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Mutually unbiased maximally entangled bases in $\mathbb{C}^d \otimes \mathbb{C}^{kd}$

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Abstract We study maximally entangled bases in bipartite systems $\mathbb{C}^d \otimes \mathbb{C}^{kd}$ ($k \in Z^+$), which are mutually unbiased. By systematically constructing maximally entangled bases, we present an approach in constructing mutually unbiased maximally entangled bases. In particular, five maximally entangled bases in $\mathbb{C}^2 \otimes \mathbb{C}^4$ and three maximally entangled bases in $\mathbb{C}^2 \otimes \mathbb{C}^6$ that are mutually unbiased are presented.

Keywords Mutually unbiased bases · Maximally entangled states · Pauli matrices

1 Introduction

Quantum entanglement is central in quantum information processing and quantum computation [1–12]. In particular, maximally entangled states play vital role in quantum information processing tasks such as perfect teleportation [5–19]. It has been proved that mixed maximally entangled states also exist when the two individual dimensions of a bipartite system are not equal [13]. A pure state $|\psi\rangle$ is said to be a $d \otimes d'$ ($d' > d$) maximally entangled state if and only if for an arbitrary given orthonormal complete basis $\{|i_A\rangle\}$ of subsystem A, there exists an orthonormal basis $\{|i_B\rangle\}$

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of subsystem B such that $|\psi\rangle$ can be written as $|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i_A\rangle \otimes |i_B\rangle$ [14]. There are many references for the bases of entangled states [15–18].

Mutually unbiased bases (MUBs) play central roles in quantum kinematics [19], quantum state tomography [20,21] and in quantifying wave–particle duality in multi-path interferometers [22]. Moreover, the importance of the mutually unbiased bases has been demonstrated in various tasks in quantum information processing, such as quantum key distribution [23], cryptographic protocols [23,24], mean king problem [25], quantum teleportation and superdense coding [26–28].

Two orthogonal bases $\mathcal{B}_1 = \{|\phi_i\rangle\}_{i=1}^{d}$ and $\mathcal{B}_2 = \{|\psi_i\rangle\}_{i=1}^{d}$ of $\mathbb{C}^d$ are said to be mutually unbiased if

$$|\langle \phi_i | \psi_j \rangle| = \frac{1}{\sqrt{d}}, \quad i, j = 1, 2, \ldots, d.$$

A set of orthonormal bases $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_m$ in $\mathbb{C}^d$ is said to be a set of mutually unbiased bases if every pair of the bases in the set is mutually unbiased.

Recent years, there are many interesting topics combining mutually unbiased bases with other bases, such as product basis (PB) [29], unextendible product basis (UPB) [30] and unextendible maximally entangled basis (UMEB) [31,32]. The UPB is a set of incomplete orthogonal product states in $\mathbb{C}^d \otimes \mathbb{C}^d$ such that whose complementary space has no product states. The UMEB is a set of less than d^2 orthogonal maximally entangled states in $\mathbb{C}^d \otimes \mathbb{C}^d$ such that whose complementary space has no maximally entangled vectors that are othogonal to all of them. In [32], two complete UMEBs that are mutually unbiased in $\mathbb{C}^2 \otimes \mathbb{C}^3$ have been presented.

Ever since the introduction of mutually unbiased bases, considerable theoretical results with useful applications have been obtained. One main concern is about the maximal number of MUBs for given dimension d. It has been shown that the maximum number $N(d)$ of MUBs in $\mathbb{C}^d$ is no more than $d + 1$ [21] and $N(d) = d + 1$ if d is a prime power. Different constructions of MUBs, especially for prime power and qubits systems, have been presented in [33–40]. Whereas d is a composite number, $N(d)$ is still unknown. Since the dimension of $\mathbb{C}^d \otimes \mathbb{C}^{kd}$ ($k \in Z^+$) is kd^2, it is still a challenging problem to study $N(kd^2)$ and construct MUBs in $\mathbb{C}^d \otimes \mathbb{C}^{kd}$.

