

$3nj$ -symbols and identities for q -Bessel functions

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Abstract The $6j$ -symbols for representations of the q -deformed algebra of polynomials on $SU(2)$ are given by Jackson's third q -Bessel functions. This interpretation leads to several summation identities for the q -Bessel functions. Multivariate q -Bessel functions are defined, which are shown to be limit cases of multivariate Askey–Wilson polynomials. The multivariate q -Bessel functions occur as $3nj$ -symbols.

Keywords Jackson's third q -Bessel function · $6j$ -symbols · $3nj$ -symbols · Multivariate q -Bessel function · Quantum algebra representations

Mathematics Subject Classification 33D45 · 33D50 · 33D80 · 81R50

1 Introduction

It is well known that Wigner's $6j$ -symbols for the $SU(2)$ group are multiples of hypergeometric orthogonal polynomials called the Racah polynomials. Similarly, $6j$ -symbols for the $SU(2)$ quantum group can be expressed in terms of q -Racah polynomials, which are q -hypergeometric orthogonal polynomials. With this interpretation, properties of $6j$ -symbols such as summation formulas and orthogonality relations lead to properties of specific families of orthogonal polynomials, see e.g., [21, 22, Chaps. 8, 14].

In this paper, we consider $6j$ -symbols for representations of the q -deformed algebra of polynomials on $SU(2)$. This algebra has as irreducible representations the trivial one, and a family of infinite-dimensional representations which disappear in the classical

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limit. The $6j$ -symbols for tensor products of three infinite-dimensional representations can be expressed in terms of Jackson's third q -Bessel functions [8]. Note that, different from the classical $6j$ -symbols, these are not polynomials. We consider three fundamental identities for $6j$ -symbols (see e.g., [1]): Racah's backcoupling identity, the Biedenharn–Elliott identity and the hexagon identity. These identities are obtained by decomposing 3- or 4-fold tensor product representations in several ways. To keep track of the order of decomposing the representations, it is convenient to identify certain vectors in the representation spaces with binary trees. Then the $6j$ -symbols can be considered as coupling coefficients between two of these trees. The identities we obtain can be interpreted as summation identities for q -Bessel functions. We remark that the hexagon identity implies that the q -Bessel functions are matrix elements of an infinite-dimensional solution of the quantum Yang–Baxter equation (or, the star-triangle equation in IRF-models), see e.g., [10], which should be of independent interest.

We also consider specific $3nj$ -symbols, which may naturally be considered as multivariate q -Bessel functions. The one variable q -Bessel functions fit into an extended Askey-scheme [15] of orthogonal q -hypergeometric functions; the original (q -)Askey-scheme [12] consists of (q -)hypergeometric orthogonal polynomials. We will show that the multivariate q -Bessel functions fit into an extended Askey-scheme of multivariate orthogonal functions of q -hypergeometric type, by showing that the multivariate q -Bessel functions can be obtained as limits of the multivariate Askey–Wilson polynomials defined by Gasper and Rahman [4], which are the q -analogs of Tratnik's multivariate Wilson polynomials [19]. The multivariate Askey–Wilson polynomials can be thought of as being on top of a scheme of multivariate orthogonal polynomials; several limit cases are considered in [4, 5, 9]. Geronimo and Iliev [7] obtained multivariate Askey–Wilson functions generalizing the multivariate Askey–Wilson polynomials, which should be on top of the extended Askey-scheme. Several families of orthogonal polynomials in this scheme and its $q = 1$ analog are connected to tensor product representations and binary coupling schemes, see e.g., Van der Jeugt [20], Rosengren [17], Scarabotti [18], and a recent result [6] by Genest et al.

This paper is organized as follows: In Sect. 2, the quantum algebra $\mathcal{A}_q(\text{SU}(2))$ and its representation theory are recalled. In Sect. 3, it is shown that the $6j$ -symbols are essentially q -Bessel functions, using a generating function for q -Bessel functions. Using binary trees, we obtain the fundamental identities for $6j$ -symbols, leading to summation formulas for the q -Bessel functions. In Sect. 4, we first define multivariate q -Bessel functions as nontrivial products of q -Bessel functions, and we prove orthogonality relations. Then we show that these multivariate q -Bessel functions occur as $3nj$ -symbols, and use this interpretation to find a summation formula.

Notations We use $\mathbb{N} = \{0, 1, 2, \dots\}$ and we use standard notation for q -hypergeometric functions as in [3].

2 The quantum algebra $\mathcal{A}_q(\text{SU}(2))$

Let $q \in (0, 1)$. The q -deformed algebra of polynomials on $\text{SU}(2)$ is the complex unital associative algebra $\mathcal{A}_q = \mathcal{A}_q(\text{SU}(2))$ generated by $\alpha, \beta, \gamma, \delta$, which satisfy the relations

$$\begin{aligned} \alpha\beta &= q\beta\alpha, & \alpha\gamma &= q\gamma\alpha, & \beta\delta &= q\delta\beta, & \gamma\delta &= q\delta\gamma, \\ \beta\gamma &= \gamma\beta, & \alpha\delta - q\beta\gamma &= 1 = \delta\alpha - q^{-1}\beta\gamma. \end{aligned} \tag{2.1}$$

\mathcal{A}_q is a Hopf- $*$ -algebra with $*$ -structure and comultiplication Δ defined on the generators by

$$\begin{aligned} \alpha^* &= \delta, & \beta^* &= -q\gamma, & \gamma^* &= -q^{-1}\beta, & \delta^* &= \alpha, \\ \Delta(\alpha) &= \alpha \otimes \alpha + \beta \otimes \gamma, & \Delta(\beta) &= \alpha \otimes \beta + \beta \otimes \delta, \\ \Delta(\gamma) &= \gamma \otimes \alpha + \delta \otimes \gamma, & \Delta(\delta) &= \delta \otimes \delta + \gamma \otimes \beta. \end{aligned} \tag{2.2}$$

$$\tag{2.3}$$

An irreducible $*$ -representation of \mathcal{A}_q is either 1-dimensional or infinite-dimensional. The infinite-dimensional irreducible $*$ -representations are labeled by $\phi \in [0, 2\pi)$, and we denote a representation by π_ϕ . The representation space of π_ϕ is $\ell^2(\mathbb{N})$. The generators $\alpha, \beta, \gamma, \delta$ act on the standard orthonormal basis $\{e_n \mid n \in \mathbb{N}\}$ of $\ell^2(\mathbb{N})$ by

$$\begin{aligned} \pi_\phi(\alpha) e_n &= \sqrt{1 - q^{2n}} e_{n-1}, \\ \pi_\phi(\beta) e_n &= -e^{-i\phi} q^{n+1} e_n, \\ \pi_\phi(\gamma) e_n &= e^{i\phi} q^n e_n, \\ \pi_\phi(\delta) e_n &= \sqrt{1 - q^{2n+2}} e_{n+1}. \end{aligned}$$

Note that, $\pi_\phi(\gamma\beta)$ is a self-adjoint diagonal operator in the standard basis.

Remark 2.1 In this paper, we consider tensor products of π_0 . We could also consider the representation $\pi_{\phi_1} \otimes \pi_{\phi_2}$, but this would not lead to more general results in this paper, because representation labels only occur in phase factors; see [8, §II.A]. The representation space of the tensor product representation is the Hilbert space completion of the algebraic tensor product of copies of $\ell^2(\mathbb{N})$.

Let $\sigma : \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ be the flip operator, the linear operator defined on pure tensors by $\sigma(v_1 \otimes v_2) = v_2 \otimes v_1$. We write

$$\pi_{12} = (\pi_0 \otimes \pi_0)\Delta, \quad \pi_{21} = \sigma\pi_{12}\sigma.$$

For three-fold tensor product representations, we write

$$\pi_{1(23)} = (\pi_0 \otimes \pi_0 \otimes \pi_0)(1 \otimes \Delta)(\Delta), \quad \pi_{(12)3} = (\pi_0 \otimes \pi_0 \otimes \pi_0)(\Delta \otimes 1)(\Delta).$$

Since Δ is coassociative, we have $\pi_{1(23)} = \pi_{(12)3}$.

