Another generalization of Euler's arithmetic function and Menon's identity

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Abstract

We define the k -dimensional generalized Euler function $\varphi_k(n)$ as the number of ordered k -tuples $(a_1, \dots, a_k) \in \mathbb{N}^k$ such that $1 \leq a_1, \dots, a_k \leq n$ and both the product $a_1 \cdots a_k$ and the sum $a_1 + \dots + a_k$ are prime to n . We investigate some of the properties of the function $\varphi_k(n)$, and obtain a corresponding Menon-type identity.

Keywords Euler's arithmetic function · Menon's identity · Asymptotic formula

Mathematics Subject Classification 11A07 · 11A25 · 11N37

1 Motivation

Jordan's arithmetic function $J_k(n)$ is defined as the number of ordered k -tuples $(a_1, \dots, a_k) \in \mathbb{N}^k$ such that $1 \leq a_1, \dots, a_k \leq n$ and the $\gcd(a_1, \dots, a_k, n) = 1$. It is well known that $J_k(n)$ is multiplicative in n and $J_k(n) = n^k \prod_{p|n} (1 - 1/p^k)$. If $k = 1$, then $J_1(n) = \varphi(n)$ is Euler's arithmetic function.

A Menon-type identity concerning the function $J_k(n)$, obtained by Nageswara Rao [7], is given by

$$\sum_{\substack{a_1, \dots, a_k=1 \\ (a_1, \dots, a_k, n)=1}}^n (a_1 - 1, \dots, a_k - 1, n)^k = J_k(n) \tau(n) \quad (n \in \mathbb{N}), \quad (1.1)$$

where $\tau(n)$ is the number of divisors of n . If $k = 1$, then (1.1) reduces to Menon's original identity [6].

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Euler's arithmetic function and Menon's identity have been generalized in various directions by several authors. See, e.g., the books [5,8], the papers [4,7,9,11,12], and their references.

The function $X(n) = \#\{(a, b) \in \mathbb{N}^2 : 1 \leq a, b \leq n, (ab, n) = (a + b, n) = 1\}$ is an analog of Euler's φ -function, and was introduced by Arai and Gakuen [2]. It was shown by Carlitz [3] that the function $X(n)$ is multiplicative and

$$X(n) = n^2 \prod_{p|n} \left(1 - \frac{1}{p}\right) \left(1 - \frac{2}{p}\right) \quad (n \in \mathbb{N}).$$

Note that if n is even, then $X(n) = 0$. The function $X(n)$ can also be given as

$$X(n) = \varphi(n)^2 \sum_{d|n} \frac{\mu(d)}{\varphi(d)} \quad (n \in \mathbb{N}),$$

μ denoting the Möbius function. The corresponding Menon-type identity

$$\sum_{\substack{a, b=1 \\ (ab, n)=(a+b, n)=1}}^n (a + b - 1, n) = X(n)\tau(n) \quad (n \in \mathbb{N}) \quad (1.2)$$

was deduced by Sita Ramaiah [9, Cor. 10.4]. In fact, (1.2) is a corollary of a more general identity involving Narkiewicz-type regular systems of divisors and k -reduced residue systems.

Recently, identity (1.2) was generalized by Ji and Wang [4,12] to residually finite Dedekind domains, by using Narkiewicz-type regular systems of divisors, and to the ring of algebraic integers, concerning Dirichlet characters modulo n , respectively. Note that in paper [12] identity (1.2) is called the “Arai–Carlitz identity.” However, Arai and Carlitz only considered the function $X(n)$ and did not deduce such an identity. We refer to (1.2) as the Sita Ramaiah identity.

It is natural to introduce and to study the following k -dimensional generalization of the function $X(n)$, and to ask if the corresponding generalization of the Sita Ramaiah identity is true for it. These were not investigated in the literature, as far as we know. For $k \in \mathbb{N}$ we define the function $\varphi_k(n)$ as

$$\varphi_k(n) := \sum_{\substack{a_1, \dots, a_k=1 \\ (a_1 \cdots a_k, n)=1 \\ (a_1 + \cdots + a_k, n)=1}}^n 1. \quad (1.3)$$

Note that $\varphi_1(n) = \varphi(n)$ is Euler's function and $\varphi_2(n) = X(n)$ of above. We investigate some of the properties of the function $\varphi_k(n)$, and obtain a corresponding Menon-type identity. Our main results are included in Sect. 2, and their proofs are presented in Sects. 3 and 4.

