

Quantum steering borders in three-qubit systems

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Abstract We discuss a family of states describing three-qubit systems in a context of quantum steering phenomena. We show that symmetric steering cannot appear between two qubits—only asymmetric steering can appear in such systems. The main aim of this paper is to discuss the possible relations between the entanglement measures and steering parameter for two-mode mixed state corresponding to the qubit–qubit subsystem. We have derived the conditions determining boundary values of the negativity parametrized by concurrence. We show that two-qubit mixed state cannot be steerable when the negativity of such state is smaller than, or equal to, its boundary value. Finally, we have found ranges of the values of the mixedness measure, parametrized by concurrence and negativity for steerable and unsteerable two-qubit mixed states.

Keywords Quantum correlations · Quantum steering · Quantum entanglement · Negativity · Concurrence · Three-qubit system

1 Introduction

Quantum steering belongs to a group of correlations which appears in quantum world and does not have its counterpart in classical physics. The concept of steering was introduced by Schrödinger in 1935 [1] as a generalization of the EPR paradox for-

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mulated by Einstein, Podolsky and Rosen [2]. In 2007 Wiseman et al. [3,4] showed that not every nonseparable states are steerable and not every nonsteerable states are nonlocal in a Bell sense. Thus, one can say that the steering represents some form of nonlocality which is, from one side, weaker than the usual Bell nonlocality, and from the other—stronger than nonseparability. Next, Oppenheimer and Weher showed that steering with uncertainty principle determine the degree of nonlocality [5]. Nowadays, steering phenomena are studied not only in space's context, but also in time's one (for such situations we call it *temporal steering*). Such phenomena were considered not only from theoretical point of view [6,7] but also first experiment studies of temporal steering were proposed and performed [8]. Steering phenomena seem to be especially relevant from the point of view of quantum information processing [9], including quantum key distribution [10].

The first experimental observation of the EPR paradox was done by Ou et al. [11], whereas experimental realization of steering was described in [8,12–18]. It should be emphasized that generation of EPR steering was considered in a context of various systems. For instance, they were three-mode optomechanical system composed of an atomic ensemble located in a cavity with an oscillating mirror [19], double-cavity optomechanical system with two separate electromagnetic fields mediated by a mechanical oscillator [20], system comprising optical cavity filled by an electromagnetic field and the mechanical oscillator [21], model involving light beams generated in time-modulated nondegenerate optical parametric oscillator [22] and many others.

To discuss steering phenomena, the criterion which was proposed by Reid [23], then developed by Cavalcanti and Reid [24], and by Walborn et al. [25] is usually applied. However, at this point one should mention very latest work by Rutkowski et al. [26] in which an experimentally feasible unbounded violation of a steering inequality was proven. The universal form of such inequality was derived there with application of the DeutschMaassenUffink entropic uncertainty relation.

In this work, we discuss steering properties of the states corresponding to a three-qubits model. In particular, we consider the relations between steering in a two-qubit subsystem and the entanglement described by the negativity and concurrence. The paper consists of two parts. In the first one, we analyze the situation when in the three-qubit system only one excitation is present, whereas in the second part we study double excited system. For the both cases, we discuss the possibility of generation symmetric steering between two qubits. We show what kind of steering (symmetric or asymmetric) can be observed in our three-qubit systems, and which type of steering will never appear for the given case. Moreover, applying entanglement measures such as concurrence and negativity, we find the conditions (upper bound) determining when unsteerable mixed states appear for qubit–qubit subsystem. In addition, we check how the mixedness of the states characterized by the linear entropy is related to the steering effect for the models discussed here. In particular, we identify ranges of the values of the entropy for steerable states described by various values of the concurrence and negativity. In this paper, we only shortly discuss cases when zero or three excitations are present—such cases lead to not interesting conclusions and correspond to trivial physical models.

2 Three-qubit system with one excitation

In this paper, we focus on the family of the states describing three-qubit systems in a context of finding conditions determining which states are steerable. We do not discuss here the influence of coupling among them leading to some particular time evolution.

In this article, we mostly concentrate on two cases. The first of them corresponds to the situation when single excitation is present in a model, whereas the second one concerns two excitations' case. Obviously, there are two other situations—when we have no excitations at all and when all three qubits are excited. However, if we assume that the discussed model of three qubits does not interact with external systems, such cases are trivial and can be neglected.

This section is devoted to the situation when single excitation is present. As the total number of photons/phonons in a physical system not interacting with external reservoirs is conserved (and equal to the unity): $\langle \hat{n} \rangle = \langle \hat{n}_1 \rangle + \langle \hat{n}_2 \rangle + \langle \hat{n}_3 \rangle = 1$, where the qubits considered here are labeled by 1, 2 and 3. For such a case, the wave function describing our model can be written as:

$$|\psi^{(I)}\rangle = C_{001}|001\rangle + C_{010}|010\rangle + C_{100}|100\rangle, \tag{1}$$

and then, the corresponding density matrix is

$$\rho^{(I)} = |\psi^{(I)}\rangle\langle\psi^{(I)}| = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & P_{001} & C_{001}^*C_{010} & 0 & C_{001}^*C_{100} & 0 & 0 & 0 & 0 \\ 0 & C_{010}^*C_{001} & P_{010} & 0 & C_{010}^*C_{100} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_{100}^*C_{001} & C_{100}^*C_{010} & 0 & P_{100} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2}$$

Obviously, C_{ijk} appearing here are the complex probability amplitudes corresponding to the states $|ijk\rangle$, and they determine probabilities $P_{ijk} = C_{ijk}^*C_{ijk}$. The symbol (I) denotes that we study the system with a single excitation.

As a parameter allowing to determine the presence and strength of the steering in the considered system, we apply that defined with application of Cavalcanti et al. inequality [27]. Such parameter can be written in terms of the boson creation and annihilation operators (\hat{a}^\dagger and \hat{a}) as

$$S_{ij} = \langle \hat{a}_i \hat{a}_j^\dagger \rangle \langle \hat{a}_i^\dagger \hat{a}_j \rangle - \left\langle \hat{a}_i^\dagger \hat{a}_i \left(\hat{a}_j^\dagger \hat{a}_j + \frac{1}{2} \right) \right\rangle, \tag{3}$$

where indices i and j label the qubits. Qubit j steers qubit i when the parameter S_{ij} takes values greater than zero. If at the same time $S_{ji} > 0$, then symmetric steering can be observed. The second type of steering is present when only one of two parameters (S_{ij} or S_{ji}) is greater than zero. For such a case, we are dealing with asymmetric steering.

For the system described by the wave function $|\psi^{(I)}\rangle$, steering parameters for each pair of qubits can be expressed by the probabilities P_{ijk} and are:

$$\begin{aligned} S_{12}^{(I)} &= P_{010}P_{100} - 0.5P_{100}, \\ S_{21}^{(I)} &= P_{010}P_{100} - 0.5P_{010}, \\ S_{13}^{(I)} &= P_{001}P_{100} - 0.5P_{100}, \\ S_{31}^{(I)} &= P_{001}P_{100} - 0.5P_{001}, \\ S_{23}^{(I)} &= P_{001}P_{010} - 0.5P_{010}, \\ S_{32}^{(I)} &= P_{001}P_{010} - 0.5P_{001}. \end{aligned} \quad (4)$$

We analyze here the system's ability to produce symmetric steering. Thus, we can show that this type of steering will never appear in systems described by the wave function $|\psi^{(I)}\rangle$. Let us consider a pair of qubits (for instance, those labeled by 1 and 2) and compare steering parameters corresponding to them: $S_{12}^{(I)}$ and $S_{21}^{(I)}$. From relations (4), we see that the qubit 2 steers qubit 1 ($S_{12}^{(I)} > 0$) when

$$S_{12}^{(I)} = (P_{010} - 0.5)P_{100} > 0 \quad (5)$$

In consequence, for such a situation the probabilities P_{010} and P_{100} should be different from zero and additionally, P_{010} must be greater than 0.5.

