Queueing Syst (2012) 71:137–149
DOI 10.1007/s11134-011-9270-x

Gaussian queues in light and heavy traffic

K. Dębicki · K.M. Kosiński · M. Mandjes

Received: 30 March 2011 / Published online: 5 January 2012

Abstract In this paper we investigate Gaussian queues in the light-traffic and in the heavy-traffic regime. Let $Q_X^{(c)} \equiv \{Q_X^{(c)}(t) : t \geq 0\}$ denote a stationary buffer content process for a fluid queue fed by the centered Gaussian process $X \equiv \{X(t) : t \in \mathbb{R}\}$ with stationary increments, $X(0) = 0$, continuous sample paths and variance function $\sigma^2(\cdot)$. The system is drained at a constant rate $c > 0$, so that for any $t \geq 0$,

$$Q_X^{(c)}(t) = \sup_{-\infty < s \leq t} \big(X(t) - X(s) - c(t-s)\big).$$

We study $Q_X^{(c)} \equiv \{Q_X^{(c)}(t) : t \geq 0\}$ in the regimes $c \to 0$ (heavy traffic) and $c \to \infty$ (light traffic). We show for both limiting regimes that, under mild regularity conditions on σ, there exists a normalizing function $\delta(c)$ such that $Q_X^{(c)}(\delta(c)\cdot)/\sigma(\delta(c))$ converges to $Q_{B_H}^{(1)}(\cdot)$ in $C[0,\infty)$, where B_H is a fractional Brownian motion with suitably chosen Hurst parameter H.

K. Dębicki
Instytut Matematyczny, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
e-mail: Krzysztof.Debicki@math.uni.wroc.pl

K.M. Kosiński (✉) · M. Mandjes
Korteweg-de Vries Institute for Mathematics, University of Amsterdam, 94248 Amsterdam, The Netherlands
e-mail: K.M.Kosinski@uva.nl

M. Mandjes
e-mail: M.R.H.Mandjes@uva.nl

K.M. Kosiński · M. Mandjes
Eurandom, Eindhoven University of Technology, 513 Eindhoven, The Netherlands

M. Mandjes
CWI, 94079 Amsterdam, The Netherlands

Keywords Gaussian processes · Heavy traffic · Light traffic · Functional limit theorems

Mathematics Subject Classification (2000) Primary 60G15 · 60F17 · Secondary 60K25

1 Introduction

A substantial research effort has been devoted to the analysis of queues with Gaussian input, often also called *Gaussian queues* [10–12]. The interest in this model can be explained from the fact that the Gaussian input model is highly flexible in terms of incorporating a broad set of correlation structures and, at the same time, adequately approximates various real-life systems. A key result in this area is [18], where it is shown that large aggregates of Internet sources converge to a fractional Brownian motion (being a specific Gaussian process).

The setting considered in this paper is that of a centered Gaussian process $X \equiv \{X(t) : t \in \mathbb{R}\}$ with stationary increments, $X(0) = 0$, continuous sample paths and variance function $\sigma^2(\cdot)$, equipped with a deterministic, linear drift with rate $c > 0$, reflected at 0:

$$Q_X^{(c)}(t) = \sup_{-\infty < s \le t} \big(X(t) - X(s) - c(t - s)\big).$$

The resulting *stationary workload process* can be regarded as a *queue* [14]. The objective of the paper is to study $Q_X^{(c)} \equiv \{Q_X^{(c)}(t) : t \ge 0\}$ in the limiting regimes $c \to 0$ (heavy traffic) and $c \to \infty$ (light traffic).

Under mild conditions on the variance function $\sigma^2(\cdot)$, $Q_X^{(c)}$ is a properly defined, almost surely (a.s.) finite stochastic process. However, if $c \to 0$, then $Q_X^{(c)}(t)$ grows to infinity (in a distributional sense), for any $t \ge 0$. The branch of queueing theory investigating *how fast* $Q_X^{(c)}$ grows to infinity (as $c \to 0$) is commonly referred to as the domain of *heavy-traffic approximations*. In many situations this regime allows manageable expressions for performance metrics that are, under 'normal' load conditions, highly complex or even intractable, see for instance the seminal paper by [9] on the classical single-server queue. Since then, a similar approach has been followed in various other settings, e.g., [5, 13, 15, 17, 19] and many other papers.

Analogously, one can ask what happens in the *light-traffic* regime, i.e., $c \to \infty$; then evidently $Q_X^{(c)}$ decreases to zero. So far, hardly any attention has been paid to the light-traffic and heavy-traffic regimes for Gaussian queues. An exception is [8], where the focus is on a special family of Gaussian processes, in a specific heavy-traffic setting. The primary contribution of the present paper concerns the analysis of $Q_X^{(c)}$ under both limiting regimes, for quite a broad class of Gaussian input processes X.

