Ramanujan J (2018) 45:719–737
https://doi.org/10.1007/s11139-016-9873-2

Sets of minimal distances and characterizations of class groups of Krull monoids

Qinghai Zhong[1]

Received: 24 June 2016 / Accepted: 14 October 2016 / Published online: 31 January 2017
© The Author(s) 2017. This article is published with open access at Springerlink.com

Abstract Let H be a Krull monoid with finite class group G such that every class contains a prime divisor. Then every non-unit $a \in H$ can be written as a finite product of atoms, say $a = u_1 \cdot \ldots \cdot u_k$. The set $\mathsf{L}(a)$ of all possible factorization lengths k is called the set of lengths of a. There is a constant $M \in \mathbb{N}$ such that all sets of lengths are almost arithmetical multiprogressions with bound M and with difference $d \in \Delta^*(H)$, where $\Delta^*(H)$ denotes the set of minimal distances of H. We study the structure of $\Delta^*(H)$ and establish a characterization when $\Delta^*(H)$ is an interval. The system $\mathcal{L}(H) = \{\mathsf{L}(a) \mid a \in H\}$ of all sets of lengths depends only on the class group G, and a standing conjecture states that conversely the system $\mathcal{L}(H)$ is characteristic for the class group. We confirm this conjecture (among others) if the class group is isomorphic to C_n^r with $r, n \in \mathbb{N}$ and $\Delta^*(H)$ is not an interval.

Keywords Krull monoids · Class groups · Arithmetical characterizations · Sets of lengths · Zero-sum sequences · Davenport constant

Mathematics Subject Classification 11B30 · 11R27 · 13A05 · 13F05 · 20M13

This work was supported by the Austrian Science Fund FWF, Project Number P28864-N35.

✉ Qinghai Zhong
qinghai.zhong@uni-graz.at
http://qinghai-zhong.weebly.com

[1] Institute for Mathematics and Scientific Computing, NAWI Graz, University of Graz, Heinrichstraße 36, 8010 Graz, Austria

1 Introduction and main results

Let H be a Krull monoid with finite class group G such that every class contains a prime divisor (holomorphy rings in global fields are such Krull monoids and more examples will be given later). Then every non-unit of H has a factorization as a finite product of atoms (or irreducible elements), and all these factorizations are unique (i.e., H is factorial) if and only if G is trivial. Otherwise, there are elements having factorizations which differ not only up to associates but also up to the order of the factors. These phenomena are described by arithmetical invariants such as sets of lengths and sets of distances. For an overview of recent developments in Factorization Theory, we refer to [3].

We recall some basic concepts and then we formulate the main results of the present paper. For a finite nonempty set $L = \{m_1, \ldots, m_k\}$ of positive integers with $m_1 < \cdots < m_k$, we denote by $\Delta(L) = \{m_i - m_{i-1} \mid i \in [2, k]\}$ the set of distances of L. If a non-unit $a \in H$ has a factorization $a = u_1 \cdot \ldots \cdot u_k$ into atoms $u_1, \ldots, u_k$, then k is called the length of the factorization, and the set $\mathsf{L}(a)$ of all possible factorization lengths k is called the set of lengths of a. Since H is Krull, every non-unit has a factorization into atoms and all sets of lengths are finite. Furthermore, all sets of lengths $\mathsf{L}(a)$ are singletons if and only if $|G| \le 2$. Suppose that $|G| \ge 3$. Then there is an element $a \in H$ with $|\mathsf{L}(a)| > 1$, and since the n-fold sumset $\mathsf{L}(a) + \cdots + \mathsf{L}(a)$ is contained in $\mathsf{L}(a^n)$, it follows that $|\mathsf{L}(a^n)| > n$ for every $n \in \mathbb{N}$. Therefore, the system $\mathcal{L}(H) = \{\mathsf{L}(a) \mid a \in H\}$ of all sets of lengths of H consists of infinitely many finite subsets of the integers, and there are arbitrarily large sets of lengths.

The set of distances $\Delta(H)$ is the union of all sets $\Delta(L)$ over all $L \in \mathcal{L}(H)$. Since the class group is finite, $\Delta(H)$ is finite, and since every class contains a prime divisor, $\Delta(H)$ is a finite interval with $\min \Delta(H) = 1$ ([13]; the maximum of $\Delta(H)$ is unknown in general, see [7,14]). The set of minimal distances $\Delta^*(H)$ is a crucial subset of $\Delta(H)$, defined as

$$\Delta^*(H) = \{\min \Delta(S) \mid S \subset H \text{ is a divisor-closed submonoid with } \Delta(S) \neq \emptyset\}.$$

It has been studied by Chapman et al. (see, e.g., [8, Chap. 6.8], [4,9,21]), and the original interest in $\Delta^*(H)$ stemmed from its occurrence in the Structure Theorem for Sets of Lengths. For convenience of the reader, we formulate the Structure Theorem and recall that the given description is best possible ([8, Chap. 4.7], [24]).

Theorem A *Let H be a Krull monoid with finite class group. Then there is a constant $M \in \mathbb{N}$ such that the set of lengths $\mathsf{L}(a)$ of any non-unit $a \in H$ is an* AAMP (*almost arithmetical multiprogression*) *with difference $d \in \Delta^*(H)$ and bound M.*

The last couple of years have seen a renewed interest in $\Delta^*(H)$ partly motivated by the Characterization Problem (which will be discussed below). Among others, the maximum of $\Delta^*(H)$ has been determined (we have $\max \Delta^*(H) = \max\{\mathsf{r}(G) - 1, \exp(G) - 2\}$ by [15]), and a better understanding of $\Delta^*(H)$ opened the door to progress in a variety of directions (e.g., [12]).

Whereas the set $\Delta(H)$ of all distances is an interval, the structure of $\Delta^*(H)$ is much more involved. A simple example shows that the interval $[1, \mathsf{r}(G) - 1]$ is contained in $\Delta^*(H)$ (Lemma 3.2) and thus $\Delta^*(H)$ is an interval if $\mathsf r(G) \ge \exp(G) - 1$. In the present paper, we further study the structure of $\Delta^*(H)$, which allows us to establish a characterization when $\Delta^*(H)$ is an interval. Here is our first main result.

Theorem 1.1 *Let H be a Krull monoid with finite class group G such that every class contains a prime divisor.*

Suppose that $|G| \ge 3$, $\exp(G) = n$, $\mathsf r(G) = r$, and let $k \in \mathbb N$ be maximal such that G has a subgroup isomorphic to C_n^k. Then

$$\begin{aligned} &[1, r-1] \cup \left\{\max\left\{1, \lfloor\frac{n}{2}\rfloor - 1\right\}\right\} \cup [\max\{1, n-k-1\}, n-2] \\ \subset\ \Delta^*(H)\ \subset &\left[1, \max\left\{r-1, \lfloor\frac{n}{2}\rfloor - 1\right\}\right] \cup [\max\{1, n-k-1\}, n-2]\,. \end{aligned}$$

In particular, the following holds:

(1) *If $r \ge \lfloor \frac{n}{2} \rfloor - 1$, then*

$$\Delta^*(H) = \left[1, \max\left\{r-1, \lfloor\frac{n}{2}\rfloor - 1\right\}\right] \cup [\max\{1, n-k-1\}, n-2]\,.$$

(2) *The following statements are equivalent*:
   (a) *$\Delta^*(H)$ is an interval.*
   (b) $\max\{1, n-k-2\} \in \Delta^*(H)$.
   (c) $n-k-2 \le \max\{r-1, \lfloor \frac{n}{2} \rfloor - 1\}$.
   (d) *$r + k \ge n - 1$ or $\big(r + k = n - 2$ and $G \cong C_{2r+2}^r\big)$.*

Thus, in particular, if $\mathsf r(G) \ge \lfloor \frac{\exp(G)}{2} \rfloor - 1$, then $\Delta^*(H)$ is completely determined. However, if $\mathsf r(G)$ is small with respect to $\lfloor \frac{\exp(G)}{2} \rfloor$, then the structure of $\Delta^*(H)$ remains open. The complexity of this case, even for cyclic groups, can be seen from a recent paper by Plagne and Schmid who studied $\Delta^*(H)$ in case of cyclic class groups [20].

In order to present our second main result, we recall the Characterization Problem for class groups. The monoid $\mathcal B(G)$ of zero-sum sequences over G is a Krull monoid with class group isomorphic to G, every class contains a prime divisor, and the systems of sets of lengths of H and that of $\mathcal B(G)$ coincide. Thus $\mathcal L(H) = \mathcal L\big(B(G)\big)$, and it is usual to set $\mathcal L(G) := \mathcal L\big(\mathcal B(G)\big)$. In particular, the system of sets of lengths of H depends only on the class group G. The associated inverse question asks whether or not sets of lengths are characteristic for the class group. More precisely, the Characterization Problem for class groups can be formulated as follows (for surveys and a detailed description of the background of this problem, see [8, Sect. 7.3], [10, p. 42], [6,23]).

> Given two finite abelian groups G and G' with $|G| \ge 3$ such that $\mathcal L(G) = \mathcal L(G')$. Does it follow that $G \cong G'$?

The system $\mathcal L(G)$ is studied with methods from Additive Combinatorics. In particular, zero-sum theoretical invariants (such as the Davenport constant or the cross number) and the associated inverse problems play a crucial role (surveys and detailed presentations of such results can be found in [8,10,17]). Most of these invariants are well understood only in a very limited number of cases (e.g., for groups of rank two, the precise value of the Davenport constant $\mathsf{D}(G)$ is known and the associated inverse problem is solved; however, if n is not a prime power and $r \ge 3$, then the precise value of the Davenport constant $\mathsf{D}(C_n^r)$ is unknown). Thus, it is not surprising that most affirmative answers to the Characterization Problem so far have been restricted to those groups where we have a good understanding of the Davenport constant. These groups include elementary 2-groups, cyclic groups, and groups of rank two (for recent progress, we refer to [11]).

The first groups, for which the Characterization Problem was solved whereas the Davenport constant is unknown, are groups of the form C_n^r, where $r, n \in \mathbb{N}$ and $r \le \frac{n+2}{6}$ [16]. Based on Theorem 1.1, we extend these results and give an affirmative answer to the Characterization Problem for all groups C_n^r for which $\Delta^*(C_n^r)$ is not an interval.