When we observe steering between the previously considered qubits, but in the opposite direction, $S_{21}^{(I)} > 0$ and hence, the inequality

$$S_{21}^{(I)} = (P_{100} - 0.5)P_{010} > 0 \quad (6)$$

should be fulfilled. Then, the both probabilities P_{010} and P_{100} should be of nonzero value too, and the probability $P_{100} > 0.5$.

Therefore, to get symmetric steering ($S_{12}^{(I)} > 0$ and $S_{21}^{(I)} > 0$) two probabilities must take values greater than 0.5 what is not possible to occur. A similar situation we get for other pairs of qubits. Then, we see that symmetric steering is impossible to obtain in the system discussed here and the generated steering is always asymmetric.

If we carefully look at Eq. (4), we see that when we assume that $P_{100} > 0.5$ two steering parameters are greater than zero— $S_{21}^{(I)}$ and $S_{31}^{(I)}$. For such situation, qubit 1 steers simultaneously the remaining two qubits, labeled by 2 and 3. Analogously, when $P_{010} > 0.5$ qubits 1 and 3 are steerable by qubit 2 ($S_{12}^{(I)} > 0$, $S_{32}^{(I)} > 0$), whereas when $P_{001} > 0.5$ qubit 3 steers those labeled by 1 and 2 ($S_{13}^{(I)} > 0$, $S_{23}^{(I)} > 0$). From the other side, if we look for the situation when two qubits steer the third one (for example, when qubits 1 and 2 steer qubit 3) two steering parameters ($S_{31}^{(I)}$ and $S_{32}^{(I)}$) should be positive, and hence, two probabilities: P_{100} and P_{010} must exceed 0.5. Obviously such situation is impossible. Therefore, for our three-qubit model, one qubit can steer two qubits, whereas two qubits cannot steer the third qubit at the same time. This fact is consistent with the monogamy results of Reid [28].

The problems related to the nonlocality seems to be one of the most interesting topics of the quantum theory and are discussed extensively in numerous papers (for instance, see [3,4,29,30] and the references quoted therein). Steering and entanglement phenomena are different types of quantum nonlocality. It is known that steerable states are the subset of the set of entangled states [3,4]. Therefore, we shall find relations among entanglement measures and the steering parameter present in our model. As the measures of two-qubit entanglement, we will apply the negativity N and concurrence C . In 2001 Verstraete et al. [31] checked the relations between the negativity and concurrence and found upper and lower bounds of the negativity’s value corresponding to a given concurrence. They proved that for two-qubit mixed states the negativity takes values smaller or equal to the corresponding concurrence C . Moreover, they have found that such negativity is always greater than or equal to $\sqrt{(1 - C)^2 + C^2} - (1 - C)$. Obviously, if we are dealing with pure states and for the Bell diagonal states, the negativity always is equal to the concurrence. In general, the negativity takes its minimal value for very special mixed states. The degree of entanglement of such states cannot be increased by any global unitary operation. The example of such states are two-qubit mixed states defined as a mixtures of Bell and separable states [32,33], named Werner states [34].

Here we will show that for steerable two-qubit mixed states the value of negativity is always larger than the value determined by its concurrence’s counterpart. As previously, we consider a pair of qubits (denoted here by 1 and 2) and calculate the both: negativity N_{12} and concurrence C_{12} describing the entanglement between them. First, we must find reduced density matrix $\rho_{12}^{(I)}$ which represents state of the two-qubit subsystem. Such matrix can be derived from the full three-qubit density matrix by tracing out one subsystem—the qubit 3.

$$\rho_{12}^{(I)} = Tr_3 \left(\rho_{123}^{(I)} \right) = \begin{bmatrix} P_{001} & 0 & 0 & 0 \\ 0 & P_{010} & C_{010}^* C_{100} & 0 \\ 0 & C_{100}^* C_{010} & P_{100} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{7}$$

The concurrence of qubit–qubit subsystem can be calculated with application of the definition proposed by Hill and Wootters [35,36]

$$C(\rho) = \max \left(\sqrt{\lambda_I} - \sqrt{\lambda_{II}} - \sqrt{\lambda_{III}} - \sqrt{\lambda_{IV}}, 0 \right). \tag{8}$$

The parameters λ_l are the eigenvalues of matrix R obtained from the relation $R = \rho \tilde{\rho}$, where $\tilde{\rho}$ is defined as $\tilde{\rho} = \sigma_y \otimes \sigma_y \rho^* \otimes \sigma_y$ and σ_y is 2×2 Pauli matrix. For the qubits 1 and 2, we have calculated eigenvalues of the matrix $R_{12} = \rho_{12} \tilde{\rho}_{12}$ and found three of four eigenvalues are zero, whereas the fourth one takes the following form

$$\lambda = 4P_{010}P_{100}. \tag{9}$$

Then, the concurrence C_{12} is given by

$$C_{12} = \sqrt{4P_{100}P_{010}}. \tag{10}$$

Next, applying the Peres–Horodecki criterion [37,38]

$$N(\rho) = \max \left(0, -2 \min_l \lambda_l \right) \tag{11}$$

where λ_l are the eigenvalues after partial transposition of the matrix ρ , we found the formulas determining the negativity for the analyzed here pairs of qubits. They can be expressed with the use of the following eigenvalues:

$$\begin{aligned} \lambda_I &= P_{001}, \\ \lambda_{II} &= P_{010}, \\ \lambda_{III} &= \frac{1}{2} \left(\sqrt{P_{001}^2 + 4P_{100}P_{010}} + P_{001} \right), \\ \lambda_{IV} &= \frac{1}{2} \left(-\sqrt{P_{001}^2 + 4P_{100}P_{010}} + P_{001} \right). \end{aligned} \tag{12}$$

We see that nonzero negativity can be obtain when eigenvalue λ_{IV} is negative—the remaining three eigenvalues are always non-negative. Then, the formula determining the negativity can be written as follows:

$$N_{12} = \sqrt{P_{001}^2 + 4P_{100}P_{010}} - P_{001}. \tag{13}$$

Afterward, we make the following replacement in Eq. (13) using relation (10): $4P_{100}P_{010}$ by C_{12}^2 . Hence, since the probability must be normalized, we obtain the formula:

$$\begin{aligned} N_{12} &= -1 + P_{010} + \frac{C_{12}^2}{4P_{010}} \\ &+ \sqrt{1 - 2P_{010} + P_{010}^2 - \frac{C_{12}^2}{2P_{010}} + \frac{1}{2}C_{12}^2 + \frac{C_{12}^4}{16P_{010}^2} + C_{12}^2} \end{aligned} \tag{14}$$

For unsteerable states, the maximal value of the steering parameter is equal to zero. Such zero value is an upper bound for not steerable states. From relation (4), for the pair of qubits 1 – 2, we see that $S_{12}^{(I)} = 0$ when $P_{010} = 0.5$, whereas $S_{21}^{(I)} = 0$ when $P_{100} = 0.5$. Thus, replacing the probability P_{010} by the boundary value 0.5 we obtain

$$N_{12} = C_{12}^2. \tag{15}$$

Analogous formulas can be derived for other pairs of qubits (1 – 3 and 2 – 3). In general, for our model with a single excitation, when two qubits are not steerable, the maximal possible value of the negativity for a given value of the concurrence is determined by the following formula:

$$N_{\text{bound}}^{(I)} = C^2, \tag{16}$$

and the states are steerable when the value of negativity becomes greater than its boundary value which is specified by Eq. (16).