We now give a somewhat more detailed introduction to the material presented in this paper. It is well known that under the assumption that $\sigma(\cdot)$ varies regularly at infinity with parameter $\alpha \in (0, 1)$, for any function δ such that $\delta(c) \to \infty$ as $c \to 0$, there is convergence to fractional Brownian motion in the heavy-traffic regime:

$$\frac{X(\delta(c)\cdot)}{\sigma(\delta(c))} \xrightarrow{d} B_\alpha(\cdot), \quad \text{as } c \to 0. \tag{1}$$

We shall show that an analogous statement holds in the light-traffic regime, that is, if $\sigma(\cdot)$ varies regularly at zero with parameter $\lambda \in (0,1)$ (i.e., $x \mapsto \sigma(1/x)$ varies regularly at infinity with parameter $-\lambda$), then for any function δ such that $\delta(c) \to 0$ as $c \to \infty$,

$$\frac{X(\delta(c)\cdot)}{\sigma(\delta(c))} \xrightarrow{d} B_\lambda(\cdot), \quad \text{as } c \to \infty. \tag{2}$$

Assuming that X satisfies some minor additional conditions, both (1) and (2) apply in $C(\mathbb{R})$, the space of all continuous functions on $\mathbb{R}$.

Our paper shows that the statements (1) and (2), which relate to the input processes, carry over to the corresponding stationary buffer content processes $Q_X^{(c)}$. That is, we identify, under specific conditions, a function $\delta(\cdot)$ such that

$$\frac{Q_X^{(c)}(\delta(c)\cdot)}{\sigma(\delta(c))} \xrightarrow{d} Q_{B_\alpha}^{(1)}(\cdot), \quad \text{as } c \to 0$$

and

$$\frac{Q_X^{(c)}(\delta(c)\cdot)}{\sigma(\delta(c))} \xrightarrow{d} Q_{B_\lambda}^{(1)}(\cdot), \quad \text{as } c \to \infty,$$

both in the space $C[0,\infty)$ of all continuous functions on $[0,\infty)$.

This paper is organized as follows. In Sect. 2 we introduce the notation and give some preliminaries. Section 3.1 presents the results for the heavy-traffic regime, whereas Sect. 3.2 covers the light-traffic regime. We give the proofs of the main theorems (i.e., Theorems 1 and 2) in Sect. 4.

2 Preliminaries

In this paper we use the following notation. By $\mathrm{id}: \mathbb{R} \to \mathbb{R}$ we shall denote the identity operator on $\mathbb{R}$, that is, $\mathrm{id}(t) = t$ for every $t \in \mathbb{R}$. We write $f(x) \sim g(x)$ as $x \to x_0 \in [0,\infty]$ when $\lim_{x\to x_0} f(x)/g(x) = 1$. Let $\mathscr{RV}_\infty(\alpha)$ and $\mathscr{RV}_0(\lambda)$ denote the class of regularly varying functions at infinity with parameter α and at zero with parameter λ, respectively. That is, for a non-negative measurable functions f, g on $[0,\infty)$, $f \in \mathscr{RV}_\infty(\alpha)$ if for all $t > 0$, $f(tx)/f(x) \to t^\alpha$ as $x \to \infty$; $g \in \mathscr{RV}_0(\lambda)$ if for all $t > 0$, $g(tx)/g(x) \to t^\lambda$ as $x \to 0$.

2.1 Spaces of continuous functions

We refer to [3] for the details of this subsection. For any $T > 0$, let $C[-T,T]$ be the space of all continuous functions $f : [-T,T] \to \mathbb{R}$. Equip $C[-T,T]$ with the topology of uniform convergence, i.e., the topology generated by the norm $\|f\|_{[-T,T]} := \sup_{t\in[-T,T]} |f(t)|$ under which $C[-T,T]$ is a separable Banach space. Therefore, by Prokhorov's theorem, weak convergence of random elements $\{X^{(c)}\}$ of $C[-T,T]$ as $c\to\infty$ is implied by convergence of finite-dimensional distributions and tightness. A family $\{X^{(c)}\}$ in $C[-T,T]$ is tight if and only if for each positive ε, there exists an a and c_0 such that

$$\mathbb{P}\big(|X^{(c)}(0)| \geq a\big) \leq \varepsilon, \quad \text{for all } c \geq c_0; \tag{3}$$

and, for any $\eta > 0$,

$$\lim_{\zeta\to 0} \limsup_{c\to\infty} \mathbb{P}\Bigg(\sup_{\substack{|t-s|\leq\zeta \\ s,t\in[-T,T]}} \big|X^{(c)}(t) - X^{(c)}(s)\big| \geq \eta \Bigg) = 0. \tag{4}$$

For notational convenience, we leave out the requirement $s,t \in [-T,T]$ explicitly in the remainder of this paper.

Finally, let $C(\mathbb{R})$ be the space of all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f_{|[-T,T]} \in C[-T,T]$ for all $T>0$. The above definitions extend in an obvious way to $C[0,T]$, $C[0,\infty)$ and convergence as $c\to 0$.

For $\gamma \geq 0$, let Ω^γ be the space of all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that $\lim_{t\to\pm\infty} f(t)/(1+|t|^\gamma) = 0$. Equip Ω^γ with the topology generated by the norm $\|f\|_{\Omega^\gamma} := \sup_{t\in\mathbb{R}} |f(t)|/(1+|t|^\gamma)$ under which Ω^γ is a separable Banach space, so that Prokhorov's theorem applies. The following property can be found in [6, Lemma 3] or [7, Lemma 4].

