The proportion of the population never hearing a rumour

Y. Duan[1] · A. Ganesh[1]

Abstract
Sudbury (J Appl Prob 22:443–446, 1985) showed for the Maki–Thompson model
of rumour spreading that the proportion of the population never hearing the rumour
converges in probability to a limiting constant (approximately equal to 0.203) as the
population size tends to infinity. We extend the analysis to a generalisation of the
Maki–Thompson model.

Keywords Rumour spreading · Epidemic processes

Mathematics Subject Classification 60G42 · 60G50

1 Introduction

The following model of rumour spreading was introducing by Maki and Thompson [7],
as a variant of an earlier model of Daley and Kendall [1]: there is a population of size n,
some of whom initially know a rumour and are referred to as infected. Time is discrete.
In each time step, an infected individual chosen uniformly at random (or arbitrarily)
contacts a member of the population chosen uniformly at random (including itself).
If this individual has not yet heard the rumour (is susceptible), then the contacted
individual becomes infected; otherwise, the contacting individual loses interest in
spreading the rumour and is termed removed (but remains in the population and can
be contacted by other infectives. In the Daley–Kendall model, if an infective contacts
another infective, both become removed, whereas, in the Maki–Thompson model,
only the initiator of the contact is removed). The process ends when there are no more
infectives. A natural question to ask is how many individuals remain susceptible at this
terminal time and consequently never hear the rumour. It was shown by Sudbury [10]
that in the large population limit of n tending to infinity, the random proportion of the

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✉ A. Ganesh
 A.Ganesh@bristol.ac.uk

[1] University of Bristol, Bristol, UK

population never hearing the rumour converges in probability to a limiting constant, approximately equal to 0.203.

In much of the literature related to the Maki–Thompson and Daley–Kendall models, the terms ignorants, spreaders and stiflers are used, respectively, for agents whom we have termed susceptible, infected and removed.

We consider the following generalisation of the Maki–Thompson model: each infective loses interest in spreading the rumour (and becomes removed) after k failed attempts, i.e. after contacting infected or removed individuals k times. Here, $k \geq 1$ is a specified constant, which is a parameter of the model; if $k = 1$, we recover the original model. Our main result is as follows.

Theorem 1 *Consider the generalisation of the Maki–Thompson model described above, parametrised by k and starting with a single infective and $n - 1$ suscepti- bles. Let S_∞ denote the number of susceptibles when the process terminates, i.e. when the number of infectives hits zero. Then,*

$$\frac{S_\infty}{n} \xrightarrow{p} y^* \text{ as } n \to \infty,$$

where y^* is the unique solution in $(0, 1)$ of the equation $(k + 1)(1 - y) = -\log y$, and logarithms are natural unless specified otherwise.

The proof is presented in the next section. We observe that $y^* = y^*(k)$ is a decreas- ing function of k and is well approximated by $e^{-(k+1)}$ for large k. This tells us that, qualitatively, the proportion of the population not hearing a rumour decays exponen- tially in the number of failed attempts before agents lose interest in spreading the rumour.

One of our main motivations for this work is that rumour spreading is used for infor- mation dissemination in many large-scale distributed algorithms in computer science; see, for example, the seminal work of Demers et al. [2]. Blockchain is a topical example of a technology which employs such algorithms [3,8]. The algorithms require a termi- nation condition in order to limit the communication overhead. The Maki–Thompson model provides such a condition, but comes at the price that approximately 20% of agents do not receive the information. This might be unacceptable in applications which require higher reliability. Therefore, it would be desirable to have a tunable trade-off between the communication overhead incurred and the reliability achieved in terms of the proportion of agents who receive the information. Theorems 1 and 2 provide such a trade-off.

Additional motivation for this work comes from the spread of information, including fake news, on online social networks. This has motivated countermeasures, such as limitations on forwarding, by social network platforms. While the match to our model is not exact, insofar as communications on such networks tend to be broadcasts rather than pairwise communications, our work nevertheless yields qualitative insights into the effectiveness of decentralised countermeasures. Extending the analysis to other communication models such as broadcasts is a topic for future work.

