Queueing Syst (2010) 64: 383–393
DOI 10.1007/s11134-010-9167-0

Global and local asymptotics for the busy period of an M/G/1 queue

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Received: 5 January 2007 / Revised: 18 January 2010 / Published online: 17 February 2010

Abstract We consider an M/G/1 queue with subexponential service times. We give a simple derivation of the global and local asymptotics for the busy period. Our analysis relies on the explicit formula for the joint distribution for the number of customers and the length of the busy period of an M/G/1 queue.

Keywords Busy period · Busy cycle · Heavy-tailed distributions

Mathematics Subject Classification (2000) 60K25 · 60G50 · 60F10

1 Introduction

Let $A_1, A_2, \dots$ and $B_1, B_2, \dots$ be independent sequences of independent and identically distributed random variables. We call $\{A_i\}$ inter-arrival times and $\{B_i\}$ service times. It is assumed throughout that $\rho := \mathbf{E}\{B_1\}/\mathbf{E}\{A_1\} < 1$, so that the system is stable. Denote also $\xi_i = B_i - A_i$ and $S_n = \sum_{i=1}^n \xi_i$. We shall denote by A, B and ξ random variables with the same distributions as A_1, B_1 and ξ_1, respectively. We use the standard notation: we denote by M/G/1 the system with exponential inter-arrival times A_i and by GI/G/1 the system with arbitrarily distributed A_i. The main interest of our paper is the (global and local) asymptotics for the distribution of the length of the busy period

$$\tau = B_1 + \cdots + B_\nu,$$

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where $\nu = \inf\{n : S_n < 0\}$ is the number of customers which arrive (and get served) during the busy period.

In [10], it was shown that in the M/G/1 case, if B has a regularly varying distribution, then

$$\mathbf{P}\{\tau > t\} \sim \mathbf{E}\{\nu\}\mathbf{P}\big\{B > (1-\rho)t\big\} \quad \text{as } t \to \infty. \tag{1}$$

This result was generalised in [11] to the case of a GI/G/1 queue and under the assumption that the tail $\mathbf{P}\{B > t\}$ is of intermediate regular variation (see the next section for a definition). Later on, it was shown in [3] and [9] that asymptotics (1) hold for the GI/G/1 model in the case when the service-time distribution belongs to another subclass of heavy-tailed distributions. This class includes, in particular, Weibull distributions with parameter $\alpha < 1/2$. The tails of the distributions considered in [3] and [9] are heavier than $e^{-\sqrt{t}}$. As is shown in [2], the latter condition is crucial for asymptotics (1) to hold.

It was shown in [6] that in an M/G/1 queue

$$\mathbf{P}(\tau > t) \sim C\frac{\mathbf{P}(X_t > 0)}{t} \quad \text{as } t \to \infty, \tag{2}$$

where C is a constant, $X_t = \sum_{i=0}^{N(t)} B_i - t$ and N is a Poisson process with intensity $\lambda = 1/\mathbf{E}A$. It was also shown in [6] that in the case when the tail of B is heavier than $e^{-\sqrt{t}}$ (and under some further conditions) the asymptotic equivalence (2) reduces to (1).

In this note, we concentrate on the M/G/1 case and have two main goals. The first (and the more important) consists in providing new results on local asymptotics for τ. These results are (not surprisingly) similar to (2):

$$\mathbf{P}\big(\tau \in (t, t+T]\big) \sim \frac{\mathbf{P}(X_t \in (0, T])}{\lambda t} \quad \text{as } t \to \infty,$$

with a fixed $T \in (0, \infty)$.

Another goal of this paper is to give a simple proof of (1) for the M/G/1 case for some classes of distributions (of service times) with tails heavier than $e^{-\sqrt{t}}$. However, our contribution here does not consist only in providing a shorter proof; we believe that our assumptions on the distribution of the service times are close to minimal. In particular, they are satisfied by all the distributions for which relation (1) has been proved earlier.

Both aims are achieved by using the explicit formula for the joint distribution of τ and ν (see e.g. [5, (4.63)]):

$$\mathbf{P}(\nu = n, \tau > t) = \int_t^\infty \frac{(\lambda u)^{n-1}}{n!} e^{-\lambda u}\mathbf{P}(B_1 + \cdots + B_n \in du). \tag{3}$$

The paper is organised as follows. In Sect. 2, we give results on local asymptotics for τ, and in Sect. 3 we present a simple proof of (1) for the M/G/1 queue.

