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Dual addition formulas: the case of continuous q-ultraspherical and q-Hermite polynomials

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This paper is dedicated to the memory of Dick Askey

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Abstract
We settle the dual addition formula for continuous q-ultraspherical polynomials as an expansion in terms of special q-Racah polynomials for which the constant term is given by the linearization formula for the continuous q-ultraspherical polynomials. In a second proof we derive the dual addition formula from the Rahman–Verma addition formula for these polynomials by using the self-duality of the polynomials. We also consider the limit case of continuous q-Hermite polynomials.

Keywords Dual addition formulas · Continuous q-ultraspherical polynomials · q-Racah polynomials · Continuous q-Hermite polynomials

Mathematics Subject Classification 33D45

1 Introduction

In this paper, as a natural continuation of our recent derivation [18] of the dual addition formula for ultraspherical polynomials, we derive the dual addition formula for continuous q-ultraspherical polynomials. We give two different proofs. The first proof is a perfect q-analogue of the derivation in [18]. Every step of the proof yields in the limit for $q \to 1$ the corresponding step in [18]. The second proof exploits the self-duality of the continuous q-ultraspherical polynomials. Then the dual addition formula easily follows from the known addition formula [22] for these polynomials.

Addition formulas are closely related to product formulas. For instance, the *addition formula for Legendre polynomials* [21, (18.18.9)]

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$$P_n(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos\phi)=P_n(\cos\theta_1)\,P_n(\cos\theta_2)$$
$$+2\sum_{k=1}^{n}\frac{(n-k)!\,(n+k)!}{2^{2k}(n!)^2}\,(\sin\theta_1)^k\,P_{n-k}^{(k,k)}(\cos\theta_1)\,(\sin\theta_2)^k\,P_{n-k}^{(k,k)}(\cos\theta_2)\cos(k\phi) \tag{1.1}$$

gives the Fourier-cosine expansion of the left-hand side as a function of ϕ. Integration with respect to ϕ over $[0,\pi]$ gives the constant term in this expansion, which is the *product formula for Legendre polynomials* [21, (18.17.6)]

$$P_n(\cos\theta_1)\,P_n(\cos\theta_2)=\frac{1}{\pi}\int_0^{\pi}P_n(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos\phi)\,d\phi. \tag{1.2}$$

Two formulas involving Legendre polynomials $P_n(x)$ (or more generally some orthogonal polynomials $p_n(x)$) are called *dual* to each other if the roles of n and x in the second formula are interchanged in comparison with the first formula. The formula dual to the product formula (1.2) is the *linearization formula*, which expands the product $P_\ell(x)P_m(x)$ in terms of Legendre polynomials $P_k(x)$. This expansion is a sum running from $k=|\ell-m|$ to $k=\ell+m$, where only terms with $\ell+m-k$ even will occur since $P_n(-x)=(-1)^nP_n(x)$. The linearization formula for Legendre polynomials is explicitly known (see [21, (18.18.22)] for $\lambda=\frac{1}{2}$ together with [21, (18.7.9)]):

$$\begin{aligned}P_\ell(x)P_m(x)=\sum_{j=0}^{\min(\ell,m)}&\frac{(\frac{1}{2})_j(\frac{1}{2})_{\ell-j}(\frac{1}{2})_{m-j}(\ell+m-j)!}{j!\,(\ell-j)!\,(m-j)!\,(\frac{3}{2})_{\ell+m-j}}\\&\times(2(\ell+m-2j)+1)\;P_{\ell+m-2j}(x),\end{aligned} \tag{1.3}$$

where $(a)_k$ is the shifted factorial, see below. Dick Askey, in his lectures at conferences, often raised the problem to find an addition type formula associated with (1.3) in a similar way as the addition formula (1.1) is associated with the product formula (1.2). The author finally solved this in [18] by recognizing the coefficient of $P_{\ell+m-2j}(x)$ in (1.3) as the weight of a special Racah polynomial [16, (9.2.1)] depending on j, and then finding the expansion of $P_{\ell+m-2j}(x)$ in terms of these Racah polynomials. More generally, the same idea worked out well in [18] for ultraspherical polynomials.

While (1.1), (1.2), (1.3), and their generalizations to ultraspherical polynomials, are formulas established long ago and staying within the realm of classical orthogonal polynomials, it is remarkable that the dual addition formula steps out from this and needs Racah polynomials, which live high up in the Askey scheme. Parallel to the Askey scheme there is the much larger q-Askey scheme[1]. Families of orthogonal polynomials in the Askey scheme are limit cases of families in the q-Askey scheme. The continuous q-ultraspherical polynomials form the family which is the q-analogue of the ultraspherical polynomials. Moreover, the q-analogues of (1.1), (1.2) and (1.3) for these polynomials are available in the literature. The continuous q-ultraspherical

[1] See http://homepage.tudelft.nl/11r49/book.html for charts of these schemes.

polynomials also have the property of *self-duality*, which is lost in the limit to $q = 1$. This notion means that, for a suitable function σ, an orthogonal polynomial $p_n(x)$ has the property that $p_n(\sigma(m)) = p_m(\sigma(n))$ $(m, n = 0, 1, \ldots)$. With all this material available there is a clear road to the derivation of the dual addition formula for these polynomials.

The contents of the paper are as follows. Section 2 summarizes the results from [18] about the dual addition formula for ultraspherical polynomials. The necessary preliminaries about special orthogonal polynomials in the q-case are given in Sect. 3. The new results of the paper appear in Sect. 4. It contains the two proofs of the dual addition formula for continuous q-ultraspherical polynomials. Finally the limit case for continuous q-Hermite polynomials is considered in Sect. 5.

Note For definition and notation of (q-)shifted factorials and (q-)hypergeometric series we follow [14, §1.2]. We will only need terminating series:

$$
{}_rF_s\left(\begin{matrix}-n, a_2, \ldots, a_r\\ b_1, \ldots, b_s\end{matrix}; z\right) := \sum_{k=0}^{n} \frac{(-n)_k}{k!} \frac{(a_2, \ldots, a_r)_k}{(b_1, \ldots, b_s)_k} z^k,
$$

$$
\begin{aligned}
&{}_r\phi_s\left(\begin{matrix}q^{-n}, a_2, \ldots, a_r\\ b_1, \ldots, b_s\end{matrix}; q, z\right)\\
&\quad := \sum_{k=0}^{n} \frac{(q^{-n}; q)_k}{(q; q)_k} \frac{(a_2, \ldots, a_r; q)_k}{(b_1, \ldots, b_s; q)_k} \left((-1)^k q^{\frac{1}{2}k(k-1)}\right)^{s-r+1} z^k.
\end{aligned}
$$

Here $(b_1, \ldots, b_s)_k := (b_1)_k \ldots (b_s)_k$ with $(b)_k := b(b+1)\cdots(b+k-1)$ the shifted factorial, and $(b_1, \ldots, b_s; q)_k := (b_1; q)_k \ldots (b_s; q)_k$ with $(b; q)_k := (1-b)(1-qb)\ldots(1-q^{k-1}b)$ the q-shifted factorial.

For formulas on orthogonal polynomials in the (q-)Askey scheme we will often refer to Chapters 9 and 14 in [16].

2 The dual addition formula for ultraspherical polynomials

Here we summarize the results of [18]. We write *ultraspherical polynomials* as

$$
R_n^{\alpha}(x) := \frac{P_n^{(\alpha,\alpha)}(x)}{P_n^{(\alpha,\alpha)}(1)} = \frac{C_n^{\left(\alpha+\frac{1}{2}\right)}(x)}{C_n^{(\alpha+\frac{1}{2})}(1)} = {}_2F_1\left(\begin{matrix}-n, n+2\alpha+1\\ \alpha+1\end{matrix}; \tfrac{1}{2}(1-x)\right), \quad (2.1)
$$

where $C_n^{(\lambda)}(x)$ is the standard notation [16, §9.8.1] for ultraspherical polynomials and $P_n^{(\alpha,\beta)}(x)$ is a Jacobi polynomial [16, §9.8].