Generalisations of the Daley–Kendall and Maki–Thompson models have been stud- ied previously. Lebensztayn et al. [5] consider a model in which a spreader i becomes

a stifler after a random number, R_i, of contacts with other spreaders or stiflers. This is more general than the model described above, where $R_i \equiv k$, a fixed constant. The analysis in [5] uses the fact that a Markovian description of this model easily reduces to a density-dependent Markov chain, to which Kurtz's theorem can be applied; this states that in the large population (hydrodynamic) limit, the trajectory of the Markov chain converges to the trajectory of the solution of an ODE, uniformly on compacts. A major disadvantage of this style of analysis is that it is only directly applicable to initial conditions in which a positive *fraction* of the population are spreaders. A separate analysis to deal with the initial phase is required if one wishes to start with a fixed *number* of initial spreaders, so that the fraction who are spreaders tends to zero in the large population limit. This difficulty is avoided by the techniques used in this paper, which are elementary as in the original paper of Sudbury [10]. Our next theorem extends the result of [5] to initial conditions with a single infective.

Theorem 2 *Consider a generalisation of the Maki–Thompson model in which the ith agent to learn the rumour stops spreading it after incurring R_i failures (instances of contacting an agent which already knew the rumour). Here, $(R_i, i = 0, 1, 2, \ldots)$ are independent and identically distributed (i.i.d.) random variables taking values in $\{0, 1, 2, \ldots\}$. Suppose that $\mu := \mathbb{E}[R_1]$ is finite, that there are initially a single infective and $n - 1$ susceptibles and that $R_0 \geq 1$. Let S_∞ denote the number of susceptibles when the process terminates, i.e. when the number of infectives hits zero. Then,*

$$\frac{S_\infty}{n} \xrightarrow{p} y^* \text{ as } n \to \infty,$$

where y^* is the unique solution in $(0, 1)$ of the equation $(\mu + 1)(1 - y) = -\log y$.

The claim of Theorem 2 coincides with that of [5][Theorem 2.3], but is made under the weaker assumption of requiring only a single initial rumour spreader rather than a positive fraction of the population being spreaders. We sketch the proof in Sect. 3, only pointing out differences from the proof of Theorem 1. Comparing Theorems 1 and 2, we see that if the number of failures tolerated by a node before it stops spreading the rumour is random, then the final proportion of nodes reached by the rumour is insensitive to the distribution of this random variable, and depends only on its mean. Thus, in order to ensure that the rumour reaches a high proportion of the population, it is not necessary for each spreader to tolerate a large number of failed attempts, but only that this be true on average.

We now briefly survey some related work. The analysis of Sudbury's model was generalised by Lefèvre and Picard [6] to obtain the joint distribution of the number reached by the rumour and the time for which the rumour was spread. Sudbury's model was generalised by Isham et al. [4] to allow for spreaders to spontaneously become stiflers, and to allow a nonzero probability that spreaders do not change state on contacting a spreader or stifler (which corresponds to the R_i having a geometric distribution in the context of Theorem 2); they further studied the model on a number of different networks rather than just the complete graph. However, they mainly focused on simulations and intuition and did not present a rigorous analysis of their model. Pittel [9] showed in the Maki–Thompson model that the proportion of nodes not

hearing the rumour, suitably centred and rescaled, converges in distribution to a normal random variable. An extension of this result to the above generalised models is an open problem.