In this paper, we first study the maximally entangled bases in arbitrary bipartite system $\mathbb{C}^d \otimes \mathbb{C}^{kd}$ ($k \in Z^+$). We provide a systematic way of constructing maximally entangled bases in $\mathbb{C}^d \otimes \mathbb{C}^{kd}$. Moreover, we present explicit constructions of mutually unbiased maximally entangled bases in $\mathbb{C}^2 \otimes \mathbb{C}^4$ and $\mathbb{C}^2 \otimes \mathbb{C}^6$.

2 Maximally entangled basis in $\mathbb{C}^d \otimes \mathbb{C}^{kd}$ ($k \in Z^+$)

Let us first consider maximally entangled basis(MEB) in $\mathbb{C}^2 \otimes \mathbb{C}^4$. Let $\{|0\rangle, |1\rangle\}$ and $\{|0'\rangle, |1'\rangle, |2'\rangle, |3'\rangle\}$ be the orthonormal bases in $\mathbb{C}^2$ and $\mathbb{C}^4$, respectively. We consider the following orthogonal basis in $\mathbb{C}^2 \otimes \mathbb{C}^4$:

$$|\phi_i^j\rangle = \frac{1}{\sqrt{2}}(\sigma_i \otimes I_4)(|0\rangle|(2j)'\rangle + |1\rangle|(2j+1)'\rangle), \quad i = 0,1,2,3; \quad j = 0,1, \tag{1}$$

where $\{\sigma_i\}_{i=1}^3$ are the Pauli matrices and $\sigma_0 = I_2$ is the 2×2 identity matrix.

It can be easily checked that the above eight states in (1) are orthogonal maximally entangled states, which constitute a MEB in $\mathbb{C}^2 \otimes \mathbb{C}^4$.

Now we generalize the above construction of MEB to the case of $\mathbb{C}^d \otimes \mathbb{C}^{kd}$ ($k \in \mathbb{Z}^+$). Let $\{|j\rangle\}_{j=0}^{d-1}$ and $\{|i'\rangle\}_{i=0}^{kd-1}$ denote the orthonormal bases of $\mathbb{C}^d$ and $\mathbb{C}^{kd}$, respectively. Consider a set of unitary matrices

$$U_{n,m} = \sum_{\ell=0}^{d-1} \omega_d^{n\ell}|\ell \oplus m\rangle\langle\ell|, \quad n,m = 0,1,\ldots,d-1, \tag{2}$$

where $\omega_d = e^{\frac{2\pi\sqrt{-1}}{d}}$, and $\ell \oplus m$ denotes $(\ell+m) \mod d$. These matrices $\{U_{n,m}\}_{n,m=0}^{d-1}$ form a basis of the operator space on $\mathbb{C}^d$ and satisfy

$$Tr(U_{n',m'}^\dagger U_{n,m}) = \mathrm{d}\delta_{n',n}\delta_{m',m}. \tag{3}$$

The above d^2 operators defined in (2) accurately corresponds to the Weyl–Heisenberg group.

Let us consider k maximally entangled states in $\mathbb{C}^d \otimes \mathbb{C}^{kd}$:

$$|\phi^j\rangle = \frac{1}{\sqrt{d}}\sum_{p=0}^{d-1}|p\rangle|(p+dj)'\rangle, \quad j = 0,1,\ldots,k-1. \tag{4}$$

Applying the unitary matrices (2) to the first space of the maximally entangled states in (4), we get kd^2 orthogonal maximally entangled states:

$$|\phi_{n,m}^{(j)}\rangle = (U_{n,m} \otimes I_{kd})|\phi^j\rangle, \quad j = 0,1,\ldots,k-1; \quad n,m = 0,1,\ldots,d-1. \tag{5}$$

Hence the above kd^2 orthogonal maximally entangled states (5) give rise to a MEB in $\mathbb{C}^d \otimes \mathbb{C}^{kd}$.