2 Local asymptotics for the busy period

In this section, we shall present our main results on local asymptotics for τ. Let $\Delta = (0, T]$ and recall that $S_n = \sum_{i=1}^{n}(B_i - A_i)$ and $X_t = \sum_{i=1}^{N(t)} B_i - t$.

We will start with the following simple result.

Theorem 2.1 *Assume that the random variable B is absolutely continuous. Then*

$$\mathbf{P}\{\nu = n\} = \frac{f_{S_n}(0)}{\lambda n} \tag{4}$$

and

$$f_\tau(t) = \frac{f_{X_t}(0)}{\lambda t}. \tag{5}$$

Here $f_\xi(x)$ denotes the density of a random variable ξ at a point x. The next result holds for random variables that are not necessarily absolutely continuous.

Theorem 2.2 *Fix a positive and finite value of T. Assume that $\mathbf{P}\{B_1 \in t + \Delta\}e^{\delta t} \to \infty$, as $t \to \infty$, for any $\delta > 0$. Then*

$$\mathbf{P}\{\tau \in t + \Delta\} \sim \frac{\mathbf{P}\{X_t \in \Delta\}}{\lambda t} \sim \frac{\mathbf{P}\{X_{[t]} \in \Delta\}}{\lambda [t]} \quad \text{as } t \to \infty,$$

where $[t]$ denotes the integer part of t.

These asymptotics are not explicit. However, the theorem above reduces the original problem to a particular case of the large-deviations problem for random walks, which is extensively studied in the literature. It turns out that some known large-deviations results (together with the theorem above) yield explicit asymptotics for the distribution of the busy period. Some of these cases will be presented below.

First, we discuss corollaries of the above theorems and then give the proofs. We start by defining two (rather large) classes of distributions.

Definition 2.1 We say that a function $f(x)$ is intermediate regularly varying if

$$\lim_{\kappa \downarrow 1} \limsup_{x\to\infty} \sup_{x \le y \le \kappa x} \left| \frac{f(y+\Delta)}{f(x+\Delta)} - 1 \right| = 0. \tag{6}$$

In particular, (6) holds when $f(x)$ is regularly varying at infinity.

Let $F(x) = \mathbf{P}(B \le x)$ be the distribution function of B and let $F(x + \Delta) = F(x + T) - F(x)$. We will use the following conditions:

(A) $\mathbf{E}B < \infty$ and $F(x + \Delta)$ is intermediate regularly varying.

(B) $\mathbf{E}B^2 < \infty$,

$$\sup_{y \le \sqrt{x}} \left| \frac{F(x - y + \Delta)}{F(x + \Delta)} - 1 \right| \to 0$$

and

$$\varepsilon_\Delta(n) \equiv \sup_{x \ge 2\sqrt{n}} \frac{\mathbf{P}\{B_1 > \sqrt{n}, B_2 > \sqrt{n}, B_1 + B_2 \in x + \Delta\}}{\mathbf{P}(B \in x + \Delta)} = o(1/n) \quad \text{as } n \to \infty.$$

Sufficient conditions for (B) to hold may be derived using, for example, Lemma B.1 and Lemma B.2 of [7]. In particular, one can see that $F(x+\Delta) \sim e^{-x^\beta}$, $\beta < 1/2$, as well as $F(x+\Delta) \sim e^{-\ln^\beta x}$, $\beta > 1$, satisfy Condition (B). The following proposition directly follows from Corollary 2.1 of [7].