2 Model and analysis

Denote by S_t the number of susceptibles present in time slot t. If at least one infective is present during this time slot, then there is an infection attempt during this time slot, which succeeds with probability S_t/n (or $S_t/(n-1)$ if an infective never contacts itself; the distinction is immaterial for large n). In that case, $S_{t+1} = S_t - 1$. Otherwise, $S_{t+1} = S_t$ and the number of failed attempts associated with the infective node which initiated the contact is incremented by 1; if its value becomes equal to k, the infective node becomes removed. We could describe this process as a Markov chain by keeping track of $I_t^0, I_t^1, \ldots, I_t^{k-1}$, which denote, respectively, the number of infective nodes which have seen $0, 1, \ldots, k-1$ failed infection attempts. A simpler Markovian representation is obtained by keeping track of I_t, the number of infection attempts available in time step t, which increases by k whenever a new node is infected. We initialise the process with $S_0 = n - 1$ and $I_0 = k$; the process terminates when I_t hits zero for the first time. If $I_t > 0$, then

$$(S_{t+1}, I_{t+1}) = \begin{cases} (S_t - 1, I_t + k), & \text{w.p. } S_t/n, \\ (S_t, I_t - 1), & \text{w.p. } 1 - (S_t/n), \end{cases} \tag{1}$$

where we use the abbreviation w.p. for "with probability".

Let T denote the random time that the process terminates, i.e. when I_t hits zero for the first time. From (1), we can see that $(k + 1)S_t + I_t + t$ remains constant for all $t \geq 0$, and therefore,

$$(k+1)S_T + T = (k+1)S_0 + I_0 + 0 = (k+1)(n-1) + k,$$

so that

$$T = \inf\{t : (k+1)(n-1-S_t) \leq t - k\} = \inf\{t : (k+1)(n-S_t) \leq t + 1\}.$$

Define $\tilde{S}_t, t = 0, 1, 2, \ldots$, to be a Markov process on the state space $\{0, 1, \ldots, n-1\}$ with transition probabilities

$$p_{s,s} = 1 - \frac{s}{n}, \quad p_{s,s-1} = \frac{s}{n}, \quad 0 \leq s \leq n, \tag{2}$$

and with initial condition $\tilde{S}_0 = n - 1$. Then, $\tilde{S}_t$ and S_t have the same transition probabilities while I_t is nonzero; hence, it is clear that we can couple the processes S_t and $\tilde{S}_t$ in such a way that they are equal until the random time T. Consequently, we can write

$$T = \inf\{t : (k+1)(n - \tilde{S}_t) \leq t + 1\}, \tag{3}$$

which relates T to a level crossing time of a lazy random walk. As the random walk $\tilde{S}_t$ is non-increasing, S_T is explicitly determined by T; we have

$$S_T = \tilde{S}_T = n - \frac{T+1}{k+1}. \tag{4}$$

While it is possible to study the random variable T directly by analysing the random walk $\tilde{S}_t$, we will follow the work of Sudbury [10] and consider a somewhat indirect approach. The random walk $\tilde{S}_t$ is exactly the same as the random walk s_k in that paper, but the level crossing required for stopping is different.

Define the filtration $\mathcal{F}_t = \sigma(\tilde{S}_u, I_u, 0 \leq u \leq t), t \in \mathbb{N}$, and notice that the random time T defined in (3) is a stopping time, i.e. the event $\{T \leq t\}$ is $\mathcal{F}_t$-measurable. Moreover, T is bounded by $(k+1)n$. Let

$$M_1(t) = \left(\frac{n}{n-1}\right)^t \tilde{S}_t, \quad M_2(t) = \left(\frac{n}{n-2}\right)^t \tilde{S}_t(\tilde{S}_t - 1).$$

The following lemma is an exact analogue of a corresponding result in [10] and follows easily from the transition probabilities in (2), so the proof is omitted.

Lemma 1 *The processes $M_1(t \wedge T)$ and $M_2(t \wedge T)$ are $\mathcal{F}_t$-martingales.*

Applying the optional stopping theorem (OST) to $M_1(t \wedge T)$, we get

$$\mathbb{E}\left[\left(\frac{n}{n-1}\right)^T \tilde{S}_T\right] = \tilde{S}_0. \tag{5}$$

We show that for large n the above random variables concentrate around their mean values and, after suitable rescaling, converge in probability.