We will consider *Racah polynomials* [16, §9.2]

$$
\begin{aligned}
&R_n\left(x(x+\gamma+\delta+1); \alpha, \beta, \gamma, \delta\right)\\
&\quad := {}_4F_3\left(\begin{matrix}-n, n+\alpha+\beta+1, -x, x+\gamma+\delta+1\\ \alpha+1, \beta+\delta+1, \gamma+1\end{matrix}; 1\right) \quad (2.2)
\end{aligned}
$$

for $\gamma = -N-1$, where $N \in \{1, 2, \ldots\}$, and for $n \in \{0, 1, \ldots, N\}$. These are orthogonal polynomials on the finite quadratic set $\{x(x+\gamma+\delta+1) \mid x \in \{0, 1, \ldots, N\}\}$:

$$\sum_{x=0}^{N} (R_m R_n)\,(x(x+\gamma+\delta+1); \alpha, \beta, \gamma, \delta)\ w_{\alpha,\beta,\gamma,\delta}(x) \\ = h_{n;\alpha,\beta,\gamma,\delta}\,\delta_{m,n} \quad (m, n \in \{0, 1, \ldots, N\})$$

with

$$w_{\alpha,\beta,\gamma,\delta}(x) = \frac{(\alpha+1)_x(\beta+\delta+1)_x(\gamma+1)_x(\gamma+\delta+1)_x}{(-\alpha+\gamma+\delta+1)_x(-\beta+\gamma+1)_x(\delta+1)_x\,x!}\,\frac{\gamma+\delta+1+2x}{\gamma+\delta+1}, \tag{2.3}$$

$$\frac{h_{n;\,\alpha,\beta,\gamma,\delta}}{h_{0;\,\alpha,\beta,\gamma,\delta}} = \frac{\alpha+\beta+1}{\alpha+\beta+2n+1}\,\frac{(\beta+1)_n(\alpha+\beta-\gamma+1)_n(\alpha-\delta+1)_n\,n!}{(\alpha+1)_n(\alpha+\beta+1)_n(\beta+\delta+1)_n(\gamma+1)_n},$$

$$h_{0;\,\alpha,\beta,\gamma,\delta} = \sum_{x=0}^{N} w_{\alpha,\beta,\gamma,\delta}(x) = \frac{(\alpha+\beta+2)_N(-\delta)_N}{(\alpha-\delta+1)_N(\beta+1)_N} \qquad (\gamma = -N-1). \tag{2.4}$$

The linearization formula for ultraspherical polynomials, see [5, (5.7)], can be written as

$$R_\ell^\alpha(x)R_m^\alpha(x) = \frac{\ell!\,m!}{(2\alpha+1)_\ell(2\alpha+1)_m} \sum_{j=0}^{\min(\ell,m)} \frac{\ell+m+\alpha+\frac12-2j}{\alpha+\frac12} \\ \times \frac{(\alpha+\frac12)_j(\alpha+\frac12)_{\ell-j}(\alpha+\frac12)_{m-j}(2\alpha+1)_{\ell+m-j}}{j!\,(\ell-j)!\,(m-j)!\,(\alpha+\frac32)_{\ell+m-j}} \\ \times R_{\ell+m-2j}^\alpha(x). \tag{2.5}$$

Assume that $\alpha > -\frac12$ and that, without loss of generality, $\ell \ge m$. By (2.3) and (2.4) formula (2.5) can be rewritten as

$$R_\ell^\alpha(x)R_m^\alpha(x) = \sum_{j=0}^{m} \frac{w_{\alpha-\frac12,\alpha-\frac12,-m-1,-\ell-\alpha-\frac12}(j)}{h_{0;\,\alpha-\frac12,\alpha-\frac12,-m-1,-\ell-\alpha-\frac12}}\,R_{\ell+m-2j}^\alpha(x) \quad (\ell \ge m). \tag{2.6}$$

This can be considered as giving the constant term of an expansion of $R_{\ell+m-2j}^{(\alpha,\alpha)}(x)$ as a function of j in terms of the following special case of Racah polynomials (2.2):

$$R_n\left(j\left(j-\ell-m-\alpha-\tfrac12\right);\ \alpha-\tfrac12, \alpha-\tfrac12, -m-1, -\ell-\alpha-\tfrac12\right) \\ = {}_4F_3\left(\begin{matrix}-n, n+2\alpha, -j, j-\ell-m-\alpha-\frac12\\ \alpha+\frac12, -\ell, -m\end{matrix}; 1\right).$$

The full expansion is the *dual addition formula for ultraspherical polynomials*:

$$
\begin{aligned}
&R^{\alpha}_{\ell+m-2j}(x)\\
&\quad=\sum_{k=0}^{\min(l,m)} \frac{\alpha+k}{\alpha+\frac12 k}\,\frac{(-\ell)_k(-m)_k\,(2\alpha+1)_k}{2^{2k}(\alpha+1)_k^2\,k!}\\
&\qquad\times\left(x^2-1\right)^k R^{\alpha+k}_{\ell-k}(x)\,R^{\alpha+k}_{m-k}(x)\\
&\qquad\times R_k\left(j(j-\ell-m-\alpha-\tfrac12);\,\alpha-\tfrac12,\alpha-\tfrac12,-m-1,-\ell-\alpha-\tfrac12\right),\\
&\qquad j\in\{0,1,\ldots,m\}.
\end{aligned}
\tag{2.7}
$$

For $j=0$ this becomes

$$
R^{\alpha}_{\ell+m}(x)=\sum_{k=0}^{\min(l,m)} \frac{\alpha+k}{\alpha+\frac12 k}\,\frac{(-\ell)_k(-m)_k(2\alpha+1)_k}{2^{2k}(\alpha+1)_k^2\,k!}\,(x^2-1)^k\,R^{\alpha+k}_{\ell-k}(x)\,R^{\alpha+k}_{m-k}(x).
\tag{2.8}
$$

Formula (2.8) was first given by Carlitz [11, (3)]. It can be rewritten as a matrix decomposition $S=LDU$ with S symmetric, L lower triangular, its transpose $U=L^{\mathsf T}$ upper triangular and D diagonal. Cagliero & Koornwinder [8, Theorem 4.1 for $\alpha=\beta$] earlier gave the inverse of the matrix L.

3 Some *q*-hypergeometric orthogonal polynomials

3.1 Askey–Wilson polynomials

We will use the following standardization and notation for *Askey–Wilson polynomials*:

$$
R_n[z]=R_n[z;a,b,c,d\,|\,q]:={}_4\phi_3\left(\begin{matrix}q^{-n},q^{n-1}abcd,az,az^{-1}\\ ab,ac,ad\end{matrix};q,q\right).
\tag{3.1}
$$

These are symmetric Laurent polynomials of degree n in z, so they are ordinary polynomials of degree n in $x:=\frac12(z+z^{-1})$. The polynomials (3.1) are related to the Askey–Wilson polynomials $p_n(x;a,b,c,d\,|\,q)$ in usual notation [7, (1.15)], [16, (14.1.1)] by

$$
R_n\left[z;a,b,c,d\,|\,q\right]=\frac{a^n}{(ab,ac,ad;q)_n}\,p_n\left(\tfrac12\left(z+z^{-1}\right);a,b,c,d\,|\,q\right).
\tag{3.2}
$$

If $|a|,|b|,|c|,|d|\le 1$ such that pairwise products of a,b,c,d are not equal to 1 and such that non-real parameters occur in complex conjugate pairs, then the Askey–Wilson polynomials are orthogonal with respect to a non-negative weight function on $x=\frac12(z+z^{-1})\in[-1,1]$. For convenience we give this orthogonality in the variable z on the unit circle, where the integrand is invariant under $z\to z^{-1}$:

$$
\int_{|z|=1} R_m[z]\,R_n[z]\,w[z]\,\frac{\mathrm{d}z}{\mathrm{i}z}=h_n\,\delta_{m,n},
\tag{3.3}
$$

where

$$w[z] = w[z; a, b, c, d; q] = \left| \frac{(z^2; q)_\infty}{(az, bz, cz, dz \,|\, q)_\infty} \right|^2, \tag{3.4}$$

$$h_0 = h_0[a, b, c, d \,|\, q] = \frac{4\pi (abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty}, \tag{3.5}$$

and where the explicit expression for h_n can be obtained from [16, (14.1.2)] together with (3.2).

