Renormalized Von Neumann entropy with application to entanglement in genuine infinite dimensional systems

Roman Gielerak[1]

Abstract

A renormalized version of the von Neumann quantum entropy (which is finite and continuous in general, infinite dimensional case) which obeys several of the natural physical demands (as expected for a "good" measure of entanglement in the case of general quantum states describing bipartite and infinite-dimensional systems) is proposed. The renormalized quantum entropy is defined by the explicit use of the Fredholm determinants theory. To prove the main results on continuity and finiteness of the introduced renormalization, the fundamental Grothendick approach, which is based on the infinite dimensional Grassmann algebra theory, is applied. Several features of majorization theory are preserved under the introduced renormalization as it is proved in this paper. This fact allows to extend most of the known (mainly, in the context of two-partite, finite-dimensional quantum systems) results of the LOCC comparison theory to the case of genuine infinite-dimensional, two-partite quantum systems.

Keywords Quantum von Neumann Entropy · Fredholm determinants · Continuous quantum system · Quantum entanglement · Schur monotonicity · Infinite dimensions

1 Introduction

1.1 General, introductory remarks

One of the basic, genuine quantum resources—that existing quantum information processing technology intensively exploits—is so-called quantum correlations [1]. For an exhaustive review of the present-day state of quantum hardware technology see, i.e. [2]. The interesting point here is that so-called continuous quantum systems (ions,

✉ Roman Gielerak
r.gielerak@issi.uz.zgora.pl

[1] Institute of Control and Computation Engineering, University of Zielona Góra, ul. Szafrana 2, 65-246 Zielona Góra, Poland

atoms, lasers, …) are becoming a very promising candidates for being a basic quantum ingredients of the future, full-scale quantum computers. This implies, that the better control on the quantum correlations in such systems may be crucial for developing these technologies. In particular, an appropriate qualitative and quantitative measures of quantum correlations have to be prepared. As it is well known, the phenomenon of quantum entanglement plays a crucial role for performing successfully several quantum protocols like teleportation, QKD protocols and many more [3, 4].

Several, qualitative and quantitative entanglement measures (obeying set of reasonable and natural—from the mathematical and information theory point of view—demands) contained in quantum states are being proposed [5, 6]. Many of them are based on the use and properties of the quantum von Neumann entropy. However, there does not exist a general straightforward passage with these mathematical formalism from the case of finite dimensional systems of qudits to the genuine, infinite-dimensional systems like ions, atoms, etc. It is the main purpose of the present paper to propose, how it is possible to fill up this gap. Majority of quantum states describing bipartite (and many partite systems as well) and infinite dimensional systems is characterized by the fact that the von Neumann entropy (and therefore, the corresponding entropy based on tangles measures for many partite systems) of the corresponding conditional quantum states (reduced density matrices) is taking infinite value [7, 8]. With the use of Fredholm determinants technique, it is possible to remove the arising infinities and thus, it is possible to extend several results known for the finite dimensional systems to the genuine, infinite dimensional continuous systems. It is the great Author's hope that the presented here mathematical technique will find, besides those included here, many other applications in the field of Quantum Information Theory.

1.2 Preliminaries

Let us consider the model of two spinless quantum particles interacting with each other and placed in three dimensional Euclidean space $\mathbb{R}^3$. Generally, the states of such quantum systems are described by the density matrices which are non-negative, of trace class operators acting on the space $\mathcal{H} = L_2(\mathbb{R}^3) \otimes L_2(\mathbb{R}^3)$, see [9, 10]. The latter is, in fact, unitary equivalent to $L_2(\mathbb{R}^6)$. In particular, any pure state can be represented (up to the global phase calibration) by the corresponding wave function $\psi(x, y) \in \mathcal{H}$; then the density matrix takes the form of the projector onto the ket vector $|\psi\rangle$.

Using Schmidt decomposition theorem, *cf* [11, Thm. 26.8], we conclude that for any pure normalized state $\psi \in \mathcal{H}$ there exist: a sequence of non-negative numbers $\{\lambda_n\}_{n=1}^{\infty}$ (called the Schmidt coefficients of ψ) satisfying the condition $\sum_{n=1}^{\infty} \lambda_n^2 = 1$ and two complete orthonormal systems of vectors $\{\varphi_n\}_{n=1}^{\infty}$, $\{\omega_n\}_{n=1}^{\infty}$ in $L_2(\mathbb{R}^3)$ such that the following equality (in the L_2 space sense):

$$\psi(x, y) = \sum_{n=1}^{\infty} \lambda_n \varphi_n(x) \omega_n(y), \tag{1.1}$$

has to be satisfied.

In particular, we call the vector ψ a separable pure state *iff* there appears only one nonzero Schmidt coefficient in the decomposition (1.1). If the number of nonzero Schmidt coefficients is finite than we say that ψ is of finite Schmidt rank pure state. In this case, one can apply the standard and the most frequently used measure of amount of entanglement included in the state ψ which is given by the von Neumann formula:

$$\mathrm{EN}(\psi) = -\sum_{n=1}^{\infty} \lambda_n^2 \log(\lambda_n^2). \tag{1.2}$$

Although, the set of finite Schmidt rank pure states of the system under consideration is dense (in the L_2-topology) on the corresponding Bloch sphere (this time infinite-dimensional and given here modulo global phase calibration for simplification of the following discussion only) denoted as $B = \{\psi \in L_2(\mathbb{R}^6) : \|\psi\| = 1\}$, it appears that also the set of infinite Schmidt rank pure states is dense there. The situation is even more complicated as it can be shown that the set of pure states for which the value of von Neumann entropy is finite is dense in B but also the set of states with infinite entropy of entanglement is dense in this Bloch sphere [7].

Similar results on densities of the infinite/finite Schmidt rank states are also valid in the proper physical L_1-topologies on the corresponding Bloch sphere. Very roughly, the reason is that in infinite dimensions there are many (too many in fact) sequences $\{\lambda_n\}$) such that: for all n, $\lambda_n \geq 0$ and $\sum_{n=1}^{\infty} \lambda_n^2 = 1$ but $\sum_{n=1}^{\infty} \lambda_n^2 \log(\lambda_n^2) = -\infty$. In other words, the set of pure states for which the entropy is finite has no internal points and this fact causes serious problems in the fundamental question on continuity of the von Neumann entropy in genuine infinite dimensional setting [7, 8]. In finite dimensions the von Neumann entropy is a non-negative, concave, lower semi-continuous and also norm continuous function defined on the set of all quantum states. A lot of fundamental results on several quantum versions of entropy, in particular, on von Neumann entropy have been obtained in the last decades, *cf* [12–20]. However, in the infinite dimensional setting, the conventionally defined von Neumann entropy is taking the value $+\infty$ on a dense subset of the space of quantum states of the system under consideration *cf* [7, 8, 11, 13, 21–26].

Nevertheless, defined in the standard way von Neumann entropy has continuous and bounded restrictions to some special (selected by some physically motivated arguments) subsets of quantum states. For example, the set of states of the system of quantum oscillators with bounded mean energy forms a set of states with finite entropy [7, 8, 27, 28]. Since, the continuity of the entropy is a very desirable property in the analysis of quantum systems, various, sufficient for continuity, conditions have been obtained up to now. The earliest one, among them, seems to be Simon's dominated convergence theorems presented in [15–17] and widely used in applications, see [12–14]. Another useful continuity condition originally appeared in [7, 8] and can be formulated as the continuity of the entropy on each subset of states characterized by bounded mean value of a given positive unbounded operator with discrete spectrum, provided that its sequence of eigenvalues has a sufficient large rate of decrease. Some special conditions yielding the continuity of the von Neumann entropy are formulated in the series of papers by Shirokov [21–26]. A stronger version of the stability property

of the set of quantum states naturally called there as strong stability was introduced by Shirokov together with some applications concerning the problem of approximations of concave (convex) functions on the set of quantum states and a new approach to the analysis of continuity of such functions has been presented there. Several other attempts and ideas to deal with the noncommutative, infinite dimensional setting were published in the current literature also. Some of them are based, on a very sophisticated, tools and methods, such as, for example theory of noncommutative (versions of) the (noncommutative) log-Sobolev spaces of operators [29].

1.3 The main idea of the paper

The main idea of the present paper is to introduce an appropriate renormalized version of the widely known von Neumann formula for the entropy in the non-commutative setting [12–14]. The notion of von Neumann entropy is one of the basic concept introduced and applied in quantum physics. However formula proposed by von Neumann works perfectly well only in the context of finite dimensional quantum systems [7, 8]. The extension to the genuine infinite-dimensional setting is not straightforward and meets several serious obstacles as mentioned in the previous sentences. Our prescription for extracting finite part of the infinite valued (which is true typically in the sense of Baire category theory) standard von Neumann formula is very simple. For this goal, let Q be a quantum state, i.e. Q is non-negative, of trace class operator defined on some separable Hilbert space $\mathcal{H}$ and such that $\text{Tr}(Q) = 1$. The standard definition of von Neumann entropy EN is given as:

$$\text{EN}(Q) = -\text{Tr}(Q \log(Q)) \tag{1.3}$$

Our renormalization proposal, denoted as FEN, is given by:

$$\text{FEN}(Q) = \text{Tr}\left((Q + 1_{\mathcal{H}}) \log(Q + 1_{\mathcal{H}})\right), \tag{1.4}$$

where $1_{\mathcal{H}}$ stands for the unit operator in $\mathcal{H}$.

Claim 1.1 *For any such Q the value* FEN(Q) *is finite.*

Proof Let $\sigma(Q) = (\tau_1, \ldots, \tau_n, \ldots)$ be sequence representing the spectrum of Q and ordered in non-increasing order (and with multiplicities included). Using the elementary inequality

$$\log(1 + x) \leq x \quad \text{for} \quad x \geq 0, \tag{1.5}$$

together with functional calculus [11, 30, 31] we have the following estimate

$$\text{FEN}(Q) = \sum_{n=1}^{\infty} (\tau_n + 1) \log(1 + \tau_n)$$

$$\leq \sum_{n=1}^{\infty} (\tau_n^2 + \tau_n) \tag{1.6}$$

$$\leq 2 \cdot \sum_{n=1}^{\infty} \tau_n \leq 2.$$

$\square$

This means that the introduced map

$$\text{FEN} : E(\mathcal{H}) \mapsto [0, \infty) \tag{1.7}$$

is finite on the space $E(\mathcal{H})$ of the quantum states on $\mathcal{H}$. The detailed mathematical study of the basic properties of the introduced here renormalization of the von Neumann entropy is the main topic of this paper. Additionally, presentation of several applications of the introduced entropy FEN and addressed to the Quantum Information Theory [3, 4, 27, 28] are also included. To achieve all these goals, the theory of Fredholm determinants as given by Grothendick [32] is intensively used in the following presentation. Also certain results from the infinite dimensional majorisation theory [33–38] have been used. A very preliminary and illustrative idea of von Neumann entropy renormalization was recently published by the Author in [39].

1.4 Organization of the paper

In the next Sect. 2, the technique of the Fredholm determinants is successfully applied to show that the proposed here renormalized version of von Neumann entropy formula in the genuine infinite-dimensional setting is finite and continuous (in the L_1-topology meaning) on the space of quantum states. Elements of the so-called multiplicative version of the standard majorization theory [3, 4, 33, 34, 40] are being introduced in Sect. 3. The main results reported there are: the rigorous proof of monotonicity of the introduced renormalization of von Neumann entropy under the semi-order relations (caused by the defined there multiplicative majorization) lifted to the space of quantum states. Additionally, an extension of the basic (in the present context) Alberti-Uhlmann theorem [33] is proved in Sect. 3. Also monotonicity of the introduced notion of renormalized von Neumann entropy under the action of a general quantum operations on quantum states is proved there. Section 4 is devoted to the study of two-partite quantum systems of infinite dimensions both (the case of one factor being finite dimensional is analysed in details see [41, 42]). In particular, the corresponding reduced density matrices are studied there and some useful formula and estimates of the corresponding renormalized entropies are included there. The particular case of pure bipartite states is analyzed from the point of view of majorization theory with the use of novel, local unitary and monotonous invariants perspective of Gram operators as introduced in another papers [43–47]. The finite dimensional results of this type, presented in [43, 44, 46], are being extended to the infinite dimensional setting there with the use

of Fredholm determinants theory [46]. At the end of this paper three appendices are attached to make this paper autonomous and also because some additional results which might be helpful in further developments of the ideas presented here are being formulated there. In appendix A, the Author have presented (after Grothendik [32], see also Simon [48]) crucial facts and estimates from the infinite dimensional Grassman algebra theory with the applications to control Fredholm determinants. Appendix B includes several results and formulas on the different types of combined Schmidt and spectral decompositions of a general bipartite quantum states. Finally in Appendix C, some useful remarks on the operator valued function $\log(1 + Q)$ are collected.