We will use the following notations: $\text{id}_k(n) = n^k$, $\mathbf{1}(n) = 1$ ($n \in \mathbb{N}$), $\omega(n)$ will denote the number of distinct prime factors of n , and “ $*$ ” the Dirichlet convolution of arithmetic functions.

2 Main results

In this paper we prove the following results.

Theorem 2.1 *For every $k, n \in \mathbb{N}$,*

$$\varphi_k(n) = \varphi(n)^k \prod_{p|n} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \cdots + (-1)^{k-1} \frac{1}{(p-1)^{k-1}} \right) \quad (2.1)$$

$$= n^k \prod_{p|n} \left(1 - \frac{1}{p} \right) \left(\left(1 - \frac{1}{p} \right)^k - \frac{(-1)^k}{p^k} \right). \quad (2.2)$$

It is a consequence of Theorem 2.1 that the function $\varphi_k(n)$ is multiplicative. Also, $\varphi_k(n) = 0$ if and only if k and n are both even. Further properties of $\varphi_k(n)$ can be deduced. Its average order is given by the next result.

Theorem 2.2 *Let $k \geq 2$ be fixed. Then*

$$\sum_{n \leq x} \varphi_k(n) = \frac{C_k}{k+1} x^{k+1} + O\left(x^k (\log x)^{k+1}\right),$$

where

$$C_k = \prod_p \left(1 + \frac{1}{p^{k+1}} \left(\left(1 - \frac{1}{p} \right) \left((p-1)^k - (-1)^k \right) - p^k \right) \right).$$

Corollary 2.3 ($k = 2$) *We have*

$$\sum_{n \leq x} X(n) = \frac{C_2}{3} x^3 + O\left(x^2 (\log x)^3\right),$$

where

$$C_2 = \prod_p \left(1 - \frac{3}{p^2} + \frac{2}{p^3} \right) \approx 0.286747.$$

We have the following generalization of Menon's identity.

Theorem 2.4 *Let f be an arbitrary arithmetic function. Then for every $k, n \in \mathbb{N}$,*

$$\sum_{\substack{a_1, \dots, a_k=1 \\ (a_1 \cdots a_k, n)=1 \\ (a_1 + \cdots + a_k, n)=1}}^n f((a_1 + \cdots + a_k - 1, n)) = \varphi_k(n) \sum_{d|n} \frac{(\mu * f)(d)}{\varphi(d)}. \quad (2.3)$$

Corollary 2.5 ($f(n) = n$) For every $k, n \in \mathbb{N}$,

$$\sum_{\substack{a_1, \dots, a_k=1 \\ (a_1 \cdots a_k, n)=1 \\ (a_1 + \cdots + a_k, n)=1}}^n (a_1 + \cdots + a_k - 1, n) = \varphi_k(n) \tau(n). \quad (2.4)$$

If $k = 1$, then (2.4) reduces to Menon's identity and if $k = 2$, then it gives the Sita Ramaiah identity (1.2).

3 Proofs of Theorems 2.1 and 2.2

We need the following lemmas.

Lemma 3.1 Let $n, d \in \mathbb{N}$, $d \mid n$, and let $r \in \mathbb{Z}$. Then

$$\sum_{\substack{a=1 \\ (a, n)=1 \\ a \equiv r \pmod{d}}}^n 1 = \begin{cases} \frac{\varphi(n)}{\varphi(d)} & \text{if } (r, d) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.1 is known in the literature, usually proved by the inclusion–exclusion principle. See, e.g., [1, Th. 5.32]. The following generalization and a different approach of proof are given in our paper [11].

Lemma 3.2 [11, Lemma 2.1] Let $n, d, e \in \mathbb{N}$, $d \mid n$, $e \mid n$ and let $r, s \in \mathbb{Z}$. Then

$$\sum_{\substack{a=1 \\ (a, n)=1 \\ a \equiv r \pmod{d} \\ a \equiv s \pmod{e}}}^n 1 = \begin{cases} \frac{\varphi(n)}{\varphi(de)} (d, e) & \text{if } (r, d) = (s, e) = 1 \text{ and } (d, e) \mid r - s, \\ 0 & \text{otherwise.} \end{cases}$$

In the case $e = 1$, Lemma 3.2 reduces to Lemma 3.1.