3.2 Continuous q-ultraspherical polynomials

The continuous q-ultraspherical polynomials are a one-parameter subfamily of the Askey–Wilson polynomials (3.1). For them we will use the following standardization and notation:

$$\begin{aligned} R_n^{\beta;q}[z] = R_n^{\beta;q}\left(\tfrac{1}{2}(z+z^{-1})\right) &:= R_n\left[z; q^{\frac14}\beta^{\frac12}, q^{\frac34}\beta^{\frac12}, -q^{\frac14}\beta^{\frac12}, -q^{\frac34}\beta^{\frac12} \,|\, q\right] \\ &= {}_4\phi_3\left(\begin{matrix} q^{-n}, \beta^2 q^{n+1}, q^{\frac14}\beta^{\frac12} z, q^{\frac14}\beta^{\frac12} z^{-1} \\ \beta q, -\beta q^{\frac12}, -\beta q \end{matrix}; q, q\right). \end{aligned} \tag{3.6}$$

The polynomials (3.6) are related to the continuous q-ultraspherical polynomials in usual notation [16, §14.10.1] by

$$R_n^{\beta;q}(x) = q^{\frac14 n}\beta^{\frac12 n} \frac{(q;q)_n}{(q\beta^2;q)_n} C_n\big(x; q^{\frac12}\beta \,|\, q\big). \tag{3.7}$$

The continuous q-ultraspherical polynomials with $\beta = q^\alpha$ tend to the ultraspherical polynomials (2.1) as $q \uparrow 1$:

$$\lim_{q\uparrow 1} R_n^{q^\alpha;q}(x) = R_n^\alpha(x).$$

In view of [14, (3.10.13)] we can represent $R_n^{\beta;q}$ by a different q-hypergeometric expression:

$$R_n^{\beta;q}[z] = {}_4\phi_3\left(\begin{matrix} q^{-\frac12 n}, q^{\frac12 n+\frac12}\beta, q^{\frac14}\beta^{\frac12} z, q^{\frac14}\beta^{\frac12} z^{-1} \\ -q^{\frac12}\beta, (q\beta)^{\frac12}, -(q\beta)^{\frac12} \end{matrix}; q^{\frac12}, q^{\frac12}\right). \tag{3.8}$$

In particular,

$$R_n^{\beta;q}\left[q^{-\frac12 m-\frac14}\beta^{-\frac12}\right] = {}_4\phi_3\left(\begin{matrix} q^{-\frac12 n}, q^{\frac12 n+\frac12}\beta, q^{-\frac12 m}, q^{\frac12 m+\frac12}\beta \\ -q^{\frac12}\beta, (q\beta)^{\frac12}, -(q\beta)^{\frac12} \end{matrix}; q^{\frac12}, q^{\frac12}\right)$$
$$(m, n = 0, 1, 2, \ldots).$$

Hence we have the *duality*

$$R_n^{\beta;q}\left[q^{-\frac{1}{2}m-\frac{1}{4}}\beta^{-\frac{1}{2}}\right] = R_m^{\beta;q}\left[q^{-\frac{1}{2}n-\frac{1}{4}}\beta^{-\frac{1}{2}}\right] \qquad (m,n=0,1,2,\ldots). \tag{3.9}$$

Note the special value

$$R_n^{\beta;q}\left[q^{\frac{1}{4}}\beta^{\frac{1}{2}}\right]=1$$

and the coefficient of the term of highest degree

$$R_n^{\beta;q}(x)=2^n\left(q^{\frac{1}{2}}\beta\right)^{\frac{1}{2}n}\frac{\left(q^{\frac{1}{2}}\beta;q\right)_n}{\left(q\beta^2;q\right)_n}x^n+\text{terms of lower degree.} \tag{3.10}$$

For $0<\beta<q^{-\frac{1}{2}}$ the polynomials $R_n^{\beta;q}(x)$ are orthogonal on $[-1,1]$ with respect to the even weight function

$$w_{\beta,q}(x):=\left(1-x^2\right)^{-\frac{1}{2}}\left|\frac{\left(e^{2i\theta};q\right)_\infty}{\left(q^{\frac{1}{2}}\beta e^{2i\theta};q\right)_\infty}\right|^2, \quad x=\cos\theta, \tag{3.11}$$

see [16, (14.10.18)]. This weight function satisfies the recurrence

$$\frac{w_{q\beta,q}(x)}{w_{\beta,q}(x)}=\left(1+q^{\frac{1}{2}}\beta\right)^2-4q^{\frac{1}{2}}\beta x^2 = 4q^{\frac{1}{2}}\beta\left(a^2-x^2\right), \qquad a=\tfrac{1}{2}\left(q^{\frac{1}{4}}\beta^{\frac{1}{2}}+q^{-\frac{1}{4}}\beta^{-\frac{1}{2}}\right). \tag{3.12}$$

We will need the difference formula

$$R_n^{\beta;q}(x)-R_{n-2}^{\beta;q}(x)=\frac{4q^{-\frac{1}{2}n+\frac{3}{2}}\beta}{\left(1+q^{\frac{1}{2}}\beta\right)(1+q\beta)}\,\frac{1-q^{n-\frac{1}{2}}\beta}{1-q\beta} \times\left(x^2-\left(\tfrac{1}{2}(q^{\frac{1}{4}}\beta^{\frac{1}{2}}+q^{-\frac{1}{4}}\beta^{-\frac{1}{2}})\right)^2\right)R_{n-2}^{q\beta;q}(x) \qquad (n\ge 2). \tag{3.13}$$

Proof of (3.13). More generally, let $w(x)=w(-x)$ be an even weight function on $[-1,1]$, let $p_n(x)=k_nx^n+\cdots$ be orthogonal polynomials on $[-1,1]$ with respect to the weight function $w(x)$, and let $q_n(x)=k_n'x^n+\cdots$ be orthogonal polynomials on $[-1,1]$ with respect to the weight function $w(x)(a^2-x^2)$ $(a\ge 1)$. Assume that p_n and q_n are normalized by $p_n(a)=1=q_n(a)$. Let $n\ge 2$. Then $p_n(x)-p_{n-2}(x)$ vanishes for $x=\pm a$. Hence $(p_n(x)-p_{n-2}(x))/(x^2-a^2)$ is a polynomial of degree

$n-2$. It is immediately seen that x^k ($k < n-2$) is orthogonal to this polynomial with respect to the weight function $w(x)(a^2 - x^2)$ on $[-1, 1]$. We conclude that

$$p_n(x) - p_{n-2}(x) = \frac{k_n}{k'_{n-2}} \left(x^2 - a^2\right) q_{n-2}(x) \qquad (n \geq 2).$$

Now specialize to the weight function (3.11) and use (3.12) and (3.10). □

From [16, (14.10.17)] we have

$$R_n^{\beta;q}(\cos\theta) = q^{\frac{1}{4}n} \beta^{\frac{1}{2}n} \frac{\left(q^{\frac{1}{2}}\beta; q\right)_n}{\left(q\beta^2; q\right)_n} \mathrm{e}^{in\theta} {}_2\phi_1 \left(\begin{matrix} q^{-n}, q^{\frac{1}{2}}\beta \\ q^{-n+\frac{1}{2}}\beta^{-1} \end{matrix}; q, q^{\frac{1}{2}}\beta^{-1}\mathrm{e}^{-2\mathrm{i}\theta}\right). \tag{3.14}$$

A limit case of (3.14) yields the *continuous q-Hermite polynomials* (see [16, (14.26.1)]):

$$H_n(\cos\theta \mid q) := \mathrm{e}^{in\theta} {}_2\phi_0 \left(\begin{matrix} q^{-n}, 0 \\ - \end{matrix}; q, q^n \mathrm{e}^{-2\mathrm{i}\theta}\right). \tag{3.15}$$

So by (3.14) and (3.15) we have

$$H_n(x \mid q) = q^{-\frac{1}{4}n} \lim_{\beta \downarrow 0} \beta^{-\frac{1}{2}n} R_n^{\beta;q}(x). \tag{3.16}$$

3.3 q-Racah polynomials

We will consider *q-Racah polynomials* [16, §14.2]

$$R_n\left(q^{-x} + \gamma\delta q^{x+1}; \alpha, \beta, \gamma, \delta \mid q\right) := {}_4\phi_3 \left(\begin{matrix} q^{-n}, q^{n+1}\alpha\beta, q^{-x}, q^{x+1}\gamma\delta \\ q\alpha, q\beta\delta, q\gamma \end{matrix}; q, q\right) \tag{3.17}$$

for $\gamma = q^{-N-1}$, where $N \in \{1, 2, \ldots\}$, and for $n \in \{0, 1, \ldots, N\}$. They are discrete cases of the Askey–Wilson polynomials (3.1). The polynomials (3.17) are orthogonal polynomials on the finite q-quadratic set $\{q^{-x} + \gamma\delta q^{x+1} \mid x \in \{0, 1, \ldots, N\}\}$:

$$\sum_{x=0}^{N} (R_m R_n)\left(q^{-x} + \gamma\delta q^{x+1}; \alpha, \beta, \gamma, \delta \mid q\right) w_{\alpha,\beta,\gamma,\delta;q}(x) = h_{n;\alpha,\beta,\gamma,\delta;q}\, \delta_{m,n} \tag{3.18}$$

with

$$w_{\alpha,\beta,\gamma,\delta;q}(x) := \frac{1 - \gamma\delta q^{2x+1}}{(\alpha\beta q)^x (1 - \gamma\delta q)} \frac{(\alpha q, \beta\delta q, \gamma q, \gamma\delta q; q)_x}{\left(q, \alpha^{-1}\gamma\delta q, \beta^{-1}\gamma q, \delta q; q\right)_x}, \tag{3.19}$$

$$\frac{h_{n;\alpha,\beta,\gamma,\delta;q}}{h_{0;\alpha,\beta,\gamma,\delta;q}} := \frac{(1 - \alpha\beta q)(q\gamma\delta)^n}{1 - \alpha\beta q^{2n+1}} \frac{(q, q\beta, q\alpha\beta\gamma^{-1}, q\alpha\delta^{-1}; q)_n}{(q\alpha, q\alpha\beta, q\gamma, q\beta\delta; q)_n}, \tag{3.20}$$

$$h_{0;\alpha,\beta,\gamma,\delta;q} := \sum_{x=0}^{N} w_{\alpha,\beta,\gamma,\delta;q}(x) = \frac{\left(q^2\alpha\beta, \delta^{-1}; q\right)_N}{\left(q\alpha\delta^{-1}, q\beta; q\right)_N} \qquad \left(\gamma = q^{-N-1}\right). \quad (3.21)$$

