Potential Analysis
https://doi.org/10.1007/s11118-024-10134-8

Stochastic Partial Differential Equations and Invariant Manifolds in Embedded Hilbert Spaces

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Received: 10 May 2022 / Accepted: 5 February 2024

Abstract
We provide necessary and sufficient conditions for stochastic invariance of finite dimensional submanifolds for solutions of stochastic partial differential equations (SPDEs) in continuously embedded Hilbert spaces with non-smooth coefficients. Furthermore, we establish a link between invariance of submanifolds for such SPDEs in Hermite Sobolev spaces and invariance of submanifolds for finite dimensional SDEs. This provides a new method for analyzing stochastic invariance of submanifolds for finite dimensional Itô diffusions, which we will use in order to derive new invariance results for finite dimensional SDEs.

Keywords Stochastic partial differential equation · Continuously embedded Hilbert spaces · Invariant manifold · Finite dimensional diffusion · Multi-parameter group · Hermite Sobolev space · Translation invariant solution

Mathematics Subject Classification (2010) 60H15 · 60H10 · 60G17

1 Introduction

The problem of finding invariant submanifolds of solutions of stochastic partial differential equations (SPDEs) arises, for example, in connection with stochastic models in finance wherein the submanifolds offer the possibility of finite dimensional realizations of the solutions which are otherwise infinite dimensional (see, for example [6–8, 16, 17, 38–41]). The problem, related to the computability of "interest rate term structure models", is also known as the "consistency problem" for such models; see [14]. In this paper we study the mathematical problem of finding invariant submanifolds for a general class of SPDEs that includes apart from quasi-semilinear and semilinear SPDEs (see, for example [13, 29, 40])

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Published online: 11 May 2024

a more recent class of SPDEs studied in [32, 33]. We will refer to this latter class as Itô type SPDEs.

In this paper, we develop a general framework, which covers the aforementioned types of SPDEs, and we present an invariance result for finite dimensional submanifolds, which generalizes existing results in this direction. In particular, the usual assumption that the volatilities must be smooth, is not required in our framework (see Theorem 3.4). Furthermore, we establish a link between invariance of submanifolds for such SPDEs in Hermite Sobolev spaces and invariance of submanifolds for finite dimensional SDEs (see Theorem 6.3). Using this connection, we will also contribute new invariance results for finite dimensional SDEs (see, in particular Theorems 6.5 and 6.13). As we will see, our results are stable under the dimension of the driving noise, which may in particular be infinite dimensional.

In order to outline our findings, let (G, H) be a pair of continuously embedded separable Hilbert spaces; this means that $G \subset H$ as sets, and that the embedding operator from $(G, \|\cdot\|_G)$ into $(H, \|\cdot\|_H)$ is continuous. Consider an SPDE of the form

$$\begin{cases} dY_t = L(Y_t)dt + A(Y_t)dW_t \\ Y_0 = y_0 \end{cases} \tag{1.1}$$

driven by a $\mathbb{R}^\infty$-Wiener process W with continuous coefficients $L : G \to H$ and $A : G \to \ell^2(H)$; we refer to Section 2 for further details. We emphasize that SPDEs of the type (1.1) in particular cover the following two types of SPDEs:

- Semilinear SPDEs of the type

$$\begin{cases} dY_t = (BY_t + \alpha(Y_t))dt + \sigma(Y_t)dW_t \\ Y_0 = y_0, \end{cases} \tag{1.2}$$

  where $B : H \supset D(B) \to H$ is a densely defined, closed operator, and $\alpha : H \to H$ and $\sigma : H \to \ell^2(H)$ are continuous mappings. Here the Hilbert space G is given by the domain $G := D(B)$, equipped with the graph norm

$$\|y\|_G = \sqrt{\|y\|_H^2 + \|By\|_H^2}, \quad y \in G, \tag{1.3}$$

  and the coefficients in (1.1) are given by $L = B + \alpha$ and $A = \sigma$. This includes SPDEs in the framework of the semigroup approach (see, for example [9, 18]), which also arise for the modeling of interest rate curves. We refer to Section 4.2 for more details.
- The above mentioned Itô type SPDEs (see [32, 33]), where the pair (G, H) of continuously embedded Hilbert spaces is given by Hermite Sobolev spaces $G = \mathscr{S}_{p+1}(\mathbb{R}^d)$ and $H = \mathscr{S}_p(\mathbb{R}^d)$ for some $p \in \mathbb{R}$, and the coefficients $L : G \to H$ and $A : G \to \ell^2(H)$ are given by second and first order differential operators of the form