Extensions of the approach to the renormalization of the von Neumann entropy presented in this paper to a very rich palette of intriguing questions, like for example renormalization of quantum relative entropy and quantum relative information notions [27, 28, 49–55], possible applications to the renormalization of the quantum entropy in the context of general Quantum Field Theory (see i.e. the recent paper on this [56]) and also possible applications to the so called Continuous Variable Quantum Information Theory [57–62] are also visible for the Author and some work on them is in progress.

2 Renormalized version of the von Neumann entropy

2.1 Some mathematical notation

Assume that $\mathcal{H}$ is a separable infinite dimensional Hilbert space.[1] In this paper, we use the following standard notation:

- $L_1(\mathcal{H})$ stands for the Banach space of trace class operators acting on $\mathcal{H}$ and equipped with the norm $\|Q\|_1 = \mathrm{Tr}\,[|Q^\dagger Q|^{1/2}]$, where $Q \in L_1(\mathcal{H})$ and the symbol † means the Hermitian conjugation,
- $L_2(\mathcal{H})$ denotes the Hilbert-Schmidt class of operators acting in $\mathcal{H}$ and with the scalar product $\langle Q|Q'\rangle_{HS} = \mathrm{Tr}\,[Q^\dagger Q']$, where $Q \in L_2(\mathcal{H})$,
- $B(\mathcal{H})$ denotes the space of the bounded operators with norm defined as the operator norm $\|\cdot\|$,
- Let $E(\mathcal{H})$ be the complete metric space of quantum states Q on the space $\mathcal{H}$, i.e. the L_1-completed intersection of the cone of non-negative, trace class operators on $\mathcal{H}$ and L_1-closed hyperplane described by the normalization condition $\mathrm{Tr}[Q] = 1$.

The convention which is used in the present paper is that always spectrum of a Hermitian Q is ordered in an non-increasing order (this is possible to achieve by performing certain unitary operation on a given operator Q).

In further discussion, we will relay on the following inequalities, *cf* [30, 31],

$$\|AB\|_1 \leq \|A\|_1 \|B\|_1, \quad A, B \in L_1(\mathcal{H}), \tag{2.8}$$

$$\|AB\|_1 \leq \|A\| \|B\|_1, \quad A \in B(\mathcal{H}), B \in L_1(\mathcal{H}); \tag{2.9}$$

the latter inequality also holds for $\|BA\|_1$ with obvious changes.

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[1] The results of this paper hold for finite dimensional Hilbert spaces as well.

The following spaces of sequences will be used in further analysis

$$C^\infty = \{\underline{a} = (a_1, ..., a_n, ...), a_n \in \mathbb{R}\}, \tag{2.10}$$

$$C_+^\infty = \{\underline{a} \in C^\infty : a_n \geq 0\}, \tag{2.11}$$

$$C_+^\infty(1) = \{\underline{a} \in C_+^\infty : \sum_{i=1}^\infty a_i = 1\}, \tag{2.12}$$

$$C^\infty(< \infty) = \{\underline{a} \in C_+^\infty : \sum_{i=1}^\infty a_i < \infty\}. \tag{2.13}$$

2.2 The renormalized von Neumann entropy

The most useful local invariants and local monotone quantities characterizing in the qualitative as well as quantitative way quantum correlations, as entanglement of states in the finite dimensional systems, are defined by means of the special versions of the entropy measures, *cf.* [3–6, 63, 64]. The von Neumann quantum entropy measure is, without a doubt, the most common tool for these purposes.

Suppose that $\underline{a} \in C^\infty$ and $a_i \neq 0$ for all i. Moreover, we assume that that the limit $\lim_{n\to\infty} \prod_{i=1}^n a_i$ exists and it is nonzero. Then, we say that the product $\prod_{i=1}^\infty a_i$ exists.

The continuity of $x \mapsto \log x$ implies the following statement.

Lemma 2.1 *Let $\underline{a} \in C^\infty(< \infty)$. Then, the product $\prod_{i=1}^\infty (1+a_i)$ exists iff $\sum_{i=1}^\infty \log(1+ a_i) < \infty$.*

Proof Let as assume that the following sequence exists

$$\pi_n = \prod_{i=1}^n (1 + a_i), \tag{2.14}$$

and it is convergent, i.e.

$$\lim_{n\to\infty} \pi_n = \pi_\infty. \tag{2.15}$$

Due to the continuity of log, it follows:

$$\lim_{n\to\infty} \log(\pi_n) = \log(\pi_\infty), \tag{2.16}$$

which is equivalent to:

$$\lim_{n\to\infty} \sum_{i=1}^n \log(1 + a_i) = \log(\pi_\infty). \tag{2.17}$$

Assuming that the sequence

$$\Sigma_n = \sum_{i=1}^{n} \log(1 + a_i), \tag{2.18}$$

is convergent, i. e.

$$\lim_{n\to\infty} \Sigma_n = \Sigma_\infty < \infty, \tag{2.19}$$

and does exist, we can write, using the continuity of exp

$$\exp(\Sigma_\infty) = \lim_{n\to\infty} \exp(\Sigma_n) = \lim_{n\to\infty} \pi_n. \tag{2.20}$$

$\square$

Lemma 2.2 *Let $\underline{a} \in C_+^\infty(1)$. Then, the product $\prod_{i=1}^{\infty}(1 + a_i)$ exists iff $\sum_{i=1}^{\infty}(1 + a_i)\log(1 + a_i) < \infty$.*

Proof The claim follows directly from

$$\log(1 + a_i) \le (1 + a_i)\log(1 + a_i) \le 2\log(1 + a_i). \tag{2.21}$$

$\square$

Let A be a compact operator in a separable Hilbert space $\mathcal{H}$ and $\sigma(A)$ stands for the discrete eigenvalues of A counted with multiplicities and ordered into non-increasing sequence. On the other hand, let $\lambda(A)$ denote singular values of A counted with multiplicities and forming non-increasing sequence. If $A \in L_1(\mathcal{H})$ then $\lambda(A) = \sigma(|A^\dagger A|^{1/2})$ and $\sum_{n=1}^{\infty} \lambda_n < \infty$. The Fredholm determinant takes the form

$$\det(I + A) = \prod_{x\in\underline{\lambda}(A)} (1 + x). \tag{2.22}$$

We remind the basic properties of the Fredholm determinants, *cf.* [32, 48] below.

Theorem 1 [32, 48] *Let $\mathcal{H}$ be a separable Hilbert space. Then*

i) *For any $\Delta \in L_1(\mathcal{H})$ the map*

$$\mathbb{C} \ni z \mapsto \det(I + z\Delta) \tag{2.23}$$

extends to the entire function which obeys the bound

$$|\det(I + z\Delta)| \le \exp(|z|\|\Delta\|_1). \tag{2.24}$$

ii) *For any maps $L_1(\mathcal{H}) \ni \Delta \mapsto \det(I + \Delta)$ and $L_1(\mathcal{H}) \ni \Delta' \mapsto \det(I + \Delta')$ the following asymptotics is true:*

$$|\det(I + \Delta) - \det(I + \Delta')| \leq \|\Delta - \Delta'\|_1 \exp(\mathcal{O}(\|\Delta\|_1 \cdot \|\Delta'\|_1)); \quad (2.25)$$

in particular det *is the Lipschitz continuous.*

iii) *The following three equivalences hold:*

$$\det(I + z\Delta) = \exp\left(\text{Tr}\left[\log(I + z\Delta)\right]\right) \quad (2.26)$$

and

$$\det(I + z\Delta) = \sum_{n=1}^{\infty} z^n \,\text{Tr}\left[\wedge^n(\Delta)\right], \quad (2.27)$$

where $\wedge^n(\Delta)$ stands for the antisymmetric tensor power of Δ, see Appendix A for more details, and

$$\det(I + z\Delta) = \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n \,\text{Tr}\left[\Delta^n\right]\right). \quad (2.28)$$

Remark 1 The last equivalence Eq. (2.28) determines so-called Pelmelj expansion with $|z| < 1$. For larger values of $|z|$, the analytic continuations are necessary to be performed.

In the Appendix A, we outline the methods of infinite dimensional Grassmann algebras (the Fermionic Fock spaces in the physical notations) as introduced in the fundamental Grothendick memoir [32].

In the further discussion, we will use the following quantity.

Definition 1 Assume that $Q \in E(\mathcal{H})$ and its spectrum $\sigma(Q) = (\lambda_1, \lambda_2, ...)$. We define

$$\text{FEN}_{\pm}(Q) = \log\left(\det(I + Q)^{\pm(I+Q)}\right), \quad (2.29)$$

where

$$\det(I + Q)^{\pm(I+Q)} = \prod_{k=1}^{\infty} (1 + \lambda_k)^{\pm(1+\lambda_k)}. \quad (2.30)$$

This means that

$$\text{FEN}_{\pm}(Q) = \pm \sum_{k=1}^{\infty} (1 + \lambda_k) \log(1 + \lambda_k). \quad (2.31)$$

In order to relate the above definition with the results formulated in Theorem 1, we introduce the following entropy-generating operators $S_\pm$.

Definition 2 For $Q \in E(\mathcal{H})$ we define

$$S_\pm(Q) = (I + Q)^{\pm(I+Q)} - 1_\mathcal{H}, \tag{2.32}$$

where $1_\mathcal{H}$ means the unit operator here in the space $\mathcal{H}$ and spectral functional calculus has been used.

Remark 2 In the standard, finite dimensional situation [12–14], the corresponding entropic operator $S_-(Q)$, looks like (informally) as

$$S_-(Q) = Q^{-Q} - 1_\mathcal{H}. \tag{2.33}$$

Our definition (2.32) is the renormalized (due to the infinite dimension of the corresponding spaces) version of it is: "$(1 + Q)^{-(1+Q)} - 1$".

One of the main results reporting on this note is the following theorem.

Theorem 2 *For any $Q \in E(\mathcal{H})$, $\mathrm{FEN}_\pm(Q)$ are finite and, moreover, $\mathrm{FEN}_\pm$ are $L_1(\mathcal{H})$ continuous on $E(\mathcal{H})$.*

The proof is based on the following sequence of Lemmas.

Let us define the scalar function

$$f_\pm(x) = (1 + x)^{\pm(1+x)} - 1, \quad \text{for} \ \ x \in [0, 1]. \tag{2.34}$$

Lemma 2.3 *i) The function $f_+(x)$ is monotonously increasing and convex on $[0, 1]$ and*

$$\begin{aligned} \inf f_+(x) &= 0, \ \text{for} \ x = 0, \\ \sup f_+(x) &= 3, \ \text{for} \ x = 1. \end{aligned} \tag{2.35}$$

ii) The function $f_-(x)$ is monotonously decreasing and concave on $[0, 1]$, and

$$\begin{aligned} \inf f_-(x) &= -0,75, \ \text{for} \ x = 1 \\ \sup f_-(x) &= 0, \quad \quad \ \text{for} \ x = 0. \end{aligned} \tag{2.36}$$

Lemma 2.4 *For any $Q \in E(\mathcal{H})$, $S_+(Q) \geq 0$ and $S_+(Q) \in L_1(\mathcal{H})$.*

Proof For $0 \leq x \leq 1$ the following estimate is valid

$$(1 + x)^{1+x} - 1 = \int_0^1 ds \, e^{s(1+x)\log(1+x)} (1 + x) \log(1 + x) \leq 2e^{2\log 2} \log(1 + x). \tag{2.37}$$

From

$$\text{Tr}\,[(1+Q)^{1+Q} - 1] = \sum_{n=0}^{\infty}((1+\lambda_n)^{1+\lambda_n} - 1) \leq 8 \sum_{n=0}^{\infty} \log(1+\lambda_n) < \infty,$$

(2.38)

where we used Theorem 1 and Lemma 2.1. □

Lemma 2.5 *For any* $Q \in E(\mathcal{H})$, $-S_-(Q) \geq 0$ *and* $S_-(Q) \in L_1(\mathcal{H})$.

Proof For $0 \leq x \leq 1$ the following estimate is valid

$$-(1+x)^{-(1+x)} + 1 = \int_0^1 ds\, e^{-(1-s)(1+x)\log(1+x)}(1+x)\log(1+x) \leq 2\log(1+x).$$

(2.39)

From

$$-\text{Tr}\,[(1+Q)^{1+Q} - 1] = -\sum_{n=0}^{\infty} -(1+\lambda_n)^{-(1+\lambda_n)} + 1 \leq 2 \sum_{n=0}^{\infty} \log(1+\lambda_n) < \infty.$$

(2.40)

where we have used Theorem 1 and Lemma 2.1. □

In order to prove that the renormalized entropy functions FEN$_\pm$ are L_1 continuous it is enough to prove that the operator valued maps $S_\pm$ are L_1 continuous. The latter is proved below.