Clearly $R_n(1+q^{-N}\delta; \alpha, \beta, q^{-N-1}, \delta \,|\, q) = 1$ while, by (3.17) and the q-Saalschütz formula [14, (1.7.2)], we can evaluate the q-Racah polynomial for $x = N$:

$$R_n\left(q^{-N} + \delta; \alpha, \beta, q^{-N-1}, \delta \,|\, q\right) = \frac{\left(q\beta, q\alpha\delta^{-1}; q\right)_n}{(q\alpha, q\beta\delta; q)_n}\,\delta^n. \qquad (3.22)$$

The backward shift operator equation [16, (14.2.10)] can be rewritten as

$$\begin{aligned} &w_{\alpha,\beta,\gamma,\delta;q}(x) R_n\left(q^{-x} + \gamma\delta q^{x+1}; \alpha, \beta, \gamma, \delta \,|\, q\right) \\ &= \frac{1 - q^2\gamma\delta}{q^{-x} - \gamma\delta q^{x+2}}\, w_{q\alpha,q\beta,q\gamma,\delta;q}(x)\, R_{n-1}\left(q^{-x} + \gamma\delta q^{x+2}; q\alpha, q\beta, q\gamma, \delta \,|\, q\right) \\ &\quad - \frac{1 - q^2\gamma\delta}{q^{-x+1} - \gamma\delta q^{x+1}}\, w_{q\alpha,q\beta,q\gamma,\delta;q}(x-1) \\ &\quad \times R_{n-1}\left(q^{-x+1} + \gamma\delta q^{x+1}; q\alpha, q\beta, q\gamma, \delta \,|\, q\right). \end{aligned} \qquad (3.23)$$

This holds for $x = 1, \ldots, N$ while for $x = 0$ (3.23) remains true if we put the second term on the right equal to 0. In the case $x = N$ the first term on the right is equal to zero because of (3.19), and the identity (3.23) can be checked by using (3.19) and (3.22).

Hence, for a function f on $\{0, 1, \ldots, N\}$ we have

$$\begin{aligned} &\sum_{x=0}^{N} w_{\alpha,\beta,\gamma,\delta;q}(x) R_n\left(q^{-x} + \gamma\delta q^{x+1}; \alpha, \beta, \gamma, \delta \,|\, q\right) f(x) \\ &\quad = \sum_{x=0}^{N-1} \frac{1 - q^2\gamma\delta}{q^{-x} - \gamma\delta q^{x+2}} \\ &\qquad \times w_{q\alpha,q\beta,q\gamma,\delta;q}(x)\, R_{n-1}\left(q^{-x} + \gamma\delta q^{x+2}; q\alpha, q\beta, q\gamma, \delta \,|\, q\right) \\ &\qquad \times (f(x) - f(x+1)). \end{aligned} \qquad (3.24)$$

4 The dual addition formula for continuous q-ultraspherical polynomials

4.1 The Rahman–Verma addition formula

The q-analogue of the product formula for ultraspherical polynomials [21, (18.17.5)] was given by Rahman & Verma [22, (1.20)]. It uses a different choice of parameter for the q-ultraspherical polynomials:

$$R_n^{\text{R-V}}[z] = R_n^{\text{R-V};a;q}[z] := R_n^{q^{-\frac{1}{2}}a^2;q}[z], \tag{4.1}$$

where we have our own notation (3.6) on the right. Then the duality (3.9) takes the form

$$R_n^{\text{R-V}}\left[q^{-\frac{1}{2}m}a^{-1}\right] = R_m^{\text{R-V}}\left[q^{-\frac{1}{2}n}a^{-1}\right] \qquad (m,n=0,1,2,\ldots)$$

or, in terms of special Askey–Wilson polynomials,

$$\begin{aligned} &R_n\left[q^{-\frac{1}{2}m}a^{-1}; a, q^{\frac{1}{2}}a, -a, -q^{\frac{1}{2}}a \,|\, q\right] \\ &\quad = R_m\left[q^{-\frac{1}{2}n}a^{-1}; a, q^{\frac{1}{2}}a, -a, -q^{\frac{1}{2}}a \,|\, q\right] \qquad (m,n=0,1,2,\ldots). \end{aligned} \tag{4.2}$$

In terms of the polynomials (4.1) and with usage of (3.4), (3.5) the Rahman–Verma product formula reads as follows:

$$R_n^{\text{R-V}}[u]\, R_n^{\text{R-V}}[v] = \int_{|z|=1} R_n^{\text{R-V}}[z] \frac{w\left[z; auv, au^{-1}v^{-1}, auv^{-1}, au^{-1}v \,|\, q\right]}{h_0\left(auv, au^{-1}v^{-1}, auv^{-1}, au^{-1}v \,|\, q\right)} \frac{\mathrm{d}z}{\mathrm{i}z}$$

with $|u|, |v| = 1$, $0 < a < 1$. This suggests an expansion

$$R_n^{\text{R-V}}[z] = \sum_{k=0}^{n} c_k\, R_k\left[z; auv, au^{-1}v^{-1}, auv^{-1}, au^{-1}v \,|\, q\right],$$

where the term c_0 equals $R_n^{\text{R-V}}[u]\, R_n^{\text{R-V}}[v]$. Indeed, [22, (1.24)] gives the addition formula

$$\begin{aligned} &R_n\left[z; a, q^{\frac{1}{2}}a, -a, -q^{\frac{1}{2}}a \,|\, q\right] \\ &= \sum_{k=0}^{n} \frac{(-1)^k q^{\frac{1}{2}k(k+1)}\left(q^{-n}, a^2, q^n a^4, q^{-1}a^4; q\right)_k}{\left(q, q^{\frac{1}{2}}a^2, -q^{\frac{1}{2}}a^2, -a^2; q\right)_k \left(q^{-1}a^4; q\right)_{2k}} \\ &\times u^{-k}\left(a^2u^2; q\right)_k R_{n-k}\left[u; q^{\frac{1}{2}k}a, q^{\frac{1}{2}(k+1)}a, -q^{\frac{1}{2}k}a, -q^{\frac{1}{2}(k+1)}a \,|\, q\right] \\ &\times v^{-k}\left(a^2v^2; q\right)_k R_{n-k}\left[v; q^{\frac{1}{2}k}a, q^{\frac{1}{2}(k+1)}a, -q^{\frac{1}{2}k}a, -q^{\frac{1}{2}(k+1)}a \,|\, q\right] \\ &\times R_k\left[z; auv, au^{-1}v^{-1}, auv^{-1}, au^{-1}v \,|\, q\right]. \end{aligned} \tag{4.3}$$

The addition formula [21, (18.18.8)] for ultraspherical polynomials can be obtained as limit case for $q \uparrow 1$ of (4.3).

4.2 The dual addition formula

As mentioned in [6, (4.18)], Rogers already gave the linearization formula for continuous q-ultraspherical polynomials in 1895. Here we refer for this formula to [4, (10.11.10)]. It can be written in notation (3.6) as

$$R_\ell^{\beta;q}(x)R_m^{\beta;q}(x)=\frac{(q;q)_\ell(q;q)_m}{\left(q\beta^2;q\right)_\ell\left(q\beta^2;q\right)_m}\sum_{j=0}^{\min(\ell,m)}\frac{1-q^{\ell+m-2j+\frac12}\beta}{1-q^{\frac12}\beta}\frac{\left(q^{\frac12}\beta;q\right)_j}{(q;q)_j}\times\frac{(q^{\frac12}\beta;q)_{\ell-j}}{(q;q)_{\ell-j}}\frac{\left(q^{\frac12}\beta;q\right)_{m-j}}{(q;q)_{m-j}}\frac{(q\beta^2;q)_{\ell+m-j}}{\left(q^{\frac32}\beta;q\right)_{\ell+m-j}}\left(q^{\frac12}\beta\right)^j\times R_{\ell+m-2j}^{\beta;q}(x). \tag{4.4}$$

By the earlier assumption $0<\beta<q^{-\frac12}$ the linearization coefficients in (4.4) are non-negative.

