the new upper bounds U_7

and U_8 are better than the upper bound U_{10} for smaller range of b but for large values of b the upper bound U_{10} is better than the upper bounds U_7 and U_8 .

For the choice $\nu = 3$ and $a = 1$, Fig. 4 illustrates the comparison between all the lower bounds with $R_3(1, b)$ in the interval $(0, 1)$. Let R_3 denote $R_3(1, b)$. Figure 4 supports the theoretical results obtained in Sect. 3.

Funding Open access funding provided by Óbuda University.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Baricz, Á., Sun, Y.: New bounds for the generalized Marcum Q -function. *IEEE Trans. Inform. Theory* **55**(7), 3091–3100 (2009)
2. Baricz, Á., Sun, Y.: Bounds for the generalized Marcum Q -function. *Appl. Math. Comput.* **217**(5), 2238–2250 (2010)
3. Baricz, Á., Ponnusamy, S., Vuorinen, M.: Functional inequalities for modified Bessel functions. *Expo Math.* **29**(4), 399–414 (2011)
4. Baricz, Á., Bisht, N., Singh, S., Vijesh, A.: The generalized Marcum function of the second kind: monotonicity patterns and tight bounds. *J. Comput. Appl. Math.* **382** (2021). Art. 113093
5. Ismail, M.E.H., Muldoon, M.E.: Monotonicity of the zeros of a cross-product of Bessel functions. *SIAM J. Math. Anal.* **9**(4), 759–767 (1978)
6. Nadarajah, S.: A modified Bessel distribution of the second kind. *Statistica* **47**(4), 405–413 (2007)
7. Olver, F.W.J., Lozier, D.W., Boisvert, R.F., Clark, C.W. (eds.): *NIST Handbook of Mathematical Functions*. Cambridge University Press, Cambridge (2010). (<http://dlmf.nist.gov/>)
8. Soni, R.P.: On an inequality for modified Bessel functions. *J. Math. Phys.* **44**, 406–407 (1965)
9. Sun, Y., Baricz, Á.: Inequalities for the generalized Marcum Q -function. *Appl. Math. Comput.* **203**(1), 134–141 (2008)
10. Yang, Z., Tian, J.F.: Monotonicity rules for the ratio of two Laplace transforms with applications. *J. Math. Anal. Appl.* **470**, 821–845 (2019)
11. Yang, Z.H., Zheng, S.Z.: The monotonicity and convexity for the ratios of modified Bessel functions of the second kind and applications. *Proc. Am. Math. Soc.* **145**(7), 2943–2958 (2017)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Affiliations

Árpád Baricz^{1,2} · Nitin Bisht³ · Sanjeev Singh³ · V. Antony Vijesh³

Nitin Bisht
nitin68bisht@gmail.com

Sanjeev Singh
snjvsng@iiti.ac.in

V. Antony Vijesh
vijesh@iiti.ac.in

- ¹ Department of Economics, Babeş-Bolyai University, 400591 Cluj-Napoca, Romania
- ² Institute of Applied Mathematics, Óbuda University, Budapest 1034, Hungary
- ³ Discipline of Mathematics, Indian Institute of Technology Indore, Indore 453552, India