Substituting (2) into (5), we can simplify the above MEB (5) as follows:

$$|\phi_{n,m}^{(j)}\rangle = \frac{1}{\sqrt{d}}\sum_{p=0}^{d-1}\omega_d^{np}|p \oplus m\rangle|(p+dj)'\rangle, \quad j = 0,1,\ldots,k-1;$$
$$n,m = 0,1,\ldots,d-1. \tag{6}$$

Let $\{|a_i'\rangle\}_{i=0}^{kd-1}$ be another orthonormal basis in $\mathbb{C}^{kd}$, which is different with $\{|i'\rangle\}_{i=0}^{kd}$. Similar to the above discussion, we can get another MEB in $\mathbb{C}^d \otimes \mathbb{C}^{kd}$:

$$|\psi_{n,m}^{(j)}\rangle = \frac{1}{\sqrt{d}} \sum_{p=0}^{d-1} \omega_d^{np} |p \oplus m\rangle |a'_{p+dj}\rangle, \quad j = 0, 1, \ldots, k-1;$$
$$n, m = 0, 1, \ldots, d-1. \tag{7}$$

Equations(6) and (7) will be useful in the next section to construct mutually unbiased maximally entangled bases in $\mathbb{C}^d \otimes \mathbb{C}^{kd}$.

3 Mutually unbiased maximally entangled bases in $\mathbb{C}^d \otimes \mathbb{C}^{kd}$

In this section, we investigate special MUBs comprised of only MEBs in $\mathbb{C}^d \otimes \mathbb{C}^{kd}$, namely we establish a method to construct mutually unbiased maximally entangled bases(MUMEBs) in $\mathbb{C}^d \otimes \mathbb{C}^{kd}$.

It is easy to show that the two MEBs (6) and (7) in $\mathbb{C}^d \otimes \mathbb{C}^{kd}$ are MUBs if they satisfy the following property

$$|\langle \phi_{n,m}^{(i)} | \psi_{x,y}^{(j)} \rangle| = \frac{1}{\sqrt{kd^2}}, \quad i, j = 0, 1, \ldots, k-1; \quad n, m, x, y = 0, 1, \ldots, d-1. \tag{8}$$

Let A denote the transition matrix from the basis $\{|i'\rangle\}_{i=0}^{kd-1}$ to the basis $\{|a'_i\rangle\}_{i=0}^{kd-1}$ in $\mathbb{C}^{kd}$, that is

$$\begin{pmatrix} |a'_0\rangle \\ |a'_1\rangle \\ \vdots \\ |a'_{(kd-1)}\rangle \end{pmatrix} = A \begin{pmatrix} |0'\rangle \\ |1'\rangle \\ \vdots \\ |(kd-1)'\rangle \end{pmatrix}, \tag{9}$$

i.e. $|a'_i\rangle = \sum_{j=0}^{kd-1} A_{ij} |j'\rangle$, A_{ij} are entries of the matrix A.

Then conditions (8) are valid if and only if A satisfies the following relations:

$$\left| \sum_{p=0}^{d-1} \omega_d^{\ell p} A_{p+dj, p \oplus q+di} \right| = \frac{1}{\sqrt{k}}, \quad i, j = 0, 1, \ldots, k-1; \quad \ell, q = 0, 1, \ldots, d-1, \tag{10}$$

Obviously, the above conditions (8) imply that the unitary matrix A is a kind of complex Hadamard matrix [41].

Let B be the transition matrix from the basis $\{|a'_i\rangle\}_{i=0}^{kd-1}$ to the third basis$\{|b'_i\rangle\}_{i=0}^{kd-1}$, i.e., $|b'_i\rangle = \sum_{j=0}^{kd-1} B_{ij} |a'_j\rangle$. Then the following MEB

$$|\lambda_{n,m}^{(j)}\rangle = \frac{1}{\sqrt{d}} \sum_{p=0}^{d-1} \omega_d^{np} |p \oplus m\rangle |b'_{p+dj}\rangle, \quad j = 0, 1, \ldots, k-1; \quad n, m = 0, 1, \ldots, d-1. \tag{11}$$

