The Ramanujan Journal (2021) 56:753–761
https://doi.org/10.1007/s11139-021-00395-x

Some double-angle formulas related to a generalized lemniscate function

Shingo Takeuchi[1]

Dedicated to Professor Tetsutaro Shibata on the occasion of his 60th birthday

Received: 20 February 2020 / Accepted: 18 January 2021 / Published online: 2 March 2021

Abstract
In this paper, we will establish some double-angle formulas related to the inverse function of $\int_0^x \mathrm{d}t/\sqrt{1-t^6}$. This function appears in Ramanujan's Notebooks and is regarded as a generalized version of the lemniscate function.

Keywords Generalized trigonometric functions · Double-angle formulas · Lemniscate function · Jacobian elliptic functions · p-Laplacian

Mathematics Subject Classification 33E05 · 34L40

1 Introduction

Let $1 < p,\ q < \infty$ and

$$F_{p,q}(x) := \int_0^x \frac{\mathrm{d}t}{(1-t^q)^{1/p}}, \quad x \in [0, 1].$$

We will denote by $\sin_{p,q}$ the inverse function of $F_{p,q}$, i.e.,

$$\sin_{p,q} x := F_{p,q}^{-1}(x).$$

The work was supported by JSPS KAKENHI Grant Number 17K05336.

✉ Shingo Takeuchi
shingo@shibaura-it.ac.jp

1 Department of Mathematical Sciences, Shibaura Institute of Technology, 307 Fukasaku, Minuma-ku, Saitama-shi, Saitama 337-8570, Japan

Clearly, $\sin_{p,q} x$ is an increasing function mapping $[0, \pi_{p,q}/2]$ to $[0, 1]$, where

$$\pi_{p,q} := 2F_{p,q}(1) = 2\int_0^1 \frac{\mathrm{d}t}{(1-t^q)^{1/p}}.$$

We extend $\sin_{p,q} x$ to $(\pi_{p,q}/2, \pi_{p,q}]$ by $\sin_{p,q}(\pi_{p,q} - x)$ and to the whole real line $\mathbb{R}$ as the odd $2\pi_{p,q}$-periodic continuation of the function. Since $\sin_{p,q} x \in C^1(\mathbb{R})$, we also define $\cos_{p,q} x$ by $\cos_{p,q} x := (\mathrm{d}/\mathrm{d}x)(\sin_{p,q} x)$. Then, it follows that

$$|\cos_{p,q} x|^p + |\sin_{p,q} x|^q = 1.$$

In case $(p, q) = (2, 2)$, it is obvious that $\sin_{p,q} x,\ \cos_{p,q} x$ and $\pi_{p,q}$ are reduced to the ordinary $\sin x,\ \cos x$ and π, respectively. This is a reason why these functions and the constant are called *generalized trigonometric functions* (with parameter (p, q)) and the *generalized* π, respectively.

The generalized trigonometric functions are well studied in the context of nonlinear differential equations (see [4,6,7] and the references given there). Suppose that u is a solution of the initial value problem of the p-Laplacian

$$-(|u'|^{p-2}u')' = \frac{(p-1)q}{p}|u|^{q-2}u, \quad u(0) = 0,\ u'(0) = 1,$$

which is reduced to the equation $-u'' = u$ of simple harmonic motion for $u = \sin x$ in case $(p, q) = (2, 2)$. Then,

$$\frac{d}{dx}(|u'|^p + |u|^q) = \left(\frac{p}{p-1}(|u'|^{p-2}u')' + q|u|^{q-2}u\right)u' = 0.$$

Therefore, $|u'|^p + |u|^q = 1$. It is possible to show that u coincides with $\sin_{p,q}$ defined as above. The generalized trigonometric functions are often applied to the eigenvalue problem of the p-Laplacian.

Now, we are interested in finding double-angle formulas for generalized trigonometric functions. It is possible to discuss addition formulas for these functions: for instance $\sin_{2,6}$ has the addition formula (3) with (2) below (see also [5] for $\sin_{4/3,4}$), but for simplicity we will not develop this point here.