Proposition 2.1 *Let either Condition* (A) *or* (B) *hold. Then*

$$\mathbf{P}\{S_n \in \Delta\} \sim n\mathbf{P}\bigl\{\xi \in n|\mathbf{E}\xi| + \Delta\bigr\} \quad \text{as } n \to \infty$$

and hence (*since* $N(t)$ *is a Poisson process*)

$$\mathbf{P}\{X_{[t]} \in \Delta\} \sim [t]\mathbf{P}\bigl\{X_1 \in [t]|\mathbf{E}X_1| + \Delta\bigr\} \quad \text{as } t \to \infty.$$

For densities, Conditions (A) and (B) can be reformulated as follows:

(A′) $\mathbf{E}B < \infty$ and $f_B(x)$ is intermediate regularly varying.
(B′) $\mathbf{E}B^2 < \infty$,

$$\sup_{y \le \sqrt{x}} \left| \frac{f(x-y)}{f(x)} - 1 \right| \to 0$$

and

$$\varepsilon(n) = \sup_{x \ge 2\sqrt{n}} \frac{\int_{\sqrt{n}}^{x-\sqrt{n}} f(x-y) f(y)\, dy}{f(x)} = o(1/n) \quad \text{as } n \to \infty.$$

Proposition 2.1, Theorems 2.1 and 2.2 immediately allow us to obtain the local asymptotics.

Corollary 2.1 *Let either Condition* (A) *or* (B) *hold for some* $T > 0$. *Then*

$$\mathbf{P}\{\tau \in t + \Delta\} \sim \mathbf{P}(B \in t - \rho t + \Delta) \quad \text{as } t \to \infty.$$

A similar result holds for densities.

Corollary 2.2 *Assume that* B *is absolutely continuous and let one of Conditions* (A′) *or* (B′) *hold. Then*

$$f_\tau(t) \sim f(t - \rho t).$$

Now we give proofs of the statements presented in this section.

Proof of Theorem 2.1 Put $a = \lambda^{-1} = \mathbf{E}A, b = \mathbf{E}B$. Then

$$\mathbf{P}\{\nu = n\} = \int_0^\infty \frac{(\lambda u)^{n-1}}{n!} e^{-\lambda u} f_B^{*n}(u)\,du$$
$$= \lambda^{-1} \int_0^\infty \frac{f_A^{*n}(u)}{n} f_B^{*n}(u)\,du = \lambda^{-1} \frac{f_{B-A}^{*n}(0)}{n}.$$

This implies the first statement of the theorem.

We proceed to prove the second statement:

$$f_\tau(t) = \sum_{n=1}^\infty \frac{(\lambda t)^{n-1}}{n!} e^{-\lambda t} f_B^{*n}(t) = \frac{1}{\lambda t} \sum_{n=1}^\infty \mathbf{P}\big\{N(t) = n\big\} f_B^{*n}(t)$$
$$= \frac{1}{\lambda t} \sum_{n=1}^\infty \mathbf{P}\big\{N(t) = n\big\} \frac{\mathbf{P}\{B_1 + \cdots + B_n \in t + du\}}{du}$$
$$= \frac{1}{\lambda t} \frac{\mathbf{P}\{\sum_{i=1}^{N(t)} B_i \in t + du\}}{du} = \frac{f_{X_t}(0)}{\lambda t}. \qquad \square$$

Proof of Theorem 2.2 We start by showing that the assumptions of the theorem imply that $\mathbf{P}\{X_t \in \Delta\} e^{\delta t} \to \infty$ for any $\delta > 0$.

Indeed, fix a value of $\delta > 0$ and assume that $\mathbf{P}\{X_1 \in t + \Delta\} \geq e^{-\delta t}$ for sufficiently large t (which is guaranteed by the assumptions of the theorem). Let $a = -\mathbf{E}X_1 > 0$. Fix also $\varepsilon > 0$ and write

$$\mathbf{P}\{X_t \in \Delta\} \geq \mathbf{P}\big\{X_{t-1} \in \big((-a-\varepsilon)t, (-a+\varepsilon)t\big], X_{t-1} + (X_t - X_{t-1}) \in \Delta\big\}$$
$$= \int_{(-a-\varepsilon)t}^{(-a+\varepsilon)t} \mathbf{P}\{X_{t-1} \in du\} \mathbf{P}\{X_1 \in -u + \Delta\}$$
$$\geq \int_{(-a-\varepsilon)t}^{(-a+\varepsilon)t} \mathbf{P}\{X_{t-1} \in du\} e^{\delta u}$$
$$\geq \mathbf{P}\big\{X_{t-1} \in \big((-a-\varepsilon)t, (-a+\varepsilon)t\big]\big\} e^{-\delta(a+\varepsilon)t}$$
$$\geq (1-\varepsilon) e^{-\delta(a+\varepsilon)t},$$

where the latter inequality follows from the Law of Large Numbers. Since $\delta > 0$ is arbitrary, this implies that $\mathbf{P}\{X_t \in \Delta\} e^{\delta t} \to \infty$ for any $\delta > 0$ as $t \to \infty$.