Proposition 1 *Let a family of random elements $\{X^{(c)}\}$ on Ω^γ be given. Suppose that the image of $\{X^{(c)}\}$ under the projection mapping $p_T:\Omega^\gamma \to C[-T,T]$ is tight in $C[-T,T]$ for all $T>0$. Then $\{X^{(c)}\}$ is tight in Ω^γ if and only if for any $\eta>0$,*

$$\lim_{T\to\infty} \limsup_{c\to\infty} \mathbb{P}\bigg(\sup_{|t|\geq T} \frac{|X^{(c)}(t)|}{1+|t|^\gamma} \geq \eta \bigg) = 0. \tag{5}$$

2.2 Fluid queues

Let $Q_X^{(c)} \equiv \{Q_X^{(c)}(t) : t \geq 0\}$ denote a stationary buffer content process for a fluid queue fed by a centered Gaussian process $X \equiv \{X(t) : t\in\mathbb{R}\}$ with stationary increments, $X(0)=0$, continuous sample paths and variance function $\sigma^2(\cdot)$. The system is drained at a constant rate $c>0$, so that for any $t\geq 0$,

$$Q_X^{(c)}(t) = \sup_{-\infty<s\leq t} \big(X(t) - X(s) - c(t-s)\big).$$

Additionally, an equivalent representation for $Q_X^{(c)}(t)$ holds [16, p. 375]:

$$Q_X^{(c)}(t) = Q_X^{(c)}(0) + X(t) - ct + \max\Big(0, \sup_{0<s<t} \big(-Q_X^{(c)}(0) - \big(X(s) - cs\big)\big)\Big). \tag{6}$$

Throughout the paper we say that X satisfies:

C: if $\sigma^2(t)|\log|t||^{1+\varepsilon}$ has a finite limit as $t \to 0$, for some $\varepsilon > 0$;
RV$_0$: if $\sigma \in \mathscr{RV}_0(\lambda)$, for $\lambda \in (0,1)$;
RV$_\infty$: if $\sigma \in \mathscr{RV}_\infty(\alpha)$, for $\alpha \in (0,1)$;
HT: if both **C** and **RV$_\infty$** are satisfied.
LT: if both **RV$_0$** and **RV$_\infty$** are satisfied.

Remark 1 In our setting (X has stationary increments), the assumption that X is continuous is equivalent to the convergence of *Dudley's integral*; see Sect. 2.3. This is immediately implied by condition **C**; see [1, Theorem 1.4]. However, the real importance of condition **C** lies in the fact that if in addition X satisfies **RV$_\infty$**, then X also belongs to Ω^γ, for every $\gamma > \alpha$. This is pointed out in Sect. 3.1. Finally, note that **C** is met under **RV$_0$**. Indeed, since $\sigma \in \mathscr{RV}_0(\lambda)$, then $t \mapsto \sigma(1/t)$ belongs to $\mathscr{RV}_\infty(-\lambda)$, thus $\sigma^2(1/t)t^\lambda \to 0$ as $t \to \infty$. Equivalently, $\sigma^2(t)t^{-\lambda} \to 0$ as $t \to 0$, implying $\lim_{t\to 0}\sigma^2(t)|\log|t||^{1+\varepsilon} = 0$, for any fixed $\varepsilon > 0$. Furthermore, **RV$_\infty$** implies that $X(t)/t \to 0$ a.s., for $t \to \pm\infty$, so that $Q_X^{(c)}$ is a properly defined stochastic process for any $c > 0$; see [7, Lemma 3]. Lastly, the assumption that X has continuous sample paths implies that σ is continuous.

Due to the stationarity of increments, all finite-dimensional distributions of X are specified by the variance function, since we have

$$\mathbb{C}\mathrm{ov}\big(X(t), X(s)\big) = \frac{1}{2}\big(\sigma^2(s) + \sigma^2(t) - \sigma^2\big(|t-s|\big)\big). \tag{7}$$

Recall that by $B_H \equiv \{B_H(t) : t \in \mathbb{R}\}$ we denote fractional Brownian motion with Hurst parameter $H \in (0,1)$, that is, a centered Gaussian process with stationary increments, continuous sample paths, $B_H(0) = 0$ and covariance function

$$\mathbb{C}\mathrm{ov}\big(B_H(t), B_H(s)\big) = \frac{1}{2}\big(|s|^{2H} + |t|^{2H} - |t-s|^{2H}\big). \tag{8}$$

As mentioned in the introduction, if $c \to 0$, then, for any t, $Q_X^{(c)}(t) \to \infty$ a.s., which is called the *heavy-traffic regime*. On the other hand, if $c \to \infty$, then $Q_X^{(c)}(t) \to 0$ a.s., which is called the *light-traffic regime*.

2.3 Metric entropy

For any $\mathbb{T} \subset \mathbb{R}$ define the *semimetric*

$$d(t,s) := \sqrt{\mathbb{E}\big|X(t) - X(s)\big|^2} = \sigma\big(|t-s|\big), \quad t, s \in \mathbb{T}.$$

We say that $S \subset \mathbb{T}$ is a ϑ-net in $\mathbb{T}$ with respect to the semimetric d, if for any $t \in \mathbb{T}$ there exists an $s \in S$ such that $d(t,s) \le \vartheta$. The metric entropy $\mathbb{H}_d(\mathbb{T}, \vartheta)$ is defined as $\log \mathbb{N}_d(\mathbb{T}, \vartheta)$, where $\mathbb{N}_d(\mathbb{T}, \vartheta)$ denotes the minimal number of points in a ϑ-net in $\mathbb{T}$ with respect to d. Later on we use the following proposition; see [2, Theorem 1.3.3] and [2, Corollary 1.3.4], respectively.

