Measurement-induced geometric measures of correlations based on the Hellinger distance and their local decoherence

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Abstract

We apply the modified Brodutch and Modi method of constructing geometric measures of correlations to obtain analytical expressions for measurement-induced geometric classical and quantum correlations based on the Hellinger distance for two-qubit Bell diagonal states. Moreover, we study dynamics of the classical and quantum correlations for these states under decoherence induced by the action of bit flip, bit-phase flip, phase flip, depolarizing and generalized amplitude damping local quantum channels and identify all possible types of the decoherence dynamics. In particular, we show that these correlations exhibit freezing, sudden change and sudden transition between classical and quantum decoherence phenomena.

Keywords Geometric measures of correlations · Classical and quantum correlations · Hellinger distance · Decoherence · Bell diagonal states

1 Introduction

In quantum information science, the problem of classification and quantification of correlations present in composite quantum systems has been extensively investigated in the last few decades [1–4]. In this regard, the most significant progress was made in the case of bipartite quantum systems that have been studied initially in the entanglement-separability paradigm, first formalized by Werner [5], under which the correlations present in a quantum state can be classified as either classical or quantum, where the latter ones are identified with entanglement that can be quantified by different entanglement measures [1].

However, due to the discovery that some quantum information processing tasks can be carried out without entanglement [6–13], it has become evident that separable

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quantum states can also have quantum correlations, other than entanglement, and therefore, the entanglement-separability paradigm should be replaced by a new one.

The paradigm shift was initiated independently by Ollivier and Zurek [14] who introduced quantum discord as an information-theoretic measure of quantum correlations in bipartite quantum systems and by Henderson and Vedral [15] who studied the problem of separation of classical and quantum correlations in such systems from an information-theoretic perspective. The problem of classification and quantification of correlations present in composite quantum systems has been extensively investigated within the information-theoretic paradigm [2,4] due to the discovery [16] that quantum discord may be the key resource in the deterministic quantum computation with one qubit (DQC1) [6].

Since quantum discord cannot be computed analytically even for arbitrary two-qubit states [2,4], an alternative information-theoretic approach to classification and quantification of correlations present in composite quantum systems has been proposed in which different types of correlations are quantified by a distance from a given quantum state to the closest state which does not have the desired property [17]. The first geometric measure of quantum correlations present in bipartite quantum systems was geometric quantum discord in which the Schatten 2-norm has been applied as the distance measure between a given quantum state and the closest zero discord state to obtain the analytical expression for geometric quantum discord for a general two-qubit state [18], which has attracted considerable attention [2–4]. However, it has been shown that geometric quantum discord based on the Schatten 2-norm cannot be regarded as a bona fide measure of quantum correlations [19] because of the lack of contractivity of the Schatten 2-norm under trace-preserving quantum channels [20]. Moreover, it has turned out that among all geometric quantum discords based on the Schatten p -norms [21] only geometric quantum discord based on the Schatten 1-norm is a bona fide measure of quantum correlations [20].

The problems with geometric quantum discord based on the Schatten 2-norm have highlighted the need for a general method of constructing bona fide measures of correlations within the information-theoretic paradigm. Recently, Brodutch and Modi [22] proposed a method in which quantum correlations present in composite quantum systems are quantified by a distance between a given multipartite quantum state and the classical-quantum state emerging from a measurement performed on the considered state, where the measurement is chosen according to a specific strategy. Moreover, classical correlations are quantified by a distance between the classical-quantum state and the completely separable state resulting from the same measurement performed on the tensor product of the states of the individual subsystems. Furthermore, Brodutch and Modi [22] identified two strategies that provide bona fide measures of classical and quantum correlations that satisfy the following necessary conditions: (i) product states have no correlations, (ii) all correlations are invariant under local unitary operations, (iii) all correlations are non-negative, and (iv) classical states have no quantum correlations. However, it has been shown that the Brodutch and Modi method should be modified as in the case of measurement-induced classical and quantum correlations based on the trace distance for two-qubit Bell diagonal states one of the two possible strategies results in the non-uniqueness of classical correlations [23]. Moreover,

the modification of the Brodutch and Modi method has been proposed to avoid the problem of non-unique results in the general case [23].

As is well known, quantum systems tend to lose their fragile quantum features due to influence of their surrounding environments that cannot be practically avoided. Therefore, dynamics of classical and quantum correlations has been studied under various types of decoherence [2,4]. In this context, it was discovered that quantum discord is more robust against decoherence than entanglement [24,25] and exhibits freezing and single sudden change phenomena [26–28]. Moreover, it was discovered that geometric quantum discord based on the Schatten 1-norm is more robust against decoherence than entanglement [29] and exhibits freezing and double sudden change phenomena [30]. Recently, the influence of various models of decoherence has been also studied in the case of quantum features such as quantum steering and non-locality [31–34].