We will only prove the asymptotic equivalence $\mathbf{P}\{\tau \in t + \Delta\} \sim \mathbf{P}\{X_t \in \Delta\}/t$ as $t \to \infty$. The proof of the equivalence $\mathbf{P}\{\tau \in t + \Delta\} \sim \mathbf{P}\{X_{[t]} \in \Delta\}/[t]$ may be given following the same lines. Fix any $\varepsilon > 0$. According to Formula (3),

$$\mathbf{P}(\tau \in t + \Delta) = \sum_{n=1}^\infty \int_t^{t+T} \frac{(\lambda u)^{n-1}}{n!} e^{-\lambda u} \mathbf{P}(B_1 + \cdots + B_n \in du)$$
$$= \sum_{n=1}^\infty \int_t^{t+T} \frac{\mathbf{P}(N(u) = n)}{\lambda u} \mathbf{P}(B_1 + \cdots + B_n \in du)$$

$$= \sum_{1\le n\le \lambda t-\varepsilon t} + \sum_{\lambda t-\varepsilon t<n\le \lambda t+\varepsilon t} + \sum_{n>\lambda t+\varepsilon t} (\cdots) \equiv S_1 + S_2 + S_3.$$

For the first sum we have:

$$S_1 \le \frac{1}{\lambda t} \sum_{1\le n\le \lambda t-\varepsilon t} \int_t^{t+T} \mathbf{P}\bigl(N(u)=n\bigr)\mathbf{P}(B_1+\cdots+B_n \in du)$$
$$\le \frac{1}{\lambda t}\mathbf{P}\bigl(N(t)\le \lambda t-\varepsilon t\bigr) \sum_{1\le n\le \lambda t-\varepsilon t} \mathbf{P}\{B_1+\cdots+B_n \in t+\Delta\} = o\bigl(e^{-\delta t}\bigr)$$

with some $\delta > 0$ as $t \to \infty$, since $N(t)$ is Poisson distributed with parameter λt. Indeed, it follows immediately from the Chernov bound that for some $\delta > 0$

$$\mathbf{P}\bigl\{\bigl|N(t)-\lambda t\bigr| \ge \varepsilon t\bigr\} = o\bigl(e^{-\delta t}\bigr) \quad \text{as } t \to \infty.$$

It can be shown in a similar way that $S_3 = o(e^{-\delta t})$ as $t \to \infty$. Let us now investigate the asymptotic behaviour (as $t \to \infty$) of the remaining sum:

$$S_2 = \bigl(1+o(1)\bigr)\frac{1}{\lambda t} \sum_{\lambda t-\varepsilon t<n\le \lambda t+\varepsilon t} \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$
$$\times \int_t^{t+T} e^{-\lambda(u-t)}\left(1+\frac{u-t}{t}\right)^n \mathbf{P}(B_1+\cdots+B_n \in du).$$

We now consider

$$e^{-\lambda(u-t)}\left(1+\frac{u-t}{t}\right)^n$$
$$= e^{-\lambda(u-t)+n\log(1+\frac{u-t}{t})} \le \exp\left\{-\lambda(u-t)+n\frac{u-t}{t}\right\}$$
$$\le \exp\bigl\{-\lambda(u-t)+(\lambda+\varepsilon)(u-t)\bigr\} = \exp\bigl\{\varepsilon(u-t)\bigr\} \le \exp\{\varepsilon T\}.$$

Here we used the facts that for the sum under consideration $n \le \lambda(1+\varepsilon)t$ and $u \le t+T$, and the inequality $\log(1+t) \le t$. In a similar way, with the help of the inequality $\log(1+t) \ge t/2, t \le 1$, one can prove that $e^{-\lambda(u-t)}(1+\frac{u-t}{t})^n \ge \exp\{-\varepsilon T/2\}$. Therefore,

$$S_2 \le \bigl(1+o(1)\bigr)\frac{e^{\varepsilon T}}{\lambda t}\mathbf{P}\bigl(X_t \in \Delta, \lambda t-\varepsilon t \le N(t) \le \lambda t+\varepsilon t\bigr)$$

and

$$S_2 \ge \bigl(1+o(1)\bigr)\frac{e^{-\varepsilon T/2}}{\lambda t}\mathbf{P}\bigl(X_t \in \Delta, \lambda t-\varepsilon t \le N(t) \le \lambda t+\varepsilon t\bigr).$$