Proposition 2 *There exists a universal constant K such that for a d-compact set $\mathbb{T}$*

$$\mathbb{E}\Big(\sup_{t\in\mathbb{T}} X(t)\Big) \le K \int_0^{\mathrm{diam}(\mathbb{T})/2} \sqrt{\mathbb{H}_d(\mathbb{T},\vartheta)}\,\mathrm{d}\vartheta$$

and for all $\zeta > 0$

$$\mathbb{E}\Bigg(\sup_{\substack{(s,t)\in\mathbb{T}\times\mathbb{T}\\ d(s,t)<\zeta}} \big|X(t)-X(s)\big|\Bigg) \le K \int_0^{\zeta} \sqrt{\mathbb{H}_d(\mathbb{T},\vartheta)}\,\mathrm{d}\vartheta.$$

The quantity $\int_0^\infty \sqrt{\mathbb{H}_d(\mathbb{T},\vartheta)}\,\mathrm{d}\vartheta$ is called the Dudley integral.

3 Main results

In this section we formulate the result for the heavy-traffic and light-traffic regime, respectively. It is emphasized that these results are highly symmetric. Let us first introduce a function δ, such that for every $c > 0$

$$\frac{c\delta(c)}{\sigma(\delta(c))} = 1. \tag{9}$$

By the continuity of σ, we can choose δ as $\delta(c) = \inf\{x > 0 : x/\sigma(x) = 1/c\}$. From the definition of δ it follows that $\delta \in \mathscr{RV}_0(1/(\alpha-1))$ under $\mathbf{RV}_\infty$ and $\delta \in \mathscr{RV}_\infty(1/(\lambda-1))$ under $\mathbf{RV}_0$.

3.1 Heavy-traffic regime

In the heavy-traffic regime we are interested in the analysis of $Q_X^{(c)}$ as $c \to 0$, under the assumption that X satisfies **HT**. The following statement follows from [7, Theorems 5 and 6].

Proposition 3 *If X satisfies* **HT**, *then*

$$\frac{X(\delta(c)\cdot)}{\sigma(\delta(c))} \xrightarrow{d} B_\alpha(\cdot), \quad \text{as } c \to 0,$$

in $C(\mathbb{R})$ and Ω^γ, for any $\gamma > \alpha$.

In fact, Theorem 3 holds for any function $\delta(c)$ such that $\delta(c) \to \infty$ as $c \to 0$. Condition **C** (which is one of the requirements of **HT**) plays a crucial role in proving tightness both in $C[-T, T]$, for some $T > 0$, and in Ω^γ.

Combining Theorem 3 with the definition of δ leads to the following statement.

Corollary 1 *If X satisfies* **HT**, *then*

$$\frac{X(\delta(c)\cdot) - c\delta(c)\,\mathrm{id}(\cdot)}{\sigma(\delta(c))} \xrightarrow{d} B_\alpha(\cdot) - \mathrm{id}(\cdot), \quad \text{as } c \to 0,$$

in $C(\mathbb{R})$.

Now we are in the position to present the main result of this subsection.

Theorem 1 *If X satisfies* **HT**, *then*

$$\frac{Q_X^{(c)}(\delta(c)\cdot)}{\sigma(\delta(c))} \xrightarrow{d} Q_{B_\alpha}^{(1)}(\cdot), \quad \text{as } c \to 0, \tag{10}$$

in $C[0,\infty)$.

We postpone the proof of Theorem 1 to Sect. 4.

Remark 2 Theorem 1 extends the findings of [8, Theorem 3.2] where, under the heavy-traffic regime, the weak convergence in $C[0,\infty)$ of $Q_X^{(c)}(\delta(c)\cdot)/\sigma(\delta(c))$ as $c \to 0$ was obtained for the class of input processes having differentiable sample paths a.s., i.e., of the form $X(t) = \int_0^t Z(s)\,\mathrm{d}s$, where $\{Z(s) : s \geq 0\}$ is a stationary centered Gaussian process whose variance function satisfies specific regularity conditions.

3.2 Light-traffic regime

In the light-traffic regime we analyze the convergence of $Q_X^{(c)}$ as $c \to \infty$, under the assumption that X satisfies **LT**. We begin by stating the counterpart of Proposition 3.

Proposition 4 *If X satisfies* $\mathbf{RV}_0$, *then*

$$\frac{X(\delta(c)\cdot)}{\sigma(\delta(c))} \xrightarrow{d} B_\lambda(\cdot), \quad \text{as } c \to \infty,$$

in $C(\mathbb{R})$. *If, moreover, X satisfies* **LT**, *then the convergence also holds in* Ω^γ, *for any* $\gamma > \max\{\lambda, \alpha\}$.

Analogously to Proposition 3, Proposition 4 holds for any function $\delta(c)$ such that $\delta(c) \to 0$ as $c \to \infty$. As in the heavy-traffic case, combining Proposition 4 with the definition of δ leads to the counterpart of Corollary 1.

Corollary 2 *If X satisfies* $\mathbf{RV}_0$, *then*

$$\frac{X(\delta(c)\cdot) - c\delta(c)\,\mathrm{id}(\cdot)}{\sigma(\delta(c))} \xrightarrow{d} B_\lambda(\cdot) - \mathrm{id}(\cdot) \quad \text{as } c \to \infty,$$

in $C(\mathbb{R})$.