The purpose of this paper is twofold. First, we apply the modified Brodutch and Modi method to solve the open problem of determining measurement-induced classical and quantum correlations based on the Hellinger distance for two-qubit Bell diagonal states [3]. Second, we study dynamics of the classical and quantum correlations for these states under decoherence induced by the action of local quantum channels such as bit flip, bit-phase flip, phase flip, depolarizing and generalized amplitude damping. In particular, we identify all possible types of dynamics of the classical and quantum correlations under decoherence induced by the action of these quantum channels.

2 Measurement-induced geometric correlations based on the Hellinger distance

In this section, we apply the modified Brodutch and Modi method of constructing geometric measures of correlations [22,23] to obtain analytical expressions for measurement-induced geometric classical and quantum correlations based on the Hellinger distance for two-qubit Bell diagonal states.

In the framework of this method, measurement-induced classical and quantum correlations present in a multipartite state ρ are quantified by:

$$C(\rho) = K[M(\rho), M(\pi_\rho)], \quad (1)$$

$$Q(\rho) = K[\rho, M(\rho)], \quad (2)$$

where $K[\rho, \tau]$ is a non-negative real-valued function of states ρ and τ that vanishes for $\tau = \rho$, $M(\rho)$ is the classical-quantum state emerging from a measurement M performed on ρ , $M(\pi_\rho)$ is the completely separable state resulting from the same measurement M performed on π_ρ being the tensor product of the states of the individual subsystems, and the measurement M is chosen according to one of the two following strategies:

- M is a non-selective rank-1 projective measurement performed on one subsystem of the multipartite system in a state ρ , and M minimizes the quantum correlations $Q(\rho)$ [22], but if the classical correlations $C(\rho)$ are not uniquely determined by

the minimization procedure, then the classical correlations $C(\rho)$ are additionally maximized over the all measurements M that minimize the quantum correlations $Q(\rho)$ [23],

- M is a non-selective rank-1 projective measurement performed on one subsystem of the multipartite system in a state ρ , and M maximizes the classical correlations $C(\rho)$ [22], but if the quantum correlations $Q(\rho)$ are not uniquely determined by the maximization procedure, then the quantum correlations $Q(\rho)$ are additionally minimized over the all measurements M that maximize the classical correlations $C(\rho)$ [23].

In the framework of the modified Brodutch and Modi method, measurement-induced geometric classical and quantum correlations based on the Hellinger distance present in a multipartite state ρ are quantified by:

$$C(\rho) = d_H(M(\rho), M(\pi_\rho)), \quad (3)$$

$$Q(\rho) = d_H(\rho, M(\rho)), \quad (4)$$

with $d_H(\rho, \tau) = 2(1 - A(\rho, \tau))$ being the Hellinger distance between states ρ and τ , adopted as a function $K[\rho, \tau]$, where $A(\rho, \tau) = \text{Tr}(\sqrt{\rho}\sqrt{\tau})$ is the quantum affinity between states ρ and τ [35], and the measurement M is chosen according to one of the two above strategies.

A two-qubit Bell diagonal state can be written in the following form [36]:

$$\rho = \frac{1}{4} \left(I \otimes I + \sum_{m=1}^3 c_m \sigma_m \otimes \sigma_m \right), \quad (5)$$

where I is the identity matrix, σ_m are the Pauli matrices, and real coefficients c_m fulfill the following inequalities:

$$0 \leq \frac{1}{4}(1 - c_1 - c_2 - c_3) \leq 1, \quad (6a)$$

$$0 \leq \frac{1}{4}(1 - c_1 + c_2 + c_3) \leq 1, \quad (6b)$$

$$0 \leq \frac{1}{4}(1 + c_1 - c_2 + c_3) \leq 1, \quad (6c)$$

$$0 \leq \frac{1}{4}(1 + c_1 + c_2 - c_3) \leq 1, \quad (6d)$$

which describe a tetrahedron of vertices $(1, 1, -1)$, $(-1, -1, -1)$, $(1, -1, 1)$ and $(-1, 1, 1)$ that represent the two-qubit Bell states [36]. Thus, there is a one-to-one correspondence between two-qubit Bell diagonal states and points within tetrahedron (6).

