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Addition and aggregative efficiency

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Abstract

The aim of this paper is to relate the condition for cost subadditivity to aggregative efficiency. In particular, we verify that subadditivity of the cost function occurs when aggregate cost efficiency is not less than the cost efficiency of the average production unit. This provides a simple way of examining the potential of merging two or more firms and of making inference about the extent of industry concentration. It also provides a simple nonparametric test for subadditivity of the cost function.

Keywords Cost subadditivity · Aggregate efficiency · Industry concentration

1 Introduction

The notion of cost subadditivity, introduced by Baumol et al. (1998), implies that the (minimum) cost of producing a number of products jointly may not be greater than the sum of (minimum) cost of producing separately mutually exclusive subsets of them. In other words, the cost function is subadditive if it is not less expensive for two or more firms to produce each a subset of a given bundle of products than it is for a single firm to produce the whole array of products. When each firm produces a single output or more generally, when we have the case of mutually orthogonal firm output vectors, cost subadditivity is a necessary and sufficient condition for scope economies (Baumol et al. 1998, p. 71) while when all firms produce some of the products and jointly with them, each one produces a unique subset of the remaining products, cost subadditivity implies economies of diversification (Grosskopf et al. 1992), reflecting cost savings that may result from an increase in the number of simultaneously produced products.

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Färe (1986) developed a nonparametric test for cost subadditivity using the relation between the dual and the primal representations of the technology. In particular, he showed that if the cost function is subadditive then efficiency measured with respect to the "sum technology" is not less than efficiency measured with respect to the "combined technology", where the former is defined by the sum of the firm's input requirement sets and the latter by the input requirement set of the combined firm which is the sum of the firms. The ratio of these two efficiency scores consist Färe's (1986) gain function, which is not less than one when the cost function is subadditive.

The aim of this paper is to re-state Färe (1986) condition for cost subadditivity in terms of aggregative efficiencies. In particular, we show that if the cost function is subadditive then aggregate cost efficiency is not less than the cost efficiency of the average production unit.

On the other hand, Maindiratta (1991) considered a non-parametric test, in the spirit of Afriat (1972), Hanoch and Rothschild (1972), Diewert and Parkan (1983), Varian (1984) and Banker and Maindiratta (1988), to examine the consistency of observed production data with cost minimization behavior under a subadditive cost function. This test requires the solution of an integer programming problem for each firm. Using our main result, namely that if the cost function is subadditive then the aggregate cost efficiency is not less than the cost efficiency of the average production unit, one can overcome the need for solving these integer programming problems as we provide an equivalent condition to those stated in Maindiratta (1991) that can directly be used to test whether a technology characterized by a subadditive cost function rationalizes the observe data.

2 The main result

Let $w \in R^J_+$ denote an input price vector, $y \in R^I_+$ an output vector and $x \in R^J_+$ an input vector, and consider a group of $k = 1,\ldots,K$ firms that use the same technology, with which they can produce either a single, some or even the whole set of a number of products, and face the same input prices. The cost function associated with w and y is subadditive if can be written as:

$$\sum_{k=1}^{K} C(w, y^k) \geq C\left(w, \sum_{k=1}^{K} y^k\right) \quad (1)$$

where $C(w, y) = min_x \{w' x{:}x \in L(y)\}$, $L(y) = \{x{:}(x, y) \in T\}$ is the input requirement set and T is the production possibility set. The input requirement set is assumed to be closed and non-empty for finite y, to satisfy strong (input and output) disposability, input convexity, continuity, and $0_I \notin L(y)$ for $y \geq 0_J$ but $0_I \in L(0_J)$.[1] Then, the cost function is continuous in w and y, non-decreasing in y, and non-decreasing, positively linearly homogenous and concave in w. From duality theory, subadditivity of the cost function is equivalent to super-additivity of the input requirement set, i.e., $L\left(\sum_{k=1}^K y^k\right) \supseteq \sum_{k=1}^K L\left(y^k\right)$ (see e.g. Färe 1986). In addition, by using Li and Ng (1995) results, one can show that $L\left(\sum_{k=1}^K y^k\right) = KL(\bar{y})$, where $\bar{y} = \left(\sum_{k=1}^K y^k\right)/K$.[2]