Theorem 1.2 *Let G and G' be finite abelian groups and let $k, k' \in \mathbb{N}$ be maximal such that G has a subgroup isomorphic to $C_{\exp(G)}^k$ and G' has a subgroup isomorphic to $C_{\exp(G')}^{k'}$. Suppose that $\mathsf{r}(G)+k \le \exp(G)-2$, $G \ncong C_{2\mathsf{r}(G)+2}^{\mathsf{r}(G)}$, and that $\mathcal{L}(G) = \mathcal{L}(G')$. Then $\exp(G) = \exp(G')$ and $k = k'$. In particular,*

(1) *if $\mathsf{r}(G) \ge \left\lfloor \frac{\exp(G)}{2} \right\rfloor + 1$, then $\mathsf{r}(G) = \mathsf{r}(G')$;*

(2) *if $G \cong C_{\exp(G)}^{\mathsf{r}(G)}$, then $G \cong G'$.*

In Sect. 2, we gather the required background both on Krull monoids as well as on Additive Combinatorics as needed in the sequel. In Sect. 3, we study structural properties of (large) minimal non-half-factorial subsets of finite abelian groups. Finally, the proofs of Theorems 1.1 and 1.2 will be provided in Sect. 4.

2 Background on Krull monoids and their sets of minimal distances

Our notation and terminology are consistent with [8,10,17]. Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $a, b \in \mathbb{Q}$, we denote by $[a, b] = \{x \in \mathbb{Z} \mid a \le x \le b\}$ the discrete, finite interval between a and b. If $A, B \subset \mathbb{Z}$ are subsets of the integers, then $A + B = \{a + b \mid a \in A, b \in B\}$ denotes their *sumset*, and $\Delta(A)$ the *set of* (*successive*) *distances* of A (that is, $d \in \Delta(A)$ if and only if $d = b - a$ with $a, b \in A$ distinct and $[a, b] \cap A = \{a, b\}$).

By a *monoid*, we mean a commutative semigroup with identity that satisfies the cancelation laws. If H is a monoid, then $H^\times$ denotes the unit group and $\mathcal{A}(H)$ the set of atoms (or irreducible elements) of H. A submonoid $S \subset H$ is called *divisor-closed* if $a \in S$, $b \in H$, and b divides a, then we have $b \in S$. A monoid H is said to be

- *atomic* if every non-unit can be written as a finite product of atoms;
- *factorial* if it is atomic and every atom is prime;
- *half-factorial* if it is atomic and $|\mathsf{L}(a)| = 1$ for each non-unit $a \in H$ (equivalently, $\Delta(H) = \emptyset$).

A monoid F is factorial with $F^\times = \{1\}$ if and only if it is free abelian. If this holds, then the set of primes $P \subset F$ is a basis of F, we write $F = \mathcal{F}(P)$, and every $a \in F$ has a representation of the form:

$$a = \prod_{p \in P} p^{\mathsf{v}_p(a)} \quad \text{with } \mathsf{v}_p(a) \in \mathbb{N}_0 \quad \text{and} \quad \mathsf{v}_p(a) = 0 \text{ for almost all } p \in P.$$

A monoid homomorphism $\theta \colon H \to B$ is called a *transfer homomorphism* if it has the following properties:

(T1) $B = \theta(H)B^\times$ and $\theta^{-1}(B^\times) = H^\times$.
(T2) If $u \in H$, $b, c \in B$ and $\theta(u) = bc$, then there exist $v, w \in H$ such that $u = vw$, $\theta(v) \simeq b$, and $\theta(w) \simeq c$.

If H and B are atomic monoids and $\theta \colon H \to B$ is a transfer homomorphism, then (see [8, Chap. 3.2])

$$\mathcal{L}(H) = \mathcal{L}(B), \quad \Delta(H) = \Delta(B), \quad \text{and} \quad \Delta^*(H) = \Delta^*(B).$$

2.1 Krull monoids

A monoid H is said to be a *Krull monoid* if it satisfies one of the following two equivalent conditions:

(a) There exists a monoid homomorphism $\varphi \colon H \to F$ into a free abelian monoid F such that $a \mid b$ in H if and only if $\varphi(a) \mid \varphi(b)$ in F.
(b) H is completely integrally closed and v-noetherian.

A detailed presentation of the theory of Krull monoids can be found in [8,18]. To recall some examples, note that an integral domain is a Krull domain if and only if its multiplicative monoid of nonzero elements is a Krull monoid. Thus, Property (b) shows that every integrally closed noetherian domain is a Krull domain. Rings of integers in algebraic number fields, holomorphy rings in algebraic function fields, and regular congruence monoids in these domains are Krull monoids with finite class group such that every class contains a prime divisor ([8, Sect. 2.11 and Examples 7.4.2]). Monoid domains and power series domains that are Krull are discussed in [2,19], and note that every class of a Krull monoid domain contains a prime divisor. For monoids of modules that are Krull and their distribution of prime divisors, we refer the reader to [1,5].

Sets of lengths in Krull monoids can be studied in the monoid of zero-sum sequences over its class group. To recall the basic concepts, let G be an additive finite abelian group and $G_0 \subset G$ a subset. An element $S = g_1 \cdot \ldots \cdot g_l \in \mathcal{F}(G_0)$ is called a *sequence* over G_0, $\sigma(S) = g_1 + \cdots + g_l$ denotes its sum, $\mathsf{k}(S) = \sum_{i=1}^{l} \frac{1}{\operatorname{ord}(g_i)} \in \mathbb{Q}_{\ge 0}$ its *cross number* of S, $|S| = l$ its length, and $\mathsf{h}(S) = \max\{\mathsf{v}_g(S) \mid g \in \operatorname{supp}(S)\}$ the maximal multiplicity of S. Since the embedding

$$\mathcal{B}(G_0) = \{S \in \mathcal{F}(G_0) \mid \sigma(S) = 0\} \hookrightarrow \mathcal{F}(G_0)$$

satisfies Property (a) above, $\mathcal{B}(G_0)$ is a Krull monoid, called the *monoid of zero-sum sequences* over G_0. Its significance for the study of general Krull monoids is summarized in the following lemma (see [8, Theorem 3.4.10 and Proposition 4.3.13]).

Lemma 2.1 *Let H be a Krull monoid with finite class group G such that every class contains a prime divisor. Then there is a transfer homomorphism $\theta\colon H \to \mathcal{B}(G)$. In particular, we have $\mathcal{L}(H) = \mathcal{L}\big(\mathcal{B}(G)\big)$ and*

$$\Delta^*(H) = \Delta^*\big(\mathcal{B}(G)\big) = \big\{ \min \Delta\big(\mathcal{B}(G_0)\big) \mid G_0 \subset G \text{ with } \Delta(\mathcal{B}(\mathrm{G}_0)) \neq \emptyset \big\}.$$

Thus, $\Delta^*(H)$ can be studied in an associated monoid of zero-sum sequences and can be tackled by methods from Additive Combinatorics. The existence of a transfer homomorphism to a monoid of zero-sum sequences is not restricted to Krull monoids, but it holds true for the so-called transfer Krull monoids and thus Theorem 1.1 holds true for transfer Krull monoids over finite abelian groups. We refer to [6] for a discussion of this concept and just mention one additional example. Let $\mathcal{O}$ be a holomorphy ring in a global field K, A a central simple algebra over K, and H a classical maximal $\mathcal{O}$-order of A such that every stably free left R-ideal is free. Then there is a transfer homomorphism from H to the monoid of zero-sum sequences over a ray class group of $\mathcal{O}$ ([25, Theorem 1.1]).

2.2 Zero-sum theory

Let G be an additive finite abelian group and $G_0 \subset G$ a subset. We denote by $\langle G_0 \rangle \subset G$ the subgroup generated by G_0. Then $G \cong C_{n_1} \oplus \cdots \oplus C_{n_r}$, where $r = \mathsf{r}(G) \in \mathbb{N}_0$ is the *rank* of G, $n_r = \exp(G)$ is the *exponent* of G, and $1 < n_1 \mid \cdots \mid n_r \in \mathbb{N}$. It is traditional to set

$$\mathcal{A}(G_0) := \mathcal{A}\big(\mathcal{B}(G_0)\big),\ \Delta(G_0) := \Delta\big(\mathcal{B}(G_0)\big), \text{ and } \Delta^*(G_0) := \Delta^*\big(\mathcal{B}(G_0)\big).$$

Clearly, the atoms of $\mathcal{B}(G_0)$ are precisely the minimal zero-sum sequences over G_0. The set $\mathcal{A}(G_0)$ is finite, and $\mathsf{D}(G_0) = \max\{|S| \mid S \in \mathcal{A}(G_0)\}$ is the *Davenport constant* of G_0. The set G_0 is called

- *half-factorial* if the monoid $\mathcal{B}(G_0)$ is half-factorial (equivalently, $\Delta(G_0) = \emptyset$);
- *non-half-factorial* if the monoid $\mathcal{B}(G_0)$ is not half-factorial (equivalently, $\Delta(G_0) \neq \emptyset$);
- *minimal non-half-factorial* if $\Delta(G_0) \neq \emptyset$ but every proper subset is half-factorial;
- an LCN-set if $\mathsf{k}(A) \geq 1$ for all $A \in \mathcal{A}(G_0)$.

The following simple result ([8, Proposition 6.7.3]) will be used throughout the paper without further mention.

Lemma 2.2 *Let G be a finite abelian group and $G_0 \subset G$ a subset. Then the following statements are equivalent*:

(a) *G_0 is half-factorial.*
(b) $\mathsf{k}(U) = 1$ *for every* $U \in \mathcal{A}(G_0)$.
(c) $\mathsf{L}(B) = \{\mathsf{k}(B)\}$ *for every* $B \in \mathcal{B}(G_0)$.

We define

$$\mathsf{m}(G) = \max\{\min \Delta(G_0) \mid G_0 \subset G \text{ is an LCN-set with } \Delta(G_0) \neq \emptyset\},$$

and we denote by $\Delta_1(G)$ the set of all $d \in \mathbb{N}$ with the following property:

For every $k \in \mathbb{N}$, there exists some $L \in \mathcal{L}(G)$ which is an AAP (almost arithmetical progression) with difference d and length $l \geq k$.