What is interesting, the qubit–qubit mixed states for which negativity takes its minimal possible value determined by the Verstraete’s formula $N_{\min} = \sqrt{(1 - C)^2 + C^2} - (1 - C)$ [31], and for which the degree of entanglement cannot be increased by any global unitary operation are not steerable. For the situation discussed here, such states are Werner states and they are mixtures of the Bell state $|\psi_1\rangle = \frac{1}{\sqrt{2}}(|01\rangle_{ij} + |10\rangle_{ij})$ and the vacuum state $|\psi_2\rangle = |00\rangle_{ij}$, represented by the following two-qubit system density matrix

$$\rho_W^{(I)} = \alpha|\psi_1\rangle\langle\psi_1| + (1 - \alpha)|\psi_2\rangle\langle\psi_2| = \begin{bmatrix} 1 - \alpha & 0 & 0 & 0 \\ 0 & \frac{1+\alpha}{2} & \frac{1-\alpha}{2} & 0 \\ 0 & \frac{1-\alpha}{2} & \frac{1+\alpha}{2} & 0 \\ 0 & 0 & 0 & \alpha \end{bmatrix}. \tag{17}$$

For the states represented by $\rho_W^{(I)}$, the concurrence and negativity are equal to

$$\begin{aligned} C(\rho_W^{(I)}) &= \alpha, \\ N(\rho_W^{(I)}) &= \sqrt{(1 - \alpha)^2 + \alpha^2} - (1 - \alpha), \end{aligned} \tag{18}$$

respectively.

To check the correctness of our formula determining the upper bound of negativity for unsteerable qubit–qubit mixed states, we have performed numerical simulations. We generated randomly $\sim 10^6$ three-qubit states defined by the density matrix $\rho^{(I)}$ (Eq. 2). Next, according to the method described previously, we have calculated two-qubit density matrix traced out third subsystem, and for each qubit–qubit state we calculated the concurrence $C(\rho_{ij}^{(I)})$ and negativity $N(\rho_{ij}^{(I)})$. The results are shown in Fig. 1, where red (upper) region corresponds to the states which are steerable, whereas the turquoise area corresponds to the unsteerable states. Black solid line represents the border defined by Eq. (16). We see that the upper bound of unsteerable states determined by our formula agrees with that obtained by numerical calculations. From the other side, the dashed line shown in Fig. 1 corresponds to the minimal values of the negativity $N_{\min} = N(\rho_W^{(I)})$. The states determined by such values of the negativity and concurrence are Werner states $\rho_W^{(I)}$.

For the same set of generated randomly states, we have shown in Fig. 2 how the value of steering parameter depends on the values of the concurrence and negativity. For a given value of the concurrence, the steering parameter takes its maximal value when $N(\rho_{ij}^{(I)}) = C(\rho_{ij}^{(I)})$; see Fig. 2a. For these situations, steering parameter can be expressed as

$$S_{ij}^{(I)} = \frac{1}{4} \left(C^2 - 1 + \sqrt{1 - C^2} \right). \tag{19}$$

In Fig. 2a, the maximal value of steering parameter is represented by black line. When the value of negativity is smaller than its concurrence’s counterpart, the steering parameter takes value smaller than that defined by Eq. (19). What is interesting, the strongest steering appears when the condition $C(\rho_{ij}^{(I)}) = N(\rho_{ij}^{(I)}) = \frac{\sqrt{3}}{2}$ is fulfilled.

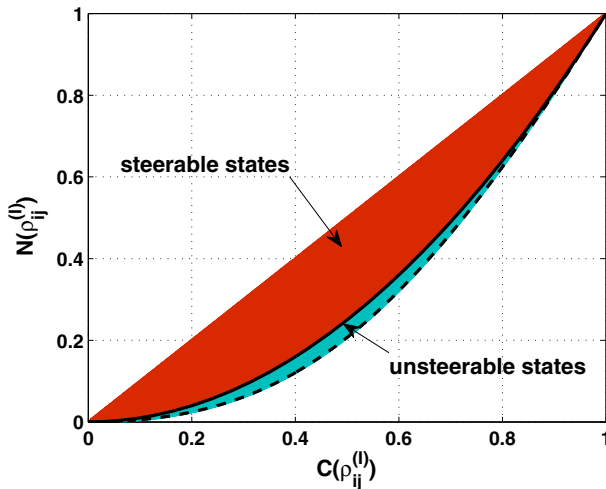


Fig. 1 The negativity $N(\rho_{ij}^{(I)})$ versus concurrence $C(\rho_{ij}^{(I)})$ for the steerable (red area) and unsteerable (turquoise area) two-qubit states described by the density matrix $\rho_{ij}^{(I)}$, calculated numerically. Black lines (solid and dashed) are plotted with application of derived here formulas (Color figure online)

For such situation, the density matrix describing system’s state takes the following form:

$$\begin{aligned} \rho_{ij}^{(I)}(S = \max) &= \left(\sqrt{\alpha}|01\rangle + \sqrt{1-\alpha}|10\rangle \right) \left(\sqrt{\alpha}\langle 01| + \sqrt{1-\alpha}\langle 10| \right) \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \alpha & \sqrt{\alpha(1-\alpha)} & 0 \\ 0 & \sqrt{\alpha(1-\alpha)} & 1-\alpha & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \end{aligned} \tag{20}$$

where α is equal to the one of two values: $\frac{1}{4}$ or $\frac{3}{4}$.

Figure 2b shows which values can reach the steering parameter for a given negativity for steerable state. The black line shown there represents maximal values of the steering parameter. For the negativity greater than $\frac{1}{\sqrt{2}}$, the range of steering parameter for steerable states is the same as for the concurrence and the parameter S_{ij} takes maximum value when $N(\rho_{ij}^{(I)}) = C(\rho_{ij}^{(I)})$. When the negativity is smaller or equal to $\frac{1}{\sqrt{2}}$, maximal value of the steering parameter is defined by the following equation:

$$S_{ij}^{(I)} = \frac{1}{16} \left(-2 + \sqrt{2} - 2N + \sqrt{2}(1 + 2N) \right) \left(-2N + \sqrt{8}N \right). \tag{21}$$

Such parameter can be found when the state of our system is described by the density matrix:

$$\rho_{ij}^{(I)}(S = \max) = \begin{bmatrix} 1-\alpha-\beta & 0 & 0 & 0 \\ 0 & \alpha & \sqrt{\alpha\beta} & 0 \\ 0 & \sqrt{\alpha\beta} & \beta & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{22}$$