Lemma 2 *Let $\tilde{S}_T$ denote the final number of susceptibles and T the random time (number of attempts to spread the rumour) after which the process terminates in a population of size n. The dependence of T and $\tilde{S}_T$ on n has been suppressed in the notation. Then,*

$$\left(\frac{n}{n-1}\right)^T \frac{\tilde{S}_T}{n} \xrightarrow{p} 1 \text{ as } n \to \infty.$$

Proof The proof is largely reproduced from [10] but is included for completeness. It proceeds by bounding the variance of the random variables of interest and invoking Chebyshev's inequality. We have by (5) that

$$\text{Var}\left\{\left(\frac{n}{n-1}\right)^T \tilde{S}_T\right\} = \mathbb{E}\left[\left(\frac{n-1}{n}\right)^{-2T} \tilde{S}_T^2\right] - \tilde{S}_0^2,$$

whereas, applying the OST to $M_2(t \wedge T)$, we get

$$\mathbb{E}\left[\left(\frac{n}{n-2}\right)^T (\tilde{S}_T^2 - \tilde{S}_T)\right] = \tilde{S}_0^2 - \tilde{S}_0.$$

Combining the last two equations, we can write

$$\mathrm{Var}\left\{\left(\frac{n}{n-1}\right)^T \tilde{S}_T\right\}$$

$$= \mathbb{E}\left[\left(\frac{n-1}{n}\right)^{-2T} \tilde{S}_T^2\right] - \mathbb{E}\left[\left(\frac{n}{n-2}\right)^T (\tilde{S}_T^2 - \tilde{S}_T)\right] - \tilde{S}_0$$

$$= \mathbb{E}\left\{\left[\left(\frac{n-1}{n}\right)^{-2T} - \left(\frac{n-2}{n}\right)^{-T}\right]\tilde{S}_T^2\right\} + \mathbb{E}\left[\left(\frac{n}{n-2}\right)^T \tilde{S}_T\right] - \tilde{S}_0.$$

Now, the first term in the above sum is negative, since $(1 - \frac{1}{n})^2 > 1 - \frac{2}{n}$. Next, since T is bounded above by $(k+1)n$, we have

$$\mathrm{Var}\left\{\left(\frac{n}{n-1}\right)^T \tilde{S}_T\right\} < \mathbb{E}\left[\left(\frac{n}{n-1}\frac{n-1}{n-2}\right)^T \tilde{S}_T\right] - \tilde{S}_0$$

$$\leq \left(\frac{n-1}{n-2}\right)^{(k+1)n} \mathbb{E}\left[\left(\frac{n}{n-1}\right)^T \tilde{S}_T\right] - \tilde{S}_0$$

$$= e^{(k+1)n \log(1+\frac{1}{n-2})} \tilde{S}_0 - \tilde{S}_0$$

$$\leq (e^{(k+1)n/(n-2)} - 1)\tilde{S}_0,$$

where we have used the fact that $\mathbb{E}\left[\left(\frac{n}{n-1}\right)^T \tilde{S}_T\right] = \tilde{S}_0$ to obtain the equality on the third line, and the inequality $\log(x) \leq x - 1$ to obtain the last inequality. Thus, we conclude that

$$\mathrm{Var}\left\{\left(\frac{n}{n-1}\right)^T \frac{\tilde{S}_T}{n}\right\} \leq \frac{(e^{(k+1)n/(n-2)} - 1)\tilde{S}_0}{n^2},$$

which tends to zero as n tends to infinity, since $\tilde{S}_0 = n - 1$. The claim of the lemma now follows from (5) and Chebyshev's inequality. $\square$

Consider the sequence of random vectors $\left(\frac{T}{n}, \frac{\tilde{S}_T}{n}\right)$, which take values in the compact set $K = [0, k+1] \times [0, 1]$; the dependence of T and $\tilde{S}_T$ on n has not been made explicit in the notation. Define $f : K \to \mathbb{R}^2$ by

$$f(x, y) = \left(\frac{x}{k+1} + y - 1, e^x y - 1\right). \tag{6}$$

Then, we see from (4) and Lemma 2 that

$$f(T/n, \tilde{S}_T/n) \xrightarrow{p} (0, 0) \text{ as } n \to \infty. \tag{7}$$

We want to use this to prove convergence in probability of the sequences T/n and $\tilde{S}_T/n$.