Let us note that if the measurement M , described by a complete set of one-dimensional orthogonal projectors $\{\Pi_{\pm}\}$, is performed on the first qubit of the two-qubit system in a Bell diagonal state, then the classical-quantum state $M(\rho)$ and the completely separable state $M(\pi_\rho)$ have the form:

$$M(\rho) = \sum_{m=+,-} (\Pi_m \otimes I) \rho (\Pi_m \otimes I), \quad (7)$$

$$M(\pi_\rho) = \frac{1}{4}(I \otimes I), \quad (8)$$

where $\Pi_\pm = \frac{1}{2}(I \pm \mathbf{n} \cdot \boldsymbol{\sigma})$, $\mathbf{n} = (n_1, n_2, n_3)$ is a real three-dimensional unit vector and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$.

In order to compute the classical and quantum correlations (3) and (4) for two-qubit Bell diagonal states, we first need to find the quantum affinity between states $M(\rho)$ and $M(\pi_\rho)$ and then the quantum affinity between states ρ and $M(\rho)$. One can show that in the case of two-qubit Bell diagonal states and the measurement M described by $\{\Pi_\pm\}$ the quantum affinity between states $M(\rho)$ and $M(\pi_\rho)$ is given by:

$$A(M(\rho), M(\pi_\rho)) = \frac{1}{\sqrt{2}} \sqrt{1 + \sqrt{1 - c_1^2 n_1^2 - c_2^2 n_2^2 - c_3^2 n_3^2}}. \quad (9)$$

Moreover, one can also show that in the case of two-qubit Bell diagonal states and the measurement M described by $\{\Pi_\pm\}$ the quantum affinity between states ρ and $M(\rho)$ is given by:

$$\begin{aligned} A(\rho, M(\rho)) = & \frac{1}{4\sqrt{2}} \sqrt{1 + \sqrt{1 - c_1^2 n_1^2 - c_2^2 n_2^2 - c_3^2 n_3^2}} \\ & \times \left[\sqrt{1 - c_1 + c_2 + c_3} \left(1 - \frac{c_1 n_1^2 - c_2 n_2^2 - c_3 n_3^2}{1 + \sqrt{1 - c_1^2 n_1^2 - c_2^2 n_2^2 - c_3^2 n_3^2}} \right) \right. \\ & + \sqrt{1 + c_1 + c_2 - c_3} \left(1 + \frac{c_1 n_1^2 + c_2 n_2^2 - c_3 n_3^2}{1 + \sqrt{1 - c_1^2 n_1^2 - c_2^2 n_2^2 - c_3^2 n_3^2}} \right) \\ & + \sqrt{1 + c_1 - c_2 + c_3} \left(1 + \frac{c_1 n_1^2 - c_2 n_2^2 + c_3 n_3^2}{1 + \sqrt{1 - c_1^2 n_1^2 - c_2^2 n_2^2 - c_3^2 n_3^2}} \right) \\ & \left. + \sqrt{1 - c_1 - c_2 - c_3} \left(1 - \frac{c_1 n_1^2 + c_2 n_2^2 + c_3 n_3^2}{1 + \sqrt{1 - c_1^2 n_1^2 - c_2^2 n_2^2 - c_3^2 n_3^2}} \right) \right]. \quad (10) \end{aligned}$$

The analytical expressions for $A(M(\rho), M(\pi_\rho))$ and $A(\rho, M(\rho))$ make it possible to obtain measurement-induced geometric classical and quantum correlations based on the Hellinger distance for two-qubit Bell diagonal states under the two possible strategies of choosing the measurement M .

2.1 Strategy 1

In the framework of the first strategy, for a given two-qubit Bell diagonal state (5) we first identify the measurements M that minimize the quantum correlations (4) and then we use these optimal measurements to compute the classical correlations (3). In other words, for a given point (c_1, c_2, c_3) of tetrahedron (6) we first identify unit vectors (n_1, n_2, n_3) that maximize the quantum affinity (10) and then we use these optimal vectors to compute the quantum affinity (9) in order to obtain the classical and quantum correlations (3) and (4).

It can be shown that under this strategy

- if $c_i^2 = c_j^2 = c_k^2$ where $i \neq j \neq k$, then all measurements M are optimal, and the classical and quantum correlations are given by:

$$C(\rho) = 2 - \sqrt{2 + 2\sqrt{1 - c_k^2}}, \quad (11a)$$

$$Q(\rho) = 2(1 - w_k), \quad (11b)$$

- if $c_i^2 = c_j^2 < c_k^2$ where $i \neq j \neq k$, then only measurements M with $n_i^2 = n_j^2 = 0$ and $n_k^2 = 1$ are optimal, and the classical and quantum correlations are given by Eqs. (11),
- if $c_i^2 < c_j^2 = c_k^2$ where $i \neq j \neq k$, then only measurements M with $n_i^2 = 0$ and $n_j^2 + n_k^2 = 1$ are optimal, and the classical and quantum correlations are given by Eqs. (11),
- if $c_i^2 < c_j^2 < c_k^2$ where $i \neq j \neq k$, then only measurements M with $n_i^2 = n_j^2 = 0$ and $n_k^2 = 1$ are optimal, and the classical and quantum correlations are given by Eqs. (11),