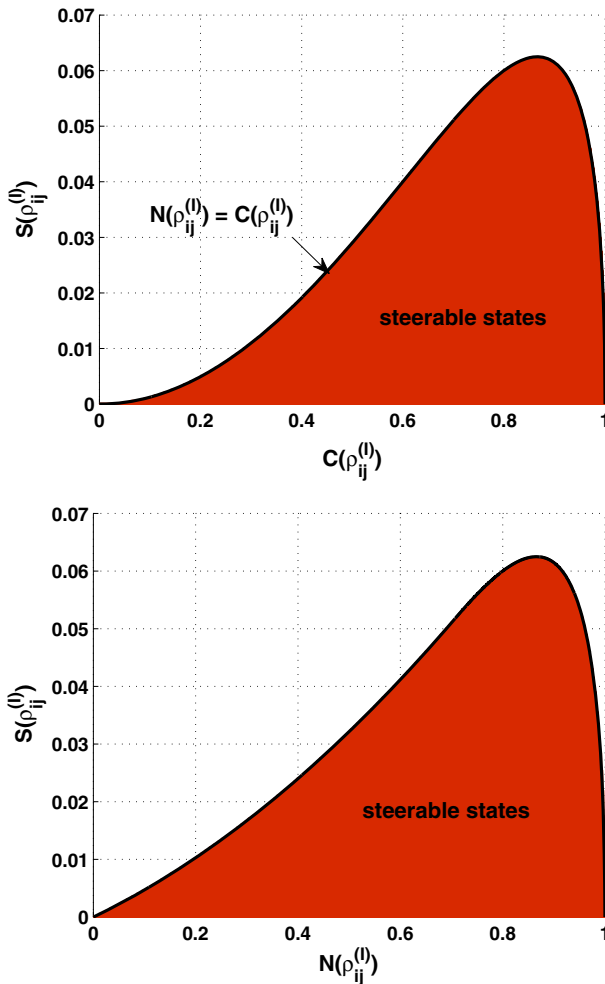


Fig. 2 Steering parameter $S_{ij}^{(I)}$ versus the concurrence $C(\rho_{ij}^{(I)})$ (top) and the negativity $N(\rho_{ij}^{(I)})$ (bottom). Red area corresponds to the set of steerable states. Black border line denotes the states for which $C(\rho_{ij}^{(I)}) = N(\rho_{ij}^{(I)})$ (Color figure online)

where the probabilities obey the relation $|\alpha - \beta| = \frac{1}{\sqrt{2}}$.

As the upper bound for the steerable states was previously found for the qubit-qubit mixed states, it seems to be natural to compare entanglement measures and steering parameter with degree of the mixedness for discussed states. As a measure of mixedness, we will apply the linear entropy defined with application of the purity [39] as:

$$E(\rho) \equiv \frac{D}{D-1} \left[1 - Tr(\rho^2) \right], \tag{23}$$

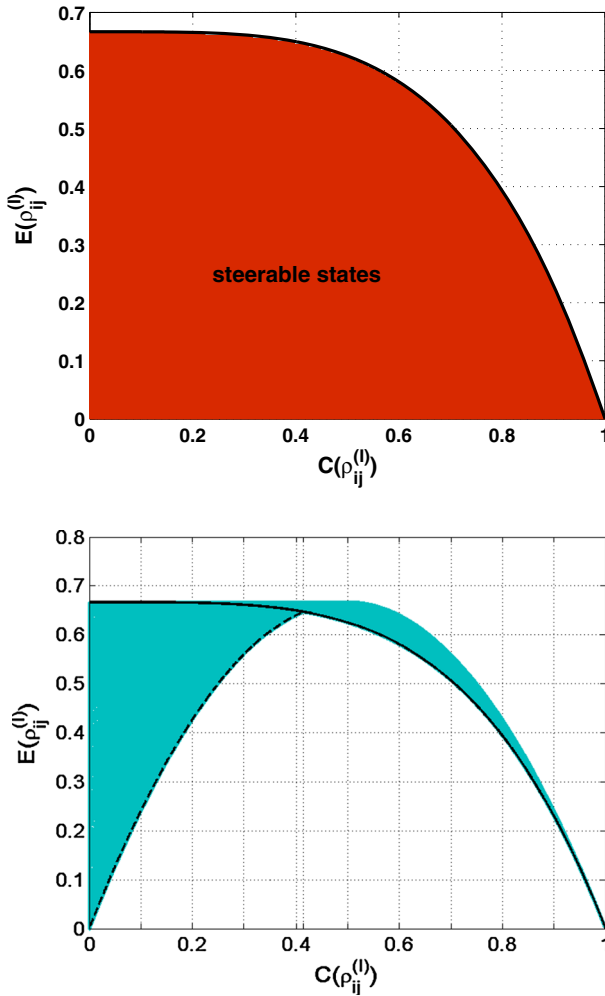


Fig. 3 Linear entropy $E(\rho_{ij}^{(I)})$ versus concurrence $C(\rho_{ij}^{(I)})$ for the both: steerable (*top*) and unsteerable (*bottom*) states. Colored areas correspond to all possible randomly chosen states, whereas solid and dashed lines were plotted with application of derived here conditions (Color figure online)

where D denotes dimension of ρ . The linear entropy takes values from zero to unity. It is equal to zero for a pure states, whereas it reaches unity for maximally mixed states. For two-qubit states, dimension $D = 4$ and linear entropy takes the following form:

$$E(\rho_{ij}) \equiv \frac{4}{3} \left[1 - \text{Tr} \left(\rho_{ij}^2 \right) \right]. \tag{24}$$

In Fig. 3, we show ranges of the linear entropy $E(\rho_{ij}^{(I)})$ for a given value of concurrence $C(\rho_{ij}^{(I)})$ for the qubit–qubit mixed states (all states are generated randomly, the same way as in the previous figure, whereas border lines were determined by the formulas derived below). Figure 3a presents results for steerable states, whereas the

results for unsteerable states are shown in Fig. 3b. We see that for some values of concurrence and linear entropy the both: steerable and unsteerable states can exist. To find upper bound of degree of mixedness for steerable state, we will compare concurrence, linear entropy and steering parameter. As previously, we consider qubits labeled by 1 and 2, and calculate linear entropy for the states described by the density matrix $\rho_{12}^{(I)}$. Thus, the entropy is given by:

$$E\left(\rho_{12}^{(I)}\right) \equiv \frac{4}{3} \left(1 - \left(P_{001}^2 + P_{010}^2 + P_{100}^2 + 2P_{010}P_{100}\right)\right). \tag{25}$$

Some of the above probabilities can be replaced by the concurrence (see Eq. 10), and applying normalization condition for the probabilities we get

$$E\left(\rho_{12}^{(I)}\right) \equiv \frac{4}{3} \left(1 - \left(1 + C_{12}^2 + \frac{C_{12}^4}{8P_{100}^2} - \frac{C_{12}^2}{2P_{100}} - 2P_{100} + 2P_{100}^2\right)\right). \tag{26}$$

Next, replacing probability P_{100} by previously derived its boundary value 0.5, we obtain

$$E\left(\rho_{12}^{(I)}\right) \equiv \frac{2}{3} - \frac{2}{3}C_{12}^4. \tag{27}$$

The derived above condition is represented by the solid line limiting the area for steerable state in Fig. 3a. Obviously, we can perform the analogous derivation for other pairs of qubits, and hence, the indices 1 and 2 can be omitted.

