



Some thoughts on the analysis of coupled queues

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1 Introduction In general, the joint analysis of multiple “coupled” queues turns out to be very hard; a good overview of available methods is given in [1]. In this paper, we share some thoughts on such analysis, based on a very simple example system, consisting of two parallel discrete-time infinite-capacity single-server queues. In either queue, the service of a customer requires exactly one slot, and customers arrive independently from slot to slot, but the numbers of customers entering queue 1 and queue 2 during a slot, are not necessarily mutually independent. Let $A(z_1, z_2)$ denote their joint probability generating function (pgf). As each queue has their own dedicated server, this “arrival correlation” is the only source of coupling between both queues. The main objective of this paper is to shed a light on the analytic derivation of the joint steady-state pgf $U(z_1, z_2)$ of the numbers of customers present (the “system contents”) in queue 1 and in queue 2.

2 Problem statement It is readily shown that $U(z_1, z_2)$ should satisfy the following functional equation:

$$[z_1 z_2 - A(z_1, z_2)]U(z_1, z_2) = A(z_1, z_2)L(z_1, z_2) , \quad (1)$$

where $L(z_1, z_2) \triangleq (z_2 - 1)U(z_1, 0) + (z_1 - 1)U(0, z_2) + (z_1 - 1)(z_2 - 1)U(0, 0)$. The main mathematical difficulty in the full analysis of the system is to determine the boundary functions $U(z_1, 0)$ and $U(0, z_2)$. Nevertheless, the marginal pgf $U_i(z)$ of the system content in queue i can be easily deduced from (1) by choosing either $\{z_1 = z, z_2 = 1\}$ or $\{z_1 = 1, z_2 = z\}$, which results in the well-known formulas (see e.g. [2])

$$U_i(z) = A_i(z) \frac{(1 - \lambda_i)(z - 1)}{[z - A_i(z)]} , \quad i = 1, 2 , \quad (2)$$

where $A_1(z) \triangleq A(z, 1)$, $A_2(z) \triangleq A(1, z)$, $\lambda_i \triangleq A'_i(1)$ and $\lambda_i < 1$ for stability.

As we shall see, the precise nature of the “kernel” $K(z_1, z_2) \triangleq z_1 z_2 - A(z_1, z_2)$, and, hence, the pgf $A(z_1, z_2)$, has a crucial impact on the difficulty level of the determination of $U(z_1, z_2)$. Some cases turn out to be easy, whereas others are extremely hard. Basically, the question we deal with in this paper is: what should $A(z_1, z_2)$ look like to make the analysis “feasible”? More context and partial answers are given in the next section.

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3 Discussion In some cases, the joint pgf $U(z_1, z_2)$ can be derived by ad hoc reasonings, which do not require the “solution” of the functional equation (1). We provide three examples.

First, assume the arrival streams in both queues are mutually independent. Then, $A(z_1, z_2) = A_1(z_1) \cdot A_2(z_2)$. Then, at any time, the system contents in both queues are also independent, and, hence, $U(z_1, z_2) = U_1(z_1) \cdot U_2(z_2)$, where $U_i(z)$ is given by (2).

Next, suppose that during any slot, the number of arrivals in one of the two queues, say queue 1, is either one or zero. In this case, $A(z_1, z_2)$ can be expressed as $A(z_1, z_2) = a_0(z_2) + z_1 a_1(z_2)$. As the server can process exactly one customer per slot, no accumulation of customers can occur in queue 1. It is easily checked that in this case,

$$U(z_1, z_2) = [a_0(z_2) + z_1 a_1(z_2)] \frac{(1 - \lambda_2)(z_2 - 1)}{z_2 - A_2(z_2)} = A(z_1, z_2) \frac{(1 - \lambda_2)(z_2 - 1)}{z_2 - A_2(z_2)}.$$

Finally, suppose that during any slot, both queues receive exactly the same number of arrivals. Then, $A(z_1, z_2) = A_1(z_1 z_2)$, and, in the steady state, the system contents of queues 1 and 2 are always identical (possibly after some initial transient period). Hence, $U(z_1, z_2) = U_1(z_1 z_2)$, where $U_1(z)$ is the known function defined in (2).

We note that in these three “simple” cases, we were also able to find the above-mentioned “solutions” for the pgf $U(z_1, z_2)$ by purely mathematical means, i.e., by just manipulating the functional equation (1), but this was not at all an easy task.

For ease of notation, we now confine ourselves to “symmetric” systems, where $A(z_1, z_2) = A(z_2, z_1)$, and, consequently, also $U(z_1, z_2) = U(z_2, z_1)$ and $U(z, 0) = U(0, z) \triangleq P(z)$. A very well-known special case of this model, often referred to as “the symmetric clocked buffered switch”, has been treated with great care in, among others, [3]. In this case, $A(z_1, z_2) = [1 - \lambda + (\lambda/2)(z_1 + z_2)]^2$. In spite of this very simple quadratic expression of $A(z_1, z_2)$, the determination of $P(z)$ (and, from this, $U(z_1, z_2)$) is extremely complicated and requires all the fine tools of complex analysis. In the end, $P(z)$ turns out to be a meromorphic function with an infinite number of poles.