From (2.3), one finds

$$\Delta(\gamma\gamma^*) = q^{-1}\Delta(\gamma\beta) = -q^{-1}(\gamma\beta \otimes \alpha\delta + \gamma\alpha \otimes \alpha\beta + \delta\beta \otimes \gamma\delta + \delta\alpha \otimes \gamma\beta).$$

Using this, eigenvectors of $\pi_{12}(\gamma\gamma^*)$ can be computed (see [8] for details): for $p \in \mathbb{Z}$ and $x \in \mathbb{N}$ define

$$e_{x,p}^{12} = \sum_{\substack{n,m \in \mathbb{N} \\ n-m=p}} C_{x,m,n} e_m \otimes e_n,$$

where we assume $e_{-n} = 0$ for $n \geq 1$, then $\pi_{12}(\gamma\gamma^*)e_{x,p}^{12} = q^{2x}e_{x,p}^{12}$. The Clebsch–Gordan coefficients $C_{x,m,n}$ can be given explicitly in terms of Wall polynomials, see [12], which are defined by

$$\begin{aligned} p_n(q^x; a; q) &= {}_2\varphi_1\left(\begin{matrix} q^{-n}, 0 \\ aq \end{matrix}; q, q^{x+1}\right) \\ &= \frac{(-a)^n q^{\frac{1}{2}n(n+1)}}{(aq; q)_n} {}_2\varphi_0\left(\begin{matrix} q^{-n}, q^{-x} \\ - \end{matrix}; q, \frac{q^x}{a}\right), \end{aligned} \tag{2.4}$$

for $n, x \in \mathbb{N}$. The second expression follows from applying transformation [3, III.8] with $b \rightarrow 0$. Note that, for $x \in \mathbb{N}$, the ${}_2\varphi_0$ -series can be considered as a polynomial in q^{-n} of degree x . This polynomial is (proportional to) an Al-Salam–Carlitz II polynomial.

Let the function $\bar{p}_n(q^x; a; q)$ be defined by

$$\bar{p}_n(q^x; a; q) = (-1)^{n+x} \sqrt{\frac{(aq)^{x-n}(aq; q)_\infty(aq; q)_n}{(q; q)_n(q; q)_x}} p_n(q^x; a; q), \tag{2.5}$$

then from the orthogonality relation for the Wall polynomials and from completeness, we obtain the orthogonality relations

$$\sum_{x \in \mathbb{N}} \bar{p}_n(q^x; a; q) \bar{p}_m(q^x; a; q) = \delta_{nm}, \quad \sum_{n \in \mathbb{N}} \bar{p}_n(q^x; a; q) \bar{p}_n(q^y; a; q) = \delta_{xy},$$

for $0 < a < q^{-1}$. The second relation corresponds to orthogonality relations for Al-Salam–Carlitz II polynomials. The coefficients $C_{x,m,n}$, $m, n \in \mathbb{N}$ are defined by

$$C_{x,m,n} = \begin{cases} \bar{p}_n(q^{2x}; q^{2(n-m)}; q^2), & n \geq m, \\ \bar{p}_m(q^{2x}; q^{2(m-n)}; q^2), & n \leq m, \end{cases}$$

and they satisfy

$$C_{x,n,m} = C_{x,m,n}, \tag{2.6}$$

which follows from the explicit expression as a ${}_2\varphi_1$ -function. Furthermore, we define $C_{x,m,n} = 0$ for $m \in -\mathbb{N}_{\geq 1}$ or $n \in -\mathbb{N}_{\geq 1}$ or $x \in -\mathbb{N}_{\geq 1}$.

The set $\{e_{x,p}^{12} \mid p \in \mathbb{Z}, x \in \mathbb{N}\}$ is an orthonormal basis for $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$. The actions of the \mathcal{A}_q -generators on this basis are given by

$$\begin{aligned} \pi_{12}(\alpha) e_{x,p}^{12} &= \sqrt{1 - q^{2x}} e_{x-1,p}^{12}, \\ \pi_{12}(\beta) e_{x,p}^{12} &= -q^{x+1} e_{x,p+1}^{12}, \\ \pi_{12}(\gamma) e_{x,p}^{12} &= q^x e_{x,p-1}^{12}, \\ \pi_{12}(\delta) e_{x,p}^{12} &= \sqrt{1 - q^{2x+2}} e_{x+1,p}^{12}, \end{aligned} \tag{2.7}$$

where $e_{-1,p}^{12} = 0$. We can also find eigenvectors $e_{x,p}^{21}$ of $\pi_{21}(\gamma\gamma^*)$ for eigenvalue q^{2x} , $x \in \mathbb{N}$:

$$e_{x,p}^{21} = \sum_{\substack{n,m \in \mathbb{N} \\ m-n=p}} C_{x,m,n} e_m \otimes e_n = e_{x,-p}^{12}.$$

3 6j-symbols and q -Bessel functions

In [8], explicit expressions for the 6j-symbols (and for more general coupling coefficients) have been found. It turns out that they are essentially q -Bessel functions. Here, we derive these results again using a more direct approach, and use this interpretation of the q -Bessel functions to obtain summation identities.

3.1 6j-symbols

In the same way as above, we can find eigenvectors of $\pi_{1(23)}(\gamma\gamma^*)$ and $\pi_{(12)3}(\gamma\gamma^*)$; for $x \in \mathbb{N}$, $p, r \in \mathbb{Z}$,

$$\begin{aligned} e_{x,p,r}^{1(23)} &= \sum_{n \in \mathbb{N}} C_{x,n,n+p} e_n \otimes e_{n+p,x-n-r}^{23} \\ &= \sum_{n,m \in \mathbb{N}} C_{x,n,n+p} C_{n+p,m,k} e_n \otimes e_m \otimes e_k, \quad n - m + k = x - r, \\ e_{x,p,r}^{(12)3} &= \sum_{k \in \mathbb{N}} C_{x,k-p,k} e_{k-p,r-x+k}^{12} \otimes e_k \\ &= \sum_{k,m \in \mathbb{N}} C_{x,k-p,k} C_{k-p,n,m} e_n \otimes e_m \otimes e_k, \quad n - m + k = x - r, \end{aligned}$$

are eigenvectors for eigenvalue q^{2x} , $x \in \mathbb{N}$. We use here the convention $e_{-n} = e_{-n,p} = 0$ for $n \in -\mathbb{N}_{\geq 1}$. The actions of the \mathcal{A}_q -generators $\alpha, \beta, \gamma, \delta$ on the eigenvectors can be obtained in the same way as in [8]

$$\begin{aligned}
 \pi_{1(23)}(\alpha)e_{x,p,r}^{1(23)} &= \sqrt{1 - q^{2x}} e_{x-1,p,r}^{1(23)}, & \pi_{(12)3}(\alpha)e_{x,p,r}^{(12)3} &= \sqrt{1 - q^{2x}} e_{x-1,p,r}^{(12)3}, \\
 \pi_{1(23)}(\beta)e_{x,p,r}^{1(23)} &= -q^{x+1} e_{x,p+1,r}^{1(23)}, & \pi_{(12)3}(\beta)e_{x,p,r}^{(12)3} &= -q^{x+1} e_{x,p+1,r}^{(12)3}, \\
 \pi_{1(23)}(\gamma)e_{x,p,r}^{1(23)} &= q^x e_{x,p-1,r}^{1(23)}, & \pi_{(12)3}(\gamma)e_{x,p,r}^{(12)3} &= q^x e_{x,p-1,r}^{(12)3}, \\
 \pi_{1(23)}(\delta)e_{x,p,r}^{1(23)} &= \sqrt{1 - q^{2x+2}} e_{x+1,p,r}^{1(23)}, & \pi_{(12)3}(\delta)e_{x,p,r}^{(12)3} &= \sqrt{1 - q^{2x+2}} e_{x+1,p,r}^{(12)3},
 \end{aligned}$$

where $e_{-1,p,r} = 0$. Note that, this corresponds exactly to the actions on the eigenvectors $e_{x,p}$.