By dividing both sides of (1) by the sum (over firms) of actual cost we get:

$$\frac{\sum_{k=1}^{K} C(w, y^k)}{w'\left(\sum_{k=1}^K x^k\right)} \geq \frac{C\left(w, \sum_{k=1}^{K} y^k\right)}{w'\left(\sum_{k=1}^K x^k\right)} \quad (2)$$

which may also be re-written as follows:

$$\sum_{k=1}^{K} s^k \left(\frac{C(w, y^k)}{w'x^k}\right) \geq \frac{C\left(w, \sum_{k=1}^{K} y^k\right)}{w'\left(\sum_{k=1}^K x^k\right)} \quad (3)$$

where $s^k = w'x^k / w'\left(\sum_{k=1}^K x^k\right)$. The left-hand side of (3) is the weighted average of firm's cost efficiency scores $C(w, y^k)/w'x^k$, which is referred to as the aggregate cost efficiency (see Färe et al. 2004; Färe and Grosskopf 2004, pp. 118).[3] The right-hand side of (3) may be viewed as the cost efficiency score of a firm using $\sum_{k=1}^K x^k$ to produce $\sum_{k=1}^K y^k$, i.e., of the combined firm.[4] Since $L\left(\sum_{k=1}^K y^k\right) = KL(\bar{y})$ we can verify that $C\left(w, \sum_{k=1}^K y^k\right) = KC(w, \bar{y})$.[5] In addition, $w'\left(\sum_{k=1}^K x^k\right) = K'\bar{x}$, where $\bar{x} = \left(\sum_{k=1}^K x^k\right)/K$. Then, by substituting these into (3) we obtain:

$$\sum_{k=1}^{K} s^k \left(\frac{C(w, y^k)}{w'x^k}\right) \geq \frac{C(w, \bar{y})}{w'\bar{x}} \quad (4)$$

That is, if the cost function is subadditive then aggregate cost efficiency is not less than the cost efficiency of the average production unit, which "is constructed by taking the arithmetic average of each amount of inputs and outputs", and it is regarded "as an arbitrary observation on the same line as the other observations" in constructing the best practice frontier (Førsund and Hjalmarsson 1979, p. 300).[6]

If one further assumes cost allocative efficiency or that all firms (including the average production unit) are equally cost allocative inefficient, (4) may be re-written in terms of the Farrell input-oriented technical efficiency $F(x, y)$ as

$$\sum_{k=1}^{K} s^k F\left(x^k, y^k\right) \geq F(\bar{x}, \bar{y}) \quad (5)$$

where $F(x, y) = \inf_{\lambda}\{\lambda > 0 : \lambda x \in L(y)\}$. Then it is clear that the efficiency score with respect to the "sum technology" is equal to the weighted average of efficiency scores with respect to the conventional technology, i.e., the left-hand side of (5), and the efficiency score with respect to the "combined

[1] 0_I and 0_J are I- and J-dimensional zero vectors, respectively.

[2] By assuming that the production possibility set is identical and convex across firms, Li and Ng (1995) showed that the production possibility set of the combined firm is equal to K times that the each individual firms. Then, one can verify that $L\left(\sum_{k=1}^K y^k\right) = \left\{\sum_{k=1}^K x^k : \left(\sum_{k=1}^K x^k, \sum_{k=1}^K y^k\right) \in KT\right\} = \left\{\sum_{k=1}^K x^k : \left(\frac{\sum_{k=1}^K x^k}{K}, \frac{\sum_{k=1}^K y^k}{K}\right) \in T\right\} = K\left\{\frac{\sum_{k=1}^K x^k}{K} : \left(\frac{\sum_{k=1}^K x^k}{K}, \frac{\sum_{k=1}^K y^k}{K}\right) \in T\right\} = K\{\bar{x} : (\bar{x}, \bar{y}) \in T\} = KL(\bar{y})$, where $\bar{x} = \left(\sum_{k=1}^K x^k\right)/K$.

[3] Since the aggregation weights are given in terms of the cost shares, i.e., the variable in the denominator of the cost efficiency measure, the underlying aggregation rule for the left-hand side of (3) is referred as the denominator rule by Färe and Karagiannis (2017).

[4] This is usually referred to as structural efficiency, see e.g. Li and Cheng (2007) and Karagiannis (2015), while it is referred to as group potential efficiency by Nesterenko and Zelenyuk (2007).

[5] The proof of this is: $C\left(w, \sum_{k=1}^K y^k\right) = \min_{\sum_{k=1}^K x^k} \left\{w'\left(\sum_{k=1}^K x^k\right) : \sum_{k=1}^K x^k \in KL(\bar{y})\right\} = \min_{\sum_{k=1}^K x^k} \left\{w'\left(\sum_{k=1}^K x^k\right) : \frac{\sum_{k=1}^K x^k}{K} \in L(\bar{y})\right\} = K \min_{\left(\sum_{k=1}^K x^k\right)/K} \left\{w'\left(\frac{\sum_{k=1}^K x^k}{K}\right) : \frac{\sum_{k=1}^K x^k}{K} \in L(\bar{y})\right\} = K \min_{\bar{x}}\{w'\bar{x} : \bar{x} \in L(\bar{y})\} = KC(w, \bar{y})$.