We have known the double-angle formulas of $\sin_{2,q},\ \sin_{q^*,q}$, and $\sin_{q^*,2}$ for $q = 2, 3, 4$ except for $\sin_{3/2,2}$, where $q^* := q/(q-1)$ (Table 1). For details for each formula, we refer the reader to [8] (after having proved the formula for $\sin_{2,3}$ in the co-authored paper [8], the author noticed that the formula has already been obtained as "$\varphi(2s)$" by Cox and Shurman [3, p. 697]). It is worth pointing out that Lemma 3.1 (resp. Lemma 3.2) below connects the parameter $(2, q)$ to (q^*, q) (resp. $(q^*, 2)$) and yields the possibility to obtain the other formula from one formula. Indeed, in this way, the formulas of $\sin_{4/3,4}$ and $\sin_{4/3,2}$ follow from that of $\sin_{2,4}$ ([10, Subsect. 3.1] and [8, Theorem 1.1], respectively), and the formula of $\sin_{2,3}$ follows from that of $\sin_{3/2,3}$ ([8, Theorem 1.2]). Nevertheless, the parameter $(3/2, 2)$ is still open because of the difficulty of the inverse problem corresponding to (10).

Table 1 The parameters for which the double-angle formulas have been obtained

q	$(q^*, 2)$	$(2, q)$	(q^*, q)
2	(2, 2) by Abu al-Wafa'	(2, 2) by Abu al-Wafa'	(2, 2) by Abu al-Wafa'
3	(3/2, 2) open	(2, 3) by Cox–Shurman	(3/2, 3) by Dixon
4	(4/3, 2) by Sato–Takeuchi	(2, 4) by Fagnano	(4/3, 4) by Edmunds et al.
6	(6/5, 2) Theorem 1.2	(2, 6) by Shinohara	(6/5, 6) Theorem 1.1

In this paper, we wish to investigate the double-angle formula of the function $\sin_{2,6} x$, whose inverse function is defined as

$$\sin_{2,6}^{-1} x = \int_0^x \frac{\mathrm{d}t}{\sqrt{1-t^6}}.$$

The function $\sin_{2,6} x$ appears as the inverse of "$H(v)$" in Ramanujan's Notebooks [1, p. 246] and is regarded as a generalized version of the lemniscate function $\sin_{2,4} x$. For the function, Shinohara [9] gives the novel double-angle formula

$$\sin_{2,6}(2x) = \frac{2\sin_{2,6} x \cos_{2,6} x}{\sqrt{1+8\sin_{2,6}^6 x}}, \quad x \in [0, \pi_{2,6}/2]. \tag{1}$$

In fact, he found (1) in "trial and error calculations" (according to private communication), but instead we will give a proof of (1) in Sect. 2. Moreover, as mentioned above, we can show the following counterparts of (1) for $\sin_{6/5,6}$ and $\sin_{6/5,2}$, respectively.

Theorem 1.1 *Let* $(p, q) = (6/5, 6)$. *Then, for* $x \in [0, \pi_{6/5,6}/4]$,

$$\sin_{6/5,6}(2x) = \frac{2^{1/6}\sin_{6/5,6} x \cos_{6/5,6}^{1/5} x \left(3+\sqrt{1+32\sin_{6/5,6}^6 x \cos_{6/5,6}^{6/5} x}\right)^{1/2}}{\left(1+32\sin_{6/5,6}^6 x\cos_{6/5,6}^{6/5} x\right)^{1/4}\left(1+\sqrt{1+32\sin_{6/5,6}^6 x\cos_{6/5,6}^{6/5} x}\right)^{1/6}}.$$

Theorem 1.2 *Let* $(p, q) = (6/5, 2)$. *Then, for* $x \in [0, \pi_{6/5,2}/2]$,

$$\sin_{6/5,2}(2x) = \sqrt{1-\left(\frac{9-8\sin_{6/5,2}^2 x - 4\sin_{6/5,2}^2 x\cos_{6/5,2}^{2/5} x}{9-8\sin_{6/5,2}^2 x + 8\sin_{6/5,2}^2 x \cos_{6/5,2}^{2/5} x}\right)^3}.$$

2 Proof of (1)

The change of variable $s = t^2$ leads to the representation

$$\sin_{2,6}^{-1} x = \frac{1}{2} \int_0^{x^2} \frac{ds}{\sqrt{s(1-s^3)}}, \quad 0 \le x \le 1.$$

The furthermore change of variable ([2, 576.00 in p. 256])

$$\operatorname{cn} u = \frac{1-(\sqrt{3}+1)s}{1+(\sqrt{3}-1)s}, \quad k^2 = \frac{2-\sqrt{3}}{4}$$

gives

$$\begin{aligned}
\sin_{2,6}^{-1} x &= \frac{1}{2} \int_0^{\operatorname{cn}^{-1}\phi(x)} \frac{((\sqrt{3}+1)+(\sqrt{3}-1)\operatorname{cn} u)^2}{2\cdot 3^{3/4} \operatorname{sn} u \operatorname{dn} u} \\
&\quad \times \frac{2\sqrt{3}\operatorname{sn} u \operatorname{dn} u}{((\sqrt{3}+1)+(\sqrt{3}-1)\operatorname{cn} u)^2}\, du \\
&= \frac{1}{2\cdot 3^{1/4}} \int_0^{\operatorname{cn}^{-1}\phi(x)} du \\
&= \frac{1}{2\cdot 3^{1/4}} \operatorname{cn}^{-1}\phi(x),
\end{aligned}$$