We need to define the following slightly more general function than $\varphi_k(n)$:

$$\varphi_k(n, m) := \sum_{\substack{a_1, \dots, a_k=1 \\ (a_1 \cdots a_k, n)=1 \\ (a_1 + \cdots + a_k, m)=1}}^n 1. \quad (3.1)$$

If $m = n$, then $\varphi_k(n, n) = \varphi_k(n)$, given by (1.3).

Lemma 3.3 (recursion formula for $\varphi_k(n, m)$) Let $k \geq 2$ and $m \mid n$. Then

$$\varphi_k(n, m) = \varphi(n) \sum_{d \mid m} \frac{\mu(d)}{\varphi(d)} \varphi_{k-1}(n, d).$$

Proof of Lemma 3.3 We have

$$\begin{aligned}
 \varphi_k(n, m) &= \sum_{\substack{a_1, \dots, a_k=1 \\ (a_1 \cdots a_k, n)=1}}^n \sum_{\substack{d|(a_1 + \dots + a_k, m)}} \mu(d) \\
 &= \sum_{d|m} \mu(d) \sum_{\substack{a_1, \dots, a_k=1 \\ (a_1 \cdots a_k, n)=1 \\ a_1 + \dots + a_k \equiv 0 \pmod{d}}}^n 1 \\
 &= \sum_{d|m} \mu(d) \sum_{\substack{a_1, \dots, a_{k-1}=1 \\ (a_1 \cdots a_{k-1}, n)=1}}^n \sum_{\substack{a_k=1 \\ (a_k, n)=1 \\ a_k \equiv -a_1 - \dots - a_{k-1} \pmod{d}}}^n 1.
 \end{aligned}$$

By using Lemma 3.1 we deduce that

$$\begin{aligned}
 \varphi_k(n, m) &= \sum_{d|m} \mu(d) \sum_{\substack{a_1, \dots, a_{k-1}=1 \\ (a_1 \cdots a_{k-1}, n)=1 \\ (a_1 + \dots + a_{k-1}, d)=1}}^n \frac{\varphi(n)}{\varphi(d)} \\
 &= \varphi(n) \sum_{d|m} \frac{\mu(d)}{\varphi(d)} \sum_{\substack{a_1, \dots, a_{k-1}=1 \\ (a_1 \cdots a_{k-1}, n)=1 \\ (a_1 + \dots + a_{k-1}, d)=1}}^n 1 \\
 &= \varphi(n) \sum_{d|m} \frac{\mu(d)}{\varphi(d)} \varphi_{k-1}(n, d),
 \end{aligned}$$

where $d \mid m$ and $m \mid n$ imply that $d \mid n$. □

Proof of Theorem 2.1 Let $k, n, m \in \mathbb{N}$ such that $m \mid n$. We show that

$$\varphi_k(n, m) = \varphi(n)^k \prod_{p|m} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \dots + (-1)^{k-1} \frac{1}{(p-1)^{k-1}} \right). \quad (3.2)$$

By induction on k . If $k = 1$, then $\varphi_1(n, m) = \varphi(n)$, by its definition (3.1). Let $k \geq 2$. Assume that (3.2) holds for $k - 1$ and prove it for k . We have, by using Lemma 3.3,

$$\begin{aligned}
 \varphi_k(n, m) &= \varphi(n) \sum_{d|m} \frac{\mu(d)}{\varphi(d)} \varphi_{k-1}(n, d) \\
 &= \varphi(n) \sum_{d|m} \frac{\mu(d)}{\varphi(d)} \varphi(n)^{k-1} \prod_{p|d} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \dots \right. \\
 &\quad \left. + (-1)^{k-2} \frac{1}{(p-1)^{k-2}} \right)
 \end{aligned}$$

$$\begin{aligned}
&= \varphi(n)^k \prod_{p|m} \left(1 + \frac{\mu(p)}{\varphi(p)} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \cdots \right. \right. \\
&\quad \left. \left. + (-1)^{k-2} \frac{1}{(p-1)^{k-2}} \right) \right) \\
&= \varphi(n)^k \prod_{p|m} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \cdots + (-1)^{k-1} \frac{1}{(p-1)^{k-1}} \right),
\end{aligned}$$

which proves formula (3.2). Now choosing $m = n$, (3.2) gives identity (2.1), which can be rewritten as (2.2). $\square$