The $6j$ -symbol (or Racah coefficient) $R_{p_1,r_1;p_2,r_2}^x$ is the (re)coupling coefficient between the two eigenvectors;

$$R_{p_1,r_1;p_2,r_2}^x = \left\langle e_{x,p_1,r_1}^{1(23)}, e_{x,p_2,r_2}^{(12)3} \right\rangle,$$

or equivalently

$$e_{x,p_1,r_1}^{1(23)} = \sum_{p_2,r_2} R_{p_1,r_1;p_2,r_2}^x e_{x,p_1,r_1}^{1(23)}. \tag{3.1}$$

We start by looking at some simple properties of R .

Proposition 3.1 *The coefficients R have the following properties:*

- (i) *Orthogonality relations:* $\sum_{p_1,r_1 \in \mathbb{Z}} R_{p_1,r_1;p_2,r_2}^x R_{p_1,r_1;p_3,r_3}^x = \delta_{p_2,p_3} \delta_{r_2,r_3}$.
- (ii) $R_{p_1,r_1;p_2,r_2}^x = R_{p_1+k,r_1;p_2+k,r_2}^x$ for $k \in \mathbb{Z}$.
- (iii) $R_{p_1,r_1;p_2,r_2}^x = R_{p_1,r_1;p_2,r_2}^{x+k}$ for $k \in \mathbb{Z}_{\geq -x}$.
- (iv) For $k, m, n \in \mathbb{N}$,

$$\begin{aligned}
 C_{x,n+p_1,n} C_{n+p_1,m,k} &= \sum_{p_2 \in \mathbb{Z}_{\leq k}} R_{p_1,r;p_2,r}^x C_{x,k-p_2,k} C_{k-p_2,m,n}, \\
 x - r &= n - m + k.
 \end{aligned}$$

- (v) *Duality:* $R_{p_1,r;p_2,r}^x = R_{-p_2,r;-p_1,r}^x$.

Note that, identity (iii) implies that R is independent of x ; therefore, we will omit the superscript ‘ x .’

Proof The coefficients R are matrix coefficients of a unitary operator, which leads to the orthogonality relations. The next two identities follow from the $*$ -structure of \mathcal{A}_q . From $\beta^* = -q\gamma$, we obtain

$$\left\langle e_{x,p_1 \pm 1,r_1}^{1(23)}, e_{x,p_2,r_2}^{(12)3} \right\rangle = \left\langle e_{x,p_1,r_1}^{1(23)}, e_{x,p_2 \mp 1,r_2}^{(12)3} \right\rangle,$$

which implies (ii). Identity (iii) follows from $\alpha^* = \delta$. Identity (iv) follows from the expansion

$$e_{x,p_1,r_1}^{1(23)} = \sum_{p_2,r_2 \in \mathbb{Z}} R_{p_1,r_1;p_2,r_2} e_{x,p_2,r_2}^{(12)3},$$

by taking inner products with $e_n \otimes e_m \otimes e_k$. The duality property follows from identity (iv). □

3.2 q -Bessel functions

Define

$$J_\nu(x; q) = x^{\frac{\nu}{2}} \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} {}_1\varphi_1 \left(\begin{matrix} 0 \\ q^{\nu+1} \end{matrix}; q, qx \right), \quad x \geq 0, \nu \in \mathbb{R}, \tag{3.2}$$

which is Jackson’s third q -Bessel function (also known as the Hahn-Exton q -Bessel function), see e.g., [16]. Note that,

$$(B; q)_\infty {}_1\varphi_1 \left(\begin{matrix} A \\ B \end{matrix}; q, Z \right) = \sum_{k=0}^\infty \frac{(A; q)_k (Bq^k; q)_\infty}{(q; q)_k} (-1)^k q^{\frac{1}{2}k(k-1)} Z^k$$

is an entire function in B , so we may take ν to be a negative integer in (3.2); in this case, we have the identity

$$J_{-n}(x; q) = (-1)^n q^{\frac{n}{2}} J_n(xq^n; q), \quad n \in \mathbb{N},$$

see [16, (2.6)]. We will use the following generating function to identify the $6j$ -symbols with q -Bessel functions.

Proposition 3.2 For $|t| < 1$,

$$\sum_{m=0}^\infty q^{-\frac{vm}{2}} J_\nu(xq^m; q) \frac{t^m}{(q; q)_m} = x^{\frac{\nu}{2}} \frac{(q^{\nu+1}; q)_\infty}{(q, t; q)_\infty} {}_1\varphi_1 \left(\begin{matrix} t \\ q^{\nu+1} \end{matrix}; q, qx \right).$$

Proof Write J_ν as a ${}_1\varphi_1$ -series, interchange the order of summation, and use summation formula [3, (II.1)];

$$\begin{aligned} \sum_{m=0}^\infty {}_1\varphi_1 \left(\begin{matrix} 0 \\ q^{\nu+1} \end{matrix}; q, xq^{m+1} \right) \frac{t^m}{(q; q)_m} &= \sum_{k=0}^\infty \frac{(-1)^k q^{\frac{1}{2}k(k-1)} (xq)^k}{(q, q^{\nu+1}; q)_k} \sum_{m=0}^\infty \frac{q^{mk} t^m}{(q; q)_m} \\ &= \sum_{k=0}^\infty \frac{(-1)^k q^{\frac{1}{2}k(k-1)} (xq)^k}{(q, q^{\nu+1}; q)_k (tq^k; q)_\infty}. \end{aligned} \quad \square$$

If $t^{-1}q^{\nu+1} \in q^{-\mathbb{N}}$, the right-hand side in the Proposition 3.2 can be written in terms of a Wall polynomial, which gives the following special case.

Corollary 3.3 For $n \in \mathbb{N}$,

$$\sum_{m=0}^{\infty} q^{-\frac{1}{2}(v-n)m} J_{v-n}(xq^m; q) \frac{q^{m(v+1)}}{(q; q)_m} = x^{\frac{1}{2}(v-n)} \frac{(qx; q)_{\infty}}{(q; q)_{\infty}} p_n(q^v; x; q).$$

Proof In Proposition 3.2 replace v by $v - n$, set $t = q^{v+1}$, and use the transformation

$${}_1\varphi_1\left(\begin{matrix} A \\ B \end{matrix}; q, Z\right) = \frac{(A, Z; q)_{\infty}}{(B; q)_{\infty}} {}_2\varphi_1\left(\begin{matrix} 0, B/A \\ Z \end{matrix}; q, A\right),$$

(which is a special case of [3, (III.4)]) and the Definition (2.4) of the Wall polynomials. □

We are now in a position to show that the $6j$ -symbols are essentially q -Bessel functions.

Proposition 3.4 For $p_1, p_2, r_1, r_2 \in \mathbb{Z}$,

$$R_{p_1, r_1; p_2, r_2} = \delta_{r_1, r_2} (-q)^{p_1 - p_2} J_{r_1}(q^{2p_1 - 2p_2}; q^2).$$

Proof We write out Proposition 3.1(iv) for $m = k = 0$, and we replace p_2 by $-p_2$,

$$\begin{aligned} \sum_{p_2 \in \mathbb{N}} R_{p_1, r; -p_2, r} \frac{(-1)^{p_2} q^{p_2(n+x+1)}}{(q^2; q^2)_{p_2}} \\ = \frac{(-1)^{p_1} q^{p_1(x-n+1)}}{(q^2; q^2)_{p_1}} p_n(q^{2x}; q^{2p_1}; q^2), \quad x - r = n, \end{aligned}$$

then the result follows from Corollary 3.3. □

3.3 Identities

Several classical identities for $6j$ -symbols for $SU(2)$ remain valid for our $6j$ -symbols. By Proposition 3.4, these can be interpreted as identities for q -Bessel functions.