The main result of this subsection is now stated as follows.

Theorem 2 *If X satisfies* **LT**, *then*

$$\frac{Q_X^{(c)}(\delta(c)\cdot)}{\sigma(\delta(c))} \xrightarrow{d} Q_{B_\lambda}^{(1)}(\cdot) \quad \text{as } c \to \infty, \tag{11}$$

in $C[0,\infty)$.

We postpone the proof of Theorem 4 and Theorem 2 to Sect. 4.

Remark 3 The assumption **LT** excludes the class of input processes of the structure $X(t) = \int_0^t Z(s)\,\mathrm{d}s$, with $\{Z(s) : s \geq 0\}$ being a centered stationary Gaussian process with continuous sample paths a.s. (since $\lambda = 1$ in this case). In [8, Theorem 4.1] it was shown that, for this class of Gaussian processes, $Q_X^{(c)}(0)/\sigma(\delta(c))$ does *not* converge weakly to $Q_{B_\lambda}^{(1)}(0)$ as $c \to \infty$.

4 Proofs

In this section we prove our results, but we start by presenting an auxiliary result.

Lemma 1 *If X satisfies* **LT**, *then for any $\epsilon > 0$, there exist constants $C, a > 0$, such that for all $x \leq a$ and $t > 0$,*

$$\frac{\sigma(tx)}{\sigma(x)} \leq C \times \begin{cases} t^\ell & t \leq 1, \\ t^u & t > 1, \end{cases}$$

where $\ell := \min\{\lambda - \epsilon, \alpha + \epsilon\}$ and $u := \max\{\alpha + \epsilon, \lambda + \epsilon\}$.

Proof Take any $\epsilon > 0$, then because $\sigma \in \mathscr{RV}_0(\lambda)$, there exists an $a \leq 1$ such that

$$\frac{\sigma(tx)}{\sigma(x)} \leq 2t^{\lambda-\epsilon}, \quad \text{for all } x \leq a \text{ and } tx \leq a. \tag{12}$$

Moreover, there exists a constant K_1 such that $\sigma(x) \geq K_1 x^{\lambda+\epsilon}$ for all $x \leq a$.

Because $\sigma \in \mathscr{RV}_\infty(\alpha)$, there exist constants $A, K_2 > 0$ such that $\sigma(x) \leq K_2 x^{\alpha+\epsilon}$ for all $x \geq A$. Because σ is continuous, we can in fact find a K_2 such that $\sigma(x) \leq K_2 x^{\alpha+\epsilon}$ for all $x \geq a$. Therefore

$$\frac{\sigma(tx)}{\sigma(x)} \leq \frac{K_2(tx)^{\alpha+\epsilon}}{K_1 x^{\lambda+\epsilon}} =: K t^{\alpha+\epsilon} x^{\alpha-\lambda}, \quad \text{for all } x \leq a \text{ and } tx \geq a.$$

Note that, if $\alpha - \lambda \geq 0$, then we have

$$\frac{\sigma(tx)}{\sigma(x)} \leq K a^{\alpha+\epsilon} t^{\alpha+\epsilon}, \quad \text{for all } x \leq a \text{ and } tx \geq a. \tag{13}$$

If $\alpha - \lambda < 0$, then

$$\frac{\sigma(tx)}{\sigma(x)} \leq K a^{\alpha-\lambda} t^{\lambda+\epsilon}, \quad \text{for all } x \leq a \text{ and } tx \geq a. \tag{14}$$

Combining (12)–(14), we conclude that there exists a constant $C > 0$, such that

$$\frac{\sigma(tx)}{\sigma(x)} \leq C \max\{t^{\lambda-\epsilon}, t^{\alpha+\epsilon}, t^{\lambda+\epsilon}\}, \quad \text{for all } x \leq a \text{ and all } t > 0.$$ □

In what follows, we will use the following notation. Let

$$X^{(c)}(t) := \frac{X(\delta(c)t)}{\sigma(\delta(c))}$$

and denote the variance of $X^{(c)}$ by $(\sigma^{(c)})^2$, that is,

$$\sigma^{(c)}(t) := \frac{\sigma(\delta(c)t)}{\sigma(\delta(c))}.$$

Proof of Theorem 4 We begin by showing the convergence in $C(\mathbb{R})$. To this end, we need to show the convergence in $C[-T,T]$ for any fixed $T>0$.

Convergence in $C[-T,T]$: From the fact that $\sigma \in \mathscr{RV}_0(\lambda)$, it is immediate that the finite-dimensional distributions of $X^{(c)}$ converge in distribution to B_λ as $c\to\infty$, cf. (7)–(8), which also implies (3). Therefore, the weak convergence of $X^{(c)}$ in $C[-T,T]$ follows after showing (4).

By the Uniform Convergence Theorem, see [4, Theorem 1.5.2], for any $t\in(0,\zeta]$, we have $\sigma^{(c)}(t)\le 2\zeta^\lambda$. Thus, Proposition 2 yields, for some universal constant $K>0$,

$$\begin{aligned}\mathbb{P}\Big(\sup_{|s-t|\le\zeta}\big|X^{(c)}(t)-X^{(c)}(s)\big|\ge\eta\Big) &\le \mathbb{P}\Big(\sup_{\sigma^{(c)}(|s-t|)\le 2\zeta^\lambda}\big|X^{(c)}(t)-X^{(c)}(s)\big|\ge\eta\Big)\\ &\le \frac{1}{\eta}\mathbb{E}\Big(\sup_{\sigma^{(c)}(|s-t|)\le 2\zeta^\lambda}\big|X^{(c)}(t)-X^{(c)}(s)\big|\Big)\\ &\le \frac{K}{\eta}\int_0^{2\zeta^\lambda}\sqrt{\mathbb{H}^{(c)}\big([-T,T],\vartheta\big)}\,\mathrm{d}\vartheta,\end{aligned}$$

where $\mathbb{H}^{(c)}([-T,T],\cdot)$ is the metric entropy induced by $\sigma^{(c)}$.