where $\operatorname{sn} u = \operatorname{sn}(u,k)$, $\operatorname{cn} u = \operatorname{cn}(u,k)$, and $\operatorname{dn} u = \operatorname{dn}(u,k)$ are the Jacobian elliptic functions (see e.g., [11, Chap. XXII] for more details), and

$$\phi(x) = \frac{1-(\sqrt{3}+1)x^2}{1+(\sqrt{3}-1)x^2}. \tag{2}$$

Thus,

$$\sin_{2,6} u = \phi^{-1}(\operatorname{cn}(2\cdot 3^{1/4}u)), \quad 0 \le u \le \pi_{2,6}/2 = K/(3^{1/4}),$$

where $K = K(k)$ is the complete elliptic integral of the first kind and

$$\phi^{-1}(x) = \sqrt{\frac{1-x}{(\sqrt{3}+1)+(\sqrt{3}-1)x}}.$$

Now, we use the addition formula of cn. For $u,\ v,\ u \pm v \in [0, K/(3^{1/4})]$,

$$\sin_{2,6}(u \pm v) = \phi^{-1}(\operatorname{cn}(\tilde{u} \pm \tilde{v})) = \phi^{-1}\left(\frac{\operatorname{cn}\tilde{u}\operatorname{cn}\tilde{v} \mp \operatorname{sn}\tilde{u}\operatorname{sn}\tilde{v}\operatorname{dn}\tilde{u}\operatorname{dn}\tilde{v}}{1-k^2\operatorname{sn}^2\tilde{u}\operatorname{sn}^2\tilde{v}}\right)$$

where $\tilde{u} := 2 \cdot 3^{1/4} u$ and $\tilde{v} := 2 \cdot 3^{1/4} v$. Recall that $\mathrm{sn}^2 x + \mathrm{cn}^2 x = 1$ and $k^2 \mathrm{sn}^2 x + \mathrm{dn}^2 x = 1$; then the last equality gives

$$\sin_{2,6}(u \pm v) = \phi^{-1}\left(\frac{\phi(U)\phi(V) \mp \sqrt{(1-\phi(U)^2)(1-\phi(V)^2)(1-k^2(1-\phi(U)^2))(1-k^2(1-\phi(V)^2))}}{1-k^2(1-\phi(U)^2)(1-\phi(V)^2)}\right), \tag{3}$$

where $U := \sin_{2,6} u$ and $V := \sin_{2,6} v$.

With $u = v$ and the observation that

$$1-\phi(U)^2 = \frac{4\sqrt{3}U^2(1-U^2)}{(1+(\sqrt{3}-1)U^2)^2},$$
$$1-k^2(1-\phi(U)^2) = \frac{1+U^2+U^4}{(1+(\sqrt{3}-1)U^2)^2},$$

this implies that

$$\sin_{2,6}(2u) = \phi^{-1}\left(\frac{\phi(U)^2-(1-\phi(U)^2)(1-k^2(1-\phi(U)^2))}{1-k^2(1-\phi(U)^2)^2}\right)$$
$$= \phi^{-1}\left(\frac{1-4(\sqrt{3}+1)U^2+8U^6+4(\sqrt{3}+1)U^8}{1+4(\sqrt{3}-1)U^2+8U^6-4(\sqrt{3}-1)U^8}\right).$$

Routine simplification now results in the formula

$$\sin_{2,6}(2u) = \frac{2U\sqrt{1-U^6}}{\sqrt{1+8U^6}} = \frac{2\sin_{2,6} u \cos_{2,6} u}{\sqrt{1+8\sin_{2,6}^6 u}},$$

and the proof is complete.

3 Proofs of theorems

To prove Theorem 1.1, we use the following multiple-angle formulas.