First of all, the orthogonality relations for the $6j$ -symbols from Proposition 3.1 are equivalent to the well-known q -Hankel orthogonality relations, see [16, (2.11)], for the q -Bessel functions J_v .

Theorem 3.5 For $n, m \in \mathbb{Z}$,

$$\sum_{x \in \mathbb{Z}} J_v(q^{x+m}; q) J_v(q^{x+n}; q) q^x = \delta_{m, n} q^{-n}.$$

To derive other identities, it is convenient to represent eigenvectors of $\gamma\gamma^*$ as binary trees; see e.g., Van der Jeugt’s lecture notes [20] for more details. We denote

$$e_{x,n_2-n_1}^{12} = \begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ n_1 \quad n_2 \end{array}$$

where $n_1, n_2, x \in \mathbb{Z}$. Equivalently, we can identify this tree with the Clebsch–Gordan coefficient C_{x,n_1,n_2} , similar as in [18]. The identity $e_{x,p}^{12} = e_{x,-p}^{21}$, which is equivalent to (2.6), is represented as

$$\begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ n_1 \quad n_2 \end{array} = \begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ n_2 \quad n_1 \end{array} \tag{3.3}$$

where $p = n_1 - n_2$. By coupling two of these, we can represent eigenvectors corresponding to threefold tensor products:

$$e_{x,p_1,r_{123}}^{1(23)} = \begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ n_1 \quad n_2 \quad n_3 \end{array} \quad e_{x,p_2,r_{123}}^{(12)3} = \begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ n_1 \quad n_2 \quad n_3 \end{array}$$

where $p'_1 = n_1 + p_1$, $p'_2 = n_3 - p_2$, and $r_{ijk} = x - n_i + n_j - n_k$ for $i, j, k \in \{1, 2, 3\}$. Now we can e.g., represent the identities $e_{x,p_1,r_{123}}^{1(23)} = e_{x,p_1,r_{132}}^{1(32)} = e_{x,-p_1,r_{231}}^{(23)1}$ by

$$\begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ n_1 \quad n_2 \quad n_3 \end{array} = \begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ n_1 \quad n_3 \quad n_2 \end{array} = \begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ n_3 \quad n_2 \quad n_1 \end{array}$$

The transition (3.1) from $e_{x,p_1,r}^{1(23)}$ to $e_{x,p_2,r}^{(12)3}$ which involves a $6j$ -symbol, which is equivalent to identity (ii) in Proposition 3.1 in terms of Clebsch–Gordan coefficients, is represented as

$$\begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ n_1 \quad n_2 \quad n_3 \end{array} \xrightarrow{R_{p'_1,p'_2}^{x,n_1,n_2,n_3}} \begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ n_1 \quad n_2 \quad n_3 \end{array}$$

where the coefficient R is given by

$$\begin{aligned}
 R_{p'_1, p'_2}^{x, n_1, n_2, n_3} &= R_{p_1, r_{123}; p_2, r_{123}} \\
 &= (-q)^{p'_1 + p'_2 - n_1 - n_3} J_{x - n_1 + n_2 - n_3}(q^{2p'_1 + 2p'_2 - 2n_1 - 2n_3}; q^2). \tag{3.4}
 \end{aligned}$$

Note that the transition from right to left involves exactly the same $6j$ -symbol. To find identities for the $6j$ -symbols, we can use the binary trees and identities for these trees as explained above, without referring to the underlying eigenvectors. We obtain the following identities, which can be considered as analogs of Racah’s backcoupling identity, the Biedenharn–Elliot (or pentagon) identity, and the hexagon identity.

Theorem 3.6 *The following identities hold:*

(i)

$$R_{p_1, p_2}^{x, n_1, n_2, n_3} = \sum_{p \in \mathbb{Z}} R_{p_1, p}^{x, n_1, n_3, n_2} R_{p, p_2}^{x, n_3, n_1, n_2},$$

or in terms of q -Bessel functions

$$J_{r_{123}}(q^{p_1 + p_2}; q) = \sum_{p \in \mathbb{Z}} J_{r_{132}}(q^{p + p_1}; q) J_{r_{312}}(q^{p + p_2}; q) q^p,$$

where $r_{ijk} = x - n_i + n_j - n_k$.

(ii)

$$R_{r_1, p_2}^{x, n_1, n_2, p_1} R_{p_1, r_2}^{x, p_2, n_3, n_4} = \sum_{p \in \mathbb{Z}} R_{p_1, p}^{r_1, n_2, n_3, n_4} R_{r_1, r_2}^{x, n_1, p, n_4} R_{p, p_2}^{r_2, n_1, n_2, n_3},$$

which in terms of q -Bessel functions is equivalent to the product formula

$$J_{v + \mu_1}(q^{P - Q}; q) J_{v + \mu_2}(q^{Q - R}; q) = \sum_{\mu \in \mathbb{Z}} A_{P, Q, R}^{\mu_1, \mu_2, \mu} J_{v + \mu}(q^{P - R}; q)$$

where $P, Q, R, v, \mu_1, \mu_2 \in \mathbb{Z}$ and

$$\begin{aligned}
 A_{P, Q, R}^{\mu_1, \mu_2, \mu} &= (-1)^{\mu_1 + \mu_2} q^{\mu - \frac{1}{2}(\mu_1 + \mu_2)} \\
 &\quad \times J_{\mu_2 - \mu_1 + P - Q}(q^{\mu - \mu_1}; q) J_{\mu_1 - \mu_2 + Q - R}(q^{\mu - \mu_2}; q).
 \end{aligned}$$

(iii)

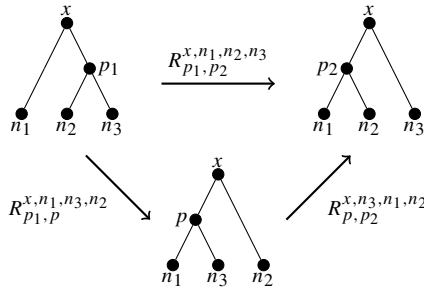
$$\begin{aligned}
 &\sum_{r \in \mathbb{Z}} R_{p_2, r}^{x, p_1, n_3, n_4} R_{p_3, p_1}^{r, n_2, n_1, n_3} R_{p_4, r}^{x, p_3, n_2, n_4} \\
 &= \sum_{r \in \mathbb{Z}} R_{r, p_1}^{x, n_1, n_2, p_2} R_{p_2, p_4}^{r, n_2, n_4, n_3} R_{r, p_3}^{x, n_1, n_3, p_4},
 \end{aligned}$$

or in terms of q -Bessel functions,

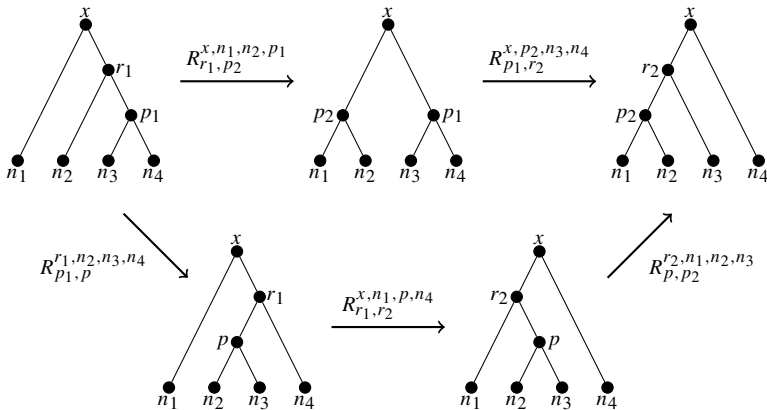
$$\sum_{r \in \mathbb{Z}} (-1)^{p_2+p_4} q^{r-n_4+\frac{1}{2}(p_2+p_4)} J_{r-n_2+n_1-n_3}(q^{p_1+p_3-n_2-n_3}; q) \\ \times J_{x-p_1+n_3-n_4}(q^{r+p_2-p_1-n_4}; q) J_{x-p_3+n_2-n_4}(q^{r+p_4-p_3-n_4}; q) \\ = \text{idem}((n_1, n_2, p_1, p_3) \leftrightarrow (n_4, n_3, p_2, p_4)).$$

Here ‘idem’ means that the same expression is inserted but with the parameters interchanged as indicated.