where coefficients w_1, w_2 and w_3 have the following form:

$$w_1 = \frac{1}{4\sqrt{2}} \sqrt{1 + \sqrt{1 - c_1^2}} \left[\left(\sqrt{1 + c_1 + c_2 - c_3} + \sqrt{1 + c_1 - c_2 + c_3} \right) \left(1 + \frac{c_1}{1 + \sqrt{1 - c_1^2}} \right) + \left(\sqrt{1 - c_1 + c_2 + c_3} + \sqrt{1 - c_1 - c_2 - c_3} \right) \left(1 - \frac{c_1}{1 + \sqrt{1 - c_1^2}} \right) \right], \quad (12a)$$

$$w_2 = \frac{1}{4\sqrt{2}} \sqrt{1 + \sqrt{1 - c_2^2}} \left[\left(\sqrt{1 - c_1 + c_2 + c_3} + \sqrt{1 + c_1 + c_2 - c_3} \right) \left(1 + \frac{c_2}{1 + \sqrt{1 - c_2^2}} \right) \right]$$

$$\begin{aligned}
& + \left(\sqrt{1 + c_1 - c_2 + c_3} + \sqrt{1 - c_1 - c_2 - c_3} \right) \left(1 - \frac{c_2}{1 + \sqrt{1 - c_2^2}} \right) \Bigg], \\
& \qquad \qquad \qquad (12b) \\
w_3 = & \frac{1}{4\sqrt{2}} \sqrt{1 + \sqrt{1 - c_3^2}} \\
& \left[\left(\sqrt{1 - c_1 + c_2 + c_3} + \sqrt{1 + c_1 - c_2 + c_3} \right) \left(1 + \frac{c_3}{1 + \sqrt{1 - c_3^2}} \right) \right. \\
& \left. + \left(\sqrt{1 + c_1 + c_2 - c_3} + \sqrt{1 - c_1 - c_2 - c_3} \right) \left(1 - \frac{c_3}{1 + \sqrt{1 - c_3^2}} \right) \right]. \\
& \qquad \qquad \qquad (12c)
\end{aligned}$$

Let us note that the classical and quantum correlations are uniquely determined under the first strategy, despite the fact that for a wide class of two-qubit Bell diagonal states there is more than one optimal measurement M , which means that for those states the classical-quantum state $M(\rho)$ cannot be uniquely determined.

2.2 Strategy 2

In the framework of the second strategy, for a given two-qubit Bell diagonal state (5) we first identify the measurements M that maximize the classical correlations (3) and then we use these optimal measurements to compute the quantum correlations (4). In other words, for a given point (c_1, c_2, c_3) of tetrahedron (6) we first identify unit vectors (n_1, n_2, n_3) that minimize the quantum affinity (9) and then we use these optimal vectors to compute the quantum affinity (10) in order to obtain the classical and quantum correlations (3) and (4).

One can show that this strategy provides the same results with regard to the classical and quantum correlations and the optimal measurements M as the first strategy considered above.

Recently, measurement-induced geometric classical and quantum correlations based on the Bures distance and the trace distance for two-qubit Bell diagonal states under the two possible strategies of choosing the measurement M have been studied in [23]. Remarkably, it turns out that for a given two-qubit Bell diagonal state the optimal measurements M are exactly the same for measurement-induced geometric classical and quantum correlations based on the Hellinger distance, the Bures distance and the trace distance under the both strategies of choosing the measurement M .

3 Dynamics of measurement-induced geometric correlations based on the Hellinger distance under local decoherence

In this section, we investigate dynamics of measurement-induced geometric classical and quantum correlations based on the Hellinger distance for two-qubit Bell diagonal

Table 1 Kraus operators for quantum channels considered in this paper, where p and γ are decoherence probabilities [37]

Channel	Kraus operators
Bit flip	$E_1 = \sqrt{1-p/2} I, E_2 = \sqrt{p/2} \sigma_1$
Bit-phase flip	$E_1 = \sqrt{1-p/2} I, E_2 = \sqrt{p/2} \sigma_2$
Phase flip	$E_1 = \sqrt{1-p/2} I, E_2 = \sqrt{p/2} \sigma_3$
Depolarizing	$E_1 = \sqrt{1-3p/4} I, E_2 = \sqrt{p/4} \sigma_1,$ $E_3 = \sqrt{p/4} \sigma_2, E_4 = \sqrt{p/4} \sigma_3$
Generalized amplitude damping	$E_1 = \sqrt{p} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, E_2 = \sqrt{p} \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix},$ $E_3 = \sqrt{1-p} \begin{pmatrix} \sqrt{1-\gamma} & 0 \\ 0 & 1 \end{pmatrix}, E_4 = \sqrt{1-p} \begin{pmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{pmatrix}$

states under decoherence induced by the action of bit flip, bit-phase flip, phase flip, depolarizing and generalized amplitude damping local quantum channels.