From now on assume without loss of generality that $\ell\geq m$. Specialization of (3.19) and (3.21) gives

$$w_{\beta q^{-\frac12},\beta q^{-\frac12},q^{-m-1},\beta^{-1}q^{-\ell-\frac12};q}(j)=\frac{\left(q^{\frac12}\beta;q\right)_{\ell+m}}{\left(q\beta^2;q\right)_{\ell+m}}\frac{(q;q)_\ell}{\left(q^{\frac12}\beta;q\right)_\ell}\frac{(q;q)_m}{(q^{\frac12}\beta;q)_m}\times\frac{1-q^{\ell+m-2j+\frac12}\beta}{1-q^{\frac12}\beta}\frac{\left(q^{\frac12}\beta;q\right)_j}{(q;q)_j}\frac{\left(q^{\frac12}\beta;q\right)_{\ell-j}}{(q;q)_{\ell-j}}\times\frac{\left(q^{\frac12}\beta;q\right)_{m-j}}{(q;q)_{m-j}}\frac{(q\beta^2;q)_{\ell+m-j}}{\left(q^{\frac32}\beta;q\right)_{\ell+m-j}}\left(q^{\frac12}\beta\right)^j$$

and

$$h_{0;\beta q^{-\frac12},\beta q^{-\frac12},q^{-m-1},\beta^{-1}q^{-\ell-\frac12};q}=\frac{\left(q\beta^2;q\right)_\ell\left(q\beta^2;q\right)_m}{\left(q\beta^2;q\right)_{\ell+m}}\frac{\left(q^{\frac12}\beta;q\right)_{\ell+m}}{\left(q^{\frac12}\beta;q\right)_\ell\left(q^{\frac12}\beta;q\right)_m}. \tag{4.5}$$

The linearization formula (4.4) can now be seen to have the equivalent concise expression

$$R_\ell^{\beta;q}(x)R_m^{\beta;q}(x)=\sum_{j=0}^m\frac{w_{\beta q^{-\frac12},\beta q^{-\frac12},q^{-m-1},\beta^{-1}q^{-\ell-\frac12};q}(j)}{h_{0;\beta q^{-\frac12},\beta q^{-\frac12},q^{-m-1},\beta^{-1}q^{-\ell-\frac12};q}}R_{\ell+m-2j}^{\beta;q}(x). \tag{4.6}$$

This identity can be considered as giving the constant term of an expansion of $R^{\beta;q}_{\ell+m-2j}(x)$ as a function of j in terms of q-Racah polynomials

$$R_k\left(q^{-j}+\beta^{-1}q^{j-\ell-m-\frac12};\beta q^{-\frac12},\beta q^{-\frac12},q^{-m-1},\beta^{-1}q^{-\ell-\frac12}\,|\,q\right).$$

The general terms of this expansion will be obtained by evaluating the sum

$$\begin{aligned}S^{\beta;q}_{k,\ell,m}(x):=&\sum_{j=0}^{m}w_{\beta q^{-\frac12},\beta q^{-\frac12},q^{-m-1},\beta^{-1}q^{-\ell-\frac12};q}(j)\,R^{\beta;q}_{\ell+m-2j}(x)\\&\times R_k\left(q^{-j}+\beta^{-1}q^{j-\ell-m-\frac12};\beta q^{-\frac12},\beta q^{-\frac12},q^{-m-1},\beta^{-1}q^{-\ell-\frac12}\,|\,q\right),\end{aligned}\tag{4.7}$$

where we still assume $l\ge m$ and where $k\in\{0,\dots,m\}$.

Theorem 4.1 *The sum (4.7) can be evaluated as*

$$\begin{aligned}S^{\beta;q}_{k,\ell,m}(x)=&\frac{\left(q^{\frac12(\ell+m+1)}\beta\right)^k\left(\beta^{-1}q^{-\ell-m+\frac12};q\right)_k}{\left(-q^{\frac12}\beta,\pm q\beta;q\right)_k}\left(\pm q^{\frac14}\beta^{\frac12}z,\pm q^{\frac14}\beta^{\frac12}z^{-1};q^{\frac12}\right)_k\\&\times\frac{(q^{2k+1}\beta^2;q)_{\ell-k}(q^{2k+1}\beta^2;q)_{m-k}}{\left(q^{2k+1}\beta^2;q\right)_{\ell+m-2k}}\\&\times\frac{\left(q^{k+\frac12}\beta;q\right)_{\ell+m-2k}}{\left(q^{k+\frac12}\beta;q\right)_{\ell-k}\left(q^{k+\frac12}\beta;q\right)_{m-k}}R^{q^k\beta;q}_{\ell-k}(x)R^{q^k\beta;q}_{m-k}(x).\end{aligned}\tag{4.8}$$

Here we use the conventions that $(\pm a;q)_n:=(a;q)_n(-a;q)_n$ *and* $x=\frac12(z+z^{-1})$.

Proof In (4.7) put $f(j):=R^{\beta;q}_{\ell+m-2j}(x)$. Then comparison of (4.7) with (3.24) gives

$$\begin{aligned}S^{\beta;q}_{k,\ell,m}(x)=&\sum_{j=0}^{m}w_{\beta q^{-\frac12},\beta q^{-\frac12},q^{-m-1},\beta^{-1}q^{-\ell-\frac12};q}(j)\\&\times R_k(q^{-j}+\beta^{-1}q^{j-\ell-m-\frac12};\beta q^{-\frac12},\beta q^{-\frac12},q^{-m-1},\beta^{-1}q^{-\ell-\frac12}\,|\,q)\,f(j)\\=&\sum_{j=0}^{m-1}\frac{1-\beta^{-1}q^{-\ell-m+\frac12}}{q^{-j}-\beta^{-1}q^{-\ell-m+j+\frac12}}w_{\beta q^{\frac12},\beta q^{\frac12},q^{-m},\beta^{-1}q^{-\ell-\frac12};q}(j)\\&\times R_{k-1}\left(q^{-j}+\beta^{-1}q^{j-\ell-m+\frac12};\beta q^{\frac12},\beta q^{\frac12},q^{-m},\beta^{-1}q^{-\ell-\frac12}\,|\,q\right)\\&\times(f(j)-f(j+1)).\end{aligned}$$

We can handle the factor $f(j)-f(j+1)$ in the right part above by using (3.13):

$$\begin{aligned} f(j)-f(j+1) &= R^{\beta;q}_{\ell+m-2j}(x)-R^{\beta;q}_{\ell+m-2j-2}(x)\\ &= \frac{4\beta^2 q^{\frac12\ell+\frac12 m+1}\left(q^{-j}-\beta^{-1}q^{-\ell-m+j+\frac12}\right)}{\left(1+q^{\frac12}\beta\right)\left(1-q^2\beta^2\right)}\\ &\quad\times\left(\tfrac14\left(q^{\frac14}\beta^{\frac12}+q^{-\frac14}\beta^{-\frac12}\right)^2-x^2\right)R^{q\beta;q}_{\ell+m-2j-2}(x). \end{aligned}$$

So, with $x=\frac12(z+z^{-1})$,

$$\begin{aligned} S^{\beta;q}_{k,\ell,m}(x) &= \frac{4\beta q^{-\frac12\ell-\frac12 m+\frac32}\left(1-\beta q^{\ell+m-\frac12}\right)}{\left(1+q^{\frac12}\beta\right)\left(1-q^2\beta^2\right)}\\ &\quad\times\left(\tfrac14\left(q^{\frac14}\beta^{\frac12}+q^{-\frac14}\beta^{-\frac12}\right)^2-x^2\right)\sum_{j=0}^{m-1} w_{\beta q^{\frac12},\beta q^{\frac12},q^{-m},\beta^{-1}q^{-\ell-\frac12};q}(j)\\ &\quad\times R_{k-1}\left(q^{-j}+\beta^{-1}q^{j-\ell-m+\frac12};\beta q^{\frac12},\beta q^{\frac12},q^{-m},\beta^{-1}q^{-\ell-\frac12}\,|\,q\right)\\ &\quad\times R^{q\beta;q}_{\ell+m-2j-2}(x)\\ &= \frac{q^{\frac12\ell+\frac12 m+\frac12}\beta\left(1-\beta^{-1}q^{-l-m+\frac12}\right)}{\left(1+q^{\frac12}\beta\right)\left(1-q^2\beta^2\right)}\\ &\quad\times\left(1+q^{\frac14}\beta^{\frac12}z\right)\left(1-q^{\frac14}\beta^{\frac12}z\right)\left(1+q^{\frac14}\beta^{\frac12}z^{-1}\right)\left(1-q^{\frac14}\beta^{\frac12}z^{-1}\right)\\ &\quad\times S^{q\beta,q}_{k-1,\ell-1,m-1}(x). \end{aligned}$$