Proof The first identity follows from



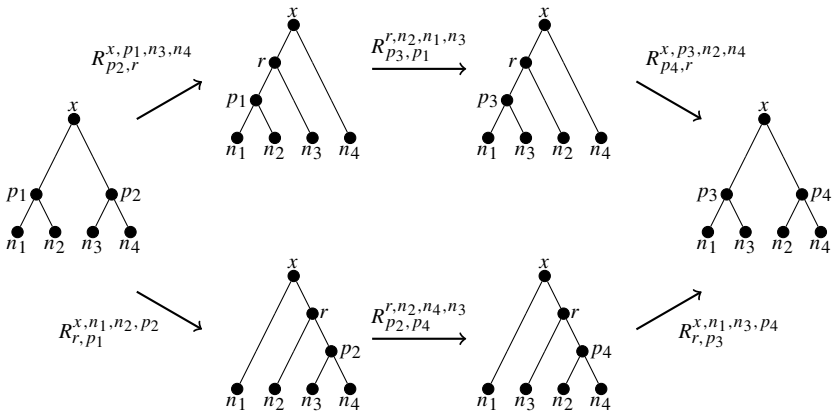
The second identity is



The corresponding identity for q-Bessel functions is obtained by substituting

$$r_1 - n_1 = P, \quad p_1 - p_2 = Q, \quad n_4 - r_2 = R, \quad v = x - n_1, \\ \mu_1 = n_2 - p_1, \quad \mu_2 = n_1 - p_2 + n_3 - n_4, \quad \mu = p - n_4.$$

The third identity is



Remark 3.7 (i) The q -Hankel transform of a function $f \in L^2(q^{\mathbb{Z}}; q^x)$ is defined by

$$(H_v f)(n) = \sum_{x \in \mathbb{Z}} f(q^x) J_v(q^{x+n}; q) q^x, \quad n \in \mathbb{Z}.$$

- Identity (i) of Theorem 3.6 shows that the q -Hankel transform maps an orthogonal basis of q -Bessel functions to another orthogonal basis of q -Bessel functions, which implies a factorization of the q -Hankel transform: $H_{r_{123}} = H_{r_{312}} H_{r_{132}}$.
- (ii) Identity (ii), the product formula for q -Bessel functions, has appeared before in the literature; representation theoretic proofs are given by Koelink in [13, Corollary 6.5] and Kalnins et al. in [11, (3.20)]. A direct analytic proof is given by Koelink and Swarttouw in [14].
- (iii) It is well known that the hexagon identity for classical $6j$ -symbols can be interpreted as a quantum Yang–Baxter equation. Here, we obtain an infinite-dimensional solution: for $u, v \in \mathbb{Z}$, define a unitary operator $\mathcal{R}(u, v) : \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z})$ by

$$\mathcal{R}(u, v)(e_{x-a} \otimes e_{b-x}) = \sum_{y \in \mathbb{Z}} R_{x,y}^{u,a,v,b} e_{b-y} \otimes e_{y-a}, \quad a, b, x \in \mathbb{Z},$$

where $\{e_x \mid x \in \mathbb{Z}\}$ is the standard orthonormal basis for $\ell^2(\mathbb{Z})$. Then the hexagon identity says that the operator \mathcal{R} satisfies

$$\mathcal{R}_{12}(u, w) \mathcal{R}_{13}(v, w) \mathcal{R}_{23}(u, v) = \mathcal{R}_{23}(u, v) \mathcal{R}_{13}(v, w) \mathcal{R}_{12}(u, w)$$

as an operator identity on $\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z})$.

4 $3nj$ -symbols and multivariate q -Bessel functions

We consider certain $3nj$ -symbols and show that these can be considered as multivariate q -Bessel functions, which are limits of the multivariate Askey–Wilson

polynomials introduced by Gasper and Rahman in [4]. In this section, we use the following notation. For $v = (v_1, v_2, \dots, v_{d-1}, v_d)$, we define $|v| = \sum_{j=1}^d v_j$ and $\hat{v} = (v_d, v_{d-1}, \dots, v_2, v_1)$. For some function $f : \mathbb{Z}^d \rightarrow \mathbb{C}$, we set

$$\sum_x f(x) = \sum_{x_d \in \mathbb{Z}} \cdots \sum_{x_1 \in \mathbb{Z}} f(x_1, \dots, x_d),$$

provided the sum converges.

4.1 Multivariate q -Bessel functions

Let $d \in \mathbb{N}_{\geq 1}$. For $v = (v_0, \dots, v_{d+1}) \in \mathbb{Z}^{d+2}$, we define q -Bessel functions in the variables $x = (x_1, \dots, x_d), \lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{Z}^d$ by

$$J_v(x, \lambda) = \prod_{j=1}^d J_{v_j - x_{j+1} - \lambda_{j-1}}(q^{x_j - x_{j+1} + \lambda_j - \lambda_{j-1}}; q), \tag{4.1}$$

where $\lambda_0 = v_0$ and $x_{d+1} = v_{d+1}$. Occasionally, we will use the notation $J_v(x, \lambda; q)$ to stress the dependence on q .

Theorem 4.1 *The multivariate q -Bessel functions have the following properties:*

(i) *Orthogonality relations:*

$$\sum_x J_v(x, \lambda) J_v(x, \lambda') q^{x_1} = \delta_{\lambda, \lambda'} q^{v_{d+1} + v_0 - \lambda_d}, \quad \lambda, \lambda' \in \mathbb{Z}^d.$$

(ii) *Self-duality:* $J_v(x, \lambda) = J_{\hat{v}}(\hat{\lambda}, \hat{x})$.

Proof The self-duality property follows directly from (4.1). The orthogonality relations follow by induction using the q -Hankel orthogonality relations from Theorem 3.5, which can be written as

$$\begin{aligned} \sum_{x_j \in \mathbb{Z}} J_{v_j - x_{j+1} - \lambda_{j-1}}(q^{x_j - x_{j+1} + \lambda_j - \lambda_{j-1}}; q) J_{v_j - x_{j+1} - \lambda_{j-1}}(q^{x_j - x_{j+1} + \lambda'_j - \lambda_{j-1}}; q) q^{x_j} \\ = \delta_{\lambda_j, \lambda'_j} q^{x_{j+1} - \lambda_j + \lambda_{j-1}}. \end{aligned} \tag{4.2}$$

Define for $k = 1, \dots, d + 1$,

$$J_v^{(k)}(x, \lambda) = \prod_{j=k}^d J_{v_j - x_{j+1} - \lambda_{j-1}}(q^{\frac{1}{2}(x_j - x_{j+1} + \lambda_j - \lambda_{j-1})}; q),$$

the empty product being equal to 1. Note that $J_v^{(1)} = J_v$ and

$$J_{v_k - x_{k+1} - \lambda_{k-1}}(q^{x_k - x_{k+1} + \lambda_k - \lambda_{k-1}}; q) J_v^{(k+1)}(x, \lambda) = J_v^{(k)}(x, \lambda). \tag{4.3}$$