Lemma 2.6 *Let* $\mathcal{H}$ *be a separable Hilbert space and let* $E(\mathcal{H})$ *be a space of quantum states on* $\mathcal{H}$. *Then the maps*

$$Q \mapsto S_\pm(Q) = (I + Q)^{\pm(I+Q)} - I,$$

(2.41)

are L_1 *continuous on* $E(\mathcal{H})$.

Proof It is enough to present essential details of the proof for the case $S_+(Q)$. Let Q and Q' be the states on $\mathcal{H}$. By the application of the Duhamel formula and equations (2.8) and (2.9) we get

$$\|S_+(Q) - S_+(Q')\|_1 \leq \sup_{0<s<1} \| \exp s \log(I+Q)\|$$
$$\cdot \| \exp(1-s)(I+Q')\log((I+Q)\|$$
$$\left(\|(I+Q)\| \cdot \| \log(I+Q) - \log(I+Q')\|_1 \cdot \| \log(I+Q)\| \cdot \|Q-Q'\|_1 \right).$$

(2.42)

To complete the proof it suffices to prove the norm continuity of the operator valued function $\log(I+Q)$. Let $Q \in E(\mathcal{H})$. Define $\tau(Q) = \sup \sigma(Q)$. Then $\tau(Q) \leq 1$ and

$\| \log(I + Q) \| = \log(1 + \tau(Q))$. Let $Q, Q' \in E(\mathcal{H})$ with $\|Q - Q'\|_1 \leq \delta < 1$. Using again formulae (2.8) and (2.9) we have

$$
\begin{aligned}
\| \log(I + Q) - \log(I + Q') \|_1 &\leq \sum_{n=1}^{\infty} \frac{1}{n} \| Q^n - (Q')^n \|_1 \\
&\leq \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{n} \| Q^{k-1}(Q - Q')(Q')^{n-k} \|_1 \\
&\leq \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{n} \tau(Q)^k \tau(Q')^{n-k-1} \| Q - Q' \|_1. \quad (2.43)
\end{aligned}
$$

Let $\tau := \max\{\tau(Q), \tau(Q')\} < 1$. Then summarizing the above reasoning, we have

$$
\| \log(I + Q) - \log(I + Q') \|_1 \leq \frac{\delta}{1 - \tau}. \tag{2.44}
$$

The analysis of the further properties, together with the analysis of the case $\tau = 1$ of the map $Q \mapsto \log(I + Q)$, we postpone to Appendix C. $\qquad\square$

Proposition 3 *Let $\mathcal{H}$ be a separable Hilbert space and let $E(\mathcal{H})$ be the space of quantum states on $\mathcal{H}$. Then, the L_∞ norms (spectral norm) of the entropy maps $S_\pm(Q) = (I + Q)^{\pm(1+Q)} - I$ are given by:*

1. $\|S_+(Q)\|_\infty = (1 + \tau_1)^{1+\tau_1} - 1$, *where* $\tau_1 = \sup(\sigma(Q))$,
2. $\sup_{Q \in E(\mathcal{H})} \|S_+(Q)\|_\infty = 3$,
3. $\|S_-(Q)\|_\infty = 1 - (1 + \tau_1)^{-(1+\tau_1)}$, *where* $\tau_1 = \sup(\sigma(Q))$,
4. $\sup_{Q \in E(\mathcal{H})} \|S_-(Q)\|_\infty = 0.75$.

Proof For the compact operators, it is known that $\|Q\|_\infty = \|Q\|$, see [30]. $\qquad\square$

Remark 3 Let us assume that $\dim(\mathcal{H}) = d$ and is finite. Then, taking a pure state Q, i.e. the state for which $\mathrm{Tr}[Q] = \mathrm{Tr}[Q^2] = 1$ it follows that the value of renormalized entropy $\mathrm{FEN}_+(Q)$ of Q which has the rank of Schmidt equal to one, is equal to $2\log(2)$ ($\mathrm{FEN}_-(Q) = -2\log(2)$) and is independent of d. Taking maximally mixed state Q with the spectral numbers $\sigma(Q) = (1/d, \ldots 1/d)$ we have $\mathrm{FEN}_+(Q) = (1 + 1/d)\log(1 + 1/d)^d$ ($\mathrm{FEN}_-(Q) = -(1 + 1/d)\log(1 + 1/d)^{-d}$) which tends monotonously, as d tends to infinity to the value 1 (resp. to the value -1).

The use of standard, not renormalized, definition of entropy of entanglement leads to the statement that it is taking values in interval $[0, \log(d)]$, which shows that there is no possible straightforward passage from the finite dimensional situation to the infinite dimensional systems. The widely used, another entropic measures of entanglement [3–6] also must be suitable renormalized in order to be applied in infinite dimensions in a way that overcome the several discontinuity and divergences problems as well problems arising in the genuine infinite dimensional cases. The results on this will be presented in a separate note.

Let $Q(n)$ be the sequence of $L_1(\mathcal{H})$ such that the n-th first eigenvalues of $Q(n)$ is equal to $1/n$ and the rest of spectrum is equal to zero. The renormalized entropy of $Q(n)$ is given by

$$\text{FEN}_+(Q(n)) = (n+1)\log\left(1 + \frac{1}{n}\right). \tag{2.45}$$

It is easy to see that $\lim_{n\to\infty} \text{FEN}_+(Q(n)) = 1$.

Theorem 4 *For any sequence of states $Q(n) \in L_1(\mathcal{H})$ as above there exists state Q^* in $E(\mathcal{H})$ and such $\text{FEN}_+(Q^*) = 1$.*

Proof For any such sequence $Q(n)$, we apply the Banach–Alaglou theorem first, concluding that the set $\{Q(n)\}$ forms *-week precompact set and therefore, in the *-weak topology $\lim Q(n)$ by subsequences do exists. However, these limits are all equal to zero. In order to obtain a non-trivial result, we use the Banach–Saks theorem which tells us that there exists a subsequence n_j such that the following Cesaro sum of $Q(n)$

$$C_M(Q) = \frac{1}{M}\sum_{j=1}^{M} Q(n_j) \tag{2.46}$$

which is strongly convergent as $M \to \infty$ to some nonzero operator $Q^* \in E(\mathcal{H})$. $\quad\square$

It would be interesting to describe in the explicit way the most mixed states i.e. the states for which the value of $\text{FEN}_+(Q) = 1$.

3 Some remarks on the majorization theory

The fundamental results obtained in Alberti and Uhlmann monograph [33] and applied so fruitfully to the quantum information theory by many researchers (see [3, 4, 33, 34, 65, 66] and references therein), are known widely today under the name (S)LOCC majorization theory (in the context of quantum information theory). Presently, this theory is pretty well understood in the context of bipartite, finite dimensional systems, (especially in the context of pure states), see [3, 4, 34]. In papers [35, 36, 38, 65, 67], successful attempts are presented in order to extend this theory to the case of bipartite, infinite dimensional systems. Below, we present some remarks which seems to be useful in this context.

For a given $\underline{a} \in C^\infty$, we apply the operation of ordering in non-increasing order and denote the result as $\underline{a}^\geq$. Of particular interest will be the image of this operation, when applied pointwise to the infinite dimensional simplex $C_+^\infty(1) := \{\underline{a} = (a_n) \in \mathbb{R}^N, a_n \geq 0, \sum_{i=1}^\infty a_i = 1\}$. This will be denoted as $C^\geq$. Let us recall some standard definitions of majorization theory. Let $\underline{a}, \underline{b} \in C^\geq$. Then, we will say that $\underline{b}$ is majorizing $\underline{a}$ iff for any n the following is satisfied

$$\sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} b_i. \tag{3.47}$$

If above assumption is fulfilled then we denote this as $\underline{a} \preccurlyeq \underline{b}$.

We will say that $\underline{b}$ majorizes multiplicatively $\underline{a}$ iff for any n the following is satisfied

$$\prod_{i=1}^{n}(a_i + 1) \le \prod_{i=1}^{n}(b_i + 1). \tag{3.48}$$

If this is true then we denote this fact as $\underline{a}$ m-$\preccurlyeq$ $\underline{b}$.

Let F be any function (continuous, but not necessarily) on the interval $[0, 1]$. The action of F on C_+^{∞} (and other spaces of sequences that do appear) will be defined $(F(a_i))$.

Recall the well-known result, see i.e. [5,6].

Lemma 3.1 *Let us assume that f is a continuous, increasing and convex function on $\mathbb{R}$. If $\underline{a} \preccurlyeq \underline{b}$ then $f(\underline{a}) \preccurlyeq f(\underline{b})$.*

It is clear from the very definitions that $\underline{a}$ m-$\preccurlyeq$ $\underline{b}$ iff $\log(\underline{a} + 1) \preccurlyeq \log(\underline{b} + 1)$.

Proposition 5 *Let $\underline{a}, \underline{b} \in C^{\ge}$ and let us assume that $\underline{a}$ m-$\preccurlyeq$ $\underline{b}$. Let f be continuous, increasing function and such that the composition $f \circ \exp(x)$ is convex on a suitable domain. Then $f(\underline{a}) \preccurlyeq f(\underline{b})$.*

Proof For fixed n we have:

$$\prod_{i=1}^{n}(a_i + 1) \le \prod_{i=1}^{n}(b_i + 1). \tag{3.49}$$

Taking log of both sides we obtain

$$\sum_{i=1}^{n} \log(a_i + 1) \le \sum_{i=1}^{n} \log(b_i + 1). \tag{3.50}$$

Applying Lemma 3.1 we obtain

$$\sum_{i=1}^{n} f(a_i) \le \sum_{i=1}^{n} f(b_i). \tag{3.51}$$

$\square$

In particular taking $f(x) = x$ we conclude

Corollary 3.2 *Let as assume that $\underline{a}, \underline{b} \in C^{\ge}$ and $\underline{a}$ m-$\preccurlyeq$ $\underline{b}$. Then $\underline{a} \preccurlyeq \underline{b}$.*

The last result says that each linear chain of the semi-order relation m-$\preccurlyeq$ in $C^{\ge}$ is contained in some linear chain of the semi-order $\preccurlyeq$. It means that the semi-order m-$\preccurlyeq$ is finer than those induced by $\preccurlyeq$.

Corollary 3.3 *Any $\preccurlyeq$-maximal element in $C^{\ge}$ is also m-$\preccurlyeq$-maximal.*

Proof If $\underline{a}$ m-$\preccurlyeq$ $\underline{b}$ then $\underline{a} \preccurlyeq \underline{b}$. Let $\underline{a}^*$ be a $\preccurlyeq$-maximal in $C^{\geq}$ and let us assume that there exists $\underline{a}^{**}$ such that $\underline{a}^*$ m-$\preccurlyeq$ $\underline{a}^{**}$ and the contradiction is present. □

To complete this subsection, we quote the infinite dimensional extension of the majorization theory applications in the context of quantum information theory.

For this goal let us consider any $Q \in E(\mathcal{H})$, where $\mathcal{H}$ is a separable Hilbert space. With any such Q, we connect a sequence $(P_{sp}(N))$ of finite dimensional projections $P_{sp}(Q)$ which we will call the spectral sequence of Q. This is defined in the following way: let $Q = \sum_{n=1}^{\infty} \tau_n E_{\phi_n}$ be the spectral decomposition of Q rewritten in such a way that eigenvalues τ_n of Q are written in non-increasing order. Then, we define $P_{sp}(Q)(n) = \oplus_{i=1}^{n} E_{\phi_n}$. Finally, we define a sequence of Gram numbers $g_n(Q)$ connected to Q:

$$\underline{g(Q_1)} = (g_n(Q) = \det(I + Q P_{sp}(Q)(n))). \tag{3.52}$$

Definition 3 Let $Q_1, Q_2 \in E(\mathcal{H})$. We will say the Q_2 m-majorizes Q_1 iff $g_n(Q_1) \leq g_n(Q_2)$ for all n. This will be written as Q_1 m-$\preccurlyeq$ Q_2.

Let $Q_1, Q_2 \in E(\mathcal{H})$. The standard definition of majorization is the following: Q_2 majorizes Q_1 iff $\underline{\sigma(Q_1)} \preccurlyeq \underline{\sigma(Q_2)}$.