Iteration gives

$$\begin{aligned} S^{\beta;q}_{k,\ell,m}(x) &= \frac{\left(q^{\frac12(\ell+m+1)}\beta\right)^k\left(\beta^{-1}q^{-\ell-m+\frac12};q\right)_k}{\left(-q^{\frac12}\beta,\pm q\beta;q\right)_k}\\ &\quad\times\left(\pm q^{\frac14}\beta^{\frac12}z,\pm q^{\frac14}\beta^{\frac12}z^{-1};q^{\frac12}\right)_k S^{q^k\beta;q}_{0,\ell-k,m-k}(x). \end{aligned}\tag{4.9}$$

By (4.7)

$$S^{\beta;q}_{0,\ell,m}(x)=h_{0;\beta q^{-\frac12},\beta q^{-\frac12},q^{-m-1},\beta^{-1}q^{-\ell-\frac12};q}R^{\beta;q}_\ell(x)R^{\beta;q}_m(x).\tag{4.10}$$

Hence, by (4.5),

$$S^{q^k\beta;q}_{0,\ell-k,m-k}(x)=h_{0;\beta q^{k-\frac12},\beta q^{k-\frac12},q^{k-m-1},\beta^{-1}q^{-\ell-\frac12};q}R^{q^k\beta;q}_{\ell-k}(x)R^{q^k\beta;q}_{m-k}(x)$$

$$= \frac{\left(q^{2k+1}\beta^2;q\right)_{\ell-k}\left(q^{2k+1}\beta^2;q\right)_{m-k}}{\left(q^{2k+1}\beta^2;q\right)_{\ell+m-2k}} \times \frac{\left(q^{k+\frac12}\beta;q\right)_{\ell+m-2k}}{\left(q^{k+\frac12}\beta;q\right)_{\ell-k}\left(q^{k+\frac12}\beta;q\right)_{m-k}} R_{\ell-k}^{q^k\beta;q}(x)R_{m-k}^{q^k\beta;q}(x).$$

Substitution of this last result in (4.9) yields (4.8). □

Remark 4.2 An integrated form of (4.8) is the same as a special case of the formula given in [3, Remark 6.5] (corrected version of [2, Remark 6.5]). Indeed, in (4.8) rewrite the three continuous q-ultraspherical polynomials in the standard notation (3.7), replace ℓ by n and β by $q^{-\frac12}\beta$, multiply both sides by $C_{m+n-2t}(x;\beta\,|\,q)$ $(0\le t\le m)$ times its weight function, and integrate both sides over $x\in[-1,1]$ (see [16, (14.10.18)]). Then write the Racah polynomial on the right-hand side by (3.17) as a balanced ${}_4\phi 3$ and apply to this Sears' transformation [14, (III.15)] (with n, a, b, c, d, e, f replaced by $k, q^{k-1}\beta^2, q^{-t}, q^{-n-m+t}\beta^{-1}, q^{-m}, q^{-n}, \beta$). We arrive at the formula in [3, Remark 6.5] for $\alpha=q^{-1}\beta$.

Note that (4.8) is not equivalent to its integrated forms if we consider these only for $0\le t\le m$. The right-hand side of (4.8) is a polynomial of degree $\ell+m$ in x, so we have to consider also the integrals for $m<t\le\frac12(\ell+m)$, of which we know a priori that they vanish. But [3, Remark 6.5] does not consider integrals for $t>m$. However, in [3, Lemma 6.4] (the case $\beta=1$, $\alpha=0$ of [3, Remark 6.5]) the integral for $t>m$ is stated to be zero.

As observed in [18, end of §4], integrated forms of the $q=1$ limit [18, (4.5)] of (4.8) coincide with special cases of [17, (2.6)].

Theorem 4.3 (Dual addition formula) *For $j\in\{0,\dots,m\}$ there is the expansion*

$$R_{\ell+m-2j}^{\beta;q}(x) = \sum_{k=0}^{\min(l,m)} q^{\frac12 k(k+\ell+m+2)}\beta^k\,\frac{1-\beta^2q^{2k}}{1-\beta^2q^k}\,\frac{\left(q^{-\ell},q^{-m},q\beta^2;q\right)_k}{(q\beta,q\beta,q;q)_k} \times \frac{\prod_{i=0}^{k-1}\left(4q^{i+\frac12}\beta x^2-\left(1+q^{i+\frac12}\beta\right)^2\right)}{\left(-q^{\frac12}\beta;q^{\frac12}\right)_{2k}^2} R_{\ell-k}^{q^k\beta;q}(x)R_{m-k}^{q^k\beta;q}(x) \times R_k(q^{-j}+\beta^{-1}q^{j-\ell-m-\frac12};\beta q^{-\frac12},\beta q^{-\frac12},q^{-m-1},\beta^{-1}q^{-\ell-\frac12}\,|\,q). \tag{4.11}$$

Proof Assume $l\ge m$. By (4.9) and (4.10)

$$S_{k,\ell,m}^{\beta;q}(x) = \frac{(-1)^k q^{\frac12 k(k-\ell-m+1)}}{\left(-q^{\frac12}\beta;q\right)_k^2\left(q^2\beta^2;q^2\right)_k^2}\,\frac{\left(q\beta^2;q\right)_{\ell+k}\left(q\beta^2;q\right)_{m+k}\left(q^{\frac12}\beta;q\right)_{\ell+m}}{\left(q^{\frac12}\beta;q\right)_{\ell+m}\left(q\beta^2;q\right)_{\ell}\left(q\beta^2;q\right)_m} \times h_{0;\beta q^{-\frac12},\beta q^{-\frac12},q^{-m-1},\beta^{-1}q^{-\ell-\frac12};q}$$

$$\times \left(\pm q^{\frac14}\beta^{\frac12}z, \pm q^{\frac14}\beta^{\frac12}z^{-1}; q^{\frac12}\right)_k R_{\ell-k}^{q^k\beta,q}(x)R_{m-k}^{q^k\beta,q}(x).$$

By Fourier-q-Racah inversion we obtain

$$\begin{aligned}
&R_{\ell+m-2j}^{\beta,q}(x)\\
&\quad=\sum_{k=0}^{m}\frac{(-1)^k q^{\frac12 k(k-\ell-m+1)}}{\left(-q^{\frac12}\beta;q\right)_k^2 (q^2\beta^2;q^2)_k^2}\,\frac{(q\beta^2;q)_{\ell+k}(q\beta^2;q)_{m+k}\left(q^{\frac12}\beta;q\right)_{\ell+m}}{\left(q^{\frac12}\beta;q\right)_{\ell+m}(q\beta^2;q)_\ell (q\beta^2;q)_m}\\
&\qquad\times\frac{h_{0;\beta q^{-\frac12},\beta q^{-\frac12},q^{-m-1},\beta^{-1}q^{-\ell-\frac12};q}}{h_{k;\beta q^{-\frac12},\beta q^{-\frac12},q^{-m-1},\beta^{-1}q^{-\ell-\frac12};q}}\left(\pm q^{\frac14}\beta^{\frac12}z, \pm q^{\frac14}\beta^{\frac12}z^{-1}; q^{\frac12}\right)_k\\
&\qquad\times R_{\ell-k}^{q^k\beta;q}(x)R_{m-k}^{q^k\beta;q}(x)\\
&\qquad\times R_k(q^{-j}+\beta^{-1}q^{j-\ell-m-\frac12};\beta q^{-\frac12},\beta q^{-\frac12},q^{-m-1},\beta^{-1}q^{-\ell-\frac12}\,|\,q).
\end{aligned}$$

Now use (3.20). □

If we put $\beta = q^\alpha$ in (4.11) and take the limit for $q \uparrow 1$ then we arrive at the dual addition formula (2.7) for ultraspherical polynomials.

For $j = 0$ (4.11) takes the form

$$\begin{aligned}
R_{\ell+m}^{\beta;q}(x) &= \sum_{k=0}^{\min(l,m)} q^{\frac12 k(k+\ell+m+2)}\beta^k\,\frac{1-\beta^2q^{2k}}{1-\beta^2q^k}\,\frac{(q^{-\ell},q^{-m},q\beta^2;q)_k}{(q\beta,q\beta,q;q)_k}\\
&\quad\times\frac{\prod_{i=0}^{k-1}\left(4q^{i+\frac12}\beta x^2-(1+q^{i+\frac12}\beta)^2\right)}{(-q^{\frac12}\beta;q^{\frac12})_{2k}^2}\,R_{\ell-k}^{q^k\beta;q}(x)R_{m-k}^{q^k\beta;q}(x). \qquad (4.12)
\end{aligned}$$

It has a similar structure as [14, Exercise 8.12]. However, the formula there expands $R_{\ell+m}^{\beta;q}(x)$ in terms of $R_{\ell-k}^{\beta;q}(x)R_{m-k}^{\beta;q}(x)$ $(k = 0, 1, \ldots, \min(\ell, m))$. A variant of (5.3) given by the specialization of [15, (9.4)] to the case of continuous q-ultraspherical polynomials is also essentially different from our formula.