in $\mathbb{C}^d \otimes \mathbb{C}^{kd}$ is mutually unbiased with (6) and (7), if and only if the matrices A and B satisfy the following relations:

$$\left| \sum_{p=0}^{d-1} \omega_d^{\ell p} B_{p+dj, p\oplus q+di} \right| = \frac{1}{\sqrt{k}}, \quad \left| \sum_{p=0}^{d-1} \omega_d^{\ell p} (BA)_{p+dj, p\oplus q+di} \right| = \frac{1}{\sqrt{k}}, \quad (12)$$

where $i, j = 0, 1, \ldots, k-1$; $\quad \ell, q = 0, 1, \ldots, d-1$.

By inductive method, more mutually unbiased MEBs can be constructed and so on. For a detailed construction of MUMEBs, we first consider the case of $\mathbb{C}^2 \otimes \mathbb{C}^4$. In the following for simplicity we denote

$$\left(|x'\rangle \right) = \begin{pmatrix} |x_0'\rangle \\ |x_1'\rangle \\ |x_2'\rangle \\ |x_3'\rangle \end{pmatrix}$$

for $x = a, b, c, d, e$, with $|e_i'\rangle = |i'\rangle$ for $i = 0, 1, 2, 3$.

By using (1) we have the first MEB in $\mathbb{C}^2 \otimes \mathbb{C}^4$. Taking the second basis $\{|a_i'\rangle\}_{i=0}^3$ in $\mathbb{C}^4$ as

$$\left(|a'\rangle \right) = A \left(|e'\rangle \right), \tag{13}$$

where

$$A = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ i & i & -i & -i \\ i & -i & i & -i \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

with $i = \sqrt{-1}$. Then the second MEB in $\mathbb{C}^2 \otimes \mathbb{C}^4$ is as follows:

$$|\psi_i^j\rangle = \frac{1}{\sqrt{2}} (\sigma_i \otimes I_4)(|0\rangle |a_{2j}'\rangle + |1\rangle |a_{2j+1}'\rangle), \quad i = 0, 1, 2, 3; \quad j = 0, 1. \tag{14}$$

It is direct to verify that the transformation matrix A satisfies the relation (10), and then the two MEBs (1) and (14) in $\mathbb{C}^2 \otimes \mathbb{C}^4$ are mutually unbiased.

The third orthonomal basis $\{|b_i'\rangle\}_{i=0}^3$ in $\mathbb{C}^4$ can be obtained by

$$\left(|b'\rangle \right) = B \left(|a'\rangle \right), \tag{15}$$

where

$$B = \frac{1}{2} \begin{pmatrix} 1 & 1 & -i & -i \\ i & i & -1 & -1 \\ i & -i & -1 & 1 \\ 1 & -1 & -i & i \end{pmatrix}$$

Hence, the third MEB in $\mathbb{C}^2 \otimes \mathbb{C}^4$ can be constructed by

$$|\lambda_i^j\rangle = \frac{1}{\sqrt{2}}(\sigma_i \otimes I_4)(|0\rangle|b'_{2j}\rangle + |1\rangle|b'_{2j+1}\rangle), \quad i = 0, 1, 2, 3; \quad j = 0, 1. \quad (16)$$

One can directly check that A and B satisfy the relations in (12). Hence the above three MEBs (1) , (14) and (16) are mutually unbiased.

The fourth and fifth MEBs in $\mathbb{C}^2 \otimes \mathbb{C}^4$ can be similarly constructed from the following orthonomal bases $\{|c'_i\rangle\}_{i=0}^3$ and $\{|d'_i\rangle\}_{i=0}^3$ in $\mathbb{C}^4$:

$$\left(|c'\rangle\right) = C\left(|b'\rangle\right), \quad \left(|d'\rangle\right) = D\left(|c'\rangle\right), \quad (17)$$

where

$$C = \frac{1}{2}\begin{pmatrix} 1 & i & -1 & -i \\ -1 & i & 1 & -i \\ 1 & i & 1 & i \\ 1 & -i & 1 & -i \end{pmatrix}; \quad D = \frac{1}{2}\begin{pmatrix} -i & -1 & -1 & -i \\ -i & -1 & 1 & i \\ 1 & i & i & 1 \\ 1 & i & -i & -1 \end{pmatrix}.$$