Note now that

$$\mathbf{P}\bigl(X_t \in \Delta, \bigl|N(t)-\lambda t\bigr| \ge \varepsilon t\bigr) \le \mathbf{P}\bigl(\bigl|N(t)-\lambda t\bigr| \ge \varepsilon t\bigr) = o\bigl(e^{-\delta t}\bigr) \quad \text{as } t \to \infty.$$

Hence, we have the following upper and lower bounds:

$$\mathbf{P}\{\tau \in t+\Delta\} \le \big(1+o(1)\big)\frac{e^{\varepsilon T}}{\lambda t}\mathbf{P}(X_t \in \Delta) + o\big(e^{-\delta t}\big) \quad \text{as } t \to \infty$$

and

$$\mathbf{P}\{\tau \in t+\Delta\} \ge \big(1+o(1)\big)\frac{e^{-\varepsilon T/2}}{\lambda t}\mathbf{P}(X_t \in \Delta) + o\big(e^{-\delta t}\big) \quad \text{as } t \to \infty.$$

Using the condition $e^{\delta t}\mathbf{P}\{X_t \in \Delta\} \to \infty$ for any $\delta > 0$ and then letting $\varepsilon \to 0$, we arrive at the statement of the theorem. □

Proof of Corollary 2.1 It follows from Proposition 2.1 and Theorem 2.2 that

$$\mathbf{P}\{\tau \in t+\Delta\} \sim \lambda^{-1}\mathbf{P}\big\{X_1 \in [t]|\mathbf{E}X_1| + \Delta\big\} = \lambda^{-1}\mathbf{P}\big\{X_1 \in [t] - \rho[t] + \Delta\big\}$$

as $t \to \infty$, since $\mathbf{E}X_1 = EN(1)\mathbf{E}B - 1 = \rho - 1$. Now we should note that

$$\begin{aligned}
&\mathbf{P}\big\{X_1 \in [t] - \rho[t] + \Delta\big\} \\
&\quad \sim \mathbf{P}\left\{\sum_{i=1}^{N(1)} B_i \in [t] - \rho[t] + \Delta\right\} \\
&\quad \sim \mathbf{E}N(1)\mathbf{P}\big(B \in [t] - \rho[t] + \Delta\big) \sim \mathbf{E}N(1)\mathbf{P}(B \in t - \rho t + \Delta) \quad \text{as } t \to \infty,
\end{aligned}$$

by the theorem on local behaviour for randomly stopped sums from [1]. Indeed, either Condition (A) or (B) implies that $F \in \mathcal{S}_\Delta$, see [1] for this theorem and definitions. □

Proof of Corollary 2.2 As is not difficult to see, Condition (A′) implies Condition (A) and Condition (B′) implies Condition (B) for any fixed T. Therefore, Corollary 2.1 implies that

$$\mathbf{P}(\tau \in t+\Delta) \sim \mathbf{P}(B \in t - \rho t + \Delta) \sim T f_B(t - \rho t). \tag{7}$$

The latter equivalence holds since by both Conditions (A′) and (B′) f_B is long-tailed, that is,

$$f_B(t+u) \sim f_B(t) \quad \text{as } t \to \infty$$

uniformly in $u \in [0, T]$. Next, it follows from the explicit formula (3) that f_τ is long-tailed as well and, therefore,

$$\mathbf{P}(\tau \in t+\Delta) \sim T f_\tau(t). \tag{8}$$

Combining (7) and (8), we arrive at the conclusion. □

Note that the results of this section hold only for some classes of distributions such that $F(x+\Delta) \gg e^{-\sqrt{x}}$. We believe that it is also possible to obtain asymptotics for the distributions with tails lighter than $e^{-\sqrt{x}}$, but one should use results for large deviations of random walks other than Proposition 2.1. There are a lot of known results in that direction; we refer the reader to a recent paper of Borovkov and Mogulskii [4] for some new results and a review.