By Potter's bound [4, Theorem 1.5.6] for any $\epsilon,\zeta>0$, $\epsilon<\lambda$ and $t\in(0,\zeta]$ and sufficiently large c (corresponding to small $\delta(c)$), we have $\sigma^{(c)}(t)\le 2t^{\lambda-\epsilon}$. Hence

$$\mathbb{H}^{(c)}\big([-T,T],\vartheta\big)\le\mathbb{H}_{\tilde d}\Big([-T,T],\frac{\vartheta}{2}\Big),$$

where $\tilde d$ is a semimetric such that $\tilde d(s,t)=|t-s|^{\lambda-\epsilon}$. The inverse of $x\mapsto x^{\lambda-\epsilon}$ is given by $x\mapsto x^{1/(\lambda-\epsilon)}$, so that

$$\mathbb{H}_{\tilde d}\big([-T,T],\vartheta\big)\le\log\Big(\frac{T}{\vartheta^{1/(\lambda-\epsilon)}}+1\Big)\le C\log\Big(\frac{1}{\vartheta}\Big),$$

for some constant $C>0$ and $\vartheta>0$ small. It follows that

$$\int_0^{2\zeta^\lambda}\sqrt{\mathbb{H}^{(c)}\big([-T,T],\vartheta\big)}\,\mathrm{d}\vartheta\le\sqrt{C}\int_0^{2\zeta^\lambda}\sqrt{\log\Big(\frac{2}{\vartheta}\Big)}\,\mathrm{d}\vartheta=2\sqrt{C}\int_{\zeta^{-\lambda}}^\infty\frac{\sqrt{\log\vartheta}}{\vartheta^2}\,\mathrm{d}\vartheta.$$

Summarizing, we have

$$\limsup_{c\to\infty}\mathbb{P}\Big(\sup_{|s-t|\le\zeta}\big|X^{(c)}(t)-X^{(c)}(s)\big|\ge\eta\Big)\le\frac{2K\sqrt{C}}{\eta}\int_{\zeta^{-\lambda}}^\infty\frac{\sqrt{\log\vartheta}}{\vartheta^2}\,\mathrm{d}\vartheta;$$

we obtain (4) by letting $\zeta\to 0$.

Convergence in Ω^γ: To show the convergence in Ω^γ, we need to verify (5). Observe that

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\geq e^k}\frac{|X^{(c)}(t)|}{1+t^\gamma}\geq\eta\right)\\
&\quad\leq\frac{1}{\eta}\sum_{j=k}^{\infty}\frac{\mathbb{E}\sup_{t\in[e^j,e^{j+1}]}|X^{(c)}(t)|}{1+e^{j\gamma}}\\
&\quad\leq\frac{1}{\eta}\sum_{j=k}^{\infty}\frac{\mathbb{E}|X^{(c)}(e^j)|}{1+e^{j\gamma}}+\frac{2}{\eta}\sum_{j=k}^{\infty}\frac{\mathbb{E}\sup_{t\in[e^j,e^{j+1}]}X^{(c)}(t)}{1+e^{j\gamma}}\\
&\quad=:I_1(k)+I_2(k).
\end{aligned}$$

$I_1(k)$ and $I_2(k)$ are dealt with separately. According to Lemma 1, for large c (that is, small $\delta(c)$), we have

$$\sigma^{(c)}(t)\leq C\times\begin{cases}t^\ell & t\leq 1,\\ t^u & t>1,\end{cases}$$

where ℓ and u can be chosen such that $\ell,u<\gamma$. Therefore,

$$I_1(k)\leq\frac{1}{\eta}\sum_{j=k}^{\infty}\frac{\sigma^{(c)}(e^j)}{1+e^{j\gamma}}\leq\frac{C}{\eta}\sum_{j=k}^{\infty}\frac{e^{ju}}{1+e^{j\gamma}},$$

and the resulting upper bound tends to zero as $k\to\infty$.

Now focus on $I_2(k)$. For some universal constant $K>0$ and because of the stationarity of the increments of X, Proposition 2 yields that $I_2(k)$ is majorized by

$$\frac{2K}{\eta}\sum_{j=k}^{\infty}\frac{\int_0^\infty\sqrt{\mathbb{H}^{(c)}([e^j,e^{j+1}],\vartheta)}\,\mathrm{d}\vartheta}{1+e^{j\gamma}}=\frac{2K}{\eta}\sum_{j=k}^{\infty}\frac{\int_0^\infty\sqrt{\mathbb{H}^{(c)}([0,e^j(e-1)],\vartheta)}\,\mathrm{d}\vartheta}{1+e^{j\gamma}}.$$

We will estimate the integrals under the sum by splitting the integration area into $\vartheta\leq 1$ and $\vartheta\geq 1$.