Next, we will find condition which corresponds to the dashed border line limiting the area for unsteerable state in Fig. 3b. States corresponding to this curve are the states for which the negativity takes its minimal value and are described by the density matrix $\rho_W^{(I)}$; see Eq. (17). For such a case, applying Eq. (18), linear entropy can be written as

$$E\left(\rho_{ij}^{(I)}\right) \equiv \frac{8}{3} \left(C - C^2\right). \tag{28}$$

Thus, when the concurrence $C < \sqrt{2} - 1$, the states are steerable, and hence, linear entropy should obey the condition $E(\rho_{ij}^{(I)}) < \frac{8}{3} (C - C^2)$. When the entropy $\frac{8}{3} (C - C^2) \leq E(\rho_{ij}^{(I)}) < \frac{2}{3} - \frac{2}{3}C^2$, discussed states can be steerable or unsteerable, whereas for $E(\rho_{ij}^{(I)}) \geq \frac{2}{3} - \frac{2}{3}C^2$ all states are unsteerable. From the other side, for the cases when the concurrence $C \geq \sqrt{2} - 1$ we always observe steering when $E(\rho_{ij}^{(I)}) < \frac{2}{3} - \frac{2}{3}C^2$. However, when $E(\rho_{ij}^{(I)}) \geq \frac{2}{3} - \frac{2}{3}C^2$ steering cannot appear. All above situations are listed in Table 1.

Figure 4 shows the same as Fig. 3, but the concurrence was replaced there by the negativity. Again, the red region represents steerable states (Fig. 4a), whereas the turquoise area corresponds to unsteerable states (Fig. 4b). When we compare Fig. 4a, b we see that for the values of linear entropy corresponding to the area confined by solid and dashed lines, the both steerable and unsteerable states can appear.

Applying Eqs. (13) and (25), we are able to determine the borders represented by solid lines in Fig. 4. The formula defining them can be written as:

Table 1 The range of linear entropy for given values of the concurrence for steerable and unsteerable states

Concurrence	Linear entropy	Generated states
$C < \sqrt{2} - 1$	$E(\rho_{ij}^{(I)}) < \frac{8}{3}(C - C^2)$	Only steerable states
	$E(\rho_{ij}^{(I)}) \geq \frac{8}{3}(C - C^2)$	Steerable or unsteerable states
	and $E(\rho_{ij}^{(I)}) < \frac{2}{3} - \frac{2}{3}C^2$	
	$E(\rho_{ij}^{(I)}) \geq \frac{2}{3} - \frac{2}{3}C^2$	Only unsteerable states
$C \geq \sqrt{2} - 1$	$E(\rho_{ij}^{(I)}) < \frac{2}{3} - \frac{2}{3}C^2$	Only steerable states
	$E(\rho_{ij}^{(I)}) \geq \frac{2}{3} - \frac{2}{3}C^2$	Only unsteerable states

$$E(\rho_{ij}^{(I)}) \equiv -\frac{2}{3}(N^2 - 1). \tag{29}$$

Thus, the states can be steerable when the entropy reaches the values smaller than those determined by (29). As we compare Fig. 4a, b, we see that such condition is a necessary, not always sufficient condition—from Fig. 4b follows that although there are states for which $E(\rho_{ij}^{(I)})$ is smaller than such boundary value (depicted in Fig. 4a), we can find there unsteerable states (corresponding to the turquoise region below solid line in Fig. 4b).

Next, we find equation describing the border of the area of unsteerable states represented by the dashed line shown in Fig. 4b. The states corresponding to the line are those described by the density matrix $\rho_W^{(I)}$ (Eq. 17). Applying relation (18) which describes the negativity and Eq. (25), we obtain the formula which expresses the linear entropy as a function of the negativity

$$E(\rho_{ij}^{(I)}) \equiv -\frac{8}{3}(-1 - N + \sqrt{2(N + N^2)})(-N + \sqrt{2(N + N^2)}). \tag{30}$$

For $N < \frac{1}{7}$ the states are steerable when the linear entropy fulfills the condition $E(\rho_{ij}^{(I)}) < -\frac{2}{3}(N^2 - 1)$. When linear entropy is greater than or equal to the value defined by Eq. (30) or is smaller than the value described by Eq. (29), the states can be the both: steerable or unsteerable (Table 2).

3 Three-qubit system with two excitations

The second case considered in this paper concerns the situation when two excitations are present in the system i.e., $\langle \hat{n} \rangle = \langle \hat{n}_1 \rangle + \langle \hat{n}_2 \rangle + \langle \hat{n}_3 \rangle = 2$. For such situation, the wave function describing the system can be written in the following form

$$|\psi^{(II)}\rangle = C_{011}|011\rangle + C_{101}|101\rangle + C_{110}|110\rangle, \tag{31}$$

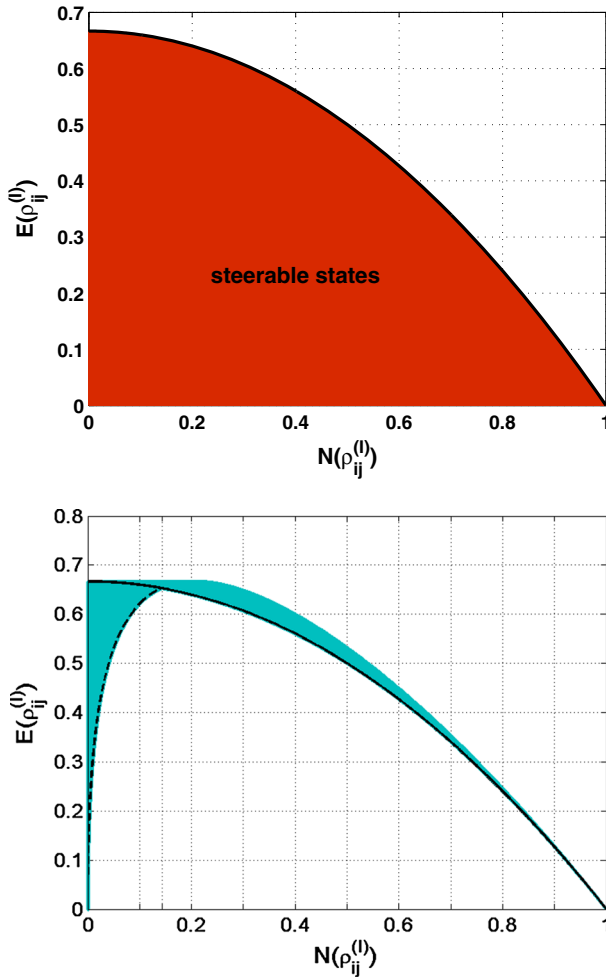


Fig. 4 The linear entropy versus negativity for randomly generated states. *Colored areas* correspond to all possible states—*red area (top)* denotes steerable, whereas *turquoise (bottom)* unsteerable states. *Solid and dashed lines* depict derived and discussed in the text boundary conditions (Color figure online)

and the corresponding density matrix is

$$\rho^{(II)} = |\psi^{(II)}\rangle\langle\psi^{(II)}| = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & P_{011} & 0 & C_{011}^* C_{101} & C_{011}^* C_{110} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{101}^* C_{011} & 0 & P_{101} & C_{101}^* C_{110} & 0 & 0 \\ 0 & 0 & 0 & C_{110}^* C_{011} & 0 & C_{110}^* C_{101} & P_{110} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{32}$$