The above examples illustrate the crucial importance of the exact form of $A(z_1, z_2)$ for the solution of the functional equation (1). A general principle method (for a slightly different problem setting), suited for all possible choices of $A(z_1, z_2)$, was discussed in chapter II of [4], but this method, “the boundary value approach” involves singular integrals, conformal mappings and additional numerical work. This raises the question: can we find (sufficient) conditions that $A(z_1, z_2)$ should meet in order to allow for an easier analysis? In the rest of this paper, we present some first ideas in this respect.

Suppose the pgf $A(z_1, z_2)$ is such that for all u on some contour Σ in the complex u -plane, and for a given real number r , there exists at least one value $z_2 = z_2(u)$ such that $(ru, z_2(u))$ and $(ru^{-1}, z_2(u))$ are zero-tuples of the kernel K , i.e., $K(ru, z_2(u)) = 0$ and $K(ru^{-1}, z_2(u)) = 0$, and are in the area of convergence of the joint pgf $U(z_1, z_2)$. Then, from (1), it follows that $L(ru, z_2(u)) = 0$ and $L(ru^{-1}, z_2(u)) = 0$, and, from this, that $(ru^{-1} - 1)P(ru) = (ru - 1)P(ru^{-1})$ for all $u \in \Sigma$. Introducing the series expansion $P(z) \triangleq \sum_{i=0}^{\infty} p(i) z^i$, and identifying equal (positive or negative) powers of the variable u on both sides of the equation, then leads to $r^2 p(2) = p(0) + p(1)$ and $r^2 p(k+1) = p(k)$, for all $k \geq 2$, from which the P -function can be readily derived:

$$P(z) = \{p(0)[r^2 + z(z-1)] + p(1)r^2 z\} / [r^2 - z],$$

i.e., a meromorphic function with one single pole. It is important to mention that (many) arrival pgfs $A(z_1, z_2)$ exist that meet the above requirement. A “quadratic example” is

$$A(z_1, z_2) = \alpha(1 + \sigma z_1^2)(1 + \sigma z_2^2) + \beta(z_1 + z_2)(1 + \sigma z_1 z_2) + [1 - \alpha(1 + \sigma)^2 - 2\beta(1 + \sigma)]z_1 z_2$$

with, in this case, $r = 1/\sqrt{\sigma}$ and $U(z_1, z_2) = [(1 - \sigma)^2 A(z_1, z_2)]/[(1 - \sigma z_1)(1 - \sigma z_2)]$.

Next, suppose that $A(z_1, z_2)$ is an arrival pgf for which an analytic solution is “feasible”, such as all the ones we discussed above. Then, it can be shown that this is also true for the arrival pgf $\hat{A}(z_1, z_2)$, defined as $\hat{A}(z_1, z_2) \triangleq [1 - D(z_1, z_2)]z_1 z_2 + D(z_1, z_2)A(z_1, z_2)$. Here the function $D(z_1, z_2)$ must be such that the new function $\hat{A}(z_1, z_2)$ is a genuine joint pgf. For instance, if $D(z_1, z_2)$ is a real positive constant not larger than 1, this requirement is certainly fulfilled, but we have found other examples as well. Moreover, if $U(z_1, z_2)$ is the joint system-contents pgf corresponding to $A(z_1, z_2)$ and $P(z) = U(z, 0) = U(0, z)$, then the corresponding results for arrival pgf $\hat{A}(z_1, z_2)$ can be shown to be

$$\hat{U}(z_1, z_2) = U(z_1, z_2) \frac{\hat{A}(z_1, z_2)}{A(z_1, z_2)} \frac{D(1, 1)}{D(z_1, z_2)}, \quad \hat{P}(z) = D(1, 1)P(z). \quad (3)$$

The main reason why this remarkable result is true is that the zero-tuples of the kernel $K(z_1, z_2) = z_1 z_2 - A(z_1, z_2)$ are also zero-tuples of the kernel $\hat{K}(z_1, z_2) = z_1 z_2 - \hat{A}(z_1, z_2)$, because $\hat{K}(z_1, z_2) = D(z_1, z_2)K(z_1, z_2)$.

We are convinced that it must be possible to construct classes of arrival pgfs $A(z_1, z_2)$ for which the analysis is “feasible”, not only for the specific model dealt with in this paper, but also for similar (possibly asymmetric or even transient) problems that give rise to other kernels. We hope that the paper will incite more research in this direction.

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