Observe that, for some constants $C_1,C_2>0$ (that is, not depending on j),

$$\begin{aligned}
\int_0^1\sqrt{\mathbb{H}^{(c)}\big([0,e^j(e-1)],\vartheta\big)}\,\mathrm{d}\vartheta&\leq\int_0^1\sqrt{\log\left(\frac{e^j(e-1)}{2\vartheta^{1/\ell}}+1\right)}\,\mathrm{d}\vartheta\\
&\leq\int_0^1\sqrt{C_1+j+\frac{1}{\ell}\log\left(\frac{1}{\vartheta}\right)}\,\mathrm{d}\vartheta\\
&=\ell e^{\ell(C_1+j)}\int_{C_1+j}^{\infty}\sqrt{\vartheta}e^{-\ell\vartheta}\,\mathrm{d}\vartheta\\
&\leq\ell e^{\ell(C_1+j)}\int_0^\infty\sqrt{\vartheta}e^{-\ell\vartheta}\,\mathrm{d}\vartheta=C_2e^{\ell j}.
\end{aligned}$$

Recall that $\ell < \gamma$, so that

$$\lim_{k\to\infty}\sum_{j=k}^{\infty}\frac{\int_0^1\sqrt{\mathbb{H}^{(c)}([0,e^j(e-1)],\vartheta)}\,\mathrm{d}\vartheta}{1+e^{j\gamma}} \le \frac{2K}{\eta}\lim_{k\to\infty}\sum_{j=k}^{\infty}\frac{C_2e^{\ell j}}{1+e^{j\gamma}} = 0.$$

So it remains to show the analogous statement for the integration interval $[1,\infty)$. Using a similar argumentation as the one above, one can show that

$$\int_1^{\infty}\sqrt{\mathbb{H}^{(c)}\big(\big[0,e^j(e-1)\big],\vartheta\big)}\,\mathrm{d}\vartheta \le C_3e^{uj},$$

for some constant $C_3 > 0$, from which the claim is readily obtained. □

Since the proof of Theorem 1 is analogous to the proof of Theorem 2, we choose to focus on the light-traffic case only.

Proof of Theorem 2 The proof consists of three steps: convergence of the one-dimensional distributions, the finite-dimensional distributions, and a tightness argument.

Step 1: Convergence of one-dimensional distributions. In this step we show that, for a fixed $t \ge 0$,

$$\frac{Q_X^{(c)}(t)}{\sigma(\delta(c))} \xrightarrow{d} Q_{B_\lambda}^{(1)}(t), \quad \text{as } c\to\infty.$$

Since $Q_X^{(c)}$ is stationary, it is enough to show the above convergence for $t = 0$ only. Observe that, due to the time-reversibility property of Gaussian processes,

$$Q_X^{(c)}(0) \stackrel{d}{=} \sup_{t\ge0}\big(X(t)-ct\big) = \sup_{t\ge0}\big(X\big(\delta(c)t\big)-c\delta(c)t\big).$$

Upon combining Corollary 2 with the continuous mapping theorem, for each $T > 0$,

$$\sup_{t\in[0,T]}\left(\frac{X(\delta(c)t)-c\delta(c)t}{\sigma(\delta(c))}\right) \xrightarrow{d} \sup_{t\in[0,T]}\big(B_\lambda(t)-t\big), \quad \text{as } c\to\infty.$$

Thus it suffices to show that

$$\lim_{T\to\infty}\limsup_{c\to\infty}\mathbb{P}\left(\sup_{t\ge T}\left(\frac{X(\delta(c)t)-c\delta(c)t}{\sigma(\delta(c))}\right)\ge\eta\right) = 0, \tag{15}$$

for any $\eta > 0$. Recall the definition of $X^{(c)}$, so that

$$\mathbb{P}\left(\sup_{t\ge T}\left(\frac{X(\delta(c)t)-c\delta(c)t}{\sigma(\delta(c))}\right)\ge\eta\right) \le \mathbb{P}\left(\sup_{t\ge T}\frac{|X^{(c)}(t)|}{\eta+t}\ge 1\right),$$

where we used (9). Proposition 4 implies that the family $\{X^{(c)}\}$ is tight in Ω^γ, for some $\gamma \le 1$. Now (15) follows from Proposition 1.

Step 2: Convergence of finite-dimensional distributions. The argumentation of this step is analogous to Step 1. First note that for any $t_i \geq 0$, $\eta_i > 0$ and $s_i < t_i$, where $i = 1, \dots, n$, for any $n \in \mathbb{N}$, it follows that

$$\begin{aligned}
&\mathbb{P}\left(\frac{Q_X^{(c)}(\delta(c)t_i)}{\sigma(\delta(c))} > \eta_i,\, i = 1, \dots, n\right) \\
&\quad = \mathbb{P}\left(\sup_{s \leq \delta(c)t_i}\left(\frac{X(\delta(c)t_i) - X(s) - c(\delta(c)t_i - s)}{\sigma(\delta(c))}\right) > \eta_i,\, i = 1, \dots, n\right) \\
&\quad \leq \mathbb{P}\left(\sup_{s \in [s_i, t_i]}\left(\frac{X(\delta(c)t_i) - X(\delta(c)s) - c\delta(c)(t_i - s)}{\sigma(\delta(c))}\right) > \eta_i, i = 1, \dots, n\right) \\
&\qquad + \sum_{i=1}^{n} \mathbb{P}\left(\sup_{s \leq s_i}\left(\frac{X(\delta(c)t_i) - X(\delta(c)s) - c\delta(c)(t_i - s)}{\sigma(\delta(c))}\right) > \eta_i\right).
\end{aligned}$$

Now the same procedure can be followed as in Step 1.