3 Global asymptotics for the busy period

In this section, we give a short proof of the known result (1). In other words, this section deals with the case $T=\infty$. Put $\overline{F}(x)=\mathbf{P}\{B>x\}$. We consider the same conditions on the distribution of B as those used in the previous section. In the case $T=\infty$, they transform into:

(A) $\mathbf{E}B<\infty$ and F is intermediate regularly varying, i.e.

$$\lim_{\kappa\downarrow 1}\limsup_{x\to\infty}\frac{\overline{F}(x)}{\overline{F}(\kappa x)}=1.$$

(B) $\mathbf{E}B^2<\infty$, $\overline{F}(x-\sqrt{x})\sim\overline{F}(x)$ and

$$\varepsilon(n)\equiv\sup_{x\ge 2\sqrt{n}}\frac{\mathbf{P}\{\xi_1>\sqrt{n},\xi_2>\sqrt{n},S_2>x\}}{\overline{F}(x)}=o\left(\frac{1}{n}\right),\quad n\to\infty.$$

The following proposition follows from Corollary 2.1 of [7].

Proposition 3.1 *Let either Condition* (A) *or* (B) *hold. Then*

$$\mathbf{P}\{S_n>0\}\sim n\overline{F}\big(n|\mathbf{E}\xi|\big)\quad as\ n\to\infty.$$

Note that the so-called square-root insensitivity condition $\overline{F}(x)\sim\overline{F}(x-\sqrt{x})$ implies that $-\ln\overline{F}(x)=o(\sqrt{x})$ as $x\to\infty$; hence, the proposition above deals with distributions whose tails are heavier than $e^{-\sqrt{x}}$.

We now state the main result of this section.

Theorem 3.1 *Let either Condition* (A) *or* (B) *hold. Then*

$$\mathbf{P}\{\tau>t\}\sim\mathbf{E}\{\nu\}\mathbf{P}\big\{B>(1-\rho)t\big\}\quad as\ t\to\infty.$$

We also need the following result.

Proposition 3.2 *Let either Condition* (A) *or* (B) *hold. Then*

$$\mathbf{P}\{\nu>t\}\sim\mathbf{E}\{\nu\}\mathbf{P}\big\{B>\big|\mathbf{E}\{\xi\}\big|t\big\}\quad as\ t\to\infty. \tag{9}$$

The proof of this proposition is rather standard (see, for instance, [8]) and is thus omitted here.

Proof of Theorem 3.1 For Condition (B), the estimate from below is a corollary of the CLT and the condition $\overline{F}(x-\sqrt{x}) \sim \overline{F}(x)$, as $x \to \infty$, a proof may be found in [11]. For Condition (A), the estimate from below follows similarly from the Law of Large Numbers.

Therefore, we will concentrate on the estimate from above. We will consider Condition (B). The proof for Condition (A) is similar. It follows from Proposition 3.2 and Condition (B) that

$$\mathbf{P}(\nu > n) \sim \mathbf{P}\big(\nu > n - \sqrt{n}\big) \quad \text{as } n \to \infty.$$

Therefore, there exists a function $l(n) \uparrow \infty$ such that

$$\mathbf{P}(\nu > n) \sim \mathbf{P}\big(\nu > n - \sqrt{n}l(n)\big) \quad \text{as } n \to \infty.$$

By the total probability formula,

$$\begin{aligned}
\mathbf{P}(\tau > t) &= P_1 + P_2 + P_3 \\
&= \mathbf{P}\left(\tau > t, \nu \le (1-\varepsilon)\frac{t}{\mathbf{E}A}\right) \\
&\quad + \mathbf{P}\left(\tau > t, (1-\varepsilon)\frac{t}{\mathbf{E}A} < \nu \le \frac{t}{\mathbf{E}A} - \sqrt{t}l(t)\right) \\
&\quad + \mathbf{P}\left(\tau > t, \nu > \frac{t}{\mathbf{E}A} - \sqrt{t}l(t)\right).
\end{aligned}$$