Thus, by definition, if G' is a further finite abelian group such that $\mathcal{L}(G) = \mathcal{L}(G')$, then $\Delta_1(G) = \Delta_1(G')$. The next proposition gathers the properties of $\Delta^*(G)$ and of $\Delta_1(G)$ which are needed in the sequel.

Proposition 2.3 *Let G be a finite abelian group with $|G| \geq 3$ and $\exp(G) = n$.*

(1) $\Delta^*(G) \subset \Delta_1(G) \subset \{d_1 \in \Delta(G) \mid d_1$ *divides some* $d \in \Delta^*(G)\}$. *In particular,* $\max \Delta^*(G) = \max \Delta_1(G)$.
(2) $\max \Delta^*(G) = \max\{\exp(G) - 2, \mathsf{m}(G)\} = \max\{\exp(G) - 2, \mathsf{r}(G) - 1\}$. *If G is a p-group, then* $\mathsf{m}(G) = \mathsf{r}(G) - 1$.
(3) *If $k \in \mathbb{N}$ is maximal such that G has a subgroup isomorphic to C_n^k, then*

$$\Delta^*(G) \subset \Delta_1(G) \subset \left[1, \max\left\{\mathsf{m}(G), \lfloor\frac{n}{2}\rfloor - 1\right\}\right] \cup [\max\{1, n-k-1\}, n-2]$$

and

$$[1, \mathsf{r}(G) - 1] \cup \left\{\max\left\{1, \left\lfloor\frac{n}{2}\right\rfloor - 1\right\}\right\} \cup [\max\{1, n-k-1\}, n-2] \subset \Delta^*(G) \subset \Delta_1(G).$$

Proof (1) follows from [8, Corollary 4.3.16] and (2) from [15, Theorem 1.1 and Proposition 3.2]. (3) In [22, Theorem 3.2], it is proved that $\Delta^*(G)$ is contained in the set given above. The set $[1, \mathsf{r}(G) - 1] \cup [\max\{1, n-k-1\}, n-2]$ is contained in $\Delta^*(G)$ by [8, Propositions 4.1.2 and 6.8.2] and $\{\max\{1, \lfloor\frac{n}{2}\rfloor - 1\}\}$ is contained in $\Delta^*(G)$ by $|G| \geq 3$ and [8, Theorem 6.8.12]. □

3 Minimal non-half-factorial subsets of finite abelian groups

Throughout this section, let G be an additive finite abelian group with $|G| \geq 3$, $\exp(G) = n$, and $\mathsf{r}(G) = r$.

The first three lemmas gather basic properties of $\Delta^*(G)$ and of non-half-factorial sets.

Lemma 3.1 *Let $G_0 \subset G$ be a subset.*

(1) *For each* $g \in G_0$,

$$\begin{aligned}\gcd\big(\{\mathsf{v}_g(B) \mid B \in \mathcal{B}(G_0)\}\big) &= \gcd\big(\{\mathsf{v}_g(A) \mid A \in \mathcal{A}(G_0)\}\big)\\ &= \min\big(\{\mathsf{v}_g(A) \mid \mathsf{v}_g(A) > 0,\, A \in \mathcal{A}(G_0)\}\big)\\ &= \min\big(\{\mathsf{v}_g(B) \mid \mathsf{v}_g(B) > 0,\, B \in \mathcal{B}(G_0)\}\big)\\ &= \min\big(\{k \in \mathbb{N} \mid kg \in \langle G_0 \setminus \{g\}\rangle\}\big)\\ &= \gcd\big(\{k \in \mathbb{N} \mid kg \in \langle G_0 \setminus \{g\}\rangle\}\big).\end{aligned}$$

In particular, $\min\big(\{k \in \mathbb{N} \mid kg \in \langle G_0 \setminus \{g\}\rangle\}\big)$ *divides* $\mathrm{ord}(g)$.

(2) *Suppose that for each two distinct elements* $h, h' \in G_0$ *we have* $h \notin \langle G_0 \setminus \{h, h'\}\rangle$. *Then, for any atom* A *with* $\mathrm{supp}(A) \subsetneq G_0$ *and any* $h \in \mathrm{supp}(A)$, *we have* $\gcd(\mathsf{v}_h(A), \mathrm{ord}(h)) > 1$.

(3) *If* G_0 *is minimal non-half-factorial, then there exists a minimal non-half-factorial subset* $G_0^* \subset G$ *with* $|G_0| = |G_0^*|$ *and a transfer homomorphism* $\theta\colon \mathcal{B}(G_0) \to \mathcal{B}(G_0^*)$ *such that the following properties are satisfied*:

   (a) *For each* $g \in G_0^*$, *we have* $g \in \langle G_0^* \setminus \{g\}\rangle$.

   (b) *For each* $B \in \mathcal{B}(G_0)$, *we have* $\mathsf{k}(B) = \mathsf{k}\big(\theta(B)\big)$.

   (c) *If* G_0^* *has the property that for each* $h \in G_0^*$, $h \notin \langle E\rangle$ *for any* $E \subsetneq G_0^* \setminus \{h\}$, *then* G_0 *also has the property.*

Proof See [15, Lemma 2.6]. □

Lemma 3.2

(1) *If* $g \in G$ *with* $\mathrm{ord}(g) \ge 3$, *then* $\mathrm{ord}(g) - 2 \in \Delta^*(G)$. *In particular,* $n - 2 \in \Delta^*(G)$.

(2) *If* $r \ge 2$, *then* $[1, r-1] \subset \Delta^*(G)$.

(3) *Let* $G_0 \subset G$ *be a subset.*

   (a) *If there exists a* $U \in \mathcal{A}(G_0)$ *with* $\mathsf{k}(U) < 1$, *then* $\min \Delta(G_0) \le \exp(G) - 2$.

   (b) *If* G_0 *is an* LCN-*set, then* $\min \Delta(G_0) \le |G_0| - 2$.

Proof See [8, Proposition 6.8.2 and Lemmas 6.8.5 and 6.8.6]. □

Lemma 3.3 *Let* $G_0 \subset G$ *be a non-half-factorial subset satisfying the following two conditions*:

(a) *There is some* $g \in G_0$ *such that* $\Delta(G_0 \setminus \{g\}) = \emptyset$.

(b) *There is some* $U \in \mathcal{A}(G_0)$ *with* $\mathsf{k}(U) = 1$ *and* $\gcd(\mathsf{v}_g(U), \mathrm{ord}(g)) = 1$.

Then $\mathsf{k}(\mathcal{A}(G_0)) \subset \mathbb{N}$ *and*

$$\min \Delta(G_0) \mid \gcd\{\mathsf{k}(A) - 1 \mid A \in \mathcal{A}(G_0)\}.$$

Note that the conditions hold if $\Delta(G_1) = \emptyset$ *for each* $G_1 \subsetneq G_0$ *and there exists some* G_2 *such that* $\langle G_2\rangle = \langle G_0\rangle$ *and* $|G_2| \le |G_0| - 2$.

Proof The first statement follows from [8, Lemma 6.8.5]. If $\Delta(G_1) = \emptyset$ for all $G_1 \subsetneq G_0$, then Condition (a) holds. Let $G_2 \subsetneq G_1 \subsetneq G_0$ with $\langle G_2\rangle = \langle G_0\rangle$. If $g \in G_1 \setminus G_2$, then $\langle G_2\rangle = \langle G_0\rangle$ implies that there is some $U \in \mathcal{A}(G_1)$ with $\mathsf{v}_g(U) = 1$, and since $G_1 \subsetneq G_0$, it follows that $\mathsf{k}(U) = 1$. □

Lemma 3.4 *Let $G_0 \subset G$ be a subset, $g \in G_0 \setminus \{0\}$, and $g \in \langle G_0 \setminus \{g\}\rangle$. Then for each prime p dividing* $\mathrm{ord}(g)$, *there exists an atom $A \in \mathcal{A}(G_0)$ with* $2 \le |\operatorname{supp}(A)| \le r+1$, $\mathsf{v}_g(A) \le \mathrm{ord}(g)/2$, $\mathsf{v}_g(A) \mid \mathrm{ord}(g)$, *and $p \nmid \mathsf{v}_g(A)$. In particular,*

(1) *if $|G_0| \ge r+2$, then there exist $s_0 < \mathrm{ord}(g)$ and $E \subsetneq G_0 \setminus \{g\}$ such that $s_0 g \in \langle E\rangle$;*
(2) *if* $\mathrm{ord}(g)$ *is a prime power, then there exists a subset $E \subset G_0 \setminus \{g\}$ with $|E| \le r$ such that $g \in \langle E\rangle$.*

Proof We set $\exp(G) = n = p_1^{k_1} \cdot \ldots \cdot p_t^{k_t}$, where $t, k_1, \ldots, k_t \in \mathbb{N}$ and $p_1, \ldots, p_t$ are distinct primes. Let $\nu \in [1, t]$ with $p_\nu \mid \mathrm{ord}(g)$. Since $g \in \langle G_0 \setminus \{g\}\rangle$, it follows that $0 \ne \frac{n}{p_\nu^{k_\nu}} g \in G_\nu = \left\langle \frac{n}{p_\nu^{k_\nu}} h \mid h \in G_0 \setminus \{g\} \right\rangle$. Obviously, G_ν is a p_ν-group. Let $E_\nu \subset G_0 \setminus \{g\}$ be minimal such that $\frac{n}{p_\nu^{k_\nu}} g \in \left\langle \frac{n}{p_\nu^{k_\nu}} E_\nu \right\rangle$. The minimality of E_ν implies that $|E_\nu| = \left|\frac{n}{p_\nu^{k_\nu}} E_\nu\right|$ and it implies that $\frac{n}{p_\nu^{k_\nu}} E_\nu$ is a minimal generating set of $G'_\nu := \left\langle \frac{n}{p_\nu^{k_\nu}} E_\nu \right\rangle$. Thus, [8, Lemma A.6.2] implies that $\left|\frac{n}{p_\nu^{k_\nu}} E_\nu\right| \le \mathsf{r}^*(G'_\nu) = \mathsf{r}(G'_\nu) \le \mathsf{r}(G_\nu)$ (note that $\mathsf{r}^*(G'_\nu)$ is the total rank of G'_ν). Putting all together, we obtain that

$$1 \le |E_\nu| = \left|\frac{n}{p_\nu^{k_\nu}} E_\nu\right| \le \mathsf{r}(G_\nu) \le r.$$

Let $d_\nu \in \mathbb{N}$ be minimal such that $d_\nu g \in \langle E_\nu\rangle$. Since $0 \ne \frac{n}{p_\nu^{k_\nu}} g \in \langle E_\nu\rangle$, it follows that $d_\nu < \mathrm{ord}(g)$. By Lemma 3.1.1, $d_\nu \mid \gcd\left(\frac{n}{p_\nu^{k_\nu}}, \mathrm{ord}(g)\right)$ and there exists an atom U_ν such that $\mathsf{v}_g(U_\nu) = d_\nu$ and $|\operatorname{supp}(U_\nu) \setminus \{g\}| \le |E_\nu| \le r$. Therefore, $|\operatorname{supp}(U_\nu)| \le r+1$, $d_\nu \mid \mathrm{ord}(g)$, and $p_\nu \nmid d_\nu$. Since $p_\nu \mid \mathrm{ord}(g)$, it follows that $d_\nu \le \mathrm{ord}(g)/2$ and $|\operatorname{supp}(U_\nu)| \ge 2$.