Step 3: Tightness in $C[0, T]$. In this step, for any $T > 0$, we show the tightness of $\{Q_X^{(c)}(\delta(c)\cdot)/\sigma(\delta(c))\}$ in $C[0, T]$. Given that we have established Step 2 already, (3) holds so we are left with proving (4), with $s, t \in [0, T]$; the remainder of the proof is devoted to settling this claim.

Stationarity of $Q_X^{(c)}$ implies that $\{Q_X^{(c)}(\delta(c)t) - Q_X^{(c)}(\delta(c)s) : t \geq s\}$ is distributed as

$$\left\{Q_X^{(c)}\big(\delta(c)(t - s)\big) - Q_X^{(c)}(0) : t \geq s\right\},$$

so that it suffices to prove (4) for $s = 0$ only. Furthermore, cf. (6),

$$\sup_{0 < t \leq \zeta}\left|Q_X^{(c)}\big(\delta(c)t\big) - Q_X^{(c)}(0)\right| \leq 2 \sup_{0 < t \leq \zeta}\left|X\big(\delta(c)t\big) - c\delta(c)t\right|.$$

From Corollary 2 it follows that

$$\sup_{0 < t \leq \zeta}\frac{|X(\delta(c)t) - c\delta(c)t|}{\sigma(\delta(c))} \xrightarrow{d} \sup_{0 < t \leq \zeta}\left|B_\lambda(t) - t\right|, \quad \text{as } c \to \infty.$$

Now notice that for $\zeta < \eta/4$, by the self-similarity of B_λ,

$$\mathbb{P}\left(\sup_{0 < t \leq \zeta}\left|B_\lambda(t) - t\right| \geq \frac{\eta}{2}\right) \leq 2\mathbb{P}\left(\sup_{0 < t \leq 1} B_\lambda(t) \geq \frac{\eta}{4}\zeta^{-\lambda}\right).$$

Now it is straightforward to conclude that the last expression tends to zero as $\zeta \to 0$. □

Acknowledgements K. Dębicki was supported by MNiSW Grant N N201 394137 (2009–2011) and by a travel grant from NWO (Mathematics Cluster STAR). K.M. Kosiński was supported by NWO grant 613.000.701.

References

1. Adler, R.J.: An Introduction to Continuity, Extrema, and Related Topics for General Gaussian Processes. Lecture Notes-Monograph Series, vol. 12 (1990). IMS
2. Adler, R.J., Taylor, J.E.: Random Fields and Geometry. Springer, Berlin (2007)
3. Billingsley, P.: Convergence of Probability Measures, 2nd edn. Wiley, New York (1999)
4. Bingham, N.H., Goldie, C.M., Teugels, J.L.: Regular Variation. Cambridge University Press, Cambridge (1987)
5. Boxma, O.J., Cohen, J.W.: Heavy-traffic analysis for the $GI/G/1$ queue with heavy-tailed distributions. Queueing Syst. **33**, 177–204 (1999)
6. Buldygin, V.V., Zaiats, V.V.: A global asymptotic normality of the sample correlogram of a stationary Gaussian process. Random Oper. Stoch. Equ. **7**, 109–132 (1999)
7. Dieker, A.B.: Conditional limit theorems for queues with Gaussian input, a weak convergence approach. Stoch. Process. Appl. **115**, 849–873 (2005)
8. Dębicki, K., Mandjes, M.: Traffic with an fBm limit: Convergence of the stationary workload process. Queueing Syst. **46**, 113–127 (2004)
9. Kingman, J.F.C.: The single server queue in heavy traffic. Proc. Camb. Philol. Soc. **57**, 902–904 (1961)
10. Mandjes, M.: Large Deviations for Gaussian Queues. Wiley, Chichester (2007)
11. Mannersalo, P., Norros, I.: A most probable path approach to queueing systems with general Gaussian input. Comput. Netw. **40**, 399–412 (2002)
12. Norros, I.: A storage model with self-similar input. Queueing Syst. **16**, 387–396 (2004)
13. Prokhorov, Ju.V.: Transition phenomena in queueing processes. I. Liet. Mat. Rink. **9**, 199–205 (1963)
14. Reich, E.: On the integrodifferential equation of Takács I. Ann. Math. Stat. **29**, 563–570 (1958)
15. Resnick, S., Samorodnitsky, G.: A heavy traffic limit theorem for workload processes with heavy tailed service requirements. Manag. Sci. **46**, 1236–1248 (2000)
16. Robert, P.: Stochastic Networks and Queues. Springer, Berlin (2003)
17. Szczotka, T., Woyczyński, W.A.: Heavy-tailed dependent queues in heavy traffic. Probab. Math. Stat. **24**, 67–96 (2004)
18. Taqqu, M.S., Willinger, W., Sherman, R.: Proof of a fundamental result in self-similar traffic modeling. Comput. Commun. Rev. **27**, 5–23 (1997)
19. Whitt, W.: Weak convergence theorems for priority queues. J. Appl. Probab. **8**, 74–94 (1971)