Proof of Theorem 2.2 Let $\varphi_k = \text{id}_k * g_k$, that is, $g_k = \varphi_k * \mu \text{id}_k$. Here the function $g_k(n)$ is multiplicative and for any prime power p^v ($v \geq 1$),

$$g_k(p^v) = \varphi_k(p^v) - p^k \varphi_k(p^{v-1}).$$

We obtain from (2.2) that for $v \geq 2$,

$$\begin{aligned}
g_k(p^v) &= p^{(v-1)k} \left(1 - \frac{1}{p} \right) \left((p-1)^k - (-1)^k \right) \\
&\quad - p^k p^{(v-2)k} \left(1 - \frac{1}{p} \right) \left((p-1)^k - (-1)^k \right) = 0,
\end{aligned} \tag{3.3}$$

and for $v = 1$,

$$\begin{aligned}
g_k(p) &= \varphi_k(p) - p^k = \left(1 - \frac{1}{p} \right) \left((p-1)^k - (-1)^k \right) - p^k \\
&= -(k+1)p^{k-1} + \cdots + (-1)^k k,
\end{aligned} \tag{3.4}$$

a polynomial in p of degree $k-1$, with leading coefficient $-(k+1)$. Actually, we have

$$-(k+1)p^{k-1} < g_k(p) < 0 \tag{3.5}$$

for every integer $k \geq 2$ and every prime $p \geq 2$. To see this, note that by Lagrange's mean value theorem,

$$k(p-1)^{k-1} < p^k - (p-1)^k < kp^{k-1},$$

and from (3.4) we deduce that

$$\begin{aligned}
g_k(p) &> \left(1 - \frac{1}{p} \right) \left(p^k - kp^{k-1} - (-1)^k \right) - p^k \\
&= -(k+1)p^{k-1} + kp^{k-2} - (-1)^k \left(1 - \frac{1}{p} \right) > -(k+1)p^{k-1}.
\end{aligned}$$

On the other hand, $p^k - (p-1)^k > 1$, $(p-1)^k - (-1)^k < p^k$ imply that

$$g_k(p) < \left(1 - \frac{1}{p}\right) p^k - p^k < 0.$$

According to (3.5), $|g_k(p)| < (k+1)p^{k-1}$ holds true for every $k \geq 2$ and every $p \geq 2$, and by (3.3) we deduce that

$$|g_k(n)| \leq (k+1)^{\omega(n)} n^{k-1} \quad (n \in \mathbb{N}).$$

To obtain the desired asymptotic formula we apply elementary arguments. We have

$$\begin{aligned} \sum_{n \leq x} \varphi_k(n) &= \sum_{d \leq x} g_k(d) \sum_{\delta \leq x/d} \delta^k \\ &= \sum_{d \leq x} g_k(d) \left(\frac{1}{k+1} \left(\frac{x}{d}\right)^{k+1} + O\left(\left(\frac{x}{d}\right)^k\right) \right) \\ &= \frac{x^{k+1}}{k+1} \sum_{d=1}^{\infty} \frac{g_k(d)}{d^{k+1}} + O\left(x^{k+1} \sum_{d>x} \frac{|g_k(d)|}{d^{k+1}}\right) + O\left(x^k \sum_{d \leq x} \frac{|g_k(d)|}{d^k}\right). \end{aligned}$$

Here the main term is $\frac{C_k}{k+1} x^{k+1}$ by using the Euler product formula. To evaluate the error terms consider the Piltz divisor function $\tau_{k+1}(n)$, representing the number of ordered $(k+1)$ -tuples $(a_1, \dots, a_{k+1}) \in \mathbb{N}^{k+1}$ such that $a_1 \cdots a_{k+1} = n$. We have $\tau_{k+1}(p^\nu) \geq \tau_{k+1}(p) = k+1$ for every prime power p^ν ($\nu \geq 1$), and $\tau_{k+1}(n) \geq (k+1)^{\omega(n)}$ for every $n \in \mathbb{N}$.