We will show that

$$\sum_{x_k \in \mathbb{Z}} \cdots \sum_{x_1 \in \mathbb{Z}} J_\nu(x, \lambda) J_\nu(x, \lambda') q^{x_1} \tag{4.4}$$

$$= \delta_{\lambda_1, \lambda'_1} \cdots \delta_{\lambda_k, \lambda'_k} q^{x_{k+1} - \lambda_k + \lambda_0} J_\nu^{(k+1)}(x, \lambda) J_\nu^{(k+1)}(x, \lambda'). \tag{4.5}$$

For $k = 1$, (4.4) follows directly from (4.2). Now assume that (4.4) holds for a certain k , then by (4.2) and (4.3),

$$\begin{aligned} &\sum_{x_{k+1} \in \mathbb{Z}} \cdots \sum_{x_1 \in \mathbb{Z}} J_\nu(x, \lambda) J_\nu(x, \lambda') q^{x_1} \\ &= \delta_{\lambda_1, \lambda'_1} \cdots \delta_{\lambda_k, \lambda'_k} \sum_{x_{k+1} \in \mathbb{Z}} J_\nu^{(k+1)}(x, \lambda) J_\nu^{(k+1)}(x, \lambda') q^{x_{k+1} - \lambda_k + \lambda_0} \\ &= \delta_{\lambda_1, \lambda'_1} \cdots \delta_{\lambda_{k+1}, \lambda'_{k+1}} J_\nu^{(k+2)}(x, \lambda) J_\nu^{(k+2)}(x, \lambda') q^{x_{k+2} - \lambda_{k+1} + \lambda_0}, \end{aligned}$$

which proves the orthogonality relations. □

Next we show that the multivariate q -Bessel functions can be considered as limit cases of multivariate Askey–Wilson polynomials. The 1-variable Askey–Wilson polynomials are defined by

$$p_n(x; a, b, c, d | q) = \frac{(ab, ac, ad; q)_n}{a^n} {}_4\phi_3 \left(\begin{matrix} q^{-n}, abcdq^{n-1}, ax, a/x \\ ab, ac, ad \end{matrix}; q, q \right),$$

which are polynomials in $x + x^{-1}$ of degree n , and they are symmetric in the parameters a, b, c, d . Using notation as in [9], the multivariate Askey–Wilson polynomials are defined as follows. Let $n = (n_1, \dots, n_d) \in \mathbb{N}^d$ and $x = (x_1, \dots, x_d) \in (\mathbb{C}^\times)^d$, then the d -variable Askey–Wilson polynomials are defined by

$$P_d(n; x; \alpha | q) = \prod_{j=1}^d p_{n_j} \left(x_j; \alpha_j q^{N_{j-1}}, \frac{\alpha_j}{\alpha_0^2} q^{N_{j-1}}, \frac{\alpha_{j+1}}{\alpha_j} x_{j+1}, \frac{\alpha_{j+1}}{\alpha_j} x_{j+1}^{-1} | q \right), \tag{4.6}$$

where $N_j = \sum_{k=1}^j n_k$, $N_0 = 0$, $\alpha = (\alpha_0, \dots, \alpha_{d+2}) \in \mathbb{C}^{d+3}$, $x_{d+1} = \alpha_{d+2}$. These are polynomials in the variables $x_1 + x_1^{-1}, \dots, x_d + x_d^{-1}$ of degree $|n| = N_d$.

Proposition 4.2 *Let $\lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{Z}^d$, $\nu = (\nu_0, \dots, \nu_{d+1}) \in \mathbb{Z}^{d+2}$ and define*

$$\alpha(m) = \left(q^{-m}, q^{\frac{1}{2}\nu_0}, q^{\frac{1}{2}\nu_1 - m}, \dots, q^{\frac{1}{2}\nu_j - jm}, \dots, q^{\frac{1}{2}\nu_d - dm}, q^{\nu_{d+1} + m} \right) \in \mathbb{C}^{d+3},$$

$$x(m) = \left(q^{-\frac{1}{2}\nu_0 - x_1 + m}, q^{\frac{1}{2}\nu_1 - \nu_0 - x_2 + m}, \dots, q^{\frac{1}{2}\nu_d - \nu_0 - x_d + m} \right) \in \mathbb{C}^d,$$

$$\lambda + m = (\lambda_1 + m, \dots, \lambda_d + m) \in \mathbb{N}^d,$$

$$C_m(x; \lambda; \alpha) = \prod_{j=1}^d q^{\left(\frac{1}{2}\nu_{j-1} - \nu_j + \nu_0 + x_{j+1} - m\right)(\lambda_j + m)} \left(q^{\nu_j - \nu_{j-1} - 2m}; q \right)_{\lambda_j + m},$$

then

$$\begin{aligned} & \lim_{m \rightarrow \infty} \frac{P_d(\lambda + m; x(m); \alpha(m) | q)}{C_m(x; \lambda; \alpha)} \\ &= (q; q)_\infty^d \left(\prod_{j=1}^d q^{\frac{1}{2}(x_{j+1} - x_j + \Lambda_{j+1} - \Lambda_j)(v_j - x_{j+1} - \Lambda_{j-1})} \right) J_\nu(x, \Lambda), \end{aligned}$$

where $\Lambda = (\Lambda_1, \dots, \Lambda_d)$ with $\Lambda_j = v_0 - \sum_{k=1}^j \lambda_k$ and $\Lambda_0 = v_0$.

Proof First we substitute

$$\begin{aligned} \alpha_0 &\mapsto q^{-m}, & n_j &\mapsto \lambda_j + m, & j &= 1, \dots, d \\ x_j &\mapsto x_j q^m, & \alpha_j &\mapsto \alpha_j q^{m(j-1)}, & j &= 1, \dots, d + 1, \end{aligned}$$

in (4.6) (recall, $x_{d+1} = \alpha_{d+2}$). The $4\varphi_3$ -part of the j th factor p_{n_j} is

$$4\varphi_3 \left(\begin{matrix} q^{-\lambda_j - m}, \alpha_{j+1}^2 q^{2(\sum_{k=1}^{j-1} \lambda_k) + \lambda_j - 1 + m}, \frac{\alpha_{j+1}}{\alpha_j} x_{j+1} x_j q^m, \frac{\alpha_{j+1} x_{j+1}}{\alpha_j x_j} q^{-m} \\ \frac{\alpha_{j+1}^2}{\alpha_j^2} q^{-2m}, \alpha_{j+1} x_{j+1} q^{2m + \sum_{k=1}^{j-1} \lambda_k}, \alpha_{j+1} x_{j+1} q^{\sum_{k=1}^{j-1} \lambda_k} \end{matrix} ; q, q \right),$$

where the empty sum equals 0. Letting $m \rightarrow \infty$, this function tends to

$$1\varphi_1 \left(\begin{matrix} 0 \\ \alpha_{j+1} x_{j+1} q^{\sum_{k=1}^{j-1} \lambda_k} \end{matrix} ; q, \frac{\alpha_j x_{j+1}}{\alpha_{j+1} x_j} q^{1 - \lambda_j} \right).$$

Finally, we substitute

$$\alpha_j \mapsto q^{\frac{1}{2}v_{j-1}}, \quad x_j \mapsto q^{\frac{1}{2}v_{j-1} - v_0 - x_j},$$