The action of local quantum channels on a two-qubit Bell diagonal state ρ can be expressed as a trace-preserving quantum operation $\mathcal{E}(\rho)$ written in the operator-sum representation [37]

$$\mathcal{E}(\rho) = \sum_{i,j} (E_i \otimes E_j) \rho (E_i \otimes E_j)^\dagger, \quad (13)$$

where Kraus operators E_k for a single quantum channel satisfy the following condition $\sum_k E_k^\dagger E_k = I$.

Let us note that if the action of local quantum channels on a two-qubit Bell diagonal state (5) preserves its Bell diagonal form, which means that

$$\mathcal{E}(\rho) = \frac{1}{4} \left(I \otimes I + \sum_{m=1}^3 c'_m \sigma_m \otimes \sigma_m \right), \quad (14)$$

then one can use Eqs. (11) and (12) with the coefficients c_m being replaced by c'_m to study dynamics of the classical and quantum correlations for two-qubit Bell diagonal states under the action of those local quantum channels.

It can be shown that in the case of the first four quantum channels listed in Table 1 the quantum operation (13) can be written in form (14) for arbitrary p , while in the case of the last quantum channel it can be done for arbitrary γ and $p = 1/2$. Moreover, it can be verified that in the case of the quantum channels listed in Table 1 the coefficients c'_m in Eq. (14) have the form provided in Table 2.

Below we present all possible types of dynamics of the classical and quantum correlations for two-qubit Bell diagonal states under decoherence induced by the action of local quantum channels listed in Table 1.

Table 2 Coefficients $c'_m = \text{Tr}[(\sigma_m \otimes \sigma_m) \mathcal{E}(\rho)]$ for quantum operations $\mathcal{E}(\rho)$ induced by the action of local quantum channels considered in this paper

Local quantum channels	c'_1	c'_2	c'_3
Bit flip	c_1	$(1-p)^2 c_2$	$(1-p)^2 c_3$
Bit-phase flip	$(1-p)^2 c_1$	c_2	$(1-p)^2 c_3$
Phase flip	$(1-p)^2 c_1$	$(1-p)^2 c_2$	c_3
Depolarizing	$(1-p)^2 c_1$	$(1-p)^2 c_2$	$(1-p)^2 c_3$
Generalized amplitude damping (with $p = 1/2$)	$(1-\gamma) c_1$	$(1-\gamma) c_2$	$(1-\gamma)^2 c_3$

3.1 Bit flip local quantum channels

Let us note that in this case a given two-qubit Bell diagonal state corresponding to point (c_1, c_2, c_3) of tetrahedron (6) evolves to another two-qubit Bell diagonal state corresponding to point $(c'_1, c'_2, c'_3) = (c_1, (1-p)^2 c_2, (1-p)^2 c_3)$ of the tetrahedron. Therefore, we can determine all possible types of dynamics of the classical and quantum correlations under decoherence induced by the action of bit flip local quantum channels using Eqs. (11) and (12) with c_1, c_2 and c_3 being replaced by $c'_1 = c_1$, $c'_2 = (1-p)^2 c_2$ and $c'_3 = (1-p)^2 c_3$.

It can be shown that in this case there are seven different types of dynamics of the classical and quantum correlations.