If $|G_0| \ge r+2$, then $|E_\nu| \le r < |G_0 \setminus \{g\}|$ implies that $E_\nu \subsetneq G_0 \setminus \{g\}$, and the assertion holds with $E = E_\nu$ and $s_0 = d_\nu$.

If $\mathrm{ord}(g)$ is a prime power, then $\mathrm{ord}(g)$ is a power of p_ν which implies that $\gcd\left(\frac{n}{p_\nu^{k_\nu}}, \mathrm{ord}(g)\right) = 1$ whence $d_\nu = 1$ and $g \in \langle E_\nu\rangle$. ◻

Lemma 3.5 *Let $G_0 \subset G$ be a minimal non-half-factorial* LCN-*set with $|G_0| \ge r+2$ such that $h \in \langle G_0 \setminus \{h\}\rangle$ for every $h \in G_0$. Suppose that for each two distinct elements $h, h' \in G_0$, we have $h \notin \langle G_0 \setminus \{h, h'\}\rangle$, and each atom $A \in \mathcal{A}(G_0)$ with* $\operatorname{supp}(A) = G_0$ *has cross number $\mathsf{k}(A) > 1$. Then* $\min \Delta(G_0) \le \lfloor \frac{n}{2} \rfloor - 1$.

Proof We choose an element $g \in G_0$. If $\mathrm{ord}(g)$ is a prime power, then there exists $E \subset G_0 \setminus \{g\}$ such that $g \in \langle E\rangle$ and $|E| \le r < |G_0| - 1$ by Lemma 3.4.2, a contradiction to the assumption on G_0. Thus, $\mathrm{ord}(g)$ is not a prime power.

Let $s \in \mathbb{N}$ be minimal such that there exists a subset $E \subsetneq G_0 \setminus \{g\}$ with $sg \in \langle E\rangle$, and by Lemma 3.4.1, we observe that $s < \mathrm{ord}(g)$. Let $E \subsetneq G_0 \setminus \{g\}$ be minimal such that $sg \in \langle E\rangle$. By Lemma 3.1.1, there is an atom V with $\mathsf{v}_g(V) = s \mid \mathrm{ord}(g)$ and $\operatorname{supp}(V) = \{g\} \cup E \subsetneq G_0$. By Lemma 3.1.2, for each $h \in \operatorname{supp}(V)$, $\mathsf{v}_h(V) \ge 2$ which implies that $s \ge 2$. Thus, there is a prime $p \in \mathbb{N}$ dividing s and hence $p \mid s \mid \operatorname{ord}(g)$. By Lemma 3.4, there exists an atom U_1 such that $|\operatorname{supp}(U_1)| \le r+1$, $\mathsf{v}_g(U_1) \mid \operatorname{ord}(g)$, and $p \nmid \mathsf{v}_g(U_1)$, and therefore $\operatorname{supp}(U_1) \subsetneq G_0$.

Let $d = \gcd(s, \mathsf{v}_g(U_1))$. Then $d < s < \mathsf{v}_g(U_1)$ and there exist $x_1 \in \left[1, \frac{\operatorname{ord}(g)}{s} - 1\right]$ and $x_2 \in \left[1, \frac{\operatorname{ord}(g)}{\mathsf{v}_g(U_1)} - 1\right]$ such that $d + \operatorname{ord}(g) = x_1 s + x_2 \mathsf{v}_g(U_1)$. Let $V^{x_1} U_1^{x_2} = g^{\operatorname{ord}(g)} \cdot W$, where $W \in \mathcal{B}(G_0)$ with $\mathsf{v}_g(W) = d$, and let W_1 be an atom dividing W with $\mathsf{v}_g(W_1) > 0$. Since $\mathsf{v}_g(W_1) \le d < s$, the minimality of s implies that $\operatorname{supp}(W_1) = G_0$ and hence $\mathsf{k}(W_1) > 1$. Since G_0 is minimal non-half-factorial, we have that $\mathsf{k}(V) = \mathsf{k}(U_1) = 1$. Therefore, there exists $l \in \mathbb{N}$ with $2 \le l < x_1 + x_2$ such that $\{l, x_1+x_2\} \subset \mathsf{L}(V^{x_1} U_1^{x_2})$. Let $W = X_1 \cdot \ldots \cdot X_{x_1+x_2}$ and $g^{\operatorname{ord}(g)} = g^{y_1} \cdot \ldots \cdot g^{y_{x_1+x_2}}$ such that $X_i g^{y_i} = V$ for each $i \in [1, x_1]$ and $X_i g^{y_i} = U_1$ for each $i \in [x_1+1, x_1+x_2]$, where $X_1, \ldots, X_{x_1+x_2} \in \mathcal{F}(G_0)$ and $y_1, \ldots, y_{x_1+x_2} \in \mathbb{N}$. If there exist distinct $i, j \in [1, x_1+x_2]$ such that $y_i = y_j = 1$, then $2\mathsf{v}_g(W) + 2 = 2d + 2 \le \mathsf{v}_g(X_i g^{y_i} X_j g^{y_j}) \le y_i + y_j + \mathsf{v}_g(W)$ which implies that $y + i + y_j \ge \mathsf{v}_g(W) + 2 \ge 3$, a contradiction. Therefore $|\{i \in [1, x_1+x_2] \mid y_i = 1\}| \le 1$. It follows that $1 + 2(x_1+x_2-1) \le \operatorname{ord}(g)$. Then

$$\min \Delta(G_0) \le x_1 + x_2 - l \le \frac{\operatorname{ord}(g)+1}{2} - 2 \le \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

□

Lemma 3.6 *Let $G_0 \subset G$ be a minimal non-half-factorial* LCN-*set with $|G_0| \ge r+2$ such that $h \in \langle G_0 \setminus \{h\} \rangle$ for every $h \in G_0$. Suppose that one of the following properties is satisfied*:

(a) *For each two distinct elements $h, h' \in G_0$, we have $h \notin \langle G_0 \setminus \{h, h'\} \rangle$, and there is an atom $A \in \mathcal{A}(G_0)$ with $\mathsf{k}(A) = 1$ and* $\operatorname{supp}(A) = G_0$.

(b) *There is a subset $G_2 \subset G_0$ such that $\langle G_2 \rangle = \langle G_0 \rangle$ and $|G_2| \le |G_0| - 2$.*

Then $\min \Delta(G_0) \le \max\{r-1, \lfloor \frac{n}{2} \rfloor - 1\}$.

Proof Assume to the contrary that $\min \Delta(G_0) \ge \max\{r, \lfloor \frac{n}{2} \rfloor\}$. Then Lemma 3.2.3.(b) implies that $|G_0| \ge 2 + \min \Delta(G_0) \ge \frac{n}{2} + 1$. If Property (a) is satisfied, then there exists some $g \in G_0$ such that $\mathsf{v}_g(A) = 1$. By Lemma 3.3, each of the two Properties (a) and (b) implies that $\mathsf{k}(U) \in \mathbb{N}$ for each $U \in \mathcal{A}(G_0)$ and

$$\min \Delta(G_0) \mid \gcd\left(\{\mathsf{k}(U) - 1 \mid U \in \mathcal{A}(G_0)\}\right).$$

We set

$$\Omega_{=1} = \{A \in \mathcal{A}(G_0) \mid \mathsf{k}(A) = 1\} \quad \text{and} \quad \Omega_{>1} = \{A \in \mathcal{A}(G_0) \mid \mathsf{k}(A) > 1\}.$$

Thus, for each $U_1, U_2 \in \Omega_{>1}$, we have

$$\begin{aligned} &\mathsf{k}(U_1) \ge \max\left\{r+1, \left\lfloor \frac{n}{2} \right\rfloor + 1\right\} \quad \text{and} \\ &\left(\text{either } \mathsf{k}(U_1) = \mathsf{k}(U_2) \text{ or } |\mathsf{k}(U_1) - \mathsf{k}(U_2)| \ge \max\left\{r, \left\lfloor \frac{n}{2} \right\rfloor\right\}\right). \end{aligned} \tag{3.1}$$

Furthermore, for each $U \in \Omega_{=1}$ we have $\mathsf{h}(U) \ge 2$ (otherwise, U would divide every atom $U_1 \in \Omega_{>1}$). We claim that

- **A1.** For each $U \in \Omega_{>1}$, there are $A_1, \ldots, A_m \in \Omega_{=1}$, where $m \le \frac{n+1}{2}$, such that $UA_1 \cdot \ldots \cdot A_m$ can be factorized into a product of atoms from $\Omega_{=1}$.

Proof of **A1** Suppose that Property (a) holds. As observed above, there exists some $g \in G_0$ such that $\mathsf{v}_g(A) = 1$. Lemma 3.4 implies that there is an atom X such that $2 \le |\operatorname{supp}(X)| \le \mathsf{r}(G) + 1$ and $1 \le \mathsf{v}_g(X) \le \operatorname{ord}(g)/2$. Since $g \notin \langle G_0 \setminus \{g,\ h\}\rangle$ for any $h \in G_0 \setminus \{g\}$, it follows that $\mathsf{v}_g(X) \ge 2$, and $|G_0| \ge r+2$ implies $\operatorname{supp}(X) \subsetneq G_0$.