Firstly, we observe that if $f(x, y) = (0, 0)$, then y solves the equation $(k+1)(1 - y) + \log y = 0$, and $x = (k+1)(1 - y)$. The function $y \mapsto (k+1)(1 - y) + \log y$ is strictly concave and is zero at $y = 1$; by considering its derivative at 1 and its value near 0, it can be seen that the function has one other zero, which lies in $(0, 1)$. Call this value y^* and define $x^* = (k+1)(1 - y^*)$. We now have the following.

Lemma 3 *Fix $\delta > 0$. Then, as n tends to infinity,*

$$\mathbb{P}\big((T/n, \tilde{S}_T/n) \notin B_\delta(0, 1) \cup B_\delta(x^*, y^*)\big) \to 0,$$

where $B_\delta(x, y)$ denotes the open ball of radius δ centred on (x, y).

Proof Suppose this is not the case. Then, there are an $\alpha > 0$ and infinitely many n such that

$$\mathbb{P}\big((T/n, \tilde{S}_T/n) \notin B_\delta(0, 0) \cup B_\delta(x^*, y^*)\big) > \alpha.$$

Since f is continuous, so is its norm. Hence, its minimum on the compact set $K \backslash \{B_\delta(0, 0) \cup B_\delta(x^*, y^*)\}$ is attained and must be strictly positive as f has no zeros other than $(0, 1)$ and (x^*, y^*). Hence, there is an $\epsilon > 0$ such that $\| f(x, y)\| > \epsilon$ whenever $(x, y) \notin B_\delta(0, 1) \cup B_\delta(x^*, y^*)$. Thus, we have shown that there are infinitely many n such that

$$\mathbb{P}(\| f(T/n, \tilde{S}_T/n)\| > \epsilon) > \alpha,$$

which contradicts (7). This proves the claim of the lemma. □

Next, define $\tau_j = \inf\{t : \tilde{S}_t = n - j\}$, $X_j = \tau_{j+1} - \tau_j$, and observe from (2) and the initial condition $\tilde{S}_0 = n - 1$ that

$$\tau_1 = 0, \quad X_j \sim Geom\Big(\frac{n-j}{n}\Big), \tag{8}$$

and that X_j, $j = 1, \ldots, n - 1$, are mutually independent; here, $\sim$ denotes equality in distribution. We also have from (3) that

$$n - \tilde{S}_T = \inf\{j : X_1 + \cdots + X_j \geq (k+1)j\}. \tag{9}$$

We now need the following elementary tail bound on the binomial distribution in order to complete the proof of Theorem 1.

Lemma 4 *Let X be binomially distributed with parameters n and p, denoted $X \sim Bin(n, p)$. Then, for any $q > p$, we have*

$$\mathbb{P}(X \geq nq) \leq \exp\Big(-n\Big[q \log \frac{q}{p} - q + p\Big]\Big).$$

Proof Recall the well-known large deviations bound,

$$\mathbb{P}(X \geq nq) \leq \exp(-nH(q; p)), \text{ where } H(q; p) = q \log \frac{q}{p} + (1 - q) \log \frac{1-q}{1-p},$$

which is a consequence of Chernoff's bound.

The claim of the lemma follows from the above inequality by noting that

$$(1 - q) \log \frac{1-p}{1-q} \leq (1 - q)\Big(\frac{1-p}{1-q} - 1\Big) = q - p,$$

which follows from the inequality $\log x \leq x - 1$.   □

Proof of Theorem 1 In view of Lemma 3, it remains only to show, for some $\epsilon \in (0, y^*)$, that $\mathbb{P}(\tilde{S}_T/n > 1 - \epsilon)$ tends to zero as n tends to infinity.