[6] Bob Chambers mentioned to us that the above manipulations are closely reminiscent of manipulations well-known from the analysis of subjectively discounted martingales and form the asset-pricing implications of the fundamental theorem of arbitrage.

technology" is equal to the efficiency score of the average production unit with respect to the conventional technology, i.e., the right-hand side of (5). Consequently, Färe's (1986) gain function may be given equivalently as:

$$G = \frac{\sum_{k=1}^{K} s^k F(x^k, y^k)}{F(\bar{x}, \bar{y})} \tag{6}$$

which given (5) is not less than one if the cost function is subadditive.

A similar relation to (1) in terms of the revenue function, i.e.,

$$\sum_{k=1}^{K} \mathrm{R}(\mathrm{p}, \mathrm{x}^k) \le \mathrm{R}\left(\mathrm{p}, \sum_{k=1}^{K} \mathrm{x}^k\right) \tag{7}$$

is given in Nesterenko and Zelenyuk (2007), where $R(p, x) = \max_{y}\{p'y : y \in P(x)\}$ is continuous in p and x, non-decreasing in x, and non-decreasing, positively linearly homogenous and convex in p, and $P(x) = \{y{:}(x, y) \in T\}$ is the producible out set that is assumed to be closed and non-empty for finite x, to satisfy strong (input and output) disposability, input convexity, continuity, and $0_J \in P(x)$ (inactivity) and $y \notin P(0_I)$ for $y \geq 0_J$ (no free lunch). Nesterenko and Zelenyuk (2007) claimed that (7) follows directly from their Lemma 2, i.e., $\sum P(x^k) \subseteq P(\sum x^k)$, without any reference to the property of additivity. At first glance, one may get the impression that their Lemma 2 and consequently, (7) holds under quite general conditions, namely the axioms stated on their p. 108: no free lunch, producing nothing is possible, boundness of the producible output set, closeness of the production possibility set and free disposability of outputs. However, this is not true: at the middle of their proof of Lemma 2 (p. 116) they noticed: "Furthermore, $\sum_{k=1}^{K} (x^{0,k}, y^{0,k}) \in \sum_{k=1}^{K} T^k \Leftrightarrow (\sum_{k=1}^{K} x^{0,k}, \sum_{k=1}^{K} y^{0,k}) \in \sum_{k=1}^{K} T^k \Leftrightarrow (X^0, Y^0) \in T^g$, where $X^0 = \sum_{k=1}^{K} x^{0,k}$". This is but the definition of the property of superadditivity of the production possibility set: since $(x^{0,k}, y^{0,k}) \in T^k$ and $\sum_{k=1}^{K} T^k = T^g$ then $(x^{0,k}, y^{0,k}) \in T^g$ for all $k = 1,\ldots,K$ and superadditivity of the T^g implies that $(\sum_{k=1}^{K} x^{0,k}, \sum_{k=1}^{K} y^{0,k}) \in T^g$ (see Färe and Primont 1995, p. 32).[7] Thus, even not mentioned by the authors their proof rests on the assumption that the production possibility set is superadditive. Moreover, one can verify that superadditivity of the production possibility set, i.e., $\sum T(x^k, y^k) \subseteq T(\sum x^k, \sum y^k)$, is equivalent to either superadditivity of the input requirement set, i.e., $\sum L(y^k) \subseteq L(\sum y^k)$ or superadditivity of the producible output set, i.e., $\sum P(x^k) \subseteq P(\sum x^k)$. If in addition one assumes that the production possibility set is convex, duality implies that the superadditivity of the input requirement set is equivalent to the subadditivity of the cost function, i.e., (1), and the supperadditivity of the producible output set is equivalent to the superadditivity of the revenue function, i.e., (7).[8] Then, from (7) and following the same steps as above we can extent our previous results to the case of a revenue function, namely

$$\sum_{k=1}^{K} m^k \left(\frac{\mathrm{R}(\mathrm{p}, \mathrm{x}^k)}{p'y^k}\right) \le \frac{\mathrm{R}(\mathrm{p}, \bar{\mathrm{x}})}{p'\bar{y}} \tag{8}$$

where $m^k = p'y^k / p'(\sum_{k=1}^{K} y^k)$ and $R(p,\ x^k)/p'y^k$ refers to firms' revenue efficiency scores. That is, if the revenue function is superadditive then aggregate revenue efficiency is not more than the revenue efficiency of the average production unit. If as before we further assume output allocative efficiency or that all firms (including the average production unit) are equally output allocative inefficient, we may re-write (8) in terms of the output-oriented technical efficiency index $E(x^k, y^k)$ as

$$\sum_{k=1}^{K} m^k E(x^k, y^k) \le \mathrm{E}(\bar{x}, \bar{y}) \tag{9}$$

where $E(x, y) = max_\theta\ \{\theta{:}\theta_y \in P(x)\}$.