1. If $0 = c_i^2 = c_j^2 = c_k^2$ where $i \neq j \neq k$, then the classical and quantum correlations both remain constant and equal to zero. This type of dynamics of the classical and quantum correlations follows directly from the fact that in this case $c_k'^2 = 0$ and $w_k = 1$.
2. If $0 = c_i^2 = c_j^2 < c_1^2$ where $i \neq j \neq 1$, then the classical correlations remain constant while the quantum correlations remain constant and equal to zero. This type of dynamics of the classical and quantum correlations follows directly from the fact that in this case $c_1'^2 = c_1^2$ and $w_1 = 1$.
3. If $0 = c_1^2 = c_i^2 < c_j^2 < 1$ where $i \neq j \neq 1$, then the classical correlations decay monotonically while the quantum correlations remain constant and equal to zero (see Fig. 1a). This type of dynamics of the classical and quantum correlations follows directly from the fact that in this case $c_j'^2 = (1-p)^4 c_j^2$ is a decreasing function of p and $w_j = 1$.
4. If $0 < c_2^2 = c_3^2 \leq c_1^2$ or $c_i^2 < c_j^2 \leq c_1^2$ where $i \neq j \neq 1$, then the classical correlations remain constant while the quantum correlations decay monotonically (see Fig. 1b). This type of dynamics of the classical and quantum correlations follows directly from the fact that in this case $c_1'^2 = c_1^2$ and w_1 is an increasing function of p .
5. If $0 = c_1^2 < c_i^2 \leq c_j^2$ where $i \neq j \neq 1$, then the classical and quantum correlations both decay monotonically (see Fig. 1c). This type of dynamics of the classical and quantum correlations follows directly from the fact that in this case $c_j'^2 = (1-p)^4 c_j^2$ is a decreasing function of p while w_j is an increasing function of p .

Fig. 1 (Color online) Dynamics of measurement-induced geometric classical and quantum correlations based on the Hellinger distance for two-qubit Bell diagonal states with **a** $c_1 = 0, c_2 = 5/7, c_3 = 0$, **b** $c_1 = -4/7, c_2 = -4/7, c_3 = -1/7$, **c** $c_1 = 0, c_2 = 3/7, c_3 = 4/7$, **d** $c_1 = 4/7, c_2 = -4/7, c_3 = 1$, $p_{sc} = 1 - \sqrt{4/7}$, **e** $c_1 = -3/7, c_2 = 5/7, c_3 = 4/7$, $p_{sc} = 1 - \sqrt{3/5}$ under decoherence induced by the action of local bit flip quantum channels

6. If $0 < c_1^2 = c_i^2 < c_j^2 = 1$ where $i \neq j \neq 1$, then a sudden change in behavior of the classical and quantum correlations occurs at $p_{sc} = 1 - \sqrt{\frac{|c_1|}{\max(|c_2|, |c_3|)}}$; namely, the classical correlations decay monotonically while the quantum correlations remain constant for $p \leq p_{sc}$ and vice versa for $p \geq p_{sc}$ (see Fig. 1d). In other words, a sudden transition between classical and quantum decoherence occurs at p_{sc} . This type of dynamics of the classical and quantum correlations is a consequence of two facts. First, in this case $c_1'^2 = c_1^2$ while $c_i'^2 = (1-p)^4 c_i^2$ and $c_j'^2 = (1-p)^4 c_j^2$ are decreasing functions of p , and moreover, $w_j = (\sqrt{1+c_1} + \sqrt{1-c_1})/2$ for $p \leq p_{sc}$, while w_1 is an increasing function of p for $p \geq p_{sc}$. Second, $c_j'^2 \geq c_1'^2$

for $p \leq p_{sc}$ and $c_j'^2 \leq c_1'^2$ for $p \geq p_{sc}$. A sudden change in decay rates of the classical and quantum correlations occurs due to the change of ordering between $c_j'^2$ and $c_1'^2$ at p_{sc} .

7. If $0 < c_1^2 < c_i^2 \leq c_j^2$ or $0 < c_1^2 = c_i^2 < c_j^2 < 1$ or $c_i^2 < c_1^2 < c_j^2$ where $i \neq j \neq 1$, then the classical and quantum correlations both decay monotonically; however, a sudden change in their decay rates occurs at $p_{sc} = 1 - \sqrt{\frac{|c_1|}{\max(|c_2|, |c_3|)}}$ (see Fig. 1e); namely, the classical and quantum correlations decay monotonically for $p \leq p_{sc}$, while the classical correlations remain constant and the quantum correlations decay monotonically for $p \geq p_{sc}$. This type of dynamics of the classical and quantum correlations is a consequence of two facts. First, in this case $c_1'^2 = c_1^2$ while $c_i'^2 = (1-p)^4 c_i^2$ and $c_j'^2 = (1-p)^4 c_j^2$ are decreasing functions of p , and moreover, w_j is an increasing function of p for $p \leq p_{sc}$, while w_1 is another increasing function of p for $p \geq p_{sc}$. Second, $c_j'^2 \geq c_1'^2$ for $p \leq p_{sc}$ and $c_j'^2 \leq c_1'^2$ for $p \geq p_{sc}$. A sudden change in decay rates of the classical and quantum correlations occurs due to the change of ordering between $c_j'^2$ and $c_1'^2$ at p_{sc} .

3.2 Bit-phase flip and phase flip local quantum channels

Let us note that the conditions for all possible types of dynamics of measurement-induced geometric classical and quantum correlations based on the Hellinger distance for two-qubit Bell diagonal states under the action of bit-phase flip and phase flip local quantum channels can be obtained from the conditions presented in Sect. 3.1 by swapping c_1 with c_2 and c_1 with c_3 , respectively.