We obtain

$$\sum_{d>x} \frac{|g_k(d)|}{d^{k+1}} \leq \sum_{d>x} \frac{(k+1)^{\omega(d)}}{d^2} \leq \sum_{d>x} \frac{\tau_{k+1}(d)}{d^2} \ll \frac{(\log x)^k}{x},$$

and

$$\sum_{d \leq x} \frac{|g_k(d)|}{d^k} \leq \sum_{d \leq x} \frac{(k+1)^{\omega(d)}}{d} \leq \sum_{d \leq x} \frac{\tau_{k+1}(d)}{d} \ll (\log x)^{k+1},$$

by using known elementary estimates on the Piltz divisor function. See, e.g., [10, Lemma 3]. This completes the proof. $\square$

4 Proof of Theorem 2.4

Let $M_k(n)$ denote the sum on the left-hand side of (2.3). We have by the convolutional identity $f = (\mu * f) * \mathbf{1}$,

$$\begin{aligned} M_k(n) &= \sum_{\substack{a_1, \dots, a_k=1 \\ (a_1 \cdots a_k, n)=1 \\ (a_1 + \cdots + a_k, n)=1}}^n \sum_{d|(a_1 + \cdots + a_k - 1, n)} (\mu * f)(d) = \sum_{d|n} (\mu * f)(d) \sum_{\substack{a_1, \dots, a_k=1 \\ (a_1 \cdots a_k, n)=1 \\ (a_1 + \cdots + a_k, n)=1 \\ a_1 + \cdots + a_k \equiv 1 \pmod{d}}}^n 1 \\ &= \sum_{d|n} (\mu * f)(d) \sum_{\substack{a_1, \dots, a_k=1 \\ (a_1 \cdots a_k, n)=1 \\ a_1 + \cdots + a_k \equiv 1 \pmod{d}}}^n \sum_{\delta|(a_1 + \cdots + a_k, n)} \mu(\delta), \end{aligned}$$

that is

$$M_k(n) = \sum_{d|n} (\mu * f)(d) \sum_{\delta|n} \mu(\delta) N_k(n, d, \delta), \quad (4.1)$$

where

$$N_k(n, d, \delta) := \sum_{\substack{a_1, \dots, a_k=1 \\ (a_1 \cdots a_k, n)=1 \\ a_1 + \cdots + a_k \equiv 1 \pmod{d} \\ a_1 + \cdots + a_k \equiv 0 \pmod{\delta}}}^n 1.$$

Next we evaluate the sum $N_k(n, d, \delta)$, where $d | n$, $\delta | n$ are fixed. If $(d, \delta) > 1$, then $N_k(n, d, \delta) = 0$, the empty sum. So, assume that $(d, \delta) = 1$. If $k = 1$, then by using Lemma 3.2 we deduce

$$N_1(n, d, \delta) := \sum_{\substack{a_1=1 \\ (a_1, n)=1 \\ a_1 \equiv 1 \pmod{d} \\ a_1 \equiv 0 \pmod{\delta}}}^n 1 = \begin{cases} \frac{\varphi(n)}{\varphi(d)} & \text{if } \delta = 1, \\ 0 & \text{otherwise,} \end{cases} \quad (4.2)$$

since for each term of the sum $\delta | a_1$ and $\delta | n$, which gives $\delta | (a_1, n) = 1$, so $\delta = 1$.