Lastly notice that as (1) is used as a condition for scope or diversification economies, (7) may be used as a condition for what we may referred to as *synergy or coordination economies* reflecting the combined or cooperative action of two or more firms that together increase each other's effectiveness and results in a higher outcome (i.e., revenue) compared to the sum of their individual achievements. This may be due to orderly arrangements of firms that create the necessary conditions of working together smoothly, easily and in a pleasing way.

[7] T^g is referred to as group potential technology in Nesterenko and Zelenyuk (2007) and as "combined technology" in our terminology.

[8] It should be clear that not all technologies are superadditive. As superadditivity means that the technology includes original and aggregated firms, it is satisfied by what Grosskopf (1986, p. 506) referred to as the *Koopmans technology*, which " … includes the sum of the observed points (and the convex combination thereof)". (Being a technology with non-increasing returns to scale, Koopmans technology also includes all radial contractions of the observed points and their convex combinations). Two other technologies satisfy the property of superadditivity when the production possibility set is convex, namely the one exhibiting global constant returns to scale (Bogetoft and Wang 2005) and the semi-additive (Ghiyashi and Cook 2021), which is essentially a Koopmans type of technology without its decreasing-returns-to-scale portion. For non-convex technologies, the property of supperadditivity is satisfied for the free replicability hull (Tulkens 1993) and the free coordination hull (Green and Cook 2004). On the other hand, technologies exhibiting ordinary non-increasing returns to scale or variable return to scale do not satisfy superadditivity.

3 Implications

In this section, we discuss three implications of the above result. *First*, the gain function in (6) provides a simple way to examine the potential of merging two or more firms producing different outputs by comparing whether the weighted average of their efficiencies is greater or less that the efficiency of their average production unit, i.e., the unit that uses the average of their inputs and produces the average of their outputs.[9] This result applies to the cases of both orthogonal and non-orthogonal output vectors and thus, it can be used to test either for economies of score or for economies of diversification. In addition, we may use (6) to identify which sub-group of firms provides the largest efficiency gains of a merger: that will be the one with the largest deviation between aggregate efficiency and the efficiency of the resulting average production unit.

On the other hand, (6) can also be related to Evans and Heckman (1984) test of conditions for natural monopoly by means of cost subadditivity. In this case, the focus is on the potential gains or losses associated with multi-firm versus single-firm configurations. The latter has an advantage under cost subadditivity and this can be examined by comparing the aggregate efficiency under alternative scenarios, within the limits of the admissible region, of hypothetical multi-firm configurations with that of the single-firm configuration, under the assumption that the sum of outputs of the former is equal to the output of the latter.

Second, Chakravarty (1992, 1998) has shown that Färe's (1986) gain function, when interpreted in terms of costs, can be regarded as a concentration index. Based on this, one may argue in terms of (6) that as concentration in an industry increases the weighted average of firms' efficiency scores tends to the efficiency of the average production unit. In other words, in more (less) concentrated industries aggregate efficiency tends to deviate less (more) from the efficiency of the average production unit. As this difference decreases, the industry is expected to be closer to the monopoly end of the monopoly to competition spectrum, and vice versa. Hence, in less (more) concentrated industries where performance heterogeneity is more pronounced, the performance of the average production unit reflects less (more) accurately aggregate performance.

All in all, economies of scope or economies diversification tend to reduce the deviations of aggregate efficiency from the efficiency of the average production unit and to increase concentration in the industry through mergers. The choice of the sub-group with the greatest impact in terms of efficiency gains is related to the extent of these differences and this provides policy makers and regulators with a practical decision-making tool.

Third, the gain function in (6) may also serve another purpose as it can be added to the conditions stated in Maindiratta (1991) to provide a set of four equivalent conditions for the observed production data to be consistent with cost minimization under a subadditive cost function. In other words, (6) may be considered as a nonparametric test for optimizing behavior with a superadditive technology. Their main advantage compared to those provided by Maindiratta (1991) is that they require no additional estimation effort except that of estimating with linear programming techniques (i.e., data envelopment analysis) the underlying technology and including the average production unit in the data set. In computational terms, this should to be compared to integer programming problems proposed by Maindiratta (1991) for the same purpose.

[9] This differs from Bogetoft and Wang (2005) harmony effect, which reflects the (technical) efficiency of a unit that uses the average of their *potential* (i.e., efficiency adjusted) inputs and produces the average of their actual outputs.

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