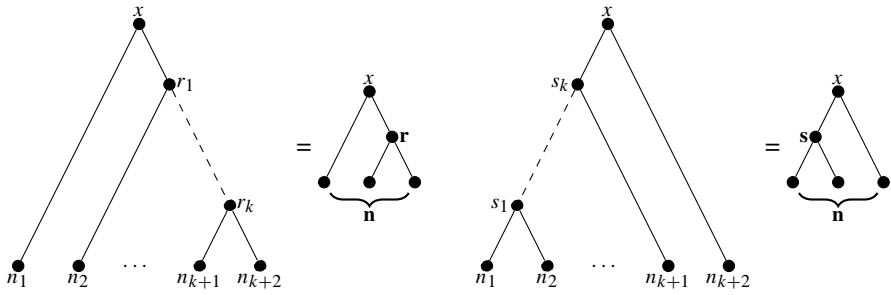
and set $v_0 - \sum_{k=1}^j \lambda_k = \Lambda_j$ for $j = 0, \dots, d$, then we have

$$1\varphi_1 \left(\begin{matrix} 0 \\ q^{v_j - x_{j+1} - \Lambda_{j-1}} \end{matrix} ; q, q^{x_{j+1} - x_j + \Lambda_j - \Lambda_{j-1}} \right),$$

which we recognize as the $1\varphi_1$ -part of the j th factor of the multivariate q -Bessel function $J_\nu(x, \Lambda)$, see (4.1). □

4.2 3nj-symbols

Let $k \in \mathbb{N}_{\geq 1}$, and let $\mathbf{r}, \mathbf{s} \in \mathbb{Z}^k, \mathbf{n} \in \mathbb{Z}^{k+2}$. We define the $3nj$ -symbols $R_{\mathbf{r}, \mathbf{s}}^{x; \mathbf{n}}$ to be the coupling coefficients between two specific binary trees corresponding to $(k + 2)$ -fold tensor product representations. We will use the following notation:



Note that, a node with a bold symbol represents several nodes, and that the label \mathbf{r} (respectively \mathbf{s}) on the right (left) of a node means that all branches ‘hang’ on the right (left) edge. The $3nj$ -symbols $R_{\mathbf{r},\mathbf{s}}^{x,\mathbf{n}}$ are defined by

$$\begin{array}{c} x \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \underbrace{\hspace{2cm}}_{\mathbf{n}} \end{array} = \sum_{\mathbf{s}} R_{\mathbf{r},\mathbf{s}}^{x,\mathbf{n}} \begin{array}{c} x \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \underbrace{\hspace{2cm}}_{\mathbf{n}} \end{array}$$

and we will denote the corresponding transition again by an arrow. Note that, for $k = 1$ we have $R_{\mathbf{r},\mathbf{s}}^{x,\mathbf{n}} = R_{r,s}^{x,n_1,n_2,n_3}$.

Proposition 4.3 *The coefficients $R_{\mathbf{r},\mathbf{s}}^{x,\mathbf{n}}$ have the following properties:*

- (i) *Orthogonality relations:* $\sum_{\mathbf{r}} R_{\mathbf{r},\mathbf{s}}^{x,\mathbf{n}} R_{\mathbf{r},\mathbf{s}'}^{x,\mathbf{n}} = \delta_{\mathbf{s},\mathbf{s}'}$
- (ii) *Duality:* $R_{\mathbf{r},\mathbf{s}}^{x,\mathbf{n}} = R_{\hat{\mathbf{s}},\hat{\mathbf{r}}}^{x,\hat{\mathbf{n}}}$.

Proof The coefficients R are the matrix coefficients of a unitary operator, which implies the orthogonality relations. The duality property is a consequence of the identity

$$\begin{array}{c} x \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \underbrace{\hspace{2cm}}_{\mathbf{n}} \end{array} = \begin{array}{c} x \\ \diagup \quad \diagdown \\ \hat{\mathbf{r}} \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \underbrace{\hspace{2cm}}_{\hat{\mathbf{n}}} \end{array}$$

which follows from repeated application of (3.3). □

Theorem 4.4 *For $i = 1, 2$ let $k_i \in \mathbb{N}_{\geq 1}$, $\mathbf{n}_i \in \mathbb{Z}^{k_i+1}$ and $\mathbf{r}_i, \mathbf{s}_i \in \mathbb{Z}^{k_i}$. Let $k = k_1 + k_2$, $\mathbf{n} = (\mathbf{n}_1, \mathbf{n}_2)$, $\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2)$, $\mathbf{s} = (\mathbf{s}_1, \mathbf{s}_2)$, then*

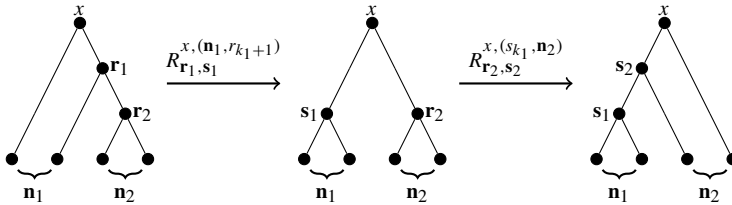
$$R_{\mathbf{r},\mathbf{s}}^{x,\mathbf{n}} = R_{\mathbf{r}_1,\mathbf{s}_1}^{x,(\mathbf{n}_1,r_{k_1+1})} R_{\mathbf{r}_2,\mathbf{s}_2}^{x,(\mathbf{s}_{k_1},\mathbf{n}_2)}.$$

As a consequence,

$$R_{\mathbf{r},\mathbf{s}}^{x,\mathbf{n}} = \prod_{j=1}^k R_{r_j,s_j}^{x,s_{j-1},n_{j+1},r_{j+1}},$$

where $s_0 = n_1$ and $r_{k+1} = n_{k+2}$.

Proof The first identity follows from



The second identity follows from repeated application of the first identity. □

From (3.4) it follows that $R_{\mathbf{r},\mathbf{s}}^{x,\mathbf{n}}$ is essentially a multivariate q -Bessel function as defined by (4.1).

Corollary 4.5 *Let $v(x, \mathbf{n}) = (n_1, x + n_2, \dots, x + n_{k+1}, n_{k+2})$, then*

$$R_{\mathbf{r},\mathbf{s}}^{x,\mathbf{n}} = (-q)^{r_1+s_k-n_1-n_{k+2}} J_{v(x,\mathbf{n})}(\mathbf{r}, \mathbf{s}; q^2).$$

Note that, this corollary and Proposition 4.3 together give a representation theoretic proof of Theorem 4.1.

Our next goal is to prove a summation identity for the multivariate q -Bessel functions. Let us first mention that by interpreting a binary tree as a product of Clebsch–Gordan coefficients, the 3nj-symbols $R_{\mathbf{r},\mathbf{s}}^{x,\mathbf{n}}$ satisfy, by definition, the formula

$$C_{x,\mathbf{r},\mathbf{n}} = \sum_{\mathbf{s}} R_{\mathbf{r},\mathbf{s}}^{x,\mathbf{n}} C_{x,\hat{\mathbf{s}},\hat{\mathbf{n}}}, \quad C_{x,\mathbf{r},\mathbf{n}} = \prod_{j=1}^{k+1} C_{r_{j-1},n_j,r_j}, \tag{4.7}$$

where $r_0 = x, r_{k+1} = n_{k+2}, s_0 = n_1, s_{k+2} = x$. The functions $C_{x,\mathbf{r},\mathbf{n}}$ can be considered as multivariate Wall polynomials, which are q -analogs of Laguerre polynomials. In this light, (4.7) is a multivariate q -analog of an identity proved by Erdélyi [2] which states that the Hankel transform maps a product of two Laguerre polynomials to a product of two Laguerre polynomials.