The corresponding MEBs in $\mathbb{C}^2 \otimes \mathbb{C}^4$ are given by

$$|\mu_i^j\rangle = \frac{1}{\sqrt{2}}(\sigma_i \otimes I_4)(|0\rangle|c'_{2j}\rangle + |1\rangle|c'_{2j+1}\rangle), \quad i = 0, 1, 2, 3; \quad j = 0, 1; \quad (18)$$

$$|v_i^j\rangle = \frac{1}{\sqrt{2}}(\sigma_i \otimes I_4)(|0\rangle|d'_{2j}\rangle + |1\rangle|d'_{2j+1}\rangle), \quad i = 0, 1, 2, 3; \quad j = 0, 1. \quad (19)$$

one can directly check that any two matrices of A, B, C, D satisfy the relations in (10) and (12); hence, the five complete MEBs (1), (14), (16), (18) and (19) are mutually unbiased.

Thus, by suitably choosing the bases in $\mathbb{C}^4$, we have presented an approach in constructing maximally entangled states that are mutually unbiased in $\mathbb{C}^2 \otimes \mathbb{C}^4$.

Remark In [17], the authors showed that a complete set of MUBs of a bipartite system contains a fixed amount of entanglement, independent on the choice of the complete set. Moreover, in [36] Klimov showed that there are four structures of MUBs in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$: (2,3,4), (3,0,6), (0,9,0) and (1,6,2), where the three numbers in a bracket represents the number of triseparable, biseparable and nonseparable bases, respectively. We do not know whether our 5 MUMEBs are extendible to 9 since we can not construct the sixth maximally entangled basis. Recently, based on different constructions, in [42] the authors presented a set of 5 MUBs in dimension 8. These MUBs are not necessary maximally entangled. But the set of these 5 MUBs is found to be unextendible. It is possible that the set of our 5 maximally entangled MUBs is also unextendible.

Nevertheless, our approach is more general than the case of multi-qubit systems. Next, to give an example which is not included in $\mathbb{C}^d \otimes \mathbb{C}^d \otimes ... \otimes \mathbb{C}^d$ systems, we present a detailed construction of MUMEBs in $\mathbb{C}^2 \otimes \mathbb{C}^6$, which is absolutely different from qubits systems.

In the following, for simplicity we denote $(|y'\rangle) = (|y'_0\rangle, |y'_1\rangle, |y'_2\rangle, |y'_3\rangle, |y'_4\rangle, |y'_5\rangle)^t$ for $y = f, g, h$ with $|f'_j\rangle = |j'\rangle$ for $j = 0, 1, 2, 3, 4, 5$, where t stands for transpose.

According to (6), we have the first MEB in $\mathbb{C}^2 \otimes \mathbb{C}^6$:

$$|\phi_{n,m}^{(j)}\rangle = \frac{1}{\sqrt{2}} \sum_{p=0}^{1} \omega_2^{np} |p \oplus m\rangle |(p+2j)'\rangle, \quad j = 0, 1, 2; \quad n, m = 0, 1. \quad (20)$$

where $\omega_2 = e^{\frac{2\pi\sqrt{-1}}{2}}$, and $p \oplus m$ denotes $(p+m) \mod 2$.