Suppose that Property (b) is satisfied. We choose an element $g \in G_0 \setminus G_2$. Then $g \in \langle G_2\rangle$ and, by Lemma 3.1.1, there is an atom A' with $\mathsf{v}_g(A') = 1$ and $\operatorname{supp}(A') \subset G_2 \cup \{g\} \subsetneq G_0$. This implies that $A' \in \Omega_{=1}$. Let $h \in G_0$ such that $\mathsf{v}_h(A') = \mathsf{h}(A')$. Since $\mathsf{h}(A') \ge 2$, we obtain that $A'^{\lceil \frac{\operatorname{ord}(h)}{\mathsf{h}(A')}\rceil} = h^{\operatorname{ord}(h)} \cdot W$, where W is a product of $\lceil \frac{\operatorname{ord}(h)}{\mathsf{h}(A')}\rceil - 1$ atoms and $\mathsf{v}_g(W) = \lceil \frac{\operatorname{ord}(h)}{\mathsf{h}(A')}\rceil$. Thus, there exists an atom X' with $2 \le \mathsf{v}_g(X') \le \lceil \frac{\operatorname{ord}(h)}{\mathsf{h}(A')}\rceil \le \frac{n}{2} + 1$.

Therefore, both properties imply that there are $A, X \in \mathcal{A}(G_0)$ and $g \in G_0$ such that $\mathsf{k}(A) = \mathsf{k}(X) = 1$, $\mathsf{v}_g(A) = 1$, and $2 \le \mathsf{v}_g(X) \le \frac{n}{2} + 1$. Let $U \in \Omega_{>1}$.

If $\operatorname{ord}(g) - \mathsf{v}_g(U) < \mathsf{v}_g(X) \le \frac{n}{2} + 1$, then

$$UA^{\operatorname{ord}(g) - \mathsf{v}_g(U)} = g^{\operatorname{ord}(g)} S,$$

where $S \in \mathcal{B}(G_0)$ and $\operatorname{ord}(g) - \mathsf{v}_g(U) \le \frac{n}{2}$. Since $\operatorname{supp}(S) \subsetneq G_0$, S is a product of atoms from $\Omega_{=1}$.

If $\operatorname{ord}(g) - \mathsf{v}_g(U) \ge \mathsf{v}_g(X)$, then

$$UX^{\lfloor \frac{\operatorname{ord}(g) - \mathsf{v}_g(U)}{\mathsf{v}_g(X)}\rfloor} A^{\operatorname{ord}(g) - \mathsf{v}_g(U) - \mathsf{v}_g(X)\cdot\lfloor \frac{\operatorname{ord}(g) - \mathsf{v}_g(U)}{\mathsf{v}_g(X)}\rfloor} = g^{\operatorname{ord}(g)} S,$$

where S is a product of atoms from $\Omega_{=1}$ (because $\operatorname{supp}(S) \subsetneq G_0$) and

$$\begin{aligned}
\lfloor \frac{\operatorname{ord}(g) - \mathsf{v}_g(U)}{\mathsf{v}_g(X)}\rfloor &+ \operatorname{ord}(g) - \mathsf{v}_g(U) - \mathsf{v}_g(X) \cdot \lfloor \frac{\operatorname{ord}(g) - \mathsf{v}_g(U)}{\mathsf{v}_g(X)}\rfloor \\
&\le \frac{\big(\operatorname{ord}(g) - \mathsf{v}_g(U)\big) - \big(\mathsf{v}_g(X) - 1\big)}{\mathsf{v}_g(X)} + \mathsf{v}_g(X) - 1 \\
&\le \frac{\operatorname{ord}(g) - \mathsf{v}_g(U) + 1}{2} \le \frac{n+1}{2}.
\end{aligned}$$

$\square$

We set

$$\Omega'_{>1} = \{A \in \mathcal{A}(G_0) \mid \mathsf{k}(A) = \min\{\mathsf{k}(B) \mid B \in \Omega_{>1}\}\} \subset \Omega_{>1},$$

and we consider all tuples $(U, A_1, \ldots, A_m)$, where $U \in \Omega'_{>1}$, $m \in \mathbb{N}$, and $A_1, \ldots, A_m \in \Omega_{=1}$, such that $UA_1 \cdot \ldots \cdot A_m$ can be factorized into a product of atoms from $\Omega_{=1}$. We fix one such tuple $(U, A_1, \ldots, A_m)$ with the property that m is minimal possible. Let

$$UA_1 \cdot \ldots \cdot A_m = V_1 \cdot \ldots \cdot V_t \quad \text{with} \quad t \in \mathbb{N} \quad \text{and} \quad V_1, \ldots, V_t \in \Omega_{=1}. \tag{3.2}$$

We observe that $\mathsf{k}(U) = t - m$ and continue with the following assertion.

- **A2** For each $\nu \in [1, t]$, we have $V_\nu \nmid UA_1 \cdot \ldots \cdot A_{m-1}$.

Proof of **A2.** Assume to the contrary that there is such a $\nu \in [1, t]$, say $\nu = 1$, with $V_1 \mid UA_1 \cdot \ldots \cdot A_{m-1}$. Then there are $l \in \mathbb{N}$ and $T_1, \ldots, T_l \in \mathcal{A}(G_0)$ such that

$$UA_1 \cdot \ldots \cdot A_{m-1} = V_1 T_1 \cdot \ldots \cdot T_l.$$

By the minimality of m, there exists some $\nu \in [1, l]$ such that $T_\nu \in \Omega_{>1}$, say $\nu = 1$. Since

$$\sum_{\nu=2}^{l} \mathsf{k}(T_\nu) = \mathsf{k}(U) + (m-1) - 1 - \mathsf{k}(T_1) \le m - 2 \le \frac{n-3}{2},$$

and $\mathsf{k}(T') \ge \frac{n}{2}$ for all $T' \in \Omega_{>1}$, it follows that $T_2, \ldots, T_l \in \Omega_{=1}$, whence $l = 1 + \sum_{\nu=2}^{l} \mathsf{k}(T_\nu) \le m - 1$. We obtain that

$$V_1 T_1 \cdot \ldots \cdot T_l A_m = UA_1 \cdot \ldots \cdot A_m = V_1 \cdot \ldots \cdot V_t,$$

and thus

$$T_1 \cdot \ldots \cdot T_l A_m = V_2 \cdot \ldots \cdot V_t.$$

The minimality of m implies that $\mathsf{k}(T_1) > \mathsf{k}(U)$. It follows that

$$\mathsf{k}(T_1) - \mathsf{k}(U) = m - 1 - l \le m - 2 \le \frac{n-3}{2} < \left\lfloor \frac{n}{2} \right\rfloor \le \mathsf{k}(T_1) - \mathsf{k}(U),$$

a contradiction. $\square$

With the minimal integer m, as fixed before **A2**, we consider all the tuples $(A_1', \ldots, A_m')$, where $A_1', \ldots, A_m' \in \Omega_{=1}$, such that $UA_1' \cdot \ldots \cdot A_m'$ can be factorized into a product of atoms from $\Omega_{=1}$. We fix one such tuple $(A_1', \ldots, A_m')$ such that $|\operatorname{supp}(A_m')|$ is minimal. For simplicity of notation, we suppose that $(A_1', \ldots, A_m') = (A_1, \ldots, A_m)$.

By Eq. (3.2), there are $X_1, Y_1, \ldots, X_t, Y_t \in \mathcal{F}(G)$ such that

$$\begin{aligned} &UA_1 \cdot \ldots \cdot A_{m-1} = X_1 \cdot \ldots \cdot X_t, \\ &A_m = Y_1 \cdot \ldots \cdot Y_t, \ \text{and} \ V_i = X_i Y_i \text{ for each } i \in [1, t]. \end{aligned}$$

Then **A2** implies that $|Y_i| \geq 1$ for each $i \in [1, t]$, and we set $\alpha = |\{i \in [1, t] \mid |Y_i| = 1\}|$. If $\alpha \leq 2m$, then

$$n \geq |A_m| = |Y_1| + \cdots + |Y_t| \geq \alpha + 2(t - \alpha) = 2t - \alpha \geq 2t - 2m,$$

and hence $\min \Delta(G_0) \leq t - 1 - m \leq \frac{n}{2} - 1$, a contradiction. Thus $\alpha \geq 2m + 1$. After renumbering if necessary, we assume that $1 = |Y_1| = \cdots = |Y_\alpha| < |Y_{\alpha+1}| \leq \cdots \leq |Y_t|$. Let $Y_i = y_i$ for each $i \in [1, \alpha]$ and

$$S_0 = \{y_1, y_2, \ldots, y_\alpha\}. \tag{3.3}$$

For every $i \in [1, \alpha]$, $V_i \mid y_i U A_1 \cdot \ldots \cdot A_{m-1}$ whence $\mathsf{v}_{y_i}(V_i) \leq 1 + \mathsf{v}_{y_i}(U A_1 \cdot \ldots \cdot A_{m-1})$ and since $V_i \nmid U A_1 \cdot \ldots \cdot A_{m-1}$, it follows that

$$\mathsf{v}_{y_i}(V_i) = \mathsf{v}_{y_i}(U A_1 \cdot \ldots \cdot A_{m-1}) + 1. \tag{3.4}$$

Assume to the contrary that there are distinct $i, j \in [1, \alpha]$ such that $y_i = y_j$. Then

$$\mathsf{v}_{y_i}(U A_1 \ \cdot \ldots \cdot A_{m-1}) + 1 = \mathsf{v}_{y_i}(V_i) = \mathsf{v}_{y_i}(X_i) + 1 = \mathsf{v}_{y_i}(V_j) = \mathsf{v}_{y_i}(X_j) + 1.$$

Since $X_i X_j \mid U A_1 \ \cdot \ldots \cdot A_{m-1}$, we infer that

$$\mathsf{v}_{y_i}(U A_1 \cdot \ldots \cdot A_{m-1}) \geq \mathsf{v}_{y_i}(X_i X_j) = \mathsf{v}_{y_i}(V_i V_j) - 2 = 2\mathsf{v}_{y_i}(U A_1 \cdot \ldots \cdot A_{m-1}),$$

which implies that $\mathsf{v}_{y_i}(U A_1 \ldots A_{m-1}) = 0$, a contradiction to $\mathrm{supp}(U) = G_0$. Thus, $|S_0| = \alpha$ and

$$|\mathrm{supp}(A_m)| \geq |S_0| = \alpha \geq 2m + 1. \tag{3.5}$$

We proceed by the following assertion.