For the 3nj-symbols $R_{\mathbf{r},\mathbf{s}}^{x,\mathbf{n}}$, there exists a multivariate analog of the Biedenharn–Elliott identity. In terms of q -Bessel functions, this gives an expansion formula for k -variable q -Bessel functions in terms of $(k - 1)$ -variable q -Bessel functions. The identity requires also another 3nj-symbol. For $\mathbf{r}, \mathbf{s} \in \mathbb{Z}^k, \mathbf{n} \in \mathbb{Z}^{k+2}, x \in \mathbb{Z}$, let $S_{\mathbf{r},\mathbf{s}}^{x,\mathbf{n}}$ be the coupling coefficient defined by

$$\begin{array}{c} x \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \\ \underbrace{\quad \quad}_n \quad \bullet \\ \underbrace{\quad \quad}_r \end{array} = \sum_{\hat{s}} S_{\mathbf{r}, \mathbf{s}}^{x, \mathbf{n}} \begin{array}{c} x \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \\ \underbrace{\quad \quad}_n \quad \bullet \\ \underbrace{\quad \quad}_s \end{array} \tag{4.7}$$

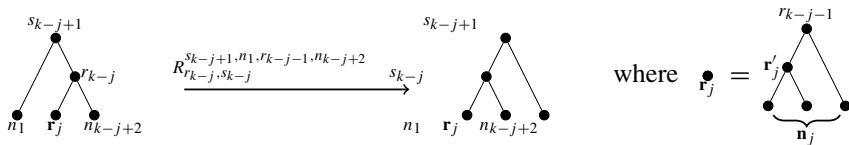
Note that, $\sum_{\hat{s}} = \sum_{s_1} \cdots \sum_{s_k}$. This $3nj$ -symbol can of course also be considered as a multivariate q -Bessel function (see the following result), but it lacks the self-duality property. Let us first express S in terms of the $6j$ -symbols.

Lemma 4.6 $S_{\mathbf{r}, \mathbf{s}}^{x, \mathbf{n}}$ is given by

$$S_{\mathbf{r}, \mathbf{s}}^{x, \mathbf{n}} = \prod_{j=1}^k R_{r_j, s_j}^{s_{j+1}, n_1, r_{j-1}, n_{j+2}},$$

with $s_{k+1} = x$ and $r_0 = n_2$.

Proof We use the transition



and where $\mathbf{r}'_j = (r_1, \dots, r_{k-j-2})$ and $\mathbf{n}_j = (n_2, \dots, n_{k-j+1})$. We set $s_{k+1} = x$ and $r_0 = n_2$, then applying this transition successively on subtrees for $j = 0, \dots, k - 1$ gives

$$S_{\mathbf{r}, \mathbf{s}}^{x, \mathbf{n}} = \prod_{j=0}^{k-1} R_{r_{k-j}, s_{k-j}}^{s_{k-j+1}, n_1, r_{k-j-1}, n_{k-j+2}}.$$

Changing the index gives the stated expression for the coupling coefficient S . □

The following identity is the multivariate analog of the Biedenharn–Elliott identity from Theorem 3.6, i.e., the $k = 2$ case gives back Theorem 3.6(ii).

Theorem 4.7 For $k \in \mathbb{N}_{\geq 2}$ let $\mathbf{r}, \mathbf{s} \in \mathbb{Z}^k$ and $\mathbf{n} \in \mathbb{Z}^{k+2}$, then

$$S_{\mathbf{r}, \mathbf{s}}^{x, \mathbf{n}} = \sum_{\mathbf{t} \in \mathbb{Z}^{k-1}} S_{(\mathbf{t}, r_1), \mathbf{s}}^{x, \mathbf{n}} R_{\mathbf{r}', \mathbf{t}}^{r_1, \mathbf{n}'},$$

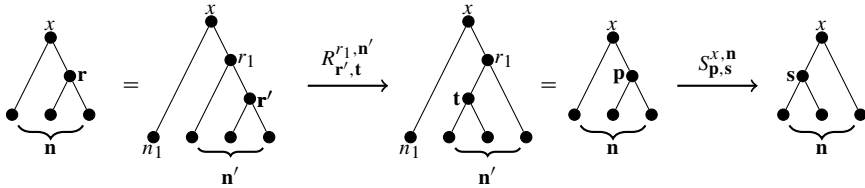
where \mathbf{v}' is obtained from \mathbf{v} by leaving out the first component. In terms of multivariate q -Bessel functions,

$$J_{\mathbf{v}(x, \mathbf{n})}(\mathbf{r}, \mathbf{s}) = \sum_{\mathbf{t} \in \mathbb{Z}^{k-1}} A_{\mathbf{t}, \mathbf{s}}^{r_1} J_{\mathbf{v}(r_1, \mathbf{n}')}(\mathbf{r}', \mathbf{t})$$

with

$$A_{\mathbf{t},\mathbf{s}}^{r_1} = (-q^{\frac{1}{2}})^{|\mathbf{t}|+|\mathbf{s}|-|\mathbf{n}|-(k-2)n_1-s_k+r_2} \times \prod_{j=1}^k J_{s_{j+1}-n_1+t_{j-1}+n_{j+2}}(q^{s_j+t_j-n_1-n_{j+2}}; q), \quad t_k = r_1.$$

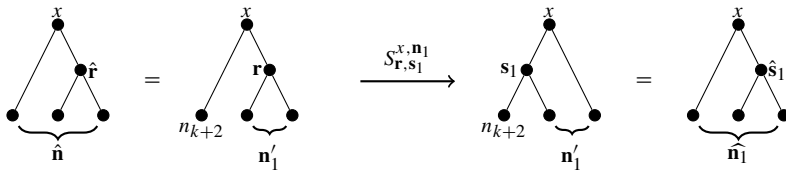
Proof This follows from the transition



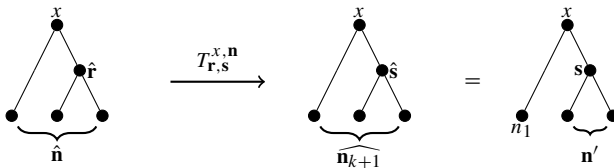
where $\mathbf{p} = (\mathbf{t}, r_1)$, and the definition of the coupling coefficients R . □

Remark 4.8 It seems that there are no analogs for the 3nj-symbols R of identities (i) and (iii) of Theorem 3.6, but there does exist an analog of Theorem 3.6(i) involving only the 3nj-symbols S which may be of interest. This is obtained as follows.

Let $\mathbf{n} \in \mathbb{Z}^{k+2}$. For $j \in \{1, 2, \dots, k + 1\}$, we define $\mathbf{n}_j = (n_{k+3-j}, \dots, n_{k+2}, n_1, \dots, n_{k+2-j})$. Furthermore, given a vector \mathbf{v} , we denote (as in Theorem 4.7) by \mathbf{v}' the vector \mathbf{v} without the first component, and we set $\mathbf{n}'_j = (\mathbf{n}_j)'$. Consider the transition



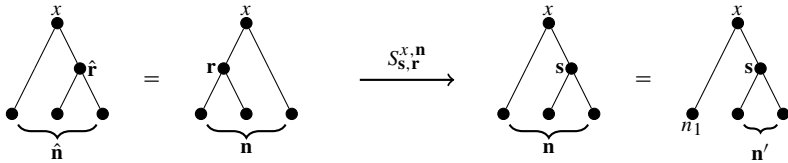
Iterating this transition $k + 1$ times shows that the coupling coefficient in the transition



is given by

$$T_{\mathbf{r},\mathbf{s}}^{x,\mathbf{n}} = \sum_{s_k} \dots \sum_{s_1} (S_{s_0,s_1}^{x,\mathbf{n}_1} \dots S_{s_k,s_{k+1}}^{x,\mathbf{n}_{k+1}}), \quad s_0 = \mathbf{r}, \quad s_{k+1} = \mathbf{s}.$$

On the other hand, by the definition of the coupling coefficient S , we have



so that

$$S_{s,r}^{x,n} = \sum_{s_k} \cdots \sum_{s_1} (S_{s_0,s_1}^{x,n_1} \cdots S_{s_k,s_{k+1}}^{x,n_{k+1}}), \quad s_0 = r, \quad s_{k+1} = s.$$

For $k = 1$, this gives back Theorem 3.6(i).

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