Lemma 3.1 ([10]) *Let* $1 < q < \infty$ *and* $q^* := q/(q-1)$. *If* $x \in [0, \pi_{2,q}/(2^{2/q})] = [0, \pi_{q^*,q}/2]$, *then*

$$\sin_{2,q}(2^{2/q}x) = 2^{2/q}\sin_{q^*,q} x \cos_{q^*,q}^{q^*-1} x, \tag{4}$$

$$\cos_{2,q}(2^{2/q}x) = \cos_{q^*,q}^{q^*} x - \sin_{q^*,q}^{q} x$$
$$= 1 - 2\sin_{q^*,q}^{q} x = 2\cos_{q^*,q}^{q^*} x - 1. \tag{5}$$

Proof of Theorem 1.1 Let $x \in [0, \pi_{6/5,6}/4]$. Applying (4) of Lemma 3.1 in case $q = 6$ with x replaced by $2x \in [0, \pi_{6/5,6}/2]$, we get

$$\sin_{2,6}(2 \cdot 2^{1/3}x) = 2^{1/3}\sin_{6/5,6}(2x)(1 - \sin_{6/5,6}^6(2x))^{1/6}. \tag{6}$$

First, we consider the case

$$0 \le x < \frac{\pi_{6/5,6}}{8}.$$

Then, since $0 \le 2\sin_{6/5,6}^6(2x) < 1$ by [10, Lemma 2.1], Eq. (6) gives

$$2\sin_{6/5,6}^6(2x) = 1 - \sqrt{1 - \sin_{2,6}^6(2 \cdot 2^{1/3}x)}.$$

Set $S = S(x) := \sin_{2,6}(2^{1/3}x)$. Using the double-angle formula (1) for $\sin_{2,6} x$, we have

$$\begin{aligned} 2\sin_{6/5,6}^6(2x) &= 1 - \sqrt{1 - \left(\frac{2S\sqrt{1-S^6}}{\sqrt{1+8S^6}}\right)^6} \\ &= 1 - \frac{\sqrt{1 - 40S^6 + 384S^{12} + 320S^{18} + 64S^{24}}}{(1+8S^6)^{3/2}} \\ &= 1 - \frac{|1 - 20S^6 - 8S^{12}|}{(1+8S^6)^{3/2}}. \end{aligned}$$

Since $0 \le S^6 < \sin_{2,6}^6(\pi_{2,6}/4) = (3\sqrt{3} - 5)/4$, evaluated by (1), we see that $1 - 20S^6 - 8S^{12} > 0$. Thus,

$$\begin{aligned} 2\sin_{6/5,6}^6(2x) &= 1 - \frac{1 - 20S^6 - 8S^{12}}{(1+8S^6)^{3/2}} \\ &= \frac{(\sqrt{1+8S^6} - 1)(\sqrt{1+8S^6} + 3)^3}{8(1+8S^6)^{3/2}} \\ &= \frac{S^6(3 + \sqrt{1+8S^6})^3}{(1+8S^6)^{3/2}(1 + \sqrt{1+8S^6})}. \end{aligned} \tag{7}$$

Therefore, by (4),

$$\sin_{6/5,6}(2x) = \frac{2^{1/6}\sin_{6/5,6} x \cos_{6/5,6}^{1/5} x \left(3 + \sqrt{1 + 32\sin_{6/5,6}^6 x \cos_{6/5,6}^{6/5} x}\right)^{1/2}}{\left(1 + 32\sin_{6/5,6}^6 x \cos_{6/5,6}^{6/5} x\right)^{1/4} \left(1 + \sqrt{1 + 32\sin_{6/5,6}^6 x \cos_{6/5,6}^{6/5} x}\right)^{1/6}}.$$

In the remaining case

$$\frac{\pi_{6/5,6}}{8} \leq x \leq \frac{\pi_{6/5,6}}{4},$$

it follows easily that $1 \leq 2\sin^6_{6/5,6}(2x) < 2$ and $1 - 20S^6 - 8S^{12} \leq 0$, hence we obtain (7) again. The proof is complete. □

To show Theorem 1.2, the following lemma is useful.

Lemma 3.2 ([5,6]) *Let* $1 < p,\ q < \infty$. *For* $x \in [0, 2]$,

$$q\pi_{p,q} = p^*\pi_{q^*,p^*},$$
$$\sin_{p,q}\left(\frac{\pi_{p,q}}{2}x\right) = \cos^{q^*-1}_{q^*,p^*}\left(\frac{\pi_{q^*,p^*}}{2}(1-x)\right).$$

Proof of Theorem 1.2 Let $x \in [0, \pi_{6/5,2}/2]$. Then, since $4x/\pi_{6/5,2} \in [0, 2]$, it follows from Lemma 3.2 that

$$\sin_{6/5,2}(2x) = \cos_{2,6}\left(\frac{\pi_{2,6}}{2}\left(1 - \frac{4x}{\pi_{6/5,2}}\right)\right) = \cos_{2,6}\left(\frac{\pi_{2,6}}{2} - \frac{2x}{3}\right).$$