Fix $\epsilon > 0$. For each $j \in \mathbb{N}$, let $Y_i^{(j)}$, $i \in \mathbb{N}$, be i.i.d. random variables with a $Geom((n-j)/n)$ distribution. Then, $Y_i^{(j)}$ stochastically dominates X_i for every $i \leq j$, and we see from (9) that

$$
\begin{aligned}
\mathbb{P}(\tilde{S}_T/n \geq 1 - \epsilon) &= \mathbb{P}(\exists j \leq \epsilon n : X_1 + \ldots + X_j \geq (k+1)j) \\
&\leq \sum_{j=1}^{\lfloor \epsilon n \rfloor} \mathbb{P}(X_1 + \ldots + X_j \geq (k+1)j) \\
&\leq \sum_{j=1}^{\lfloor \epsilon n \rfloor} \mathbb{P}(Y_1^{(j)} + \ldots + Y_j^{(j)} \geq (k+1)j) \\
&\leq \sum_{j=1}^{\lfloor \epsilon n \rfloor} \mathbb{P}\left(Bin\left((k+1)j - 1, \frac{n-j}{n}\right) \leq j - 1\right).
\end{aligned}
$$

We can rewrite the above as

$$
\begin{aligned}
\mathbb{P}(\tilde{S}_T/n \geq 1 - \epsilon) &\leq \sum_{j=1}^{\lfloor \epsilon n \rfloor} \mathbb{P}\left(Bin\left((k+1)j - 1, \frac{j}{n}\right) \geq kj\right) \\
&\leq \sum_{j=1}^{\lfloor \epsilon n \rfloor} \mathbb{P}\left(Bin\left((k+1)j, \frac{j}{n}\right) \geq kj\right).
\end{aligned}
$$

Hence, it follows from Lemma 4 that, for $\epsilon < \frac{k}{k+1}$, we have

$$
\begin{aligned}
\mathbb{P}(\tilde{S}_T/n \geq 1 - \epsilon) &\leq \sum_{j=1}^{\lfloor \epsilon n \rfloor} \exp\left(-(k+1)j\left[\frac{k}{k+1} \log \frac{kn}{(k+1)j} - \frac{k}{k+1} + \frac{j}{n}\right]\right) \\
&\leq \sum_{j=1}^{\lfloor \epsilon n \rfloor} \exp\left(-kj \log \frac{kn}{(k+1)j} + kj\right) \\
&\leq \sum_{j=1}^{\lfloor \frac{k}{k+1} \frac{\sqrt{n}}{e} \rfloor - 1} n^{-kj/2} + \sum_{j=\lfloor \frac{k}{k+1} \frac{\sqrt{n}}{e} \rfloor}^{\lfloor \epsilon n \rfloor} e^{-kj}.
\end{aligned}
$$

It is easy to see that both sums above vanish as n tends to infinity. This completes the proof of the theorem.   □

3 Analysis of generalised model

In this section, we outline the proof of Theorem 2. As it largely follows the same lines as the proof of Theorem 1, we only highlight the differences.

We start with a single infected agent who knows the rumour and has $R_0 \geq 1$ available failed infection attempts, after which it stops spreading the rumour. Let the processes S_t, I_t and $\tilde{S}_t$ be as in Sect. 2; S_t denotes the number of susceptibles, and I_t the number of available failed infection attempts, at the end of time step t, while $\tilde{S}_t$ is a Markov chain with the same probability law as S_t and coupled to be the same as S_t until the rumour-spreading process terminates. The only difference is that the $\tilde{S}_t$ process continues to evolve after rumour spreading terminates, and so its value at any t can be defined without reference to the process I_t and whether it has hit zero. In Sect. 2, the I_t process increased by k at each successful infection event and decreased by 1 for each failed attempt. Here, it increases by a random amount R_i upon each successful infection. But the S_t and $\tilde{S}_t$ processes have the same transition probabilities as in Sect. 2, and so Lemmas 1 and 2 continue to hold.