For the second MEB in $\mathbb{C}^2 \otimes \mathbb{C}^6$, we take the basis $\{|g'_j\rangle\}_{j=0}^{5}$ in $\mathbb{C}^6$ as

$$(|g'\rangle) = X (|f'\rangle), \quad (21)$$

where

$$X = \frac{1}{\sqrt{6}} \begin{pmatrix} iv^* & -iv^* & iv^* & -iv^* & iv^* & -iv^* \\ v^* & v^* & v^* & v^* & v^* & v^* \\ i & -i & iv & -iv & iv^* & -iv^* \\ 1 & 1 & v & v & v^* & v^* \\ i & -i & iv^* & -iv^* & iv & -iv \\ 1 & 1 & v^* & v^* & v & v \end{pmatrix},$$

with $v = -\frac{1}{2} + \frac{\sqrt{3}i}{2}$, $*$ denotes conjugate. Then the second MEB in $\mathbb{C}^2 \otimes \mathbb{C}^6$ has the form:

$$|\psi_{n,m}^{(j)}\rangle = \frac{1}{\sqrt{2}} \sum_{p=0}^{1} \omega_2^{np} |p \oplus m\rangle |g'_{p+2j}\rangle, \quad j = 0, 1, 2; \quad n, m = 0, 1. \quad (22)$$

It is direct to verify that the transformation matrix X satisfies the relation (10), and then the two MEBs (20) and (22) in $\mathbb{C}^2 \otimes \mathbb{C}^6$ are mutually unbiased.

The third orthonomal basis $\{|h'_j\rangle\}_{j=0}^{5}$ in $\mathbb{C}^6$ can be obtained by

$$(|h'\rangle) = Y (|g'\rangle), \quad (23)$$

where

$$Y = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ i & -i & i & -i & i & -i \\ 1 & 1 & v & v & v^* & v^* \\ i & -i & iv & -iv & iv^* & -iv^* \\ 1 & 1 & v^* & v^* & v & v \\ i & -i & iv^* & -iv^* & iv & -iv \end{pmatrix}.$$

Then the third MEB in $\mathbb{C}^2 \otimes \mathbb{C}^6$ can be constructed by

$$|\lambda_{n,m}^{(j)}\rangle = \frac{1}{\sqrt{2}} \sum_{p=0}^{1} \omega_2^{np} |p \oplus m\rangle |h'_{p+2j}\rangle, \quad j = 0, 1, 2; \quad n, m = 0, 1. \qquad (24)$$

One can directly check that X and Y satisfy the relations in (12). Therefore the above three MEBs (20), (22) and (24) in $\mathbb{C}^2 \otimes \mathbb{C}^6$ are mutually unbiased.

It would be interesting to mention that no more than 3 MUBs are known in dimension 12 (despite 13 is the upper bound), and then our construction of three MUMEBs (20), (22) and (24) in $\mathbb{C}^2 \otimes \mathbb{C}^6$ are exactly a breakthrough, which is also different from those in qubits systems. In fact, our approach applies to general bipartite systems $\mathbb{C}^d \otimes \mathbb{C}^{kd}$ ($k \in Z^+$). Such constructions of orthonormal bases of MEBS by applying local unitaries (Weyl-Heisenberg group) have been adopted in [18], corresponding to the case $\mathbb{C}^d \otimes \mathbb{C}^{kd}$ with $k = 1$.

4 Conclusion and discussion

We have provided an explicit construction of maximally entangled basis in arbitrary bipartite spaces $\mathbb{C}^d \otimes \mathbb{C}^{kd}$ ($k \in Z^+$). Based on such bases, we have established a method to construct mutually unbiased maximally entangled bases in $\mathbb{C}^d \otimes \mathbb{C}^{kd}$ ($k \in Z^+$). As detailed examples, we have constructed five mutually unbiased maximally entangled bases in $\mathbb{C}^2 \otimes \mathbb{C}^4$ and three mutually unbiased maximally entangled bases in $\mathbb{C}^2 \otimes \mathbb{C}^6$.

The problem we have investigated about maximally entangled basis is different from that of unextendible maximally entangled basis. There are still many open problems related to maximally entangled basis and mutually unbiased maximally entangled bases, such as the construction of mutually unbiased bases that are comprised of one maximally entangled basis and one unextendible maximally entangled basis in $\mathbb{C}^d \otimes \mathbb{C}^{kd}$ ($k \in Z^+$) or $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ ($d \neq d'$), as well as to the roles played by such bases in information processing.