First,

$$\begin{aligned}
P_1 &= \mathbf{P}\left(B_1 + \cdots + B_\nu > t, \nu \le (1-\varepsilon)\frac{t}{\mathbf{E}A}\right) \\
&\le \mathbf{P}\left(\sum_{1 \le n \le (1-\varepsilon)\frac{t}{\mathbf{E}A}} B_n > t, \sum_{1 \le n \le (1-\varepsilon)\frac{t}{\mathbf{E}A}} A_n > t\right) \\
&\le \mathbf{P}\left(\sum_{1 \le n \le \frac{t}{\mathbf{E}A}} B_n > t\right)\mathbf{P}\left(\sum_{1 \le n \le (1-\varepsilon)\frac{t}{\mathbf{E}A}} A_n > t\right) \\
&= o(1/t)\mathbf{P}\left(\sum_{1 \le n \le \frac{t}{\mathbf{E}A}} B_i > t\right), \quad t \to \infty,
\end{aligned}$$

where that latter equivalence follows from the fact that, when $\mathbf{E}A^2 < \infty$,

$$\mathbf{P}\left(\sum_{i=1}^{n}(A_i - \mathbf{E}A_i) > \varepsilon n\right) = o(1/n)$$

as $n \to \infty$. Then,

$$P_2 = \sum_{(1-\varepsilon)\frac{t}{\mathbf{E}A} < n \le \frac{t}{\mathbf{E}A} - \sqrt{t}l(t)} \int_t^\infty \frac{\mathbf{P}(N(u) = n-1)}{n} \mathbf{P}(B_1 + \cdots + B_n \in du).$$

For $u \ge n-1$, probability $\mathbf{P}(N(u) = n-1)$ is non-increasing in u. Therefore, $\mathbf{P}(N(u) = n-1) \le \mathbf{P}(N(t) = n-1)$, and this fact implies that

$$\begin{aligned} P_2 &\le \sum_{(1-\varepsilon)\frac{t}{\mathbf{E}A} < n \le \frac{t}{\mathbf{E}A} - \sqrt{t}l(t)} \frac{\mathbf{P}(N(t) = n-1)}{n} \mathbf{P}(B_1 + \cdots + B_n > t) \\ &\le \frac{\mathbf{P}\left(\sum_{1 \le n \le \frac{t}{\mathbf{E}A}} B_n > t\right)}{(1-\varepsilon)\frac{t}{\mathbf{E}A}} \mathbf{P}\left((1-\varepsilon)\frac{t}{\mathbf{E}A} \le N(t) \le \frac{t}{\mathbf{E}A} - t^{1/2}l(t)\right). \end{aligned}$$

By the Central Limit Theorem for renewal processes,

$$\mathbf{P}\left((1-\varepsilon)\frac{t}{\mathbf{E}A} \le N(t) \le \frac{t}{\mathbf{E}A} - t^{1/2}l(t)\right) = o(1)$$

as $t \to \infty$ and, therefore,

$$P_2 = o(1/t)\mathbf{P}\left(\sum_{1 \le n \le \frac{t}{\mathbf{E}A}} B_i > t\right), \quad t \to \infty.$$

As a result, we have

$$P_1 + P_2 = o(1/t)\mathbf{P}\left(\sum_{1 \le n \le \frac{t}{\mathbf{E}A}} B_n > t\right) = o\big(\mathbf{P}(B > t - \rho t)\big), \quad t \to \infty,$$

where the latter follows from Corollary 2.1 of [7]. For the third term,

$$P_3 \le \big(1 + o(1)\big)\mathbf{P}\left(\nu > \frac{t}{\mathbf{E}A}\right) = \big(1 + o(1)\big)\mathbf{E}\{\nu\}\mathbf{P}(B > t - \rho t)$$

as $t \to \infty$ (see (9)). Finally, we have

$$\mathbf{P}(\tau > t) \le \big(1 + o(1)\big)\mathbf{E}\{\nu\}\mathbf{P}(B > t - \rho t), \quad t \to \infty.$$ □

Acknowledgements The authors would like to thank Serguei Foss for drawing their attention to this problem and Onno Boxma for a number of useful discussions and for many suggestions that helped improving the manuscript. The authors are also grateful to the anonymous referees for their thorough reading of the paper and many valuable remarks. The research of both authors was supported by the Dutch BSIK project (*BRICKS*) and the EURO-NGI project.

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