Thus,

$$\sin_{6/5,2} 2x = \sqrt{1 - \sin^6_{2,6}\left(\frac{\pi_{2,6}}{2} - \frac{2x}{3}\right)}. \tag{8}$$

The function $\sin_{2,6}$ has the addition formula (3). Letting $u = \pi_{2,6}/2$ and $v = 2x/3$, we have

$$\sin_{2,6}\left(\frac{\pi_{2,6}}{2} - \frac{2x}{3}\right) = \phi^{-1}(-\phi(V)) = \sqrt{\frac{1-V^2}{1+2V^2}}, \tag{9}$$

where $V := \sin_{2,6}(2x/3)$. Applying (9) to the right-hand side of (8), we obtain

$$\sin_{6/5,2} 2x = \sqrt{1 - \left(\frac{1 - \sin^2_{2,6}(2x/3)}{1 + 2\sin^2_{2,6}(2x/3)}\right)^3}.$$

Let $f(x) := \sin_{6/5,2} x$ and $g(x) := \sin_{2,6}(2x/3)$. Then

$$f(2x) = \sqrt{1 - \left(\frac{1 - g(x)^2}{1 + 2g(x)^2}\right)^3}. \tag{10}$$

Therefore, it is easy to see that

$$g(x) = \sqrt{\frac{1 - (1 - f(2x)^2)^{1/3}}{1 + 2(1 - f(2x)^2)^{1/3}}}. \tag{11}$$

On the other hand, by (1) with x replaced with $x/2$, we see that $g(x)$ satisfies

$$g(x) = \frac{2g(x/2)\sqrt{1 - g(x/2)^6}}{\sqrt{1 + 8g(x/2)^6}}.$$

Applying (11) with x replaced with $x/2$ to the right-hand side, we obtain

$$g(x) = \frac{2f(x)(1 - f(x)^2)^{1/6}}{\sqrt{9 - 8f(x)^2}}. \tag{12}$$

Substituting (12) into (10), we can express $f(2x)$ in terms of $f(x)$, i.e.,

$$f(2x) = \sqrt{1 - \left(\frac{9 - 8f(x)^2 - 4f(x)^2(1 - f(x)^2)^{1/3}}{9 - 8f(x)^2 + 8f(x)^2(1 - f(x)^2)^{1/3}}\right)^3}.$$

Since $1 - f(x)^2 = \cos_{6/5,2}^{6/5} x$, the proof is complete. □

Acknowledgements The author would like to thank Professor Kazunori Shinohara and anonymous reviewers for their valuable comments and suggestions to improve the quality of the paper.

References

1. Berndt, B.C.: Ramanujan's Notebooks, Part IV. Springer, New York (1994)
2. Byrd, P.F., Friedman, M.D.: Handbook of Elliptic Integrals for Engineers and Scientists, 2nd edn revised. Die Grundlehren der mathematischen Wissenschaften, Band 67. Springer, New York (1971)
3. Cox, D.A., Shurman, J.: Geometry and number theory on clovers. Am. Math. Mon. **112**(8), 682–704 (2005)
4. Drábek, P., Manásevich, R.: On the closed solution to some nonhomogeneous eigenvalue problems with p-Laplacian. Differ. Integral Equ. **12**, 773–788 (1999)
5. Edmunds, D.E., Gurka, P., Lang, J.: Properties of generalized trigonometric functions. J. Approx. Theory **164**(1), 47–56 (2012)
6. Kobayashi, H., Takeuchi, S.: Applications of generalized trigonometric functions with two parameters. Commun. Pure Appl. Anal. **18**(3), 1509–1521 (2019)

7. Lang, J., Edmunds, D.E.: Eigenvalues, Embeddings and Generalised Trigonometric Functions. Lecture Notes in Mathematics, vol. 2016. Springer, Heidelberg (2011)
8. Sato, S., Takeuchi, S.: Two double-angle formulas of generalized trigonometric functions. J. Approx. Theory **250**, 105322 (2020)
9. Shinohara, K.: Addition formulas of leaf functions according to integral root of polynomial based on analogies of inverse trigonometric functions and inverse lemniscate functions. Appl. Math. Sci. **11**(52), 2561–2577 (2017)
10. Takeuchi, S.: Multiple-angle formulas of generalized trigonometric functions with two parameters. J. Math. Anal. Appl. **444**(2), 1000–1014 (2016)
11. Whittaker, E.T., Watson, G.N.: A course of modern analysis, An Introduction to the General Theory of Infinite Processes and of Analytic Functions; with an Account of the Principal Transcendental Functions. Reprint of the 4th edn, p. 1996. Cambridge University Press, Cambridge Mathematical Library, Cambridge (1927)