3.3 Depolarizing local quantum channels

Let us note that in this case a given two-qubit Bell diagonal state corresponding to point (c_1, c_2, c_3) of tetrahedron (6) evolves to another two-qubit Bell diagonal state corresponding to point $(c'_1, c'_2, c'_3) = ((1-p)^2 c_1, (1-p)^2 c_2, (1-p)^2 c_3)$ of the tetrahedron. Therefore, we can determine all possible types of dynamics of the classical and quantum correlations under decoherence induced by the action of depolarizing local quantum channels using Eqs. (11) and (12) with c_1, c_2 and c_3 being replaced by $c'_1 = (1-p)^2 c_1, c'_2 = (1-p)^2 c_2$ and $c'_3 = (1-p)^2 c_3$.

It can be shown that in this case there are three different types of dynamics of the classical and quantum correlations.

1. If $0 = c_i^2 = c_j^2 = c_k^2$ where $i \neq j \neq k$, then the classical and quantum correlations both remain constant and equal to zero. This type of dynamics of the classical and quantum correlations follows directly from the fact that in this case $c_k'^2 = 0$ and $w_k = 1$.
2. If $0 = c_i^2 = c_j^2 < c_k^2$ where $i \neq j \neq k$, then the classical correlations decay monotonically while the quantum correlations remain constant and equal to zero (see Fig. 2a). This type of dynamics of the classical and quantum correlations

Fig. 2 (Color online) Dynamics of measurement-induced geometric classical and quantum correlations based on the Hellinger distance for two-qubit Bell diagonal states with **a** $c_1 = -4/7$, $c_2 = 0$, $c_3 = 0$, **b** $c_1 = -6/7$, $c_2 = -6/7$, $c_3 = -6/7$ under decoherence induced by the action of local depolarizing quantum channels

follows directly from the fact that in this case $c_k'^2 = (1 - p)^4 c_k^2$ is a decreasing function of p and $w_k = 1$.

3. If $c_i^2 < c_j^2 < c_k^2$ or $0 < c_i^2 = c_j^2 < c_k^2$ or $c_i^2 < c_j^2 = c_k^2$ or $0 < c_i^2 = c_j^2 = c_k^2$ where $i \neq j \neq k$, then the classical and quantum correlations both decay monotonically (see Fig. 2b). This type of dynamics of the classical and quantum correlations follows directly from the fact that in this case $c_k'^2 = (1 - p)^4 c_k^2$ is a decreasing function of p , while w_k is an increasing function of p .

3.4 Generalized amplitude damping local quantum channels with $p = 1/2$

Let us note that in this case a given two-qubit Bell diagonal state corresponding to point (c_1, c_2, c_3) of tetrahedron (6) evolves to another two-qubit Bell diagonal state corresponding to point $(c'_1, c'_2, c'_3) = ((1 - \gamma)c_1, (1 - \gamma)c_2, (1 - \gamma)^2 c_3)$ of the tetrahedron. Therefore, we can determine all possible types of dynamics of the classical and quantum correlations under decoherence induced by the action of amplitude damping local quantum channels (with $p = 1/2$) using Eqs. (11) and (12) with c_1, c_2 and c_3 being replaced by $c'_1 = (1 - \gamma)c_1$, $c'_2 = (1 - \gamma)c_2$ and $c'_3 = (1 - \gamma)^2 c_3$.

It can be shown that in this case there are four different types of dynamics of the classical and quantum correlations.

1. If $0 = c_i^2 = c_j^2 = c_k^2$ where $i \neq j \neq k$, then the classical and quantum correlations both remain constant and equal to zero. This type of dynamics of the classical and quantum correlations follows directly from the fact that in this case $c_k'^2 = 0$ and $w_k = 1$.
2. If $0 = c_i^2 = c_j^2 < c_k^2$ where $i \neq j \neq k$, then the classical correlations decay monotonically while the quantum correlations remain constant and equal to zero (see Fig. 3a). This type of dynamics of the classical and quantum correlations follows directly from the fact that in this case $c_k'^2 = (1 - \gamma)^r c_k^2$ with $r = 2$ if $k = 1, 2$ and $r = 4$ if $k = 3$ is a decreasing function of γ and $w_k = 1$.