$$L(y) := \frac{1}{2} \sum_{i,j=1}^d (\langle \sigma, y \rangle \langle \sigma, y \rangle^\top)_{ij} \partial_{ij}^2 y - \sum_{i=1}^d \langle b_i, y \rangle \partial_i y, \tag{1.4}$$

$$A^j(y) := - \sum_{i=1}^d \langle \sigma_i^j, y \rangle \partial_i y, \quad j \in \mathbb{N}. \tag{1.5}$$

  where $b_i \in \mathscr{S}_{-(p+1)}(\mathbb{R}^d)$ for $i = 1, \dots, d$ and $\sigma_i^j \in \mathscr{S}_{-(p+1)}(\mathbb{R}^d)$ for $i = 1, \dots, d$ and $j \in \mathbb{N}$, and where $\langle \cdot, \cdot \rangle$ denotes the dual pair on $\mathscr{S}_{-(p+1)}(\mathbb{R}^d) \times \mathscr{S}_{p+1}(\mathbb{R}^d)$. We refer to Section 5.3 for further details. Concerning Hermite Sobolev spaces, we refer to [5, App. B]. At this point, let us mention that the Hermite Sobolev spaces $(\mathscr{S}_p(\mathbb{R}^d))_{p \in \mathbb{R}}$ are separable Hilbert spaces, which are between the Schwartz space $\mathscr{S}(\mathbb{R}^d)$ of rapidly decreasing functions and its dual space $\mathscr{S}'(\mathbb{R}^d)$, the so-called space of tempered distributions. In view of the above definitions (1.4) and (1.5), let us emphasize that for all $p, q \in \mathbb{R}$ with $q \leq p$ we have the pair $(\mathscr{S}_p(\mathbb{R}^d), \mathscr{S}_q(\mathbb{R}^d))$ consists of continuously embedded Hilbert spaces, and that the partial derivatives ∂_i on the Schwartz space $\mathscr{S}(\mathbb{R}^d)$ can be extended to continuous linear operators $\partial_i : \mathscr{S}_{p+\frac{1}{2}}(\mathbb{R}^d) \to \mathscr{S}_p(\mathbb{R}^d)$ for any $p \in \mathbb{R}$.

Let $\mathscr{M} \subset H$ be a finite dimensional C^2-submanifold of H. We are interested in local invariance of $\mathscr{M}$, which means that for each starting point $y_0 \in \mathscr{M}$ there exists a local solution Y to the SPDE (1.1) with $Y_0 = y_0$ such that $Y^\tau \in \mathscr{M}$, where the positive stopping time $\tau > 0$ denotes the lifetime of Y. Let us first recall a known result for semilinear SPDEs of the type (1.2). If $\sigma^j \in C^1(H)$ for each $j \in \mathbb{N}$, then the following statements are equivalent:

(i) $\mathscr{M}$ is locally invariant for the semilinear SPDE (1.2).
(ii) We have

$$\mathscr{M} \subset D(B), \tag{1.6}$$

$$\sigma^j|_{\mathscr{M}} \in \Gamma(T\mathscr{M}), \quad j \in \mathbb{N}, \tag{1.7}$$

$$B|_{\mathscr{M}} + \alpha|_{\mathscr{M}} - \frac{1}{2}\sum_{j=1}^{\infty} D\sigma^j \cdot \sigma^j|_{\mathscr{M}} \in \Gamma(T\mathscr{M}). \tag{1.8}$$

Here $\Gamma(T\mathscr{M})$ denotes the space of all vector fields on $\mathscr{M}$; that it, the space of all mappings $A : \mathscr{M} \to H$ such that $A(y) \in T_y\mathscr{M}$ for each $y \in \mathscr{M}$, where $T_y\mathscr{M}$ denotes the tangent space to $\mathscr{M}$ at y. Furthermore, for each $j \in \mathbb{N}$ we denote by $D\sigma^j \cdot \sigma^j|_{\mathscr{M}}$ the mapping $y \mapsto D\sigma^j(y)\sigma^j(y)$, $y \in \mathscr{M}$.