References

1. Vedral, V.: The role of relative entropy in quantum information theory. Rev. Mod. Phys. **74**, 197–234 (2002)
2. Plenio, M.B., Virmani, S.: An introduction to entanglement measures. Quant. Inf. Comput. **7**, 1–51 (2007)
3. Horodecki, R., Horodecki, P., Horodecki, M., Horodecki, K.: Quantum entanglement. Rev. Mod. Phys. **81**, 865–942 (2009)

4. Benhelm, J., Kirchmair, G., Roos, C.F., Blatt, R.: Towards fault-tolerant quantum computing with trapped ions. Nat. Phys. **4**, 463–480 (2008)
5. Bennett, C.H., Brassard, G., Crepeau, C., Jozsa, R., Peres, A., Wootters, W.K.: Teleporting an unknown quantum state via dual classical and Einstein–Podolsky–Rosen channels. Phys. Rev. Lett. **70**, 1895–1899 (1993)
6. Zhang, Q., Goebel, A., Wagenknecht, C., Chen, Y.A., Zhao, B., Yang, T., Mair, A., Schmiedmayer, J., Pan, J.W.: Radio-frequency dressed state potential for manipulation neutral atoms. Nat. Phys. **2**, 678–682 (2006)
7. Modlawska, J., Grudka, A.: Nonmaximally entangled states can be better for multiple linear optical teleportation. Phys. Rev. Lett. **100**, 110503-1–110503-4 (2008)
8. Ishizaka, S., Hiroshima, T.: Quantum teleportation scheme by selecting one of multiple output ports. Phys. Rev. A **79**, 042306-1–042306-13 (2009)
9. Noh, C., Chia, A., Nha, H., Collett, M.J., Carmichael, H.J.: Quantum teleportation of the temporal fluctuations of light. Phys. Rev. Lett. **102**, 230501-1–230501-4 (2009)
10. Bennett, C.H., Wiesner, S.J.: Communication via one- and two-particle operators on Einstein–Podolsky–Rosen states. Phys. Rev. Lett. **69**, 2881–2884 (1992)
11. Barreiro, J.T., Wei, T.C., Kwiat, P.G.: Beating the channel capacity limit for linear photonic superdense coding. Nat. Phys. **4**, 282–286 (2008)
12. Bennett, C.H., DiVincenzo, D.P.: Quantum information and computation. Nature **404**, 247–255 (2000)
13. Li, Z.G., Zhao, M.J., Fei, S.M., Fan, H., Liu, W.M.: Mixed maximally entangled states. Quant. Inf. Comput. **12**(1–2), 0063–0073 (2012)
14. Peres, A.: Quantum Theory: Concepts and Methods. Kluwer Academic Publishers, Dordrecht (1995)
15. Karimipour, V., Memarzadeh, L.: Equientangled bases in arbitrary dimensions. Phys. Rev. A **73**, 012329-1–012329-5 (2006)
16. Gheorghiu, V., Looi, S.Y.: Construction of equally entangled bases in arbitrary dimensions via quadratic Gauss sums and graph states. Phys. Rev. A **81**, 062341-1–062341-7 (2010)
17. Wiesniak, M., Paterek, T., Zeilinger, A.: Entanglement in mutually unbiased bases. New J. Phys. **13**, 053047-1–053047-25 (2011)
18. Fan, H.: Distinguishability and indistinguishability by local operations and classical communication. Phys. Rev. Lett. **92**, 177905-1–177905-4 (2004)
19. Durt, T., Englert, B.-G., Bengtsson, I., Zyczkowski, K.: On mutually unbiased bases. Int. J. Quant. Inf. **8**, 535–640 (2010)
20. Ivanović, I.D.: Geometrical description of quantal state determination. J. Phys. A **14**, 3241–3245 (1981)
21. Wootters, W.K., Fields, B.D.: Optimal state-determination by mutually unbiased measurements. Ann. Phys. **191**, 363–381 (1989)
22. Englert, B.-G., Kaszlikowski, D., Kwek, L.C., Chee, W.H.: Wave–particle duality in multi-path interferometers: general concepts and three-path interferometers. Int. J. Quant. Inf. **6**, 129–157 (2008)
23. Cerf, N.J., Bourennane, M., Karlsson, A., Gisin, N.: Security of quantum key distribution using d-level systems. Phys. Rev. Lett. **88**, 127902-1–127902-4 (2002)
24. Brierley, S.: Quantum key distribution highly sensitive to eavesdropping. arXiv:0910.2578
25. Aharonov, Y., Englert, B.-G., Naturforsch, Z.: The mean king's problem: spin 1. Section A–A. J. Phys. Sci. **56**(1–2), 16–19 (2001)
26. Durt, T.: If 1 = 2 ⊕ 3, then 1 = 2 ⊙ 3: Bell states, finite groups, and mutually unbiased bases, a unifying approach. arXiv:quant-ph/0401046
27. Klimov, A.B., Sych, D., Sánchez-Soto, L.L., Leuchs, G.: Continuum percolation of isotropically oriented circular cylinders. Phys. Rev. A **79**, 052101-1–052101-4 (2009)
28. Revzen, M.: Maximally entangled states via mutual unbiased collective bases. Phys. Rev. A **81**, 012113-1–012113-7 (2010)
29. McNulty, D., Weigert, S.: The limited role of mutually unbiased product bases in dimension 6. J. Phys. A Math. Theor. **45**, 102001-1–102001-6 (2012)
30. Bennett, C.H., DiVincenzo, D.P., Mor, T., Shor, P.W., Smolin, J.A., Terhal, B.M.: Unextendible product bases and bound entanglement. Phys. Rev. Lett. **82**, 5385–5388 (1999)
31. Bravyi, S., Smolin, J.A.: Unextendible maximally entangled bases. Phys. Rev. A **84**, 042306-1–042306-3 (2011)
32. Chen, B., Fei, S.M.: Unextendible maximally entangled bases and mutually unbiased bases. Phys. Rev. A **88**, 034301-1–034301-4 (2013)
33. Wootters, W.K.: Picturing qubits in phase space. IBM J. Res. Dev. **48**, 99–110 (2004)