The derivation of the expression in (3) for the time at which rumour spreading stops relied on the fact that each infected individual has exactly k available failed infection attempts and hence that $(k+1)S_t + I_t + t$ is constant over time. This is no longer true when the ith infected individual stops spreading the rumour after a random number, R_i, of failed attempts. Instead, we obtain the following analogue of (3) for the time T when the process terminates:

$$T = \inf\{t \geq 0 : R_0 + \cdots + R_{n-1-\tilde{S}_t} \leq t - (n - 1 - \tilde{S}_t)\}. \tag{10}$$

To see this, note that as each infection reduces the number of susceptibles by 1, $n - 1 - \tilde{S}_t$ is the number of infections that have occurred up to time t. Hence, $t - (n - 1 - \tilde{S}_t)$ is the number of failed infection attempts up to this time, while $R_0 + \ldots + R_{n-1-\tilde{S}_t}$ is the total budget of failed attempts from all agents infected up to this time. If $R_i \equiv k$, then (10) reduces to (3).

Next, recalling the definitions $\tau_j = \inf\{t : \tilde{S}_t = n - j\}$ and $X_j = \tau_{j+1} - \tau_j$, we see that Eq. (8) continues to hold; however, (9) needs to be modified, in light of (10), as follows:

$$T = \tau_{j^*}, \quad \tilde{S}_T = n - j^*, \quad \text{where } j^* = \inf\left\{j : \sum_{k=0}^{j-1} R_k \leq \tau_j - (j-1)\right\}. \tag{11}$$

For $n \in \mathbb{N}$, define $\phi_n : [0, 1] \to \mathbb{R}_+^2$ by

$$\phi_n(x) = (\phi_n^1(x), \phi_n^2(x)) = \frac{1}{n}\left(\tau_{\lfloor xn \rfloor}, \sum_{k=0}^{\lfloor xn \rfloor - 1} R_k\right). \tag{12}$$

Now, by the functional law of large numbers, ϕ_n^2 converges in probability, in the space $L^\infty([0, 1])$, to ϕ^2, given by $\phi^2(x) = x\mu$, where $\mu = \mathbb{E}[R_1]$. We also observe that, for

$x \in [0, 1)$,

$$\frac{\mathbb{E}[\tau_{\lfloor xn \rfloor}]}{n} = \sum_{k=1}^{\lfloor xn \rfloor - 1} \frac{\mathbb{E}[X_j]}{n} = \sum_{k=1}^{\lfloor xn \rfloor - 1} \frac{1}{n - j} \rightarrow -\log(1 - x), \text{ as } n \rightarrow \infty.$$

Using this, it can be shown that ϕ_n^1 converges in probability, in the space $L^\infty([0, \beta])$ for any $\beta < 1$, to ϕ^1, where $\phi^1(x) = -\log(1 - x)$. Thus, ϕ_n converges in probability to $\phi = (\phi^1, \phi^2)$.

Now, consider the sequence $(T, \tilde{S}_T)/n$, where the dependence of T and $\tilde{S}_T$ on n has not been made explicit in the notation. This sequence takes values in the compact set $[0, 1] \times [0, 1]$ and hence contains convergent subsequences. Let (x, y) be the limit of such a subsequence. Observe from (11) that, along this subsequence,

$$\frac{1}{n} \sum_{k=0}^{j^* - 1} R_k = \frac{1}{n} \sum_{k=0}^{n - 1 - \tilde{S}_T} R_k \rightarrow \mu(1 - y),$$

while

$$\frac{\tau_{j^*} - j^* + 1}{n} = \frac{T - (n - \tilde{S}_t - 1)}{n} \rightarrow x + y - 1.$$