Proposition 6 *Let $\mathcal{H}$ be separable Hilbert space and let $Q_1, Q_2 \in E(\mathcal{H})$ be such that $S_+(Q_1)$ m-$\preccurlyeq$ $S_+(Q_2)$. Then*

1. $\mathrm{FEN}_+(Q_1) \leq \mathrm{FEN}_+(Q_2)$,
2. $\mathrm{FEN}_-(Q_1) \geq \mathrm{FEN}_-(Q_2)$,
3. $\mathrm{FEN}_+(Q_1) = \mathrm{FEN}_+(Q_2)$ iff $\sigma(Q_1) = \sigma(Q_2)$,
4. $\mathrm{FEN}_-(Q_1) = \mathrm{FEN}_-(Q_2)$ iff $\sigma(Q_1) = \sigma(Q_2)$.

Proof The point (1) and (2) follows from the fact that majorisation in the sense of Definition 3 is equivalent to the m-majorisation of the considered entropy generating operators from which follows, using Corollary 3.2, that they are also in the standard majorisation relation.

More details for this: let $\sigma(Q_1) = (\lambda_n)$ and $\sigma(Q_2) = (\mu_n)$. Then, $\sigma(S_+(Q_1)) = ((1 + \lambda_k)^{1+\lambda_k} - 1)$ and similarly for $\sigma(S_+(Q_2)) = ((1 + \mu_k)^{1+\mu_k} - 1)$. It follows from Corollary 3.2:

$$((1 + \lambda_k)^{1+\lambda_k}) \preccurlyeq ((1 + \mu_k)^{1+\mu_k}). \tag{3.53}$$

Using the fact that log is convex, it follows that

$$((1 + \lambda_k) \log(1 + \lambda_k)) \preccurlyeq ((1 + \mu_k) \log(1 + \mu_k)). \tag{3.54}$$

Application the standard, finite dimensional arguments leads to the inequalities:

$$\mathrm{FEN}_+(Q_1 P_{sp}(n)) \leq \mathrm{FEN}_+(Q_2 P_{sp})(n)).$$

Using the L_1 convergence $\lim_{n\to\infty} P_{sp}(Q)(n) = Q$ and the continuity of FEN$_\pm$ the proof of (1) follows. The proof of (2) is almost identical to that for (1).

To prove (3) and (4) let us introduce the following interpolation: if $\sum_{n=1}^\infty \lambda_n E_{\phi_n}$, resp. $\sum_{n=1}^\infty \mu_n E_{\omega_n}$ are the spectral decompositions of Q_1, resp. Q_2 then

$$Q(t) = \sum_{n=1}^\infty (t\lambda_n + (1-t)\mu_n) E_{\phi_n}. \tag{3.55}$$

It is easy to see that assuming $Q_1 \preccurlyeq Q_2$

$$\sigma(Q_1) \preccurlyeq \sigma(Q(t)) \preccurlyeq \sigma(Q_2), \tag{3.56}$$

from which we conclude that if FEN$(Q_1) =$ FEN(Q_2) then FEN$(Q(t)) =$ const. It is not difficult to prove that

$$\text{FEN}(Q(t)) = \sum_{n=1}^\infty (1 + t\lambda_n + (1-t)\mu_n) \log(1 + t\lambda_n + (1-t)\mu_n), \tag{3.57}$$

as function of t is smooth. Calculating the second derivative of its we find

$$\frac{d^2}{dt^2}\text{FEN}(Q(t)) = \sum_{n=1}^\infty \frac{(\lambda_n - \mu_n)^2}{1 + t\lambda_n + (1-t)\mu_n} = 0. \tag{3.58}$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Before we present (after [33, 35, 36] and with minor modifications) infinite dimensional generalization of the fundamental in this context Alberti-Uhlmann theorem, we briefly recall some definitions.

A completely positive map Φ on a von Neumann algebra $L_\infty(\mathcal{H})$ is said to be normal if Φ is continuous with respect to the ultraweak ($\ast$-weak) topology. Normal completely positive contractive maps on $B(\mathcal{H})$ are characterized by the theorem of Kraus which says that Φ is a normal completely positive map if and only if there exists at least one sequence $(A_i)_{i=1,\dots}$ of bounded operators in $L_\infty(\mathcal{H})$ such that for any $Q \in L_\infty(\mathcal{H})$:

$$\Phi(Q) = \sum_{i=1}^\infty A_i Q A_i^\dagger, \tag{3.59}$$

where

$$\sum_{i=1}^\infty A_i A_i^\dagger \leq I_\mathcal{H}, \tag{3.60}$$

and where the limits are defined in the strong operator topology. A normal completely positive map Φ which is trace preserving is called a quantum channel. If a normal completely positive map Φ satisfies $\Phi(I_{\mathcal{H}}) \leq I_{\mathcal{H}}$ then called a quantum operation. A quantum operation Φ is called unital iff $\Phi(I_{\mathcal{H}}) = I_{\mathcal{H}}$ which is equivalent to $\sum_{i=1}^{\infty} A_i A_i^{\dagger} = I_{\mathcal{H}}$ for some Krauss decomposition of Φ.

A quantum operation Φ is called bistochastic operation if it is both trace preserving and unital. Central notion for us is the notion of a mixed unitary operation.

A quantum operation Φ is called a (finite) mixed unitary operation iff there exists a (finite) ensemble $\{U_i\}_{i=1:n}$ of unitary operators on $\mathcal{H}$ and a (finite) sequence $p_i \in [0, 1]$ such that $\sum_{i=1}^{n} p_1 = 1$ and

$$\Phi(Q) = \sum_{i=1}^{n} p_i U_i Q U_i^{\dagger}. \tag{3.61}$$

Theorem 7 *Let $\mathcal{H}$ be a separable Hilbert space and let $Q_1, Q_2 \in E(\mathcal{H})$. Assume that Q_1 m-$\preccurlyeq Q_2$. Then, there exists a sequence (Φ_n) of mixed unitary operations and a limiting bi-stochastic operation Φ^* such that the sequence of states $\Phi_n(Q_2)$ is L_1-convergent to $\Phi^*(Q_2) = Q_1$.*

Proof The only essential difference comparing to the original formulation of this result [33, 35, 36] is that instead of $\preccurlyeq$ type majorisation m-$\preccurlyeq$ is used. □

Also the following result is true.

Theorem 8 *Let $\mathcal{H}$ be a separable Hilbert space and let Φ be any quantum operation acting on $E(\mathcal{H})$. Then*

$$\det(I_{\mathcal{H}} + \Phi(Q)) \leq \det(I_{\mathcal{H}} + Q). \tag{3.62}$$

Proof Let T be any non-expansive linear operator acting on $\mathcal{H}$—this means that the operator norm of T, $\|T\| \leq 1$. Using the Grothendick formula (A.7) and the following reasoning:

$$\mathrm{Tr}\,[\wedge^n(T Q T^{\dagger})] = \sum_{i_{1_{<\cdots<i_n}}} \langle i_1 \ldots i_n | (T Q T^{\dagger})^{\otimes n} |i_1 \ldots i_n\rangle$$

$$= \sum_{i_{1<\cdots<i_n}} \prod_{k=1:n} \langle i_k| T Q T^{\dagger} |i_k\rangle \tag{3.63}$$

$$\leq \sum_{i_{1<\cdots<i_n}} \langle i_1 \ldots i_n| Q^{\otimes n} |i_1 \ldots i_n\rangle = \mathrm{Tr}[\wedge^n Q],$$

where we have used the assumption that the norm T of is not bigger than 1 and positivity of Q.

Now, let us assume that we have a pair of bounded operators T_1, T_2 and such that $T_1 T_1^{\dagger} + T_2 T_2^{\dagger} \leq I_{\mathcal{H}}$. For $Q \in E(\mathcal{H})$:

$$\mathrm{Tr}[\wedge^n(T_1 Q T_1^{\dagger} + T_2 Q T_2^{\dagger})] = \sum_{i_1<\cdots<i_n} \prod_{k=1:n} \langle i_k|(T_1 T_1^{\dagger} + T_2 T_2^{\dagger})Q|i_k\rangle$$

$$\leq \sum_{i_1 < \cdots < i_n} \langle i_1 \ldots i_n | Q^{\otimes n} | i_1 \ldots i_n \rangle$$

$$= \mathrm{Tr}[\wedge^n(Q)]. \tag{3.64}$$

Now, the general case follows by application of Krauss representation theorem for quantum operations (3.59) and some elementary inductive and continuity arguments.
□

Several additional results on renormalized version of von Neumann entropy, in particular on the invariance and monotonicity properties of von Neumann entropy in the infinite dimensional setting of conditional entropies, are included in [55].

4 The case of tensor product of states

4.1 Renormalized Kronecker products

Let us recall the finite dimensional formula for computing determinant of tensor product of matrices.

Lemma 4.1 (Kronecker formula) *Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be a pair of finite dimensional Hilbert spaces with dimension N_A, and resp. N_B. Then, for any $Q_A \in L(\mathcal{H}_A)$ and $Q_B \in L(\mathcal{H}_B)$ the following formula is valid*

$$\det(Q_A \otimes Q_B) = \left(\det(Q_A) \right)^{N_B} \cdot \left(\det(Q_B) \right)^{N_A}. \tag{4.65}$$

Proof (quick-argument based). Let stands I_A, respectively I_B stands for the unit operators in the corresponding spaces $\mathcal{H}$. Then

$$Q_A \otimes Q_B = (I_A \otimes Q_B)(Q_A \otimes I_B) \tag{4.66}$$

from which it follows easily the Kronecker formula (4.65).   □

If one of the factors in (4.65) is infinite dimensional and the determinant (absolute value of) of the corresponding matrix Q is strictly bigger than one (or strictly smaller than one) then the value det of the product (4.65) is infinite, respectively equal to zero.

In order to understand better this problem, we define renormalized Kronecker product

$$(I_A + Q_A) \otimes_r (I_B + Q_B) := I_{\mathcal{H}} + Q_A \otimes Q_B \tag{4.67}$$

which formally can be written as:

$$(I_A + Q_A) \otimes_r (I_B + Q_B) := (I_A + Q_A) \otimes (I_B + Q_B) - Q_A \otimes I_B - I_A \otimes Q_B. \tag{4.68}$$

Proposition 9 *Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be a pair of separable Hilbert spaces of an arbitrary dimensions $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and let $Q_A \in E(\mathcal{H}_A)$ and $Q_B \in E(\mathcal{H}_B)$. Then the map*

$$z \mapsto \det(I_{\mathcal{H}} + zQ_A \otimes Q_B) \tag{4.69}$$

defines an entire function in the complex plane and such that the following estimate is valid:

$$|\det(I_{\mathcal{H}} + zQ_A \otimes Q_B)| \le \exp(|z|). \tag{4.70}$$

The proof is an immediate consequence of the Theorem 1 (i) and Lemma (4.2) below.

Lemma 4.2 *Let $Q_A \in E(\mathcal{H}_A)$ and $Q_B \in E(\mathcal{H}_B)$. Then $Q_A \otimes Q_B \in E(\mathcal{H}_A \otimes \mathcal{H}_B)$.*

Proof Recall that the spectrum $\sigma(Q_A \otimes Q_B)$ is given by

$$\sigma(Q_A \otimes Q_B) = (\lambda\mu, \lambda \in \sigma(Q_A), \mu \in \sigma(Q_B)) \tag{4.71}$$

from which it follows:

$$\mathrm{Tr}[Q_A \otimes Q_B] = \mathrm{Tr}[Q_A] \cdot \mathrm{Tr}[Q_B] = 1. \tag{4.72}$$

This completes the proof. □

Another renormalization of the tensor product can be achieved by the use of infinite dimensional Grassmann algebras as we have outlined in the Appendix A to this note. For this goal let us define

$$(I_{\mathcal{H}_A} + Q_A) \otimes_{fr} (I_{\mathcal{H}_B} + Q_B) := (I_{\mathcal{H}_A} + Q_A) \wedge (I_{\mathcal{H}_A} + Q_B), \tag{4.73}$$

where $\wedge$ stands for skew (antisymmetric) tensor product and the right hand side here is defined as a one particle operator in the skew Grassmann algebras built on $\mathcal{H}_A$ and $\mathcal{H}_B$, see Appendix A. Using the unitary isomorphism map J in between the antisymmetric product of fermionic Fock spaces build on the spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ (see Appendix A and the Theorem 27) and the antisymmetric Fock build on the space $\mathcal{H}_\oplus = \mathcal{H}_A \oplus \mathcal{H}_B$, we can define

$$\det\left((I_A + Q_A) \otimes_{fr} (I_B + Q_B)\right) := \det\left(I_{\mathcal{H}_A \oplus \mathcal{H}_B} + Q_A \oplus Q_B\right). \tag{4.74}$$

Theorem 10 *Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be a pair of separable Hilbert spaces of an arbitrary dimensions and $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and let $Q_A \in L_1(\mathcal{H}_A)$ and $Q_B \in L_1(\mathcal{H}_B)$. Then, the map*

$$z \to \det(I_{\mathcal{H}_A \oplus \mathcal{H}_B} + zQ_A \oplus Q_B) \tag{4.75}$$

defines an entire function in the complex plane and such that the following estimate is valid:

$$|\det(I_{\mathcal{H}} + zQ_A \oplus Q_B)| \leq \exp(|z|(\|Q_A\|_1 + \|Q_B\|_1). \tag{4.76}$$

Proof As we have proved in the Theorem (27), the right hand side of (4.75) is equal to the product $\det(I_A + Q_A)\det(I_B + Q_B)$. Having this, the claim of this theorem follows by a straightforward application of Theorem 1 (i). □

Remark 4 For an interesting paper on the influence of quantum statistics on the entanglement see i.e. [68].