Just as with (2.7), formula (4.12) can be rewritten as a matrix decomposition $S = LDU$ with S symmetric, L lower triangular, its transpose $U = L^\mathsf{T}$ upper triangular and D diagonal. Aldenhoven [1, Theorem 1.1] earlier gave the inverse of the matrix L.

4.3 A second proof of the dual addition formula

We will now show that the addition formula (4.3) and the dual addition formula (4.11) coincide when both formulas are suitably restricted in their x or z variable. This will follow from the duality (4.2).

In (4.11) put $\beta = q^{-\frac{1}{2}}a^2$, $x = \frac{1}{2}(z+z^{-1})$ and use (4.1). Then the dual addition formula takes the form

$$\begin{aligned}
&R_{\ell+m-2j}\left[z; a, q^{\frac{1}{2}}a, -a, -q^{\frac{1}{2}}a \mid q\right]\\
&\quad= \sum_{k=0}^{m}(-1)^k q^{\frac{1}{2}k(k+\ell+m+1)} a^{2k}\,\frac{1-a^4q^{2k-1}}{1-a^4q^{k-1}}\,\frac{(q^{-\ell}, q^{-m}, a^4; q)_k}{\left(q^{\frac{1}{2}}a^2, q^{\frac{1}{2}}a^2, q; q\right)_k}\\
&\qquad\times \frac{(a^2z^2, a^2z^{-2}; q)_k}{\left(-a^2; q^{\frac{1}{2}}\right)_{2k}^2}\\
&\qquad\times R_{\ell-k}\left[z; q^{\frac{1}{2}k}a, q^{\frac{1}{2}(k+1)}a, -q^{\frac{1}{2}k}a, -q^{\frac{1}{2}(k+1)}a \mid q\right]\\
&\qquad\times R_{m-k}\left[z; q^{\frac{1}{2}k}a, q^{\frac{1}{2}(k+1)}a, -q^{\frac{1}{2}k}a, -q^{\frac{1}{2}(k+1)}a \mid q\right]\\
&\qquad\times R_k\left(q^{-j}+q^{j-\ell-m}a^{-2}; q^{-1}a^2, q^{-1}a^2, q^{-m-1}, q^{-\ell}a^{-2} \mid q\right).
\end{aligned} \tag{4.13}$$

Since both sides of (4.13) are symmetric Laurent polynomials in z, verification of the identity for $z = q^{-\frac{1}{2}n}a^{-1}$ $(n = m, m+1, m+2, \ldots)$ will settle the identity for all z. Thus put $z = q^{-\frac{1}{2}n}a^{-1}$ in (4.13) and use the duality (4.2) in the polynomials $R_{\ell+m-2j}$, $R_{\ell-k}$ and R_{m-k} occurring in (4.13). Furthermore, use (3.17) and (3.1) in order to substitute

$$\begin{aligned}
&R_k(q^{-j}+q^{j-\ell-m}a^{-2}; q^{-1}a^2, q^{-1}a^2, q^{-m-1}, q^{-\ell}a^{-2} \mid q)\\
&\quad= {}_4\phi_3\left(\begin{matrix} q^{-k}, q^{k-1}a^k, q^{-j}, q^{j-\ell-m}a^{-2} \\ a^2, q^{-\ell}, q^{-m}\end{matrix}; q, q\right)\\
&\quad= R_k\left[q^{-\frac{1}{2}(\ell+m-2j)}a^{-1}; q^{-\frac{1}{2}(\ell+m)}a^{-1}, q^{\frac{1}{2}(\ell+m)}a^3, q^{\frac{1}{2}(\ell-m)}a, q^{\frac{1}{2}(m-\ell)}a \mid q\right].
\end{aligned}$$

We obtain

$$\begin{aligned}
&R_n\left[q^{-\frac{1}{2}(\ell+m-2j)}a^{-1}; a, q^{\frac{1}{2}}a, -a, -q^{\frac{1}{2}}a \mid q\right]\\
&\quad= \sum_{k=0}^{n}(-1)^k q^{\frac{1}{2}k(k+\ell+m+1)} a^{2k}\,\frac{1-a^4q^{2k-1}}{1-a^4q^{k-1}}\,\frac{(q^{-\ell}, q^{-m}, a^4; q)_k}{\left(q^{\frac{1}{2}}a^2, q^{\frac{1}{2}}a^2, q; q\right)_k}\,\frac{(q^{-n}, q^na^4; q)_k}{\left(-a^2; q^{\frac{1}{2}}\right)_{2k}^2}\\
&\qquad\times R_{n-k}\left[q^{-\frac{1}{2}\ell}a^{-1}; q^{\frac{1}{2}k}a, q^{\frac{1}{2}(k+1)}a, -q^{\frac{1}{2}k}a, -q^{\frac{1}{2}(k+1)}a \mid q\right]\\
&\qquad\times R_{n-k}\left[q^{-\frac{1}{2}m}a^{-1}; q^{\frac{1}{2}k}a, q^{\frac{1}{2}(k+1)}a, -q^{\frac{1}{2}k}a, -q^{\frac{1}{2}(k+1)}a \mid q\right]\\
&\qquad\times R_k\left[q^{-\frac{1}{2}(\ell+m-2j)}a^{-1}; q^{-\frac{1}{2}(\ell+m)}a^{-1}, q^{\frac{1}{2}(\ell+m)}a^3, q^{\frac{1}{2}(\ell-m)}a, q^{\frac{1}{2}(m-\ell)}a \mid q\right].
\end{aligned} \tag{4.14}$$

Because of the factor $(q^{-m}; q)_k$ on the right-hand side and since $n \geq m$, there was no harm to replace m by n as the upper bound of the summation.

On the other hand, for integers j, m, ℓ such that $0 \le j \le m \le \ell$ and $m \le n$, substitute $z = q^{-\frac{1}{2}(\ell+m-2j)}a^{-1}$, $u = q^{-\frac{1}{2}\ell}a^{-1}$, $v = q^{-\frac{1}{2}m}a^{-1}$ in (4.3) in order to obtain

$$\begin{aligned}
&R_n\left[q^{-\frac{1}{2}(\ell+m-2j)}a^{-1}; a, q^{\frac{1}{2}}a, -a, -q^{\frac{1}{2}}a \,|\, q\right] \\
&\quad = \sum_{k=0}^{n} \frac{(-1)^k q^{\frac{1}{2}k(k+\ell+m+1)} a^{2k} \left(q^{-n}, q^{-\ell}, q^{-m}, a^2, q^n a^4, q^{-1}a^4; q\right)_k}{\left(q, q^{\frac{1}{2}}a^2, -q^{\frac{1}{2}}a^2, -a^2; q\right)_k \left(q^{-1}a^4; q\right)_{2k}} \\
&\qquad \times R_{n-k}\left[q^{-\frac{1}{2}\ell}a^{-1}; q^{\frac{1}{2}k}a, q^{\frac{1}{2}(k+1)}a, -q^{\frac{1}{2}k}a, -q^{\frac{1}{2}(k+1)}a \,|\, q\right] \\
&\qquad \times R_{n-k}\left[q^{-\frac{1}{2}m}a^{-1}; q^{\frac{1}{2}k}a, q^{\frac{1}{2}(k+1)}a, -q^{\frac{1}{2}k}a, -q^{\frac{1}{2}(k+1)}a \,|\, q\right] \\
&\qquad \times R_k\left[q^{-\frac{1}{2}(\ell+m-2j)}a^{-1}; q^{-\frac{1}{2}(\ell+m)}a^{-1}, q^{\frac{1}{2}(\ell+m)}a^3, q^{\frac{1}{2}(\ell-m)}a, q^{\frac{1}{2}(m-\ell)}a \,|\, q\right].
\end{aligned} \tag{4.15}$$

An easy computation shows that (4.14) can be rewritten as (4.15). Thus we have shown that the addition formula (4.3) implies the dual addition formula (4.11).

5 A limit to continuous *q*-Hermite polynomials

This section gives the q-analogue of the results in [18, §5]. The treatment given here is completely parallel to the one given there.