Hence, it can be shown from (11) that (x, y) satisfies the equation $\mu(1 - y) = x + y - 1$, i.e. $x = (\mu + 1)(1 - y)$. Moreover, we see from Lemma 2 that $e^x y = 1$. Comparing these two equations with (6), we see that (7) continues to hold, with k replaced by μ in the definition of f. Following this, it can be verified that Lemma 3 continues to hold, with (x^*, y^*) denoting the unique nonzero solution of the equation $f(x, y) = (0, 0)$, where k has been replaced by μ in the definition of f given in (6). Thus, all the lemmas used in the proof of Theorem 1 continue to hold, with the modification that k has been replaced by μ. Thus, the proof of Theorem 2 will follow along the same lines as that of Theorem 1, provided we can rule out the solution $(0, 0)$ of the equation $f(x, y) = (0, 0)$ as a subsequential limit point for the sequence $(T, n - \tilde{S}_T)/n$. In other words, we need to show that the proportion of susceptibles in the limit cannot be equal to 1.

We now provide an intuitive explanation for why this is so, i.e. why the proportion of nodes hearing the rumour cannot converge to zero with positive probability, provided $R_0 \geq 1$, as assumed in the statement of Theorem 2. The initial node knowing the rumour continues spreading it until it encounters a node which already knows the rumour. As the target for rumour spreading is chosen uniformly at random at each step, it follows from the birthday paradox that the initial source spreads the rumour to some strictly positive random multiple of $\sqrt{n}$ nodes before losing interest. Next, for a fixed $\epsilon > 0$, the spread of the rumour until at least ϵn nodes learn it dominates a branching process in which the mean number of offspring is $\mathbb{P}(R_1 \geq 1)/\epsilon$; this is the mean number of nodes contacted by an infective (spreader) until it contacts an informed node (of which there are fewer than ϵn), but only counting those for which $R_i \geq 1$, as nodes with $R_i = 0$ will not spread the rumour. If $\epsilon > 0$ is chosen small enough, then this branching process is supercritical. As its initial population size is

a multiple of $\sqrt{n}$, its extinction probability is vanishing in n. But non-extinction of the branching process implies that at least ϵn agents learn the rumour in the rumour-spreading process. This shows that a strictly positive fraction of the population learns the rumour, with high probability.

This completes the sketch of the proof of Theorem 2.

References

1. Daley, D., Kendall, D.G.: Stochastic rumours. J. Inst. Math. Appl. **1**, 42–55 (1965)
2. Demers, A., Greene, D., Hauser, C., Irish, W., Larson, J., Shenker, S., Sturgis, H., Swinehart D., Terry, D.: Epidemic algorithms for replicated database maintenance, In: Proceedings of ACM Symposium on Principles of Distributed Computing (PODC) (1987)
3. Gopalan, A., Sankararaman, A., Walid, A., Vishwanath, S.: Stability and scalability of blockchain systems. Proc. ACM Meas. Anal. Comput. Syst. **4**(2), 1–35 (2020)
4. Isham, V., Kaczmarska, J., Nekovee, M.: Spread of information and infection on finite random networks. Phys. Rev. E **83**, 046128 (2011). https://doi.org/10.1103/PhysRevE.83.046128
5. Lebensztayn, E., Machado, F.P., Rodriguez, P.M.: On the behaviour of a rumour process with random stifling. Environ. Mod. Soft. **26**, 517–522 (2011)
6. Lefèvre, C., Picard, P.: Distribution of the final extent of a rumour process. J. Appl. Prob. **31**, 244–249 (1994)
7. Maki, D.P., Thompson, M.: Mathematical Models and Applications. Prentice-Hall, Englewood Cliffs, N.J. (1973)
8. Patsonakis, C., Roussopoulos, M.: Revisiting asynchronous rumor spreading in the blockchain era, In: Proceedings of IEEE Conference on Parallel and Distributed Systems (ICPADS), pp. 284–293 (2019)
9. Pittel, B.: On a Daley-Kendall model of random rumours. J. Appl. Prob. **27**, 14–27 (1990)
10. Sudbury, A.: The proportion of the population never hearing a rumour. J. Appl. Prob. **22**, 443–446 (1985)