Lemma 4.1 (Recursion formula for $N_k(n, d, \delta)$) *Let $k \geq 2$, $d | n$, $\delta | n$, $(d, \delta) = 1$. Then*

$$N_k(n, d, \delta) = \frac{\varphi(n)}{\varphi(d)\varphi(\delta)} \sum_{j|d} \mu(j) \sum_{t|\delta} \mu(t) N_{k-1}(n, j, t). \quad (4.3)$$

Proof of Lemma 4.1 We have

$$N_k(n, d, \delta) = \sum_{\substack{a_1, \dots, a_{k-1}=1 \\ (a_1 \cdots a_{k-1}, n)=1}}^n \sum_{\substack{a_k=1 \\ (a_k, n)=1 \\ a_k \equiv 1 - a_1 - \cdots - a_{k-1} \pmod{d} \\ a_k \equiv -a_1 - \cdots - a_{k-1} \pmod{\delta}}}^n 1.$$

Using that $(d, \delta) = 1$ and applying Lemma 3.2 we deduce that

$$\begin{aligned}
 N_k(n, d, \delta) &= \sum_{\substack{a_1, \dots, a_{k-1}=1 \\ (a_1 \cdots a_{k-1}, n)=1 \\ (a_1 + \cdots + a_{k-1} - 1, d)=1 \\ (a_1 + \cdots + a_{k-1}, \delta)=1}}^n \frac{\varphi(n)}{\varphi(d)\varphi(\delta)} \\
 &= \frac{\varphi(n)}{\varphi(d)\varphi(\delta)} \sum_{\substack{a_1, \dots, a_{k-1}=1 \\ (a_1 \cdots a_{k-1}, n)=1}}^n \sum_{j|(a_1 + \cdots + a_{k-1} - 1, d)} \mu(j) \sum_{t|(a_1 + \cdots + a_{k-1}, \delta)} \mu(t) \\
 &= \frac{\varphi(n)}{\varphi(d)\varphi(\delta)} \sum_{j|d} \mu(j) \sum_{t|\delta} \mu(t) \sum_{\substack{a_1, \dots, a_{k-1}=1 \\ (a_1 \cdots a_{k-1}, n)=1 \\ a_1 + \cdots + a_{k-1} \equiv 1 \pmod{j} \\ a_1 + \cdots + a_{k-1} \equiv 0 \pmod{t}}}^n 1 \\
 &= \frac{\varphi(n)}{\varphi(d)\varphi(\delta)} \sum_{j|d} \mu(j) \sum_{t|\delta} \mu(t) N_{k-1}(n, j, t).
 \end{aligned}$$

□

Lemma 4.2 Let $k \geq 2$, $d \mid n$, $\delta \mid n$, $(d, \delta) = 1$. Then

$$\begin{aligned}
 N_k(n, d, \delta) &= \frac{\varphi(n)^k}{\varphi(d)\varphi(\delta)} \prod_{p|d} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \cdots + (-1)^{k-1} \frac{1}{(p-1)^{k-1}} \right) \\
 &\quad \times \prod_{p|\delta} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \cdots + (-1)^{k-2} \frac{1}{(p-1)^{k-2}} \right). \quad (4.4)
 \end{aligned}$$

Proof of Lemma 4.2 By induction on k . If $k = 2$, then by the recursion (4.3) and (4.2),

$$\begin{aligned}
 N_2(n, d, \delta) &= \frac{\varphi(n)}{\varphi(d)\varphi(\delta)} \sum_{j|d} \mu(j) \sum_{t|\delta} \mu(t) N_1(n, j, t) \\
 &= \frac{\varphi(n)}{\varphi(d)\varphi(\delta)} \sum_{j|d} \mu(j) \sum_{\substack{t|\delta \\ t=1}} \mu(t) \frac{\varphi(n)}{\varphi(j)} \quad (4.5)
 \end{aligned}$$

$$= \frac{\varphi(n)^2}{\varphi(d)\varphi(\delta)} \sum_{j|d} \frac{\mu(j)}{\varphi(j)} = \frac{\varphi(n)^2}{\varphi(d)\varphi(\delta)} \prod_{p|d} \left(1 - \frac{1}{p-1} \right). \quad (4.6)$$