- **A3.** $|\mathrm{supp}(A_m)| \leq r + 1$.

Proof of **A3** Assume to the contrary that $|\mathrm{supp}(A_m)| \geq r + 2$. We fix one element $g' \in S_0$. Let $s_0 \in \mathbb{N}$ be minimal such that there exists a subset $E \subsetneq \mathrm{supp}(A_m) \setminus \{g'\}$ such that $s_0 g' \in \langle E \rangle$. By $|\mathrm{supp}(A_m)| \geq r + 2$, Lemma 3.4 (applied to the subset $\mathrm{supp}(A_m) \subset G_0$) implies that $s_0 < \mathrm{ord}(g')$. Let E be a minimal subset with this property. Thus, by Lemma 3.1.1, there exists an atom A' with $\mathsf{v}_{g'}(A') = s_0$ and $\mathrm{supp}(A') = \{g'\} \cup E \subsetneq \mathrm{supp}(A_m) \subset G_0$ which implies that $\mathsf{k}(A') = 1$.

If $s_0 = 1$, then we assume that $g' = y_1$. Since $\mathsf{v}_{y_1}(V_1) = \mathsf{v}_{y_1}(U A_1 \cdot \ldots \cdot A_{m-1}) + 1$ by Eq. 3.4 and $V_1 \mid U A_1 \cdot \ldots \cdot A_{m-1} \cdot y_1$, we obtain that $|\mathrm{supp}(U A_1 \cdot \ldots \cdot A_{m-1} \cdot A'(V_1)^{-1})| < |G_0|$ and hence $U A_1 \cdot \ldots \cdot A_{m-1} \cdot A'$ can be factorized into a product of atoms from $\Omega_{=1}$, a contradiction to the minimality of $|\mathrm{supp}(A_m)|$.

Suppose $s_0 \geq 2$. We distinguish two cases:

CASE 1: $|\mathrm{supp}(A') \cap S_0| \geq m + 1$.

We may suppose that $\{y_1, \ldots, y_{m+1}\} \subset \mathrm{supp}(A') \cap S_0$. Then $V_1 \cdot \ldots \cdot V_{m+1} \mid U A_1 \cdot \ldots \cdot A_{m-1} A'$ and $\mathsf{k}(U A_1 \cdot \ldots \cdot A_{m-1} A'(V_1 \cdot \ldots \cdot V_{m+1})^{-1}) < \mathsf{k}(U)$. By the minimality of $\mathsf{k}(U)$, we have that $UA_1 \cdot \ldots \cdot A_{m-1}A'$ can be factorized into a product of atoms from $\Omega_{=1}$, a contradiction to the minimality of $|\operatorname{supp}(A_m)|$.

CASE 2: $|\operatorname{supp}(A') \cap S_0| \le m$.

Let p be a prime dividing s_0. Lemma 3.4 (applied to the subset $\operatorname{supp}(A_m) \subset G_0$) implies that there exists an atom $A'_p \in \mathcal{A}(\operatorname{supp}(A_m))$ such that $|\operatorname{supp}(A'_p)| \le r+1 < |\operatorname{supp}(A_m)|$ and $p \nmid \mathsf{v}_{g'}(A'_p)$.

Let $d = \gcd(s_0, \mathsf{v}_{g'}(A'_p))$. Then $d < s_0$ and

$$dg' \in \langle s_0 g', \mathsf{v}_{g'}(A'_p)g' \rangle \subset \langle (\operatorname{supp}(A') \cup \operatorname{supp}(A'_p)) \setminus \{g'\} \rangle.$$

Thus, by minimality of s_0, we have $\operatorname{supp}(A_m)\setminus\{g'\} = \big(\operatorname{supp}(A'')\cup\operatorname{supp}(A'_p)\big)\setminus\{g'\}$. It follows that

$$\begin{aligned}|\operatorname{supp}(A'_p) \cap S_0| &\ge |S_0 \setminus \operatorname{supp}(A')| \ge |S_0| - |\operatorname{supp}(A') \cap S_0| \\ &\ge 2m+1-m = m+1.\end{aligned}$$

Similar to CASE 1, $UA_1 \cdot \ldots \cdot A_{m-1}A'_p$ can be factorized into a product of atoms from $\Omega_{=1}$, a contradiction to the minimality of $|\operatorname{supp}(A_m)|$. $\square$

We consider all tuples $T = (X_1, Y_1, \ldots, X_t, Y_t)$, where $X_1, Y_1, \ldots, X_t, Y_t \in \mathcal{F}(G)$, such that

$$\begin{aligned}&UA_1 \cdot \ldots \cdot A_{m-1} = X_1 \cdot \ldots \cdot X_t,\\ &A_m = Y_1 \cdot \ldots \cdot Y_t, \text{ and } V_i = X_iY_i \text{ for each } i \in [1,t].\end{aligned}$$

After renumbering if necessary, we can assume that $|Y_i| = 1$ for each $i \in [1, s_1]$, $|Y_i| = 2$ and $\operatorname{supp}(Y_i) = 1$ for each $i \in [s_1+1, s_2]$, $|Y_i| = 2$ and $\operatorname{supp}(Y_i) = 2$ for each $i \in [s_2+1, s_3]$, and $|Y_i| \ge 3$ for each $i \in [s_3+1, t]$, where $s_1, s_2, s_3 \in [0, t]$. Let $F_1(T) = \operatorname{supp}(Y_1 \cdot \ldots \cdot Y_{s_1})$, $F_2(T) = \operatorname{supp}(Y_{s_1+1} \cdot \ldots \cdot Y_{s_2})$, $F_3(T) = \operatorname{supp}(Y_{s_2+1} \cdot \ldots \cdot Y_{s_3})$, and $F_4(T) = \operatorname{supp}(Y_{s_3+1} \cdot \ldots \cdot Y_t)$.

Now, we fix one such tuple $T = (X_1, Y_1, \ldots, X_t, Y_t)$ such that $\big(\alpha_T = |\{i \in [1,t] \mid |Y_i| = 1\}|, |F_1(T) \cap F_3(T)|\big) \in (\mathbb{N}_0^2, +)$ is minimal with respect to lexicographic order.

- **A4.** There exists a subset $\{g_1, \ldots, g_\ell\} \subset \operatorname{supp}(A_m)$ with $\ell \le r-m$ such that $UA_1 \cdot \ldots \cdot A_{m-1}g_1^{\operatorname{ord} g_1} \cdot \ldots \cdot g_\ell^{\operatorname{ord}(g_\ell)}$ can be factorized into a product of atoms from $\Omega_{=1}$.

Proof of **A4** If $F_1(T) \cap F_4(T) \ne \emptyset$, there exist $i \in [1, s_1]$ and $j \in [s_3+1, t]$ such that $Y_i \cap Y_j = \{y_i\}$, where $Y_i = \{y_i\}$. By Eq. 3.4, $\mathsf{v}_{y_i}(X_i) \ge 1$. Let $X'_i = X_iy_i^{-1}$, $Y'_i = Y_iy_i$, $X'_j = X_jy_i$, $Y'_j = Y_jy_i^{-1}$ and substitute X_i, Y_i, X_j, Y_j with X'_i, Y'_i, X'_j, Y'_j in the tuple $T = (X_1, Y_1, \ldots, X_t, Y_t)$. Thus, we get a new tuple T' such that $\alpha_{T'} = \alpha_T - 1$, a contradiction to the minimality of α_T. Thus $F_1(T) \cap F_4(T) = \emptyset$.

If $F_1(T) \cap F_3(T) \ne \emptyset$, there exist $i \in [1, s_1]$ and $j \in [s_2+1, s_3]$ such that $Y_i \cap Y_j = \{y_i\}$, where $Y_i = \{y_i\}$. Let $Y_j = \{y_i, y_j\}$, where $y_j \ne y_i$. By Eq. 3.4, $\mathsf{v}_{y_i}(X_i) \ge 1$. Let $X'_i = X_iy_i^{-1}$, $Y'_i = Y_iy_i$, $X'_j = X_jy_i$, $Y'_j = Y_jy_i^{-1}$ and substitute

X_i, Y_i, X_j, Y_j with X'_i, Y'_i, X'_j, Y'_j in the tuple $T = (X_1, Y_1, \ldots, X_t, Y_t)$. Thus, we get a new tuple T' such that $\alpha_{T'} = \alpha_T$, $|F_1(T') \cap F_3(T')| = |F_1(T) \cap F_3(T)| - 1$, a contradiction to the minimality of $\big(\alpha_T = |\{i \in [1,t] \mid |Y_i| = 1\}|, |F_1(T) \cap F_3(T)|\big)$. Thus $F_1(T) \cap F_3(T) = \emptyset$.

Suppose that $|F_1(T) \cap F_2(T)| \ge m$. Then let $\{g_1, \ldots, g_m\} \subset F_1(T) \cap F_2(T)$ and $Y_i = g_i$, $Y_{s_1+i} = g_i^2$, for each $i \in [1,m]$. Hence

$$\prod_{i \in [1,m]} (V_i V_{s_1+i}) \mid U A_1 \cdot \ldots A_{m-1} g_1^{\mathrm{ord}(g_1)} \cdot \ldots \cdot g_m^{\mathrm{ord}(g_m)},$$

and

$$\mathsf{k}\left(U A_1 \cdot \ldots A_{m-1} g_1^{\mathrm{ord}(g_1)} \cdot \ldots \cdot g_m^{\mathrm{ord}(g_m)} \left(\prod_{i \in [1,m]} (V_i V_{s_1+i}) \right)^{-1} \right) = \mathsf{k}(U) - 1.$$

It follows by the minimality of $\mathsf{k}(U)$ that $U A_1 \cdot \ldots A_{m-1} g_1^{\mathrm{ord}(g_1)} \cdot \ldots \cdot g_m^{\mathrm{ord}(g_m)}$ can be factorized into a product of atoms from $\Omega_{=1}$. Note that $r+1 \ge |\operatorname{supp}(A_m)| \ge 2m+1$ by **A3** and Eq. 3.5. We have that $\ell = m \le r - m$.