Table 2 The ranges of the values of linear entropy and negativity for steerable and unsteerable states

Negativity	Linear entropy	Generated states
$N < \frac{1}{7}$	$E(\rho_{ij}^{(I)}) < -\frac{8}{3} \left(-1 - N + \sqrt{2(N + N^2)} \right) \left(-N + \sqrt{2(N + N^2)} \right)$	Only steerable states
	$E(\rho_{ij}^{(I)}) \geq -\frac{8}{3} \left(-1 - N + \sqrt{2(N + N^2)} \right) \left(-N + \sqrt{2(N + N^2)} \right)$	Steerable or unsteerable states
	and $E(\rho_{ij}^{(I)}) < -\frac{2}{3} (N^2 - 1)$	
$N \geq \frac{1}{7}$	$E(\rho_{ij}^{(I)}) \geq -\frac{2}{3} (N^2 - 1)$	Only unsteerable states
	$E(\rho_{ij}^{(I)}) < -\frac{2}{3} (N^2 - 1)$	Only steerable states
	$E(\rho_{ij}^{(I)}) \geq -\frac{2}{3} (N^2 - 1)$	Only unsteerable states

Analogously as in the previous section, we can derive the expressions describing steering parameters for each pair of qubits expressed by the probabilities as:

$$\begin{aligned}
 S_{12}^{(II)} &= P_{011} P_{101} + \frac{3}{2} P_{011} + P_{101} - \frac{3}{2} \\
 S_{21}^{(II)} &= P_{011} P_{101} + \frac{3}{2} P_{101} + P_{011} - \frac{3}{2} \\
 S_{13}^{(II)} &= P_{011} P_{110} + \frac{3}{2} P_{011} + P_{110} - \frac{3}{2} \\
 S_{31}^{(II)} &= P_{011} P_{110} + \frac{3}{2} P_{110} + P_{011} - \frac{3}{2} \\
 S_{23}^{(II)} &= P_{101} P_{110} + \frac{3}{2} P_{101} + P_{110} - \frac{3}{2} \\
 S_{32}^{(II)} &= P_{101} P_{110} + \frac{3}{2} P_{110} + P_{101} - \frac{3}{2}
 \end{aligned} \tag{33}$$

Next, to check the system’s ability to produce symmetric steering, we consider the first and second qubits and compare two steering parameters $S_{12}^{(II)}$ and $S_{21}^{(II)}$ (due to the symmetry of the model we omit here discussion of other pairs of qubits). From relations (33), we see that qubit 2 steers qubit 1 ($S_{12}^{(II)} > 0$) when the probabilities P_{011} and P_{101} are of nonzero values and additionally, $P_{011} P_{101} + \frac{3}{2} P_{011} + P_{101} > \frac{3}{2}$. When steering in opposite direction is present (i.e., $S_{21}^{(II)} > 0$), the both probabilities P_{010} and P_{100} should be greater than zero, and $P_{011} P_{101} + \frac{3}{2} P_{101} + P_{011} > \frac{3}{2}$. To satisfy the conditions for the presence of steering in two directions, the following inequalities should be fulfilled:

$$\begin{aligned}
 0 < P_{011} \leq \frac{1}{2} \quad P_{101} > \frac{3 - 3P_{011}}{2 + 2P_{011}} \\
 \text{or} \\
 \frac{1}{2} < P_{011} \leq 1 \quad P_{101} > \frac{3 - 2P_{011}}{3 + 2P_{011}}.
 \end{aligned} \tag{34}$$

However, as the sum of probabilities P_{010} and P_{100} can not be greater than 1, conditions (34) will never be satisfied. Thus, symmetric steering never appears in the double excited system of three qubits—the discussed here case and for the double excited three-qubit system the generated steering is always asymmetric.

As we have shown in the previous section, for the case of single excited system one qubit can steer two others simultaneously. For double excited systems, such type of steering can not appear. Let us analyze situations when the first qubit steers two other. For such a case, two steering parameters must be greater than zero $S_{21}^{(II)} > 0$, $S_{21}^{(II)} > 0$. From relations (33), we obtain that two conditions must be satisfied simultaneously: $P_{011}P_{101} + \frac{3}{2}P_{101} + P_{011} > \frac{3}{2}$ and $P_{011}P_{110} + \frac{3}{2}P_{110} + P_{011} > \frac{3}{2}$. Unfortunately, the system of these two inequalities for the probabilities has no solution and, in consequence, one subsystem cannot steers two others.

Next, we concentrate on the situation when two qubits steer the third one—for example, when qubits 1 and 3 steer qubit 2. For such situation parameters $S_{21}^{(II)}$ and $S_{23}^{(II)}$ should be greater than zero. In consequence, $P_{011}P_{101} + \frac{3}{2}P_{101} + P_{011} > \frac{3}{2}$ and $P_{101}P_{110} + \frac{3}{2}P_{101} + P_{110} > \frac{3}{2}$, and it leads to the inequalities:

$$-\frac{1}{2} < P_{011} < 0 \quad \frac{3-2P_{011}}{3+2P_{011}} < P_{101} < \frac{1}{4}(3-2P_{011}) + 14\sqrt{1-28P_{011}+4P_{011}^2}, \tag{35}$$

where the probabilities are negative and the solution is not physical. Therefore, for double excited system, the situation when two qubits steer the third one at the same time cannot be possible.

Again, as for the single excited system, we can find boundary values of the negativity for unsteerable states. Thus, we consider a pair of qubits (labeled her as 1 and 2) and calculate two entanglement’s measures: the negativity and concurrence. To do it, we find the reduced density matrix $\rho_{12}^{(II)}$ which represents a state of the two-qubit subsystem

$$\rho_{12}^{(II)} = Tr_3 \left(\rho_{123}^{(II)} \right) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & P_{011} & C_{011}^* C_{101} & 0 \\ 0 & C_{101}^* C_{011} & P_{101} & 0 \\ 0 & 0 & 0 & P_{110} \end{bmatrix}. \tag{36}$$

Next, we calculate the concurrence

$$C_{12}^{(II)} = \sqrt{4P_{011}P_{101}}, \tag{37}$$

and the negativity

$$N_{12}^{(II)} = \sqrt{P_{110}^2 + 4P_{011}P_{101}} - P_{110}. \tag{38}$$

We know that for unsteerable states the maximal value of the steering parameter is equal to zero. Knowing that the sum of all probabilities should be normalized and applying conditions for the steering (Eqs. 37, 38), we obtain the following formula

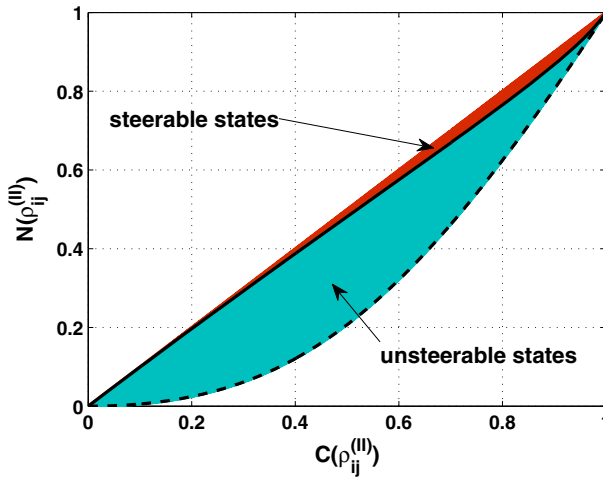


Fig. 5 The negativity $N(\rho_{ij}^{(II)})$ versus concurrence $C(\rho_{ij}^{(II)})$ for 10^6 randomly chosen three-qubit states. Red region corresponds to the steerable states, whereas turquoise to unsteerable ones (Color figure online)

which allows to determine maximal possible value of the negativity for a given value of the concurrence for unsteerable states

$$N_{\text{bound}}^{(II)}(C) = \frac{1}{24} \left(6 - 5C^2 - A + \sqrt{2} \sqrt{13C^4 - 6(-6 + A) + 5C^2(48 + A)} \right), \tag{39}$$

where $A = \sqrt{C^4 - 36C^2 + 36}$. The states are steerable when negativity takes values greater than the boundary negativity defined by above equation. Obviously, analogous equations can be derived for other pairs of qubits.