Another interesting implication of Theorem 27 seems to be the following observation.

Theorem 11 *Let $\mathcal{H} = \oplus_{i=1}^{N}\mathcal{H}_i$ and $Q \in L_1(\mathcal{H})$ and such that $Q = \oplus \lambda_i Q_i$, where $Q_i \in E(\mathcal{H}_i)$ for all $i = 1, ..., \lambda_i \geq 0, \sum_i^N \lambda_i = 1$.*
Then $Q \in E(\mathcal{H})$ and

$$\text{FEN}_{\pm}(Q) = \sum_{i=1}^{N} \text{FEN}_{\pm}(\lambda_i Q_i). \tag{4.77}$$

Proof Let us observe that the renormalized entropy operators $S_{\pm}$ can be decomposed as:

$$S_{\pm}(Q) = \oplus_{i=1}^{N} S_{\pm}(\lambda_i Q_i) = \oplus_{i=1}^{N}\left[(I_{\mathcal{H}_i} + \lambda_i Q_i)^{\pm(I_{\mathcal{H}_i} + \lambda_i Q_i)} - I_{\mathcal{H}_i}\right]. \tag{4.78}$$

Therefore, using Theorem A.4, we obtain

$$\begin{aligned}
\text{FEN}_{\pm}(Q) &= \log \det(I_{\mathcal{H}} + S_{\pm}(Q)) \\
&= \log\left(\prod_{i=1}^{N} \det(I_{\mathcal{H}_i} + S_{\pm}(\lambda_i Q_i))\right) \\
&= \sum_{i=1}^{N} \text{FEN}_{\pm}(\lambda_i Q_i).
\end{aligned} \tag{4.79}$$

□

Also the following result seems to be interesting.

Theorem 12 *Let $\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B$ be a separable Hilbert space and let Φ be a separable quantum operation on $\mathcal{H}$, i.e. $\Phi = \Phi^A \otimes \Phi^B$, where Φ^A, resp. Φ^B are local quantum operations. Then for any $Q \in E(\mathcal{H})$:*

$$\det\left(1_{\mathcal{H}} + \Phi(Q)\right) \leq \det(1_{\mathcal{H}} + Q). \tag{4.80}$$

Proof Let $K_i^A, i = 1, \ldots$, resp. $K_j^B, j = 1, \ldots$ be the families of operators giving the Kraus representations, for

$$\Phi^A(A) = \sum_{i=1} K_i^A A K_i^{A\dagger}, \tag{4.81}$$

and, resp.

$$\Phi^B(A) = \sum_{i=1} K_i^B A K_i^{B\dagger}. \tag{4.82}$$

Then, for any $Q \in E(\mathcal{H})$:

$$\Phi(Q) = \sum_{i,j} K_i^A \otimes K_j^B(Q) K_i^{A\dagger} \otimes K_j^{B\dagger}. \tag{4.83}$$

Taking into account that

$$\sum_{i,j} \left(K_i^A \otimes K_j^B \right) \left(K_i^A \otimes K_j^B \right)^\dagger = \left(\sum_i K_i^A \cdot K_i^{A\dagger} \right) \otimes \left(\sum_{i=1} K_i^B \cdot K_i^{B\dagger} \right)$$

$$\leq 1_{\mathcal{H}_A} \otimes 1_{\mathcal{H}_B} = 1_{\mathcal{H}}, \tag{4.84}$$

the proof follows as the proof of Theorem 8. $\qquad\qquad\square$

4.2 Reduced density matrices—the bipartite case

Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ be the tensor product of two separable Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ of arbitrary dimensions. In this section, we assume that both spaces $\mathcal{H}_A, \mathcal{H}_B$ are infinite dimensional (everything works also in finite dimensional situations [41], and also in situation for which only one of the spaces $\mathcal{H}_i$ is finite dimensional as well [41]).

Let Q^A (respectively Q^B) be the corresponding reduced density matrices obtained from Q by tracing out the corresponding degrees of freedom. Then $Q^A \geq 0$, $\mathrm{Tr}_{\mathcal{H}_A}[Q^A] = 1$, and identically in the case of Q^B. As is well known the spectrum $\sigma(Q^A) = (\lambda_n)$ is purely discrete (we are presenting it always with the corresponding multiplicities and in nonincreasing order) and in general different from the spectrum of Q^B in the case of mixed states. For more on this see below and the Appendix B. In the case when, as in the introduction, $Q = |\Psi\rangle\langle\Psi|$ for some $\Psi \in \mathcal{H}$ the spectrum of Q^A and Q^B are equal to each other and equal to the list of squared Schmidt coefficients of the corresponding Schmidt decomposition of the vector Ψ [3, 4, 69]. The same is valid for the Hilbert–Schmidt level reduced density matrices when we consider these type of Schmidt decompositions of a given $Q \in \mathcal{H}$, see Appendix B and [44, 46].

Let us recall now some well-known facts on the reduced density matrices. Let $Q \in E(\mathcal{H})$. Let $\{|i\rangle\}$ be an arbitrary complete orthonormal system of vectors in $\mathcal{H}_B$.

Then, we have canonical unitary equivalence

$$\mathcal{H}_A \otimes \mathcal{H}_B \cong \oplus_i \mathcal{H}_A \otimes |i\rangle, \tag{4.85}$$

where $\cong$ means that $\varphi \in \mathcal{H}$ is decomposed as $|\varphi\rangle \cong \oplus_i (I_{\mathcal{H}_A} \otimes |i\rangle\langle i|)|\varphi\rangle$.

Then for any $A \in B(\mathcal{H})$, we can write:

$$A = \left(\sum_{i=1}^{\infty} 1_{\mathcal{H}_A} \otimes |i\rangle\langle i| \right) (A) \left| \left(\sum_{i=1}^{\infty} 1_{\mathcal{H}_A} \otimes |i\rangle\langle i| \right) \right| = \sum_{i,j}^{\infty} A_{ij}, \tag{4.86}$$

where

$$A_{ij} = (1_{\mathcal{H}_A} \otimes |i\rangle\langle i|)(A)(1_{\mathcal{H}_A} \otimes |i\rangle\langle i|), \tag{4.87}$$

is the bounded linear map from $\mathcal{H}_A \otimes |i\rangle$ to $\mathcal{H}_A \otimes |j\rangle$.

Using the Krauss decomposition Theorem 3.59, we have the following observation: the linear and bounded map

$$\begin{aligned} &\text{Tr}_B : L_1(\mathcal{H}) \mapsto L_1(\mathcal{H}_A), \\ &A \mapsto \text{Tr}_B(A) \cong \sum_{i=1}^{\infty} A_{ii}, \end{aligned} \tag{4.88}$$

named partial trace map is a quantum operation in the sense of the previously introduced definition in Sect. 3.

Theorem 13 *Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be a pair of separable Hilbert spaces of an arbitrary dimensions $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and let $Q \in E(\mathcal{H})$ and let $Q^A = \text{Tr}_B(Q) \in E(\mathcal{H}_A)$ and $Q^B = \text{Tr}_A(Q) \in E(\mathcal{H}_B)$ be the corresponding reduced density matrices. Then:*

$$\begin{aligned} FEN_-(Q^A) &\leq FEN_-(Q), \\ FEN_-(Q^B) &\leq FEN_-(Q). \end{aligned} \tag{4.89}$$

Proof Follows from the formula 4.88 which demonstrates that the operations of taking partial traces are quantum operations and application of Theorem 8. □

Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ be a bipartite separable Hilbert space and let $Q \in E(\mathcal{H})$. It is well known that the spectrum of Q counted with multiplicities, denoted $\sigma(Q) = (\lambda_1, ...)$ is purely discrete and the following spectral decomposition holds:

$$Q = \sum_{n=1}^{\infty} \lambda_i |\Psi_n\rangle\langle\Psi_n|, \tag{4.90}$$

where the orthogonal (and normalized) system of eigenfunctions $|\Psi_n\rangle$ of Q forms a complete system. Each eigenfunction $|\Psi_n\rangle$ can be expanded further by the use of the Schmidt decomposition:

$$|\Psi_n\rangle = \sum_{i=1}^{\infty} \tau_i^n |\psi_i^n \otimes \phi_i^n\rangle, \tag{4.91}$$

where $\tau_i^n \geq 0 \sum_{n=1}^{\infty} (\tau_i^n)^2 = 1$ and the systems $\{\psi_i^n\}$ and $\{\phi_i^n\}$ form the complete orthonormal systems in $\mathcal{H}_A$ and, respectively, $\mathcal{H}_B$. Using (4.91) and (4.90), we can compute the corresponding reduced density matrices

$$Q^B = \mathrm{Tr}_A \left[\sum_{i=1}^{\infty} \lambda_n |\Psi_n\rangle\langle\Psi_n| \right] = \sum_{n=1}^{\infty} \lambda_n Q_n^B, \tag{4.92}$$

where the operators

$$Q_n^B = \sum_{i=1}^{\infty} |\tau_n|^2 |\phi_i^n\rangle\langle\phi_i^n| \tag{4.93}$$

are the states on $\mathcal{H}_B$. Similarly, for the reduced density matrix connected to the observer localized with $\mathcal{H}_A$:

$$Q^A = \mathrm{Tr}_B(Q) = \mathrm{Tr}_B \left[\sum_n \lambda_n |\Psi_n\rangle\langle\Psi_n| \right] = \sum_{n=1}^{\infty} \lambda_n Q_n^A, \tag{4.94}$$

where $Q_n^A = \sum_{i=1}^{\infty} |\tau_i^n|^2 |\psi_i^n\rangle\langle\psi_i^n|$ are states on $\mathcal{H}_A$. The obtained systems of operators $\{Q_n^A\}$ and $\{Q_n^B\}$ consist of bounded non-negative self-adjoint, local operators of class $L_1(\mathcal{H}_A)$, respectively of class $L_1(\mathcal{H}_B)$ and therefore they are locally measurable. In particular the squares of the Schmidt coefficients τ_i^n of the Schmidt decompositions of the eigenfunctions of the parent state Q are observable (measurable locally) quantities.

Proposition 14 *Let* $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ *be a bipartite separable Hilbert space and let* $Q \in E(\mathcal{H})$. *Let* (Q^A, Q^B) *be the corresponding reduced density matrices and let*

$$Q^A(n) = \sum_{i=1}^{\infty} |\tau_i^n|^2 |\psi_i^n\rangle\langle\psi_i^n|. \tag{4.95}$$

And corr.

$$Q^B(n) = \sum_{i=1}^{\infty} |\tau_i^n|^2 |\phi_i^n\rangle\langle\phi_i^n|. \tag{4.96}$$

Then, for any n:

1. $G(Q^A(n)) = \det(I_{\mathcal{H}_A} + Q^A(n)) = \prod_{j=1}^{\infty} (1 + (\tau_j^n)^2) \leq e,$
2. *The value* $G(Q^A(n))$ *is invariant under the action of unitary group, for any unitary map* $U \in \mathcal{H}_A$:

$$G(U Q^A(n) U^\dagger) = G(Q^A(n)) \tag{4.97}$$

3. *The value $G(Q^A(n))$ is not increasing under the action of any local quantum operation Φ acting on $E(\mathcal{H}_A)$:*

$$G(\Phi(Q^A(n))) \leq G(Q^A(n)) \tag{4.98}$$

Identical facts are valid for the reduced density matrices $Q^B(n)$.

Proof Obvious. □

Remark 5 The list $\Gamma(Q) = (r_j^n)$ associated with Q is locally $SU(\mathcal{H}_A) \otimes SU(\mathcal{H}_B)$ matrix valued invariant of Q (after taking care on the localisation in this $2d$ table of the corresponding Schmidts numbers). Therefore, any scalar functions build on Γ will define a locally-unitary invariant of Q. Some of them are additionally also monotonous under the action of the local quantum operations and therefore are promising candidates for being a "good" [3–6] quantitative measures of quantum correlations included in Q. More on this is reported elsewhere [43, 45].