Suppose that $|F_1(T) \cap F_2(T)| \le m-1$. Then $|F_1(T) \setminus F_2(T)| \ge m+2$. Since $F_1(T) \cap F_4(T) = \emptyset$ and $F_1(T) \cap F_3(T) = \emptyset$, we let $\{g_1, \ldots, g_{m+2}\} \subset F_1(T) \setminus (F_2(T) \cup F_3(T) \cup F_4(T))$ and $\operatorname{supp}(A_m) \setminus \{g_1, \ldots, g_{m+2}\} = \{h_1, \ldots, h_\ell\}$, where $\ell \le r - 1 - m$. We assume that $Y_i = g_i$ for each $i \in [1, m+2]$. Therefore

$$\prod_{i \in [m+3,t]} V_i \mid U A_1 \cdot \ldots A_{m-1} h_1^{\mathrm{ord}(h_1)} \cdot \ldots \cdot h_\ell^{\mathrm{ord}(h_\ell)},$$

and

$$\begin{aligned} &\mathsf{k}\left(U A_1 \cdot \ldots A_{m-1} h_1^{\mathrm{ord}(h_1)} \cdot \ldots \cdot h_\ell^{\mathrm{ord}(h_\ell)} \left(\prod_{i \in [m+3,t]} V_i \right)^{-1} \right) \\ &\quad = \mathsf{k}(U) + m - 1 + \ell - (t - m - 2) \le r \le \mathsf{k}(U) - 1. \end{aligned}$$

It follows by the minimality of $\mathsf{k}(U)$ that $U A_1 \cdot \ldots A_{m-1} g_1^{\mathrm{ord}(g_1)} \cdot \ldots \cdot g_m^{\mathrm{ord}(g_m)}$ can be factorized into a product of atoms from $\Omega_{=1}$. $\square$

By **A4**, we consider all $I \in [1, m-1]$ and $J \in [1, \ell]$ such that $U \prod_{i \in I} A_i \prod_{j \in J} g_j^{\mathrm{ord}(g_j)}$ can be factorized into a product of atoms from $\Omega_{=1}$. We fix such I and J with $|I| + |J|$ being minimal. Then $|I| + |J| \le m - 1 + \ell \le r - 1$. Since $J \ne \emptyset$, we choose $j_0 \in J$ and hence $U \prod_{i \in I} A_i \prod_{j \in J \setminus \{j_0\}} g_j^{\mathrm{ord}(g_j)}$ cannot be factorized into a product of atoms from $\Omega_{=1}$ by the minimality of $|I| + |J|$.

Now, we consider all tuples $(U', A'_1, \ldots, A'_{m'-1}, g)$, where $U' \in \Omega'_{>1}$, $m' \in \mathbb{N}$, $A'_1, \ldots, A'_{m'-1} \in \Omega_{=1}$, and $g \in G_0$ such that $U' A'_1 \cdot \ldots A'_{m'-1} g^{\mathrm{ord}(g)}$ can be factorized into a product of atoms from $\Omega_{=1}$ and $U' A'_1 \cdot \ldots A'_{m'-1}$ cannot be factorized into a product of atoms from $\Omega_{=1}$. We fix one such tuple $(U', A_1', \ldots, A_{m'-1}', g)$ with m' being minimal. Thus $m' \le |I| + |J| \le r - 1$. Let

$$U' A_1' \cdot \ldots A_{m'-1}' g^{\mathrm{ord}(g)} = W_1 \cdot \ldots \cdot W_{t'}, \text{ where } W_1, \ldots, W_{t'} \in \Omega_{=1},$$

and we claim that

- **A5.** For each $\nu \in [1, t']$, we have $W_\nu \nmid U' A_1' \cdot \ldots \cdot A_{m'-1}'$.

Proof of **A5** Assume to the contrary that there is such a $\nu \in [1, t']$, say $\nu = 1$, with $W_1 \mid U' A_1' \cdot \ldots \cdot A_{m'-1}'$. Then there are $l \in \mathbb{N}$ and $T_1, \ldots, T_l \in \mathcal{A}(G_0)$ such that

$$U' A_1' \cdot \ldots \cdot A_{m'-1}' = W_1 T_1 \cdot \ldots \cdot T_l.$$

Since $U' A_1' \cdot \ldots A_{m'-1}'$ cannot be factorized into a product of atoms from $\Omega_{=1}$, there exists some $\nu \in [1, l]$ such that $T_\nu \in \Omega_{>1}$, say $\nu = 1$, and $T_1 \cdot \ldots \cdot T_l$ cannot be factorized into a product of atoms from $\Omega_{=1}$. Since

$$\sum_{\nu=2}^{l} \mathsf{k}(T_\nu) = \mathsf{k}(U') + (m' - 1) - 1 - \mathsf{k}(T_1) \le m' - 2 \le r - 3,$$

and $\mathsf{k}(T') \ge r + 1$ for all $T' \in \Omega_{>1}$, it follows that $T_2, \ldots, T_l \in \Omega_{=1}$, whence $l = 1 + \sum_{\nu=2}^{l} \mathsf{k}(T_\nu) \le m' - 1$. We obtain that

$$W_1 T_1 \cdot \ldots \cdot T_l g^{\mathrm{ord}(g)} = U' A_1' \cdot \ldots \cdot A_{m'-1} g^{\mathrm{ord}(g)} = W_1 \cdot \ldots \cdot W_{t'},$$

and thus

$$T_1 \cdot \ldots \cdot T_l g^{\mathrm{ord}(g)} = W_2 \cdot \ldots \cdot W_{t'}.$$

Since $T_1 \cdot \ldots \cdot T_l$ cannot be factorized into a product of atoms from $\Omega_{=1}$, we obtain that $\mathsf{k}(T_1) > \mathsf{k}(U)$ by the minimality of m'. It follows that

$$\mathsf{k}(T_1) - \mathsf{k}(U') = m' - 1 - l \le m' - 2 \le r - 3 < r \le \mathsf{k}(T_1) - \mathsf{k}(U),$$

a contradiction. □

Let $U' A_1' \cdot \ldots \cdot A_{m'-1}' = X_1' \cdot \ldots \cdot X_{t'}'$ and $g^{\mathrm{ord}(g)} = g^{y_1} \cdot \ldots \cdot g^{y_{t'}}$ such that $W_i = X_i' g^{y_i}$ for each $i \in [1, t']$. By **A5**, we obtain that $y_i \ge 1$ for all $i \in [1, t']$. If $|\{i \in [1, t'] \mid y_i = 1\}| \ge 2$, say $y_1 = y_2 = 1$, then $\mathsf{v}_g(W_1) = \mathsf{v}_g(W_2) = 1 + \mathsf{v}_g(U' A_1' \cdot \ldots \cdot A_{m'-1}')$ by **A5** and hence $\mathsf{v}_g(X_1 X_2) = \mathsf{v}_g(W_1) + \mathsf{v}_g(W_2) - 2 = 2\mathsf{v}_g(U' A_1' \cdot \ldots \cdot A_{m'-1}') \ge \mathsf{v}_g(U' A_1' \cdot \ldots \cdot A_{m'-1}') + \mathsf{v}_g(X_1 X_2)$, a contradiction. Thus, $|\{i \in [1, t'] \mid y_i = 1\}| \le 1$ and hence $1 + 2(t' - 1) \le \mathrm{ord}(g) \le n$. It follows that

$$\mathsf{k}(U') = t' - m' \le \frac{n+1}{2} - 1 \le \left\lfloor \frac{n}{2} \right\rfloor,$$

a contradiction. □

Proposition 3.7 *We have* $\mathsf{m}(G) \le \max\{r-1, \lfloor \frac{n}{2} \rfloor - 1\}$.

Proof Let $G_0 \subset G$ be a non-half-factorial LCN-set. We have to prove that

$$\min \Delta(G_0) \le \max \left\{ r-1, \left\lfloor \frac{n}{2} \right\rfloor - 1 \right\}.$$

If $G_1 \subset G_0$ is non-half-factorial, then $\min \Delta(G_0) = \gcd \Delta(G_0) \mid \gcd \Delta(G_1) = \min \Delta(G_1)$. Thus, we may suppose that G_0 is minimal non-half-factorial. By Lemma 3.1.3.(a), we may suppose that $g \in \langle G_0 \setminus \{g\} \rangle$ for all $g \in G_0$.

If $|G_0| \le r+1$, then $\min \Delta(G_0) \le |G_0| - 2 \le r-1$ by Lemma 3.2.3. Thus, we may suppose that $|G_0| \ge r+2$ and we distinguish two cases.

CASE 1: There exists a subset $G_2 \subset G_0$ such that $\langle G_2 \rangle = \langle G_0 \rangle$ and $|G_2| \le |G_0| - 2$.

Then Lemma 3.6 implies that $\min \Delta(G_0) \le \max\{r-1, \lfloor \frac{n}{2} \rfloor - 1\}$.

CASE 2: Every subset $G_1 \subset G_0$ with $|G_1| = |G_0| - 1$ is a minimal generating set of $\langle G_0 \rangle$.

Then, for each $h \in G_0$, $G_0 \setminus \{h\}$ is half-factorial and $h \notin \langle G_0 \setminus \{h, h'\} \rangle$ for any $h' \in G_0 \setminus \{h\}$. It follows that Lemma 3.5 and Lemma 3.6 imply that $\min \Delta(G_0) \le \max\{r-1, \lfloor \frac{n}{2} \rfloor - 1\}$. $\square$

4 Proofs of the main theorems

In this section, we give the proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1 Let H be a Krull monoid with finite class group G where $|G| \ge 3$ and every class contains a prime divisor. We set $\exp(G) = n$, $\mathsf{r}(G) = r$, and let $k \in \mathbb{N}$ be maximal such that G has a subgroup isomorphic to C_n^k. By Lemma 2.1, it suffices to prove the assertions for the Krull monoid $\mathcal{B}(G)$.

Propositions 2.3.3 and 3.7 immediately imply the required inclusions for $\Delta^*(G)$, namely that

$$\begin{aligned} &[1, r-1] \cup \left\{ \max \left\{ 1, \lfloor \frac{n}{2} \rfloor - 1 \right\} \right\} \cup [\max\{1, n-k-1\}, n-2] \\ &\subset \Delta^*(G) \subset \left[1, \max \left\{ r-1, \lfloor \frac{n}{2} \rfloor - 1 \right\} \right] \cup [\max\{1, n-k-1\}, n-2]. \end{aligned} \tag{4.1}$$

It remains to verify the in particular statements.