The correctness of the analytical formulas derived here we check by numerical simulation. We generated randomly 10^6 three-qubit states defined by the density matrix $\rho^{(II)}$ (Eq. 32). Next, we calculated the elements of two-qubit density matrix tracing out the third subsystem, and then we calculated the concurrence $C(\rho_{ij}^{(II)})$ and the negativity $N(\rho_{ij}^{(II)})$ for each qubit–qubit subsystems. The results are shown in Fig. 5, where red and turquoise regions correspond to the steerable and unsteerable states, respectively. Solid black line in Fig. 5 represents the border between steerable and unsteerable states defined by Eq. (39). We see that the derived here analytical formula determining upper bound of unsteerable states is correct. What is interesting, the area of region corresponding to the steerable states is much smaller than its counterpart corresponding to unsteerable states. We see that for three-qubit system with double excitation steerability is more difficult to obtain than for the case considered in previous section—”amount” of states for which steering is generated is smaller than for three-qubit system with single excitation. Dashed line shown in Fig. 5 corresponds to the situations when negativity takes its minimal possible values $N_{\text{min}} = \sqrt{(1 - C)^2 + C^2} - (1 - C)$ [31]. For such a case, the states belong to the family of Werner states which are mixtures of the Bell state $|\psi_1\rangle = \frac{1}{\sqrt{2}}(|01\rangle_{ij} + |10\rangle_{ij})$ and the state $|\psi_2\rangle = |11\rangle_{ij}$ and can be represented by the following two-qubit density matrix

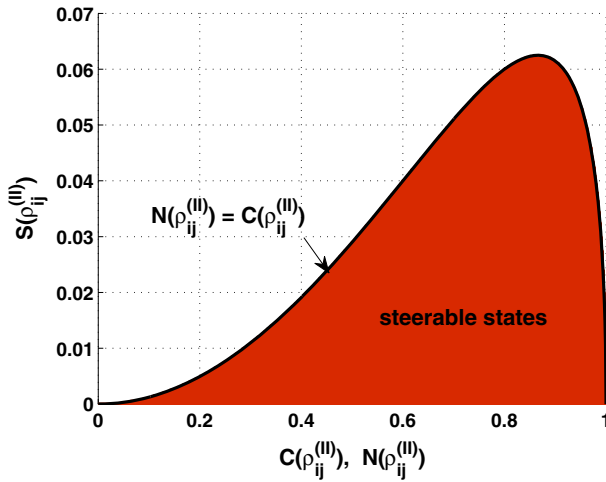


Fig. 6 Simulations of steerable two-qubit states $\rho_{ij}^{(II)}$ for the steering parameter $S_{ij}^{(II)}$ versus concurrence $C(\rho_{ij}^{(II)})$ or negativity $N(\rho_{ij}^{(II)})$

$$\rho_W^{(II)} = \alpha |\psi_1\rangle\langle\psi_1| + (1 - \alpha) |\psi_2\rangle\langle\psi_2| = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha}{2} & \frac{\alpha}{2} & 0 \\ 0 & \frac{\alpha}{2} & \frac{\alpha}{2} & 0 \\ 0 & 0 & 0 & 1 - \alpha \end{bmatrix}. \tag{40}$$

For states described by the density matrix $\rho_W^{(II)}$, concurrence and negativity are equal

$$\begin{aligned} C(\rho_W^{(II)}) &= \alpha, \\ N(\rho_W^{(II)}) &= \sqrt{(1 - \alpha)^2 + \alpha^2} - (1 - \alpha). \end{aligned} \tag{41}$$

Next, we analyze the ranges of possible values of the steering parameter according to a given concurrence or negativity for steerable states. From Fig. 6, we see that for steerable states such range for a given concurrence is the same as that for the negativity. What is important, the same as for the system with a single excitation, the steering parameters for given values of the concurrence take here their maximal values when $N(\rho_{ij}^{(II)}) = C(\rho_{ij}^{(II)})$ and such values are equal to those defined by Eq. (19). The strongest steering appears for $C(\rho_{ij}^{(II)}) = N(\rho_{ij}^{(II)}) = \frac{\sqrt{3}}{2}$, and the density matrix describing the system’s state takes the form presented by matrix (20).

To find how the mixedness is related to the entanglement in our system, we inspect ranges of the values of linear entropy $E(\rho_{ij}^{(II)})$ for a given concurrence (Fig. 7) and negativity (Fig. 8), respectively. The red regions shown in figures correspond to steerable states, whereas the turquoise ones denote the regions of unsteerable states. For the both situations, we see that for some values of the linear entropy and concurrence/negativity we observe only steerable states. To find the upper bound of steerable states determined by the degree of mixedness for a given concurrence, we consider qubits 1 and 2 and calculate the linear entropy for the state $\rho_{12}^{(II)}$

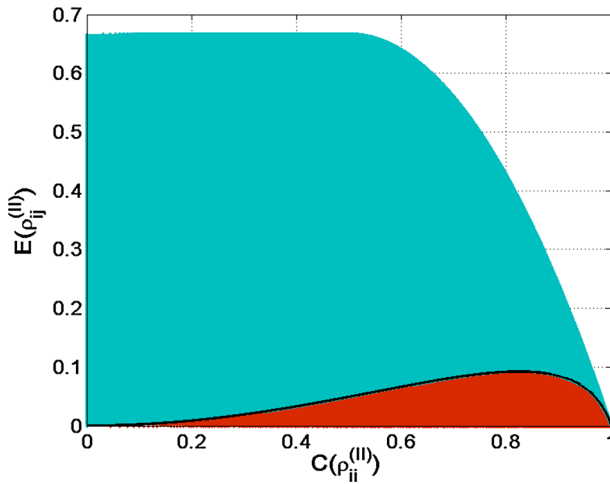


Fig. 7 Phase diagram for the steerable (*red region*) and unsteerable (*turquoise region*) 10^6 randomly chosen two-qubit states described by the density matrix $\rho_{ij}^{(II)}$. *Solid line* denotes the border between two considered here classes of states (Color figure online)