Another approach to certain version of reduced density matrices structure is based on the use of the Schmidt decomposition method in the Hilbert-Schmidt space of operators build on the space $\mathcal{H}_A \otimes \mathcal{H}_B$. Some details are presented in appendix B and in paper [44].

Systematic and much wider applications of the obtained forms of the reduced density matrices will be presented in an another publications (under preparations now).

4.3 The case of pure states

Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ be a bipartite, separable Hilbert space and let $Q \in E(\mathcal{H})$ be such that $\text{tr}(Q^2) = 1$. Then, there exists an unique, normalized vector $|\Psi\rangle \in \mathcal{H}$ such that $Q = |\Psi\rangle\langle\Psi|$.

Let $\{e_i^A, i = 1, \dots\}$, resp. $\{e_j^B, j = 1, \dots\}$ be some complete orthonormal systems in $\mathcal{H}_A$, resp. in $\mathcal{H}_B$.

Then, we can write:

$$|\Psi\rangle = \sum_{i,j=1}^{\infty} \Psi_{ij} |e_i^A\rangle \otimes |e_j^B\rangle \tag{4.99}$$

where $\Psi_{ij} = \langle e_j^B \otimes e_i^A | \Psi \rangle$.

We start with the Schmidt decomposition (essentially SVD decomposition, see i.e. Thm. 26.8 in [11]) in the infinite dimensional setting.

Theorem 15 *For any unit vector $|\Psi\rangle \in \mathcal{H}$ there exist*

- *a sequence of non-negative numbers τ_n (called the Schmidt coefficients of Ψ) and such that $\sum_{n=1}^{\infty} \tau_n^2 = 1$,*

- *two, complete orthonormal systems of vectors $\{\phi_n\}$ in $\mathcal{H}_A$ and $\{\omega_n\}$ in $\mathcal{H}_B$ such that the following equality (in the L_2-space sense) holds:*

$$|\Psi\rangle = \sum_{n=1}^{\infty} \tau_n |\phi_n\rangle |\omega_n\rangle. \tag{4.100}$$

The decomposition 4.100 is called the Schmidt decomposition of $|\Psi\rangle$. The expansion formula 4.100 can be rewritten as:

$$|\Psi\rangle = \sum_{i=1}^{\infty} |e_i^A\rangle |F_i^B\rangle \tag{4.101}$$

where

$$|F_i^B\rangle = \sum_{j=1}^{\infty} \Psi_{ij} |e_j^B\rangle \tag{4.102}$$

and also

$$|\Psi\rangle = \sum_{j=1}^{\infty} |F_j^A\rangle |e_j^B\rangle, \tag{4.103}$$

where

$$|F_j^A\rangle = \sum_{i=1}^{\infty} \Psi_{ij} |e_j^A\rangle. \tag{4.104}$$

Let us define pair of linear maps $J^A : \mathcal{H}_A \to \mathcal{H}_B$, resp. $J^B : \mathcal{H}_B \to \mathcal{H}_A$ by the following

$$J^A : |e_i^A\rangle \to |F_i^B\rangle \tag{4.105}$$

and then extended by linearity and continuity to the whole $\mathcal{H}_A$. In an identical way the map J^B is defined. Both of the introduced operators J are bounded as can be seen by simple arguments. Now, we define a pair of operators which plays an important role in the following

$$\Delta^A(\Psi) : J^{A^\dagger} J^A : \mathcal{H}_A \to \mathcal{H}_A \tag{4.106}$$

and similarly

$$\Delta^B(\Psi) : J^{B^\dagger} J^B : \mathcal{H}_B \to \mathcal{H}_B \tag{4.107}$$

Some elementary properties of the introduced operators Δ^A and Δ^B are collected in the following proposition.

Proposition 16 *The operators Δ^A and Δ^B have the following properties:*

(RDM1) They both are non-negative and bounded $\|\Delta^A\|_1 = \|\Delta^B\|_1 = 1$.
(RDM2) The nonzero parts of the spectra of Δ^A and Δ^B coincides and are equal to squares τ_n^2 of nonzero Schmidt numbers in (4.100).
(RDM3) In particular the following formulas are valid:

$$\Delta^A|\phi_n\rangle = \tau_n^2|\phi_n\rangle,$$

$$\Delta^B|\omega_n\rangle = \tau_n^2|\omega_n\rangle,$$

which means that the kets $|\phi_n\rangle$ are eigenvectors of the reduced density matrix Q^A, and similarly for Q^B.

The interesting observation is that the explicit Gram matrix nature (it is well known fact [66] that any (semi)-positive matrix has a Gram matrix structure) of the operators Δ can be flashed on.

Proposition 17 *Let*

$$|\Psi\rangle = \sum_{i,j=1}^{\infty} \Psi_{ij}|e_i^A\rangle \otimes |e_j^B\rangle \in \mathcal{H}, \tag{4.108}$$

be given. Then, the matrix elements of the corresponding operators Δ, given in the product base $|e_i^A\rangle \otimes |e_j^B\rangle$ are given by the formulas below

$$\Delta_{ij}^A(\Psi) = \langle e_j^A|\Delta^A e_i^A\rangle_{\mathcal{H}_A} = \langle F_j^B|F_i^B\rangle_{\mathcal{H}_B} \tag{4.109}$$

and similarly

$$\Delta_{ij}^B(\Psi) = \langle e_j^B|\Delta^B e_i^B\rangle_{\mathcal{H}_B} = \langle F_j^A|F_i^A\rangle_{\mathcal{H}_A} \tag{4.110}$$

where the corresponding vectors F are given by (4.102) and (4.104).

In the finite dimensional case the following, nice geometrical picture is known [43]. Let $\{v_i, i = 1, ..d\}$ be a system of linearly independent vectors in the space C^d, where $d = d$. Let us build on these vectors a d dimensional parallelepiped. Then, the Euclidean volume of this parallelepiped is equal to the determinant of the Gram matrix built on these vectors. The matrix elements of this Gram matrix are given by the scalar products $\langle v_i|v_j\rangle$ for $i, j = 1 : d$. Under the condition that the sum of the lengths of the spanning vectors v_i is equal to 1 the parallelepiped which has the maximal volume is that which is spanned by the system of orthogonal vectors of equal length. In this particular case, the corresponding Gram matrix elements are equal to $(1/d)\delta_{ij}$. In a general case, the volume of the parallelepiped spanned by the vectors forming some

square matrix columns (or rows) can be estimated from above be several inequalities. The Hadamard inequality saying that this volume is no bigger than the product of the lengths of the spanning vectors v_i is the best known among them. For more on this see [43].

On the basis of results and facts presented in previous sections, we can define the following quantity (in fact entire function of z) that will be called gramian function of the state $|\Psi\rangle$.

$$G(\Psi)(z) = \det(I_A + z\Delta^A(\Psi)) = \det(I_B + z\Delta^B(\Psi)) = \prod_{n=1}^{\infty}(1 + z\tau_n^2).$$

(4.111)

In particular case when $z = 1$ the value of the gramian function G of state $|\Psi\rangle$ will be called the gramian volume of $|\Psi\rangle$ and denoted as $G(\Psi)$. The logarithm of the gramian volume will be called the logarithmic (gramian) volume of $|\Psi\rangle$ and denoted as $g(\Psi)$. Using (4.111), it follows that

$$g(\Psi) = \sum_{n=1}^{\infty}\log(1 + \tau_n^2).$$

(4.112)

Proposition 18 *Let*

$$|\Psi\rangle = \sum_{i,j=1}^{\infty}\Psi_{ij}|e_i^A\rangle \otimes |e_j^B\rangle \in \mathcal{H}.$$

(4.113)

Then, the gramian volume $G(\Psi)$ has the following properties:

1. *For any $|\Psi\rangle$: $2 \le G(\Psi) \le e$.*
2. *$G(\Psi) = 2$ iff Ψ is a separable state, i.e. Schmidt rank of Ψ is equal to 1.*
3. *Let $\mathcal{U}(\mathcal{H})$ be a multiplicative group of unitary operators acting in the Hilbert space $\mathcal{H}$. Then, the gramian volume of $|\Psi\rangle$ is invariant under the action on of the local unitary groups $\mathcal{U}(\mathcal{H}_A) \otimes \mathcal{U}(\mathcal{H}_B)$.*
4. *Let $\Phi_{A(B)}$ be any local quantum operation on the local space $\mathcal{H}_A$ (resp. $\mathcal{H}_B$). Then*

   (a) *$G((\Phi_A \otimes I_B)(\Psi)) \le G(\Psi)$,*
   (b) *$G((I_A \otimes \Phi_B)(\Psi)) \le G(\Psi)$,*
   (c) *$G((\Phi_A \otimes \Phi_B)(\Psi)) \le G(\Psi)$.*

We can see that the Gramian volume of pure states is locally invariant (under the local unitary operations action) quantity. And what is also important, we have proved that the gramian volume defined in (4.111) do not increase under the action of any separable quantum operation. This is why we think that the Gramian volume might be a very good candidate for the entanglement measure included in pure quantum states.

Sometimes it is more useful to use logarithmic Gramian volume g instead of the Gramian volume G. Some basic properties of the logarithmic volume g are contained in the following Theorem.

Theorem 19 *Let*

$$|\Psi\rangle = \sum_{i,j=1}^{\infty} \Psi_{ij} |e_i^A\rangle \otimes |e_j^B\rangle \in \mathcal{H}. \tag{4.114}$$

Then, the logarithmic Gramian volume $g(\Psi)$ has the following properties;

1. For any $|\Psi\rangle$

$$g(\Psi) = \sum_{n=1}^{\infty} \log(1 + \tau_n^2), \tag{4.115}$$

where τ_n are the Schmidt numbers of $|\Psi\rangle$.
2. For any $|\Psi\rangle$:

$$\log(2) \leq g(\Psi) \leq 1. \tag{4.116}$$

3. $g(\Psi) = \log(2)$ iff Ψ is a separable state, i.e. Schmidt rank of Ψ is equal to 1.
4. Let $\mathcal{U}(\mathcal{H})$ be a multiplicative group of unitary operators acting in the Hilbert space $\mathcal{H}$. Then, the logarithmic gramian volume of $|\Psi\rangle$ is invariant under the action on of the local unitary groups

$$\mathcal{U}(\mathcal{H}_A) \otimes I_B, \quad I_A \otimes \mathcal{U}(\mathcal{H}_B), \quad \mathcal{U}(\mathcal{H}_A) \otimes \mathcal{U}(\mathcal{H}_B) \tag{4.117}$$

5. Let $\Phi_{A(B)}$ be any local quantum operation on the local space $\mathcal{H}_A$ (resp. $\mathcal{H}_B$). Then

(a) $g((\Phi_A \otimes I_B)(\Psi)) \leq g(\Psi)$,
(b) $g((I_A \otimes \Phi_B)(\Psi)) \leq g(\Psi)$,
(c) $g((\Phi_A \otimes \Phi_B)(\Psi)) \leq g(\Psi)$.

Similar results are true for the renormalized von Neumann entropies. For this goal, let us recall the definitions of entropies generating operators:

$$S_-(\Psi) = (I_{\mathcal{H}_A} + \Delta^A(\Psi))^{-(I_{\mathcal{H}_A} + \Delta^A(\Psi))} - I_{\mathcal{H}_A} = \sum_{n=1}^{\infty} \left((1 + \tau_n^2)^{-(1+\tau_n^2)} - 1 \right) |\phi_n\rangle\langle\phi_n|. \tag{4.118}$$

From which we obtain an estimate

Lemma 4.3 *For any pure state $|\Psi\rangle \in \mathcal{H}$ the renormalized entropy generator defined as $S_-(\Psi)$ in (4.118) obeys the bound:*

$$\|S_-(\Psi)\|_1 \leq 2\|\Delta^A\|_1 = 2. \tag{4.119}$$

Proof We have used the following, rough estimate:

$$|1 - (1 + \tau_n^2)^{-(1+\tau_n^2)}| \leq (1 + \tau_n^2) \log(1 + \tau_n^2) \leq (1 + \tau_n^2)\tau_n^2 \qquad (4.120)$$

From which the bound (4.119) follows immediately.   □

Theorem 20 *Let* $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, *where* $\mathcal{H}_A$ *and* $\mathcal{H}_B$ *are separable Hilbert spaces. Then, for any pure state* $|\Psi\rangle \in E(\mathcal{H})$ *the renormalized entropy defined as*

$$\text{FEN}(\Psi) = \log(\det(\mathrm{I}_A + S_-(\Psi)) = \sum_{n=1}^{\infty} (1 + \tau_n^2) \log(1 + \tau_n^2), \qquad (4.121)$$

is finite, L_1*-continuous on* $E(Q)$ *and bounded by:*

$$0 \leq \text{FEN}(\Psi) \leq 2. \qquad (4.122)$$

Theorem 21 *Let* $\mathcal{U}(\mathcal{H})$ *be a multiplicative group of unitary operators acting in the Hilbert space* $\mathcal{H}$. *Then, the renormalized entropy of* $|\Psi\rangle$ *is invariant under the action on of the local unitary groups* $\mathcal{U}(\mathcal{H}_A) \otimes \mathrm{I}_B$, $\mathrm{I}_A \otimes \mathcal{U}(\mathcal{H}_B)$ *and also* $\mathcal{U}(\mathcal{H}_A) \otimes \mathcal{U}(\mathcal{H}_B)$.