Hence, the formula is true for $k = 2$. Assume it holds for $k - 1$, where $k \geq 3$. Then we have, by the recursion (4.3),

$$\begin{aligned}
N_k(n, d, \delta) &= \frac{\varphi(n)}{\varphi(d)\varphi(\delta)} \sum_{j|d} \mu(j) \sum_{\substack{t|\delta \\ (t,j)=1}} \mu(t) \frac{\varphi(n)^{k-1}}{\varphi(j)\varphi(t)} \\
&\quad \times \prod_{p|j} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \cdots + (-1)^{k-2} \frac{1}{(p-1)^{k-2}} \right) \\
&\quad \times \prod_{p|t} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \cdots + (-1)^{k-3} \frac{1}{(p-1)^{k-3}} \right),
\end{aligned}$$

where the condition $(t, j) = 1$ can be omitted, since $j | d$, $t | \delta$ and $(d, \delta) = 1$. We deduce that

$$\begin{aligned}
N_k(n, d, \delta) &= \frac{\varphi(n)^k}{\varphi(d)\varphi(\delta)} \sum_{j|d} \frac{\mu(j)}{\varphi(j)} \prod_{p|j} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \cdots \right. \\
&\quad \left. + (-1)^{k-2} \frac{1}{(p-1)^{k-2}} \right) \\
&\quad \times \sum_{t|\delta} \frac{\mu(t)}{\varphi(t)} \prod_{p|t} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \cdots + (-1)^{k-3} \frac{1}{(p-1)^{k-3}} \right) \\
&= \frac{\varphi(n)^k}{\varphi(d)\varphi(\delta)} \prod_{p|d} \left(1 - \frac{1}{p-1} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \cdots \right. \right. \\
&\quad \left. \left. + (-1)^{k-2} \frac{1}{(p-1)^{k-2}} \right) \right) \\
&\quad \times \prod_{p|\delta} \left(1 - \frac{1}{p-1} \left(\frac{1}{p-1} + \frac{1}{(p-1)^2} - \cdots \right. \right. \\
&\quad \left. \left. + (-1)^{k-3} \frac{1}{(p-1)^{k-3}} \right) \right),
\end{aligned}$$

giving (4.4), which completes the proof of Lemma 4.2. $\square$

Now we continue the evaluation of $M_k(n)$. According to (4.1) and Lemma 4.2, we have

$$\begin{aligned}
M_k(n) &= \sum_{d|n} (\mu * f)(d) \sum_{\substack{\delta|n \\ (\delta,d)=1}} \mu(\delta) \frac{\varphi(n)^k}{\varphi(d)\varphi(\delta)} \prod_{p|d} \left(1 - \frac{1}{p-1} \right. \\
&\quad \left. + \frac{1}{(p-1)^2} - \cdots + (-1)^{k-1} \frac{1}{(p-1)^{k-1}} \right) \\
&\quad \times \prod_{p|\delta} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \cdots + (-1)^{k-2} \frac{1}{(p-1)^{k-2}} \right)
\end{aligned}$$

$$\begin{aligned}
 &= \varphi(n)^k \sum_{d|n} \frac{(\mu * f)(d)}{\varphi(d)} \prod_{p|d} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \dots \right. \\
 &\quad \left. + (-1)^{k-1} \frac{1}{(p-1)^{k-1}} \right) \\
 &\quad \times \sum_{\substack{\delta|n \\ (\delta, d)=1}} \frac{\mu(\delta)}{\varphi(\delta)} \prod_{p|\delta} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \dots \right. \\
 &\quad \left. + (-1)^{k-2} \frac{1}{(p-1)^{k-2}} \right),
 \end{aligned}$$

where the inner sum is

$$\begin{aligned}
 &\prod_{\substack{p|n \\ p \nmid d}} \left(1 + \frac{\mu(p)}{\varphi(p)} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \dots + (-1)^{k-2} \frac{1}{(p-1)^{k-2}} \right) \right) \\
 &= \prod_{p|n} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \dots + (-1)^{k-1} \frac{1}{(p-1)^{k-1}} \right) \\
 &\quad \times \prod_{p|d} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \dots + (-1)^{k-1} \frac{1}{(p-1)^{k-1}} \right)^{-1}.
 \end{aligned}$$

This leads to

$$\begin{aligned}
 M_k(n) &= \varphi(n)^k \prod_{p|n} \left(1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} - \dots + (-1)^{k-1} \frac{1}{(p-1)^{k-1}} \right) \\
 &\quad \times \sum_{d|n} \frac{(\mu * f)(d)}{\varphi(d)} \\
 &= \varphi_k(n) \sum_{d|n} \frac{(\mu * f)(d)}{\varphi(d)},
 \end{aligned}$$

by using (2.1), finishing the proof of Theorem 2.4.

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