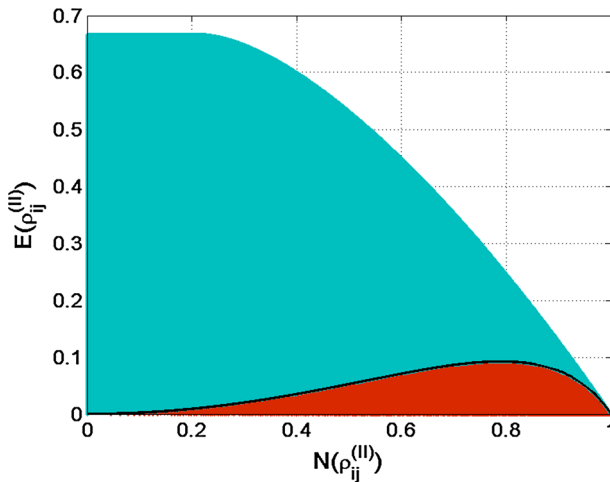


Fig. 8 The same as in Fig. 7 but for the negativity instead of concurrence

$$E(\rho_{12}^{(II)}) \equiv \frac{8}{3} \left(-P_{011}^2 + P_{011} - P_{101}^2 + P_{101} - 2P_{011}P_{101} \right). \tag{42}$$

then, applying formula (37) we can express (42) in the following form:

$$E(\rho_{12}^{(II)}) = -\frac{(-6 + A_{12} + C_{12}^2)}{6(-18 + C_{12}^2 + A_{12})^2} \times \left(-240 + 56A_{12} + 236C_{12}^2 - 18A_{12}C_{12}^2 - 36C_{12}^4 + A_{12}C_{12}^4 + C_{12}^6 \right), \tag{43}$$

where $A_{12} = \sqrt{C_{12}^4 - 36C_{12}^2 + 36}$. Obviously, for other pairs of qubits we obtain analogous results, and hence, the indices appearing in (43) can be omitted. Moreover, from Figs. 7 and 8 we see that not only the areas of regions corresponding to the steerable states are smaller than those for unsteerable ones, but also the values of entropy for a given concurrence and negativity are greater for unsteerable states than those for steerable ones.

As Fig. 8 depicts the same situation as that shown in Fig. 7 but for the negativity instead of concurrence, we can derive the formula determining the border between two discussed here families of the states as a function of the negativity. Thus, Eq. (43) can be expressed by the negativity as

$$E\left(\rho_{ij}^{(II)}\right) = \frac{54 - 15N - 50N^2 + 23N^3 + 2N^4 - 2N^5 + A(-9 + N + 2N^2)}{-54 + 63N - 24N^2 + 3N^3}, \quad (44)$$

where $A = \sqrt{N^4 + 14N^3 - 35N^2 - 12N + 36}$.

At this point, we would like to mention two other possible situations—when there are no excitations in the system or three excitations are present. For such cases, the system remains in the state $|000\rangle$ or $|111\rangle$, respectively. Since we assumed here that all qubits do not interact with other systems, we deal here with trivial situations for which all steering parameters remain equal to zero or are negative. In consequence, for all states describing such systems the steering phenomenon does not appear.

It would be also desirable to mention here the problem of physical realization of the ideas presented in the last two sections. As we discuss here three-qubits states, we are looking for physical models which could be good candidates for systems described by such states. In particular, they should be two-level systems such as two-level atoms [40], two-state spin systems [41] or systems which can be treated as the so-called quantum scissors [42] (systems generally described in infinite Hilbert space but under some conditions their evolution remains closed within a finite set of the states—here we deal with two states $|0\rangle$ and $|1\rangle$). Such scissors can lead to photon(phonon) blockade [43, 44] effects, and hence, considered system can evolve within two states. The examples of such scissors system can be found in a group of the optical Kerr or Kerr-like systems [45–48]. Another interesting proposals are those related to the atoms trapped in optical lattices [49] (even in 3D lattices [50]), superconducting circuits [51, 52], quantum dot systems [53, 54], optomechanical models [19, 55] and other. What is important, the systems mentioned here were not only considered theoretically but also were applied in various experiments. Thus, measurements of qubits in molecular single-ion magnet were presented in [56], whereas three-qubit entanglement in a superconducting circuit was achieved experimentally by DiCarlo et al. [57] and by Neely et al. [58]. From other side, Neumann et al. presented the creation of (bipartite and tripartite) entangled quantum states in a tiny quantum register built from individual ^{13}C nuclei located in a diamond lattice [59]. The blockade effects have also been observed successfully in experiments. In particular, one should mention at this point Santa Barbara experiment [60] in which photon blockade has been observed for a single atom trapped inside optical cavity [61]. Moreover, quantum scissors system which was based on the Kerr

media effect, leading to the single-photon states generation, has been accomplished and experimentally observed by Kirchmair et al. [62].

If we look at the results concerning two models discussed here (those comprising one or two excitations), we can ask which of them is more feasible experimentally and which can lead to more interesting results. When we compare the map presented in Fig. 1 with its counterpart from Fig. 5, we see that the range of steerable states is wider for the situation when we deal with systems involving only single excitation. Thus, such systems seem to be more interesting and promising from the point of view of steerable states generation. From other side, if we look at the current progress in the field of experiments, systems for which single-photon (phonon) blockade can appear, seem to be more attainable experimentally than those where two or more photons blockade can be achieved. Additionally, for systems based on the trapped atoms, manipulation of a single atom seems to be easier than of groups of them. For the situation when we are dealing with two atoms, we should avoid the situation when both of them will be located at the same site. Since our system should be described in only two states basis (that defined by one-atom and no-atoms states), maximally one atom can be located at a given site. Therefore, models with single excitations seem to be more promising and interesting than those with two excitations.

4 Summary

In this work, we discussed the properties of a family of three-qubit states, especially in the context of quantum steerability. In particular, we have concentrated on two cases—(i) one qubit was in its upper state (one excitation was present in physical system) and (ii) two qubits were excited. Applying the parameters based on the Cavalcanti inequality, we analyzed the possibility of appearance of the steering between two qubits for the both situations.

We have shown that for the discussed model symmetric steering cannot appear—only asymmetric one can be observed. Moreover, we proved that for the single excited model one qubit can simultaneously steer two qubits. However, this type of steering we cannot observe for double excited cases. Additionally, it was proved that two qubits cannot steer simultaneously the remaining one.

It is known that the value of negativity for two-qubit system can not be greater than that for the corresponding concurrence. The negativity always exceeds or is equal to the bound found by Verstraete et al. [31]. In this paper, we have also compared the values of negativity, concurrence, linear entropy characterizing mixedness of the state and steering parameter. Such parameters were calculated for the states corresponding to the two-qubit subsystem. As a result, we have found the upper bound of unsteerable two-qubit mixed state for the states corresponding to the model with single and double excitations. We showed that if the negativity is greater than its boundary value, which was derived here, the discussed states are steerable. Moreover, we have found a condition when the considered here states are characterized by the strongest steering; see Eq. (20). It was also shown here that for a given value of concurrence the steering parameter reaches its maximal value when the negativity is equal to that of

concurrence. In addition, maximal values of steering parameter for a given concurrence (Eq. 19) and for given negativity (Eq. 21) were found here.

Finally, we discussed how mixedness of the considered here states is related to the steering phenomenon. We have found linear entropy and conditions defining bounds between the regions corresponding to the steerable states and unsteerable ones. Those conditions were determined by formulas for the linear entropy parametrized by (i) the concurrence and (ii) the negativity. All derived here conditions were derived for the both cases—states with single and two excitations.

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