Theorem 22 *Let* $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, *where* $\mathcal{H}_A$ *and* $\mathcal{H}_B$ *are separable Hilbert spaces. Then, for any pure state* $|\Psi\rangle \in E(\mathcal{H})$ *the renormalized entropy defined as*

$$\text{FEN}(\Psi) = \log(\det(\mathrm{I}_A + S_-(\Psi)), \qquad (4.123)$$

is non-increasing under the action of any local quantum operation $\Psi_{A(B)}$ *on the local space* $\mathcal{H}_A$ *(resp.* $\mathcal{H}_B$).

Finally, we mention the monotonicity of the renormalised Entropy with respect to the by majorization relation introduced semi-order. Details will be presented elsewhere [46].

Data availability Data or code sharing was not applicable to this article as no datasets or code were generated or analysed during the study presented in this article.

Declarations

Appendix A: Fermionic Fock space aspects

Let $\mathcal{H}$ be a separable Hilbert space and let the system of vectors $\{e_n\}_{n=1,\dots}$ form a complete orthonormal system (i.e. the orthonormal base) in $\mathcal{H}$. Then, the system $\{e_{i_1} \otimes \cdots \otimes e_{i_n}\}$ forms an orthonormal base in $\mathcal{H}^{\otimes n}$. The free Fock space over $\mathcal{H}$ is defined as:

$$\Gamma(\mathcal{H}) = \oplus_{n=0}^{\infty} \mathcal{H}^{\otimes n} \tag{A.1}$$

where $\mathcal{H}^{\otimes 0} = \mathcal{C}$.

The anti-symmetrization operator $\wedge^n$ on the n-fold summand of Eq. (A.1):

$$\wedge^n (f_1 \otimes \cdots \otimes f_n) = \frac{1}{n!} \sum_{\pi \in S_n} (-1)^s (\pi) f_{\pi(1)} \otimes \cdots \otimes f_{\pi(n)} \equiv f_1 \wedge \cdots \wedge f_n, \tag{A.2}$$

where S_n stands for symmetric group of order n and $s(\pi)$ stands for the parity of π.

Operator $\wedge^n$ is then extended by linearity and continuity and normalized properly to be the orthogonal projector acting in the free Fock space and with the range which is called the fermionic Fock space over $\mathcal{H}$ and denoted as

$$\wedge (\Gamma(\mathcal{H})) = \oplus_{n=0}^{\infty} \wedge^n (\mathcal{H}^{\otimes n}). \tag{A.3}$$

In particular the system $\{e_{i_1} \wedge \cdots \wedge e_{i_n}\}$ forms an orthonormal base in $\wedge(\mathcal{H}^{\otimes n})$.

Lemma A.1 *For any tensors $F = f_1 \otimes \cdots \otimes f_n$ and $G = g_1 \otimes \cdots \otimes g_n$, we have*

$$\langle f_1 \wedge \cdots \wedge f_n | g_1 \wedge \cdots \wedge g_n \rangle = \frac{1}{(n!)^2} \sum_{\pi, \pi' \in S_n} (-1)^{s(\pi)+s(\pi')} \prod_{i=1}^{n} \langle f_{\pi(i)} | g_{\pi'(i)} \rangle$$

$$= \frac{1}{n!} \det(\mathcal{R}(FF|GG)), \tag{A.4}$$

where $\mathcal{R}(FF|GG)$ is the relative Gramian matrix build on $FF = \{f_1, \ldots, f_n\}$ and $GG = \{g_1, \ldots, g_n\}$, see [43] for the corresponding definitions.

The fermionic Fock space over $\mathcal{H}$ is defined as

$$\Gamma_{as}(\mathcal{H}) \cong \oplus_{n=1}^{\infty} \wedge^n (\mathcal{H}), \tag{A.5}$$

i.e. $\Psi = \oplus_{n=1}^{\infty} |\psi_n\rangle \in \Gamma_{as}(\mathcal{H})$, $|\psi_n\rangle \in \wedge^n(\mathcal{H})$ iff $\sum_{n=0}^{\infty} n! \|\psi_n\|^2 < \infty$.

Remark 6 If $\dim(\mathcal{H}) = d < \infty$ then $\wedge^n(\mathcal{H}) = \emptyset$ for $n > d$ and $\dim(\wedge^n(\mathcal{H})) = \binom{d}{n}$ for $n \leq d$. The corresponding antisymmetric Fock spaces in this situation are used for describing fermionic, discrete degrees of freedom.

Let $T \in B(\mathcal{H})$. Then, we lift the action of T onto the Fock space(s) as

$$\Gamma(T) : f_1 \otimes \cdots \otimes f_n \to T f_1 \otimes \cdots \otimes T f_n, \tag{A.6}$$

and similarly for $f_1 \wedge \cdots \wedge f_n$ case.

Let us collect here some well-known facts:

Proposition 23 *Let $T \in B(\mathcal{H})$, then*

1. $\Gamma(T) \in B(\Gamma_*(\mathcal{H}))$ *(where * stands for empty sign or as),*
2. *for $T, S \in B(\mathcal{H})$),*
3. $\Gamma(TS) = \Gamma(T)\Gamma(S)$.

Proposition 24 *Let $T \in L_1(\mathcal{H})$, then*

1. $\sigma(T^{\otimes n}) = (\lambda_{i_1} \cdots \lambda_{i_n}, \ \lambda_i \in \sigma(T))$,
2. $\sigma(T^{\wedge n}) = \left(\lambda_{i_1} \cdots \lambda_{i_n}, \ i_1 < \cdots < i_n, \ \lambda_i \in \sigma(T)\right)$,
3. $\mathrm{Tr}[T^{\otimes n}] = (\mathrm{Tr}[T])^n$,
4. $\mathrm{Tr}[T^{\wedge n}] = \sum_{i_1 < \cdots < i_n} \lambda_{i_1} \cdots \lambda_{i_n} = \frac{1}{n!}(\mathrm{Tr}[T])^n$ *(for $T \geq 0$).*

Corollary A.2 *Let $T \in L_1(\mathcal{H})$, then*

1. $\|T^{\otimes n}\|_1 \leq \|T\|_1^n$,
2. $\|T^{\wedge n}\|_1 \leq \frac{1}{n!}\|T\|_1^n$,
3. $\Gamma(T) \in L_1(\Gamma(\mathcal{H}))$ *if $\|T\|_1 < 1$,*
4. $\Gamma_{as}(T) \in L_1(\Gamma_{as}(\mathcal{H}))$ *iff $\|T\|_1 < \infty$.*

Now, we are in the position to prove the Grothendick result about the possibility to determine the Fredholm determinant of infinite-dimensional matrices by the second-quantisation mathematics methods.

Theorem 25 (Grothendick) *Let $T \in L_1(\mathcal{H})$, then*

$$\det(I_{\mathcal{H}} + T) = \sum_{n=0}^{\infty} \mathrm{Tr}[\wedge^n(T)]. \tag{A.7}$$

From the Grothendick Theorem 25 it is possible to prove in a relatively easy way [48] the quoted in Sect. 2 results on Fredholm determinants and to conclude several other implications of his ingenious approach [32]. The key point is the following observation.

Lemma A.3 *Let $Q \in E(\mathcal{H})$ and $\sigma(Q) = (\lambda_1, \dots)$ be the spectrum of Q. Then*

1. $\mathrm{Tr}[Q^{\otimes n}] = \sum_{i_1, \dots, i_n} \lambda_{i_1} \cdots \lambda_{i_n} = 1$,
2. $\mathrm{Tr}[\wedge^n(Q)] \leq \frac{1}{n!}\mathrm{Tr}[Q^{\otimes n}] = \frac{1}{n!}$.

Proposition 26 *Let $\mathcal{H}_\oplus = \mathcal{H}_A \oplus \mathcal{H}_B$ be a two-particles separable Hilbert space. Then, with the convention that = means temporary the unitary equivalence of the corresponding spaces the following is true:*

1. $\Gamma(\mathcal{H}) = \Gamma(\mathcal{H}_A) \otimes \Gamma(\mathcal{H}_B)$,
2. $\Gamma_{as}(\mathcal{H}) = \Gamma_{as}(\mathcal{H}_A) \wedge \Gamma_{as}(\mathcal{H}_B)$,
3. $\Gamma_{sym}(\mathcal{H}) = \Gamma_{sym}(\mathcal{H}_A) \otimes_{sym} \Gamma_{sym}(\mathcal{H}_B)$.

Proof (sketch of): We concentrate only on the fermionic case (2). The reason is that it is the case which is relevant for the purposes of the present note. The point (1) is true from the general fact that all separable, infinite dimensional spaces are unitary isomorphic to each other. The case of bosonic space is similar to that of fermionic space.

We construct a special unitary map J from the space $\Gamma_{as}(\mathcal{H}_A) \wedge \Gamma_{as}(\mathcal{H}_B)$ to the space $\Gamma_{as}(\mathcal{H})$. For this goal let us observe that:

$$
\Gamma_{as}(\mathcal{H}_A) \wedge \Gamma_{as}(\mathcal{H}_B) = \left(\oplus_{N=0}^\infty \wedge^N(\mathcal{H}_A) \right) \wedge \left(\oplus_{N=0}^\infty \wedge^N(\mathcal{H}_B) \right)
$$

$$
= \oplus_{N=0}^\infty \left(\oplus_{n+m=N} \left(\wedge^n(\mathcal{H}_A) \right) \wedge \left(\wedge^m(\mathcal{H}_B) \right) \right). \tag{A.8}
$$

So, a typical N-particles vector Ψ_N looks in this space like: $\Psi_N = f_1 \wedge \cdots \wedge f_n \wedge g_1 \wedge \cdots \wedge g_m$, where $n + m = N$ and $f_i \in \mathcal{H}_A$ and $g_j \in \mathcal{H}_B$. For such a vector, we define

$$
J(\Psi_N) = (f_1 0) \wedge \cdots \wedge (f_n 0) \wedge (0 g_1) \wedge \cdots \wedge (0 g_m) \tag{A.9}
$$

which is a vector from $\wedge^N(\mathcal{H}_A \oplus \mathcal{H}_B)$. It easy to check that J preserves the norm. Extending J by linearity and continuity argument, we construct the unitary map

$$
J : \Gamma_{as}(\mathcal{H}_A) \wedge \Gamma_{as}(\mathcal{H}_B) \to \Gamma(\mathcal{H}_\oplus). \tag{A.10}
$$

$\square$

Let $T_A \in B(\mathcal{H}_A)$ and respectively $T_B \in B(\mathcal{H}_B)$. Then, $\Gamma(T_A) \in B(\Gamma_{as}(\mathcal{H}_A))$ and resp. $\Gamma(T_B) \in B(\Gamma_{as}(\mathcal{H}_B))$. In particular, if T_A and T_B are of trace class then also $T_A \oplus T_B$ is trace class and moreover:

$$\mathrm{Tr}[T_A \oplus T_B] = \mathrm{Tr}[T_A] + \mathrm{Tr}[T_B]. \tag{A.11}$$

Note that

$$T_A \oplus T_B = \begin{bmatrix} T_A & 0 \\ 0 & T_B \end{bmatrix}, \tag{A.12}$$

as an operator acting in $\mathcal{H}_A \oplus \mathcal{H}_B$.