34. Gibbons, K.S., Hoffman, M.J., Wootters, W.K.: Discrete phase space based on finite fields. Phys. Rev. A **70**, 062101-1–062101-23 (2004)
35. Klimov, A.B., Romero, J.L., Björk, G., Sánchez-Soto, L.L.: Geometrical approach to mutually unbiased bases. J. Phys. A Math. Theor. **40**, 3987–3998 (2007)
36. Lawrence, J., Brukner, Č., Zeilinger, A.: Mutually unbiased binary observable sets on N qubits. Phys. Rev. A **65**, 032320-1–032320-5 (2002)
37. Romero, J.L., Björk, G., Klimov, A.B., Sánchez-Soto, L.L.: Structure of the sets of mutually unbiased bases for N qubits. Phys. Rev. A **72**, 062310-1–062310-8 (2005)
38. Björk, G., Romero, J.L., Klimov, A.B., Sánchez-Soto, L.L.: Mutually unbiased bases and discrete Wigner functions. J. Opt. Soc. Am. B **24**, 371–378 (2007)
39. Ghiu, I.: A new method of construction of all sets of mutually unbiased bases for two-qubit systems. J. Phys. Conf. Ser. **338**, 012008-1–012008-7 (2012)
40. Ghiu, I.: Generation of all sets of mutually unbiased bases for three-qubit systems. Phys. Scr. **T153**, 014027-1–014027-6 (2013)
41. Tadej, W., Zyczkowski, K.: A concise guide to complex Hadamard matrices. Open Syst. Inf. Dyn. **13**, 133–177 (2006)
42. Mandayam, P., Bandyopadhyay, S., Grassl, M., Wootters, W.K.: Unextendible mutually unbiased bases from Pauli classes. Quant. Inf. Comput. **14**(9–10), 823–844 (2014)