We will do a rescaling in the dual addition formula (4.11) such that we can take the limit for $\beta \downarrow 0$. For this purpose observe that the q-Racah polynomial (3.17) has limits

$$\begin{aligned}
&\lim_{\beta\downarrow 0} \beta^j R_n\left(q^{-j} + \beta^{-1}q^{-m-l+j-\frac{1}{2}};\ \beta q^{-\frac{1}{2}}, \beta q^{-\frac{1}{2}}, q^{-m-1}, \beta^{-1}q^{-l-\frac{1}{2}}; q\right) \\
&\quad = \frac{\left(q^{-n}; q\right)_j}{\left(q^{-l}, q^{-m}; q\right)_j} q^{\left(j-m-l-\frac{1}{2}\right)j}, \\
&\quad \lim_{\beta\downarrow 0} \beta^n R_n\left(q^{-j} + \beta^{-1}q^{-m-l+j-\frac{1}{2}};\ \beta q^{-\frac{1}{2}}, \beta q^{-\frac{1}{2}}, q^{-m-1}, \beta^{-1}q^{-l-\frac{1}{2}}; q\right) \\
&\quad = \frac{(q^{-j}; q)_n}{(q^{-l}, q^{-m}; q)_n} q^{(j-m-l-\frac{1}{2})n},
\end{aligned} \tag{5.1}$$

where $l, m \ge \max(j, n)$. Otherwise said,

$$\begin{aligned}
&R_n\left(q^{-j} + \beta^{-1}q^{-m-l+j-\frac{1}{2}};\ \beta q^{-\frac{1}{2}}, \beta q^{-\frac{1}{2}}, q^{-m-1}, \beta^{-1}q^{-l-\frac{1}{2}}; q\right) \\
&\quad = O\left(\beta^{-\min(n,j)}\right)
\end{aligned}$$

as $\beta \downarrow 0$ with the order constant given in (5.1).

Now, in (4.11), multiply both sides by $\beta^{-\frac{1}{2}(l+m-2j)}$ and let $\beta \downarrow 0$. By (3.16) and (5.1) we obtain for $l \ge m$ that

$$\begin{aligned} q^{(l+m-j)j} \left(q^{-l}, q^{-m}; q\right)_j H_{l+m-2j}(x \mid q) \\ = \sum_{k=j}^{m} (-1)^k q^{k(l+m)} q^{-\frac{1}{2}k(k-1)} \left(q^{-l}, q^{-m}; q\right)_k \\ \times H_{l-k}(x \mid q) H_{m-k}(x \mid q) \frac{(q^{-k}; q)_j}{(q; q)_k}, \end{aligned} \tag{5.2}$$

which may be called the dual addition formula for continuous q-Hermite polynomials. When written equivalently as

$$\begin{aligned} H_{l+m-2j}(x \mid q) = \sum_{k=j}^{m} (-1)^{k-j} q^{(k-j)\left(l+m-2j+\frac{1}{2}\right)} q^{-\frac{1}{2}(k-j)^2} \\ \times \frac{(q^{-l+j}, q^{-m+j}; q)_{k-j}}{(q; q)_{k-j}} H_{l-k}(x \mid q) H_{m-k}(x \mid q), \end{aligned}$$

it is seen to be equivalent to its special case $j = 0$, which can be written as

$$H_{l+m}(x \mid q) = \sum_{k=0}^{\min(l,m)} (-1)^k q^{\frac{1}{2}k(k-1)} (q; q)_k \begin{bmatrix} l \\ k \end{bmatrix}_q \begin{bmatrix} m \\ k \end{bmatrix}_q H_{l-k}(x \mid q) H_{m-k}(x \mid q), \tag{5.3}$$

where

$$\begin{bmatrix} l \\ k \end{bmatrix}_q := \frac{(q; q)_l}{(q; q)_k (q; q)_{l-k}}$$

is a q-binomial coefficient. Formula (5.3) was first given, with two different proofs, by Carlitz [9, (1.8)], [10, (3)]. Here our $H_n(x \mid q)$ is related to Carlitz' $H_n(z)$ by $H_n(\frac{1}{2}(z + z^{-1}) \mid q) = z^{-n} H_n(z^2)$. An (essentially different) variant of (5.3) is given in [15, (9.5)].

The $q = 1$ limit of (5.3) is [13, 10.13(36)], which goes back to Nielsen (1918) and Burchnall (1941), see [15, §1] for historical details.

Note that (5.3) gives a matrix decomposition $S = LDU$ with S symmetric, L lower triangular, its transpose $U = L^{\mathsf{T}}$ upper triangular and D diagonal. Aldenhoven [1, Corollary 5.2] earlier gave the inverse of the matrix L.

Next we want to consider the limit as $\beta \downarrow 0$ of (4.8) with $S_{k,\ell,m}^{\beta;q}$ given by (4.7). Recall that (4.8) together with (4.7) is the dual of (4.11) in the sense of Fourier-q-Racah inversion. Observe from (3.19)–(3.21) that

$$\lim_{\beta \downarrow 0} \beta^{-j} w_{\beta q^{-\frac{1}{2}}, \beta q^{-\frac{1}{2}}, q^{-m-1}, \beta^{-1} q^{-l-\frac{1}{2}}; q}(j) = \frac{(q^{-l}, q^{-m}; q)_j}{(q; q)_j} q^{(l+m-j+\frac{3}{2})j}, \tag{5.4}$$

$$\lim_{\beta\downarrow 0}\beta^n\, h_{n;\beta q^{-\frac12},\beta q^{-\frac12},q^{-m-1},\beta^{-1}q^{-l-\frac12};q}=\frac{(q;q)_n}{(q^{-l},q^{-m};q)_n}\,q^{-(l+m+\frac12)n}. \tag{5.5}$$

In (4.8) multiply both sides by $\beta^{-\frac12(l+m)+k}$ and let $\beta\downarrow 0$. By (3.16), (5.1) and (5.4) we obtain for $l\ge m$ that

$$\begin{aligned}&\sum_{j=k}^{m} q^{j(l+m)}q^{-j^2}(q^{-l},q^{-m};q)_j\,H_{l+m-2j}(x\,|\,q)\,\frac{(q^{-j};q)_k q^{j(k+1)}}{(q;q)_j}\\ &\quad=(-1)^k q^{k(l+m)}q^{-\frac12 k(k-1)}\left(q^{-l},q^{-m};q\right)_k H_{l-k}(x\,|\,q)H_{m-k}(x\,|\,q).\end{aligned} \tag{5.6}$$

When written equivalently as

$$\begin{aligned}&\sum_{j=k}^{m} q^{(j-k)(l+m-2k)}q^{-(j-k)(j-k-1)}\,\frac{\left(q^{-(l-k)},q^{-(m-k)};q\right)_{j-k}}{(q;q)_{j-k}}\,H_{l+m-2j}(x\,|\,q)\\ &\quad=H_{l-k}(x\,|\,q)H_{m-k}(x\,|\,q),\end{aligned}$$

it can be seen, just as with (5.2), to be equivalent to its special case $k=0$, which can be written as

$$\sum_{j=0}^{\min(l,m)}(q;q)_j\begin{bmatrix}l\\ j\end{bmatrix}_q\begin{bmatrix}m\\ j\end{bmatrix}_q H_{l+m-2j}(x\,|\,q)=H_l(x\,|\,q)H_m(x\,|\,q). \tag{5.7}$$

This is the linearization formula for continuous q-Hermite polynomials, see [4, (10.11.17)].

Just as with (4.11) and (4.8), the identities (5.2) and (5.6) can be obtained from each other by a Fourier type inversion. This no longer involves an orthogonal system as the q-Racah polynomials but a biorthogonal system implied by the *biorthogonality relation*

$$\sum_{j=0}^{\infty}\frac{(q^{-n};q)_j}{(q;q)_j}\,\frac{(q^{-j};q)_k\,q^{j(k+1)}}{(q;q)_k}=\delta_{k,n} \tag{5.8}$$

(see Carlitz [12, Theorem 2] or Krattenthaler [20, (1.2)] for $a_j=1, b_j=0$). Note that the above sum in fact runs from $j=k$ to n. For $k<n$ formula (5.8) is also equivalent to ${}_1\phi_0(q^{k-n};-;q,q)=\sum_{j=0}^{n-k}\frac{(q^{k-n};q)_j}{(q;q)_j}\,q^j=0$.

The biorthogonality (5.8) is also a limit case of the q-Racah orthogonality relation (3.18). Indeed, replace $\alpha,\beta,\gamma,\delta$ by $\beta q^{-\frac12},\beta q^{-\frac12},q^{-m-1},\beta^{-1}q^{-l-\frac12}$, multiply both sides of (3.18) by β^n, let $\beta\downarrow 0$, and use (5.1), (5.4) and (5.5).

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