(1) If $r \ge \lfloor \frac{n}{2} \rfloor - 1$, then $[1, \max\{r-1, \lfloor \frac{n}{2} \rfloor - 1\}] \subset [1, r-1] \cup \{\max\{1, \lfloor \frac{n}{2} \rfloor - 1\}\}$. Therefore, $\Delta^*(G) = [1, \max\{r-1, \lfloor \frac{n}{2} \rfloor - 1\}] \cup [\max\{1, n-k-1\}, n-2]$ by Eq. 4.1.

(2) (a) $\Rightarrow$ (b) Suppose that $\Delta^*(G)$ is an interval. Since $\max\{1, n-k-2\} \le \max\{r-1, n-2\} = \max \Delta^*(G)$, we obtain that $\max\{1, n-k-2\} \in \Delta^*(G)$.

(b) $\Rightarrow$ (c) Suppose that $\max\{1, n-k-2\} \in \Delta^*(G)$. If $n-k-2 \le 0$, then $n-k-2 \le \max\{r-1, \lfloor \frac{n}{2} \rfloor - 1\}$. If $n-k-2 \ge 1$, then $n-k-2 \in \Delta^*(G) \subset [1, \max\{r-1, \lfloor \frac{n}{2} \rfloor - 1\}] \cup [n-k-1, n-2]$ by Eq. 4.1. Therefore $n-k-2 \le \max\{r-1, \lfloor \frac{n}{2} \rfloor - 1\}$.

(c) $\Rightarrow$ (d) Suppose that $n-k-2 \le \max\{r-1, \lfloor \frac{n}{2} \rfloor - 1\}$. Therefore $n-k-2 \le r-1$ or $r \le n-k-2 \le \lfloor \frac{n}{2} \rfloor - 1$. If $n-k-2 \le r-1$, then $r+k \ge n-1$. If $r \le n-k-2 \le \lfloor \frac{n}{2} \rfloor - 1$, then $n-r-2 \le n-k-2 \le \lfloor \frac{n}{2} \rfloor - 1 \le \frac{n}{2} - 1$ and $r \le \lfloor \frac{n}{2} \rfloor - 1 \le \frac{n}{2} - 1$. It follows that $n-2 = n-r-2+r \le \frac{n}{2}-1+\frac{n}{2}-1 = n-2$ which implies that $n-r-2 = n-k-2 = \frac{n}{2} - 1$ and $r = \frac{n}{2} - 1$. Therefore $r = k$, $n = 2r+2$, and hence $G \cong C_{2r+2}^r$.

(d) $\Rightarrow$ (a) If $r+k = n-2$ and $G \cong C_{2r+2}^r$, then $\Delta^*(G) = [1, 2r]$ is an interval by 1. If $r+k \ge n-1$, then $r \ge \lfloor \frac{n}{2} \rfloor$ and hence $\Delta^*(G) = [1, r-1] \cup [\max\{1, n-k-1\}, n-2]$ is an interval by (1). □

Proof of Theorem 1.2 Let G and G' be finite abelian groups with $\exp(G) = n$ and $\mathsf{r}(G) = r$. Let $k, k' \in \mathbb{N}$ be maximal such that G has a subgroup isomorphic to C_n^k and G' has a subgroup isomorphic to $C_{\exp(G')}^{k'}$. Suppose that

$$r + k \le n - 2, \quad G \ncong C_{2r+2}^r, \quad \text{and that} \quad \mathcal{L}(G) = \mathcal{L}(G').$$

By our assumption and Theorem 1.1.2, we have that $\Delta^*(G)$ is not an interval, $n-k-2 \notin \Delta^*(G)$, and $n-k-2 \ge \max\{r, \lfloor \frac{n}{2} \rfloor\}$. By Proposition 2.3, we obtain that $\max \Delta_1(G) = \max \Delta^*(G) = \max\{r-1, n-2\} = n-2$, $n-k-2 \notin \Delta_1(G)$, and $n-k-1 \in \Delta_1(G)$. Note that $\mathsf{D}(G) = \mathsf{D}(G')$ and $\Delta_1(G) = \Delta_1(G')$ (see [8, Proposition 7.3.1]). Then $\max \Delta_1(G') = \max\{\mathsf{r}(G') - 1, \exp(G') - 2\} = \max \Delta_1(G) = n - 2$, $n-k-2 \notin \Delta_1(G')$, $n-k-1 \in \Delta_1(G')$. If $\mathsf{r}(G') \ge \exp(G') - 1$, then $\Delta_1(G') = [1, \mathsf{r}(G') - 1]$ by Proposition 2.3, a contradiction. It follows that $\exp(G') = n$ by $\max \Delta_1(G') = \exp(G') - 2$. Suppose that $k' \ge k+1$. Then $n-k-2 \in [n-k'-1, n-2] \subset \Delta_1(G') = \Delta_1(G)$, a contradiction. Suppose that $k' \le k-1$. Then $n-k-1 \notin [n-k'-1, n-2]$ and hence $n-k-1 \in [1, \max\{\mathsf{r}(G')-1, \lfloor \frac{n}{2} \rfloor - 1\}]$. If $n-k-1 \le \mathsf{r}(G') - 1$, then $n-k-2 \in [1, \mathsf{r}(G') - 1] \subset \Delta_1(G') = \Delta_1(G)$, a contradiction. Otherwise $n-k-1 \le \lfloor \frac{n}{2} \rfloor - 1$, a contradiction to $n-k-2 \ge \lfloor \frac{n}{2} \rfloor$. It follows that $k = k'$.

In particular, if $r \ge \lfloor \frac{n}{2} \rfloor + 1$, then $[1, r-1] \cup [n-k-1, n-2] = \Delta_1(G) = \Delta_1(G')$ and hence $[1, \mathsf{r}(G')] \subset [1, r-1] \subset [1, \max\{\mathsf{r}(G') - 1, \lfloor \frac{n}{2} \rfloor - 1\}]$. Therefore, by $r \ge \lfloor \frac{n}{2} \rfloor + 1$, we obtain that $\mathsf{r}(G') = r$.

If $\mathsf{r}(G) = k$, then $G \cong C_n^r$ is a subgroup of G'. Thus $\mathsf{D}(G) = \mathsf{D}(G')$ implies that $G \cong G'$. □

Acknowledgements Open access funding provided by University of Graz.

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

References

1. Baeth, N.R., Geroldinger, A.: Monoids of modules and arithmetic of direct-sum decompositions. Pac. J. Math. **271**, 257–319 (2014)

2. Chang, G.W.: Every divisor class of Krull monoid domains contains a prime ideal. J. Algebr. **336**, 370–377 (2011)
3. Chapman, S.T., Fontana, M., Geroldinger, A., Olberding, B. (eds.): Multiplicative Ideal Theory and Factorization Theory. Proceedings in Mathematics and Statistics, vol. 170. Springer, Berlin (2016)
4. Chapman, S.T., Schmid, W.A., Smith, W.W.: On minimal distances in Krull monoids with infinite class group. Bull. Lond. Math. Soc. **40**, 613–618 (2008)
5. Facchini, A.: Krull monoids and their application in module theory. In: Facchini, A., Fuller, K., Ringel, C.M., Santa-Clara, C. (eds.) Algebras, Rings and their Representations, pp. 53–71. World Scientific, Hackensack (2006)
6. Geroldinger, A.: Sets of lengths. arXiv:1509.07462
7. Geroldinger, A., Grynkiewicz, D.J., Schmid, W.A.: The catenary degree of Krull monoids I. J. Théor. Nombres Bordx. **23**, 137–169 (2011)
8. Geroldinger, A., Halter-Koch, F.: Non-unique factorizations. Algebraic, combinatorial and analytic theory. In: Pure and Applied Mathematics, vol. 278. Chapman & Hall/CRC, Boca Raton (2006)
9. Geroldinger, A., Hamidoune, Y.O.: Zero-sumfree sequences in cyclic groups and some arithmetical application. J. Théor. Nombres Bordx. **14**, 221–239 (2002)
10. Geroldinger, A., Ruzsa, I.: Combinatorial number theory and additive group theory. In: Advanced Courses in Mathematics—CRM Barcelona. Birkhäuser, Basel (2009)
11. Geroldinger, A., Schmid, W.A.: A characterization of class groups via sets of lengths. arXiv:1503.04679
12. Geroldinger, A., Schmid, W.A.: The system of sets of lengths in Krull monoids under set addition. Rev. Mat. Iberoam. **32**, 571–588 (2016)
13. Geroldinger, A., Yuan, P.: The set of distances in Krull monoids. Bull. Lond. Math. Soc. **44**, 1203–1208 (2012)
14. Geroldinger, A., Zhong, Q.: The catenary degree of Krull monoids II. J. Aust. Math. Soc. **98**, 324–354 (2015)
15. Geroldinger, A., Zhong, Q.: The set of minimal distances in Krull monoids. Acta Arith. **173**, 97–120 (2016)
16. Geroldinger, A., Zhong, Q.: A characterization of class groups via sets of lengths II, J. Théor. Nombres Bordx. (to appear)
17. Grynkiewicz, D.J.: Structural Additive Theory. Developments in Mathematics. Springer, New York (2013)
18. Halter-Koch, F.: Ideal Systems. An Introduction to Multiplicative Ideal Theory. Marcel Dekker, New York (1998)
19. Kim, H., Park, Y.S.: Krull domains of generalized power series. J. Algebr. **237**, 292–301 (2001)
20. Plagne, A., Schmid, W.A.: On congruence half-factorial Krull monoids with cyclic class group (submitted)
21. Schmid, W.A.: Differences in sets of lengths of Krull monoids with finite class group. J. Théor. Nombres Bordx. **17**, 323–345 (2005)
22. Schmid, W.A.: Arithmetical characterization of class groups of the form $\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ via the system of sets of lengths. Abh. Math. Semin. Univ. Hambg. **79**, 25–35 (2009)
23. Schmid, W.A.: Characterization of class groups of Krull monoids via their systems of sets of lengths: a status report. In: Adhikari, S.D., Ramakrishnan, B. (eds.) Number Theory and Applications. Proceedings of the International Conferences on Number Theory and Cryptography, pp. 189–212. Hindustan Book Agency, New Delhi (2009)
24. Schmid, W.A.: A realization theorem for sets of lengths. J. Number Theory **129**, 990–999 (2009)
25. Smertnig, D.: Sets of lengths in maximal orders in central simple algebras. J. Algebr. **390**, 1–43 (2013)