Fig. 3 (Color online) Dynamics of measurement-induced geometric classical and quantum correlations based on the Hellinger distance for two-qubit Bell diagonal states with **a** $c_1 = 0, c_2 = 0, c_3 = -1$, **b** $c_1 = -3/7, c_2 = -2/7, c_3 = 2/7$, **c** $c_1 = 2/7, c_2 = -2/7, c_3 = 3/7, \gamma_{sc} = 1/3$ under decoherence induced by the action of local generalized amplitude damping quantum channels with $p = 1/2$

3. If $c_i^2 < c_3^2 < c_j^2$ or $c_3^2 < c_i^2 < c_j^2$ or $0 < c_3^2 = c_i^2 < c_j^2$ where $i \neq j \neq 3$ or $c_i^2 < c_j^2 = c_k^2$ or $0 < c_i^2 = c_j^2 = c_k^2$ where $i \neq j \neq k$, then the classical and quantum correlations both decay monotonically (see Fig. 3b). This type of dynamics of the classical and quantum correlations follows directly from the fact that in this case $c_j'^2 = (1 - \gamma)^r c_j^2$ with $r = 2$ if $j = 1, 2$ and $r = 4$ if $j = 3$ is a decreasing function of γ , while w_j is an increasing function of γ .
4. If $c_i^2 < c_j^2 < c_3^2$ or $0 < c_i^2 = c_j^2 < c_3^2$ where $i \neq j \neq 3$, then the classical and quantum correlations both decay monotonically; however, a sudden change in their decay rates occurs at $\gamma_{sc} = 1 - \frac{\max(|c_1|, |c_2|)}{|c_3|}$ (see Fig. 3c). This type of dynamics of the classical and quantum correlations is a consequence of two facts. First, in this case $c_i'^2 = (1 - \gamma)^2 c_i^2$, $c_j'^2 = (1 - \gamma)^2 c_j^2$ and $c_3'^2 = (1 - p)^4 c_3^2$ are decreasing functions of γ , and moreover, w_3 is an increasing function of γ for $\gamma \leq \gamma_{sc}$, while w_j is another increasing function of γ for $\gamma \geq \gamma_{sc}$. Second, $c_3'^2 \geq c_j'^2$ for $\gamma \leq \gamma_{sc}$ and $c_3'^2 \leq c_j'^2$ for $\gamma \geq \gamma_{sc}$. A sudden change in decay rates of the classical and quantum correlations occurs due to the change of ordering between $c_3'^2$ and $c_j'^2$ at γ_{sc} .

As we see measurement-induced geometric classical and quantum correlations based on the Hellinger distance for two-qubit Bell diagonal states exhibit a freezing phenomenon under decoherence induced by the action of bit flip, bit-phase flip and phase flip local quantum channels. Moreover, a single sudden change in the clas-

sical and quantum correlations occurs under decoherence induced by the action of bit flip, bit-phase flip, phase flip and generalized amplitude damping (with $p = 1/2$) local quantum channels. Interestingly, in the case of Bell diagonal states with $|c_1| \geq |c_2|$, $|c_3|$ classical correlations remain constant, while quantum correlations decay monotonically under the action of bit flip local quantum channels both for the entropic measures of correlations considered in [28] and for the geometric measures of correlations studied in this paper. A similar observation holds for Bell diagonal states with $|c_2| \geq |c_1|$, $|c_3|$ and $|c_3| \geq |c_1|$, $|c_2|$ under the action of bit-phase flip and phase flip local quantum channels, respectively.

Finally, in the case of decoherence induced by the action of bit flip, bit-phase flip and phase flip local quantum channels measurement-induced geometric classical and quantum correlations based on the Hellinger distance for two-qubit Bell diagonal states exhibit a sudden transition between classical and quantum decoherence. Interestingly, in the case of Bell diagonal states with $|c_1| < 1$, $c_2 = \mp c_1$, $c_3 = \pm 1$ a sudden transition between classical and quantum decoherence occurs under the action of bit flip local quantum channels both for the entropic measures of correlations considered in [26] and for the geometric measures of correlations studied in this paper. A similar observation holds for Bell diagonal states with $c_1 = \pm 1$, $|c_2| < 1$, $c_3 = \mp c_2$ and $c_1 = \pm 1$, $c_2 = \mp c_3$, $|c_3| < 1$ under the action of bit-phase flip and phase flip local quantum channels, respectively.

4 Conclusion

We have applied the modified Brodutch and Modi method of constructing geometric measures of correlations to obtain analytical expressions for measurement-induced geometric classical and quantum correlations based on the Hellinger distance for two-qubit Bell diagonal states. Moreover, we have considered dynamics of the classical and quantum correlations for these states under decoherence induced by the action of several local quantum channels and identified all possible types of the decoherence dynamics. In particular, we have shown that these correlations exhibit freezing, sudden change and sudden transition between classical and quantum decoherence phenomena.

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