Theorem 27 *Let* $\mathcal{H} = \mathcal{H}_A \oplus \mathcal{H}_B$ *be a bipartite separable Hilbert space and let* $T_A \in L_1(\mathcal{H}_A)$ *and* $T_B \in L_1(\mathcal{H}_B)$. *The following formula is valid:*

$$\det(1 + T_A)\det(1 + T_B) = \det(1 + T_A \oplus T_B). \tag{A.13}$$

Proof By the use of Grothendick Theorem 25 and Proposition 26 pt.2, i.e. the use of the unitary map J to transport the operator $\Gamma(T_A \oplus T_B)$ onto the skew tensor product $\Gamma_{as}(\mathcal{H}_A) \wedge \Gamma_{as}(\mathcal{H}_B)$. □

Corollary A.4 *Let* $\mathcal{H}$ *be a separable Hilbert space and such that* $\mathcal{H} = \oplus_{i=1}^{\infty} \mathcal{H}_i$ *and let* $T \in L_1(\mathcal{H})$ *be of the form:* $T = \oplus_{i=1}^{\infty} T_i$ *(which implies that* $\sum_{i=0}^{\infty} \|T_i\|_1 < \infty$*)) and* $T_i(\mathcal{H}_i) \subseteq \mathcal{H}_i$, *for all* i. *Then*

$$\det(I_{\mathcal{H}} + T) = \prod_{i=1}^{\infty} \det(I_{\mathcal{H}_i} + T_i) \tag{A.14}$$

Proof Let P_N be the orthogonal projector in $\mathcal{H}$ onto the subspace $\mathcal{H}_N = \oplus_{i=1}^{N}$. Applying in the inductive way (which is possible due to associativity of the procedures used to prove Theorem 27), it follows that the following is true for any finite N:

$$\det(I_{\mathcal{H}} + T P_N) = \prod_{i=1}^{N} \det(I_{\mathcal{H}_i} + T_i). \tag{A.15}$$

The existence of the $\lim_N$ (l.h.s of A) follows from the L_1 continuity of the Fredholm determinant formula (2.24) from Sect. 2. □

Example 1 Let $\mathcal{H} = \Gamma_{as}(h)$, where h is some separable Hilbert space. Let T_n for $n = 1, \ldots$ be a sequence of trace class operators defined on and reduced by $\wedge^n(h)$. Then, the operator $TT = \oplus_{n=1}^{\infty} T_n$ is continuous (on the fermionic Fock space) and of the trace class iff $\sum_{n=1}^{\infty} \|T_n\|_1 < \infty$. From Theorem 27, we learn that:

$$\det(I_{\Gamma_{as(h)}} + TT) = \prod_{i=1}^{\infty} \det(I_{\mathcal{H}_i} + T_i). \tag{A.16}$$

In a particular case of a given one-particle operator of trace class $T \in L_1(h)$ and defining $T_n = \wedge^n(T)$, we obtain

$$\det(1 + \Gamma_{as}(T)) = \prod_{i=1}^{\infty} \det(I_{\wedge^i(h)} + \wedge^i(T)). \tag{A.17}$$

The following result might be also of some interest.

Theorem 28 *Let $\mathcal{H}$ be a separable Hilbert space and let $A \in L_1(\mathcal{H})$ and $B \in L_1(\mathcal{H})$. Then, the following formula is valid:*

$$\det(1 + A)\det(1 + B) = \det\left((1 + A)(1 + B)\right). \tag{A.18}$$

Proof If both A and B are of finite range the proof of (28) follows from the corresponding finite dimensional matrix calculus. Using the fact that finite range operators are dense in $L_1(\mathcal{H})$ and L_1-continuity of $\det(1 + \cdots)$ the proof follows. □

Appendix B: Schmidt decompositions

Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ be a bipartite separable Hilbert space. Then, the space $L_2(\mathcal{H})$ is canonically isomorphic with the space $L_2(\mathcal{H}_A) \otimes L_2(\mathcal{H}_B)$ as is well known. In particular, if the system of operators $\{E_i^A \ldots\}$, and resp. $\{E_j^B \ldots\}$ is complete and orthonormal in $L_2(\mathcal{H}_A)$, resp. in $L_2(\mathcal{H}_B)$, then the system $\{Ei^A \otimes E_j^B\}$ is complete orthonormal system in $L_2(\mathcal{H})$.

Theorem 29 *Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ be a bipartite separable Hilbert space and let $Q \in E(\mathcal{H})$. Then, there exist—a system of non-negative numbers (τ_n), $\sum_{n=1}^{\infty} \tau_n^2 = ||Q||_2^2$ called the canonical (L_2-space) Schmidt numbers of Q and such that—two complete, orthonormal systems of L_2-class of operators $\{\Omega_n^A\} \subset L_2(\mathcal{H}_A)$, resp. $\{\Omega_n^B\} \subset L_2(\mathcal{H}_B)$ such that:*

$$Q = \sum_n \tau_n \Omega_n^A \otimes \Omega_n^B \tag{B.1}$$

Let $He(\mathcal{H})$ be the real Hilbert space of L_2-class and additionally Hermitian operators acting in the space $\mathcal{H}$. In particular, $E(\mathcal{H})$ is subset of $He(\mathcal{H})$. As the SVD theorem and the spectral theorem are still valid in the space $He(\mathcal{H})$ [70, 71], we can decompose any state Q in this space in the spirit of Schmidt decomposition.

Theorem 30 *Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ be a bipartite separable Hilbert space and let $Q \in E(\mathcal{H})$. Then, there exist—a system of non-negative numbers $(\tau_n^\star)$, called the Hermitian, Schmidt numbers of Q and such that $\sum_n (\tau_n^\star)^2 = \|Q\|_{He}^2$ and two complete (in the corresponding He spaces), orthonormal systems of L_2-class Hermitian operators $\{\Omega_n^{\star A}\} \subset He(L_2(\mathcal{H}))$, resp. $\{\Omega_n^{\star B}\} \subset He(L_2(\mathcal{H}))$ such that:*

$$Q = \sum_{n=1}^{\infty} \tau_n^\star \Omega_n^{\star A} \otimes \Omega_n^{\star B}. \tag{B.2}$$

Remark 7 Whether the Schmidt numbers of both expansions are identical or not is not clear for us. Also the operators Ω appearing in Theorems 29 and 30 are different in general. In particular, all the operators appearing in (B.2) are Hermitian.

Corollary B.1 *If all the operators appearing in Eq. (B.2) are non-negative then Q is separable.*

Proof If dimensions of the spaces $\mathcal{H}_A$ and $\mathcal{H}$ are both finite then the proof follows from the very definition of separability. For $N < \infty$, we define (modulo normalization) using expansion (B.2) the following separable states:

$$Q^N = \sum_{n=1:N} \tau_n^\star \Omega_n^{\star A} \otimes \Omega_n^{\star B}. \tag{B.3}$$

The sequence Q^N tends in the L_2 topology to the limiting state Q. Therefore, we conclude that Q belongs to the L_2 closure of the set of separable states. But Q belongs to $E(\mathcal{H})$ from the very assumptions made on it. □

As it is well known the Schmidt decompositions (B.1) and (B.2) can be used in finite dimensions to test the presence of entanglement in Q. For this, let us recall the well known realignment criterion: if Q is separable then the sum of the corresponding canonical Schmidt numbers τ is not bigger than 1 [72, 73].

For other, generalized version of this criterion see [74–79]. The infinite dimensional applications are also possible and are reported in a separate note [46].

With the help of these expansions, the following formulas for the corresponding reduced density operators (RDM) on the local L_2-spaces are derived

Corollary B.2 *Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ be a bipartite separable Hilbert space and let $Q \in E(\mathcal{H})$. Then L_2-RDM of $|Q\rangle\langle Q| \in L_2(L_2(\mathcal{H}))$ are given by*

$$QQ^A = \mathrm{Tr}_{L_2(\mathcal{H}_B)}(|Q\rangle\langle Q|) = \sum_{n=1}^{\infty} \tau_n^2 |\Omega_n^A\rangle\langle\Omega_n^A|, \tag{B.4}$$

and

$$QQ^B = \mathrm{Tr}_{L_2(\mathcal{H}_A)}(|Q\rangle\langle Q|) = \sum_{n=1}^{\infty} \tau_n^2 |\Omega_n^B\rangle\langle\Omega_n^B|. \tag{B.5}$$

The Hermitian version of this expansion is given:

Corollary B.3 *Let* $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ *be a bipartite separable Hilbert space and let* $Q \in E(\mathcal{H})$. *Then* HeL_2-*RDM of* $|Q\rangle\langle Q| \in He(He(\mathcal{H}))$ *are given by*

$$QQ^A = \text{Tr}_{L_2(\mathcal{H}_B)}(|Q\rangle\langle Q|) = \sum_{n=1}^{\infty} \tau_n^{\star 2} |\Omega_n^{\star A}\rangle\langle\Omega_n^{\star A}|, \tag{B.6}$$

and

$$QQ^B = \text{Tr}_{L_2(\mathcal{H}_A)}(|Q\rangle\langle Q|) = \sum_{n=1}^{\infty} \tau_n^{\star 2} |\Omega_n^{\star B}\rangle\langle\Omega_n^{\star B}|. \tag{B.7}$$

The operator $|Q\rangle\langle Q|$ acts in the Hilbert-Schmidt space of operators acting in $\mathcal{H}$ as an orthogonal projector. The spaces of operators acting on the space of states $E(\mathcal{H})$ are called often the space of superoperators. From the physical point of view, the most important class of superoperators are those which are completely positive and trace preserving [3, 4, 64]. Such superoperators are called quantum channels. From our considerations, it follows that any superoperator from $He2(He2(\mathcal{H}))$ can be decomposed similarly to the decompositions (B.5)-(B.8).

Appendix C: Operator valued (renormalized) map $(Q \rightarrow \log(I_{\mathcal{H}} + Q))$

Several useful properties of the map $Q \rightarrow \log(I_{\mathcal{H}} + Q)$ will be collected in this supplement. To start with let us consider non-negative $Q \in L(\mathcal{H})$. Using the spectral theorem, we can define operator $\log(I_{\mathcal{H}} + Q)$.

Proposition 31 *The map* $\log(1_{\mathcal{H}} + .)$ *with values in* $L_1^+(\mathcal{H})$ *is well defined on* $L_1^+(\mathcal{H})$ *and moreover, for* $Q \in L_1^+(\mathcal{H})$:

1. $\| \log(1_{\mathcal{H}} + Q)\|_1 \leq \|Q\|_1$,
2. *the map* $Q \rightarrow \log(1_{\mathcal{H}} + Q)$ *is operator monotone map,*
3. *The map* $\log(1_{\mathcal{H}} + .)$ *as defined on the cone* $L_1^+(\mathcal{H})$ *is strictly operator concave function which means the following for any* $Q_1, Q_2 \in L_1^+(\mathcal{H})$, *any* $\tau \in (0, 1)$:

$$\log(1_{\mathcal{H}} + \tau Q_1 + (1 - \tau)Q_2) \geq \tau \log(1_{\mathcal{H}} + Q_1) + (1 - \tau)\log(1_{\mathcal{H}} + Q_2). \tag{C.1}$$

For any $Q \in E(\mathcal{H})$:

$$\text{Tr}[\log(1_{\mathcal{H}} + Q)] \leq 1. \tag{C.2}$$

Lemma C.1 *For any* $Q_1, Q_2 \in L_1^+(\mathcal{H})$, *the strong Frechet directional derivative of the map* $\log(1_H + \ldots)$ *in the point* Q_1 *and in the direction to* Q_2 *is given by the*

formula:

$$\nabla Q_2(\log(1_H + \ldots)(Q_1) = \int_0^\infty dx(1_h + Q_1 + x)^{-1}Q_2(1_H + Q_1) + x)^{-1}.$$

(C.3)

Theorem 32 *[C] For any Q_1, $Q_2 \in L_1^+(\mathcal{H})$, the following estimate is valid*

$$\|\log(1_H + Q_1) - \log(1_H + Q_2)\|_1 \le o(1)\|Q_1 - Q_2\|_1.$$

(C.4)

Proof All the formulated here results are valid in the finite dimensional setting. The corresponding infinite dimensional results follows by performing the finite dimensional approximations and then performing the passage (controllable by the L_1-continuity) to limiting cases. □

C.1 Continuation of the proof of Theorem 2

The case $\tau = 1$.

If $\tau = 1$ then it follows that Q or Q' or both one are pure states. Assume that Q, Q' are both pure states. Then, there exist two unit vectors $|\psi\rangle$ and $|\theta\rangle$ such that $Q = |\psi\rangle\langle\psi|$ and $Q' = |\theta\rangle\rangle\theta|$. From the idempotency of Q and Q' it follows:

$$\log(1 + Q) = \log(2) \cdot |\psi\rangle\langle\psi|,$$

(C.5)

and

$$\log(1 + Q') = \log(2) \cdot |\theta\rangle\rangle\theta|,$$

(C.6)

from which

$$\|\log(1 + Q) - \log(1 + Q')\|_1 \le o(1)\|\psi - \theta\|.$$

(C.7)

If Q' is not pure but $Q \in \partial E(\mathcal{H})$ then $\delta = \|Q'\| < 1$. Taking Q' such that $\|Q' - |\psi\rangle\langle\psi|\| = \rho \le \delta$ the proof follows by repeating almost literally the arguments as in (2.43).

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