For this result we refer to [13, 29]; see also [15], where the more general situation with jump-diffusions and submanifolds with boundary has been treated. In [13], the conditions (1.7) and (1.8) above are called "Nagumo type consistency" conditions. However the term $\frac{1}{2}\sum_{j=1}^{\infty} D\sigma^j \cdot \sigma^j$ in condition (1.8) can also be viewed as a "Stratonovich" correction term, which requires smoothness of the volatilities σ^j, $j \in \mathbb{N}$.

When dealing with the more general SPDE (1.1), the smoothness of the coefficients A^j, $j \in \mathbb{N}$ becomes problematic, since they are defined between two different Hilbert spaces $A^j : G \to H$. In particular, for Itô type SPDEs with coefficients of the form (1.4) and (1.5), the volatilities A^j, $j \in \mathbb{N}$ are typically not of class C^1 (see Remark 5.8). Therefore, one of the principal challenges that we deal with in this paper is to find a suitable generalization of condition (1.8) for these SPDEs.

This leads to a geometric framework where we consider (G, H)-submanifolds. More precisely, a C^2-submanifold $\mathscr{M}$ of H is called a (G, H)-submanifold of class C^2 if $\mathscr{M} \subset G$ and $\tau_H \cap \mathscr{M} = \tau_G \cap \mathscr{M}$, where τ_H and τ_G denote the topologies of H and G. In our main result we will show that for such a submanifold $\mathscr{M}$ the following statements are equivalent:

(i) $\mathscr{M}$ is locally invariant for the SPDE (1.1).
(ii) We have

$$A^j|_{\mathscr{M}} \in \Gamma(T\mathscr{M}), \quad j \in \mathbb{N}, \tag{1.9}$$

$$[L|_{\mathscr{M}}]_{\Gamma(T\mathscr{M})} - \frac{1}{2}\sum_{j=1}^{\infty}[A^j|_{\mathscr{M}}, A^j|_{\mathscr{M}}]_{\mathscr{M}} = [0]_{\Gamma(T\mathscr{M})}. \tag{1.10}$$

We refer to Theorem 3.4 for the precise result and further details. The condition (1.10) is an equation in the quotient space $A(\mathscr{M})/\Gamma(T\mathscr{M})$, where $A(\mathscr{M})$ denotes the space of all mappings $A : \mathscr{M} \to H$. Furthermore, for each $j \in \mathbb{N}$ the element $[A^j|_{\mathscr{M}}, A^j|_{\mathscr{M}}]_{\mathscr{M}}$ arises from the quadratic variation term in Itô's formula, when we realize the solutions Y of the SPDE (1.1) on $\mathscr{M}$ as the image $Y = \phi(X)$ of a finite dimensional process X and a local parametrization $\phi : V \to U \cap \mathscr{M}$ of the submanifold $\mathscr{M}$; we refer to Definition 3.2 for more details. The advantage in this formulation is clearly that it does not require smoothness of the vector fields A^j, $j \in \mathbb{N}$, which is also seen in subsequent results; see, for example Theorem 3.16.

In particular, our main result applies to semilinear SPDEs of the type (1.2), where σ is only assumed to be continuous. Recalling that $G = D(B)$ endowed with the graph norm (1.3), in this situation we will show that for a finite dimensional C^2-submanifold $\mathscr{M}$ of H the following statements are equivalent:

(i) $\mathscr{M}$ is locally invariant for the semilinear SPDE (1.2).
(ii) $\mathscr{M}$ is a (G, H)-submanifold of class C^2, which is locally invariant for the semilinear SPDE (1.2).
(iii) $\mathscr{M}$ is a (G, H)-submanifold of class C^2, and we have

$$\sigma^j|_{\mathscr{M}} \in \Gamma(T\mathscr{M}), \quad j \in \mathbb{N}, \tag{1.11}$$

$$[(B + \alpha)|_{\mathscr{M}}]_{\Gamma(T\mathscr{M})} - \frac{1}{2} \sum_{j=1}^{\infty} [\sigma^j|_{\mathscr{M}}, \sigma^j|_{\mathscr{M}}]_{\mathscr{M}} = [0]_{\Gamma(T\mathscr{M})}. \tag{1.12}$$

Furthermore, if $\sigma^j \in C^1(H)$ for each $j \in \mathbb{N}$, then condition (1.12) is equivalent to (1.8). We refer to Theorem 4.14 and Remark 4.15 for further details. These findings are a consequence a more general result for so-called quasi-semilinear SPDEs, which we establish in this paper; see Theorem 4.9.

Note that in the aforementioned result for semilinear SPDEs we only assume that $\mathscr{M}$ is a finite dimensional C^2-submanifold of H, whereas in our main result we assume that $\mathscr{M}$ is a (G, H)-submanifold of class C^2. Indeed, as the previous equivalences (i)–(iii) show, for semilinear SPDEs the submanifold $\mathscr{M}$ is automatically a (G, H)-submanifold in case of local invariance, which is due to the fact that $G = D(B)$ endowed with the graph norm (1.3).

Our main result also applies to Itô type SPDEs (1.1), where the coefficients are of the form (1.4) and (1.5), and where we recall that $G = \mathscr{S}_{p+1}(\mathbb{R}^d)$ and $H = \mathscr{S}_p(\mathbb{R}^d)$ for some $p \in \mathbb{R}$. Then, for any $\Phi \in G$ the submanifold

$$\mathscr{M} = \{\tau_x \Phi : x \in \mathbb{R}^d\}$$

is locally invariant for the SPDE (1.1), where $(\tau_x)_{x \in \mathbb{R}^d}$ denotes the group of translation operators on H. Here, for any $x \in \mathbb{R}^d$ the translation operator τ_x is defined by extending the translation $\tau_x : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$ given by

$$(\tau_x \varphi)(y) := \varphi(y - x) \quad \text{for all } y \in \mathbb{R}^d$$

to an operator $\tau_x : \mathscr{S}'(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ by duality as

$$\langle \tau_x \Phi, \varphi \rangle := \langle \Phi, \tau_{-x} \varphi \rangle \quad \text{for all } \Phi \in \mathscr{S}'(\mathbb{R}^d) \text{ and } \varphi \in \mathscr{S}(\mathbb{R}^d).$$

The aforementioned result shows that the solutions to the Itô type SPDE (1.1) are translation invariant; that is, we have $Y = \tau_X \Phi$ for some $\mathbb{R}^d$-valued diffusion X; see also [32].

We will generalize this result to SPDEs (1.1) with a general pair of continuously embedded Hilbert spaces (G, H) as follows. Let $T = (T(t))_{t \in \mathbb{R}^d}$ be a multi-parameter C_0-group on

H; that is, T is a family of continuous linear operators $T(t) \in L(H)$ such that the following conditions are fulfilled:

(1) $T(0) = \mathrm{Id}$.
(2) We have $T(t+s) = T(t)T(s)$ for all $t, s \in \mathbb{R}^d$.
(3) For each $x \in H$ the *orbit map* $\xi_x : \mathbb{R}^d \to H$, $\xi_x(t) := T(t)x$ is continuous.

We refer to [5, App. A] for further details about multi-parameter C_0-groups. Moreover, let $\mathscr{N}$ be an m-dimensional C^2-submanifold of $\mathbb{R}^d$ for some $m \leq d$, and consider the submanifold

$$\mathscr{M} = \{T(t)y_0 : t \in \mathscr{N}\} \tag{1.13}$$

for some $y_0 \in G$. Denoting by $\psi : \mathbb{R}^d \to H$ the orbit map $\psi(t) := T(t)y_0$ for $t \in \mathbb{R}^d$, we will show that the following statements are equivalent:

(i) $\mathscr{M}$ is locally invariant for the SPDE (1.1).
(ii) $\mathscr{N}$ is locally invariant for the $\mathbb{R}^d$-valued SDE

$$\begin{cases} dX_t = \bar{b}(X_t)dt + \bar{\sigma}(X_t)dW_t \\ X_0 = x_0, \end{cases}$$

where $\bar{\sigma} : \mathscr{N} \to \ell^2(\mathbb{R}^d)$ and $\bar{b} : \mathscr{N} \to \mathbb{R}^d$ are the unique solutions of the equations

$$L|_{\mathscr{M}} = \frac{1}{2} \sum_{i,j=1}^{d} (\bar{\sigma}\bar{\sigma}^{\top})_{ij} \circ \psi^{-1}|_{\mathscr{M}} \, B_{ij}|_{\mathscr{M}} + \sum_{i=1}^{d} \bar{b}_i \circ \psi^{-1}|_{\mathscr{M}} \, B_i|_{\mathscr{M}}, \tag{1.14}$$

$$A^j|_{\mathscr{M}} = \sum_{i=1}^{d} \bar{\sigma}_i^j \circ \psi^{-1}|_{\mathscr{M}} \, B_i|_{\mathscr{M}}, \quad j \in \mathbb{N}. \tag{1.15}$$

We refer to Theorem 5.2 for the precise statement. Note that the structures of the coefficients in (1.4) and (1.5) are particular cases of (1.14) and (1.15). This result is a consequence of a more general result for arbitrary (G, H)-submanifolds, which we establish in this paper; see Theorem 3.12. Moreover, we will show that the structure (1.13) of the submanifold $\mathscr{M}$ appears naturally with coefficients of the kind (1.14) and (1.15) in case of local invariance; see Theorem 5.6 for the precise result.

Diffusions on manifolds in $\mathbb{R}^d$ is a well studied topic (see for a partial list [11, 12, 20–22, 37]). In this paper, we will also establish new invariance results for finite dimensional diffusions, which come as a consequence of Theorem 3.4. More precisely, consider an $\mathbb{R}^d$-valued diffusion of the type

$$\begin{cases} dX_t = b(X_t)dt + \sigma(X_t)dW_t \\ X_0 = x_0 \end{cases} \tag{1.16}$$

with coefficients $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \ell^2(\mathbb{R}^d)$, and let $\mathscr{N}$ be a submanifold of $\mathbb{R}^d$.

Before we proceed, let us mention that for finite dimensional diffusions of the type (1.16) there are also invariance results in the situation where $\mathscr{N}$ is a closed subset of $\mathbb{R}^d$; see in particular the two works [1, 2] and the references therein, such as [3, 4], where Nagumo-type conditions on the second order normal cone are provided, and [10], where conditions on the first order normal cone with the Stratonovich correction term are provided. We will compare our upcoming findings with those from the aforementioned papers later on.

Our essential assumption in the present paper is that the coefficients of the SDE (1.16) belong to a Hermite Sobolev space with sufficient regularity. More precisely, we assume that for some $q > \frac{d}{4}$ we have $b_i \in \mathscr{S}_q(\mathbb{R}^d)$ for $i = 1, \ldots, d$ and $\sigma_i^j \in \mathscr{S}_q(\mathbb{R}^d)$ for

$i = 1, \dots, d$ and $j \in \mathbb{N}$. In view of the Sobolev embedding theorem for Hermite Sobolev spaces (see [5, Thm. B.19]) this essentially means that the components are continuous, with some restrictions on the growth of the functions, but do not need to satisfy any smoothness conditions. Let $\mathscr{N}$ be an m-dimensional C^2-submanifold of $\mathbb{R}^d$ for some $m \leq d$. We set $G := \mathscr{S}_{-q}(\mathbb{R}^d)$, $H := \mathscr{S}_{-(q+1)}(\mathbb{R}^d)$, define the coefficients of the SPDE (1.1) as (1.4), (1.5) with $p := -(q+1)$, and consider the submanifold

$$\mathscr{M} := \{\delta_x : x \in \mathscr{N}\},$$

where δ_x denotes the Dirac distribution at point x. Then the following statements are equivalent:

(i) $\mathscr{M}$ is locally invariant for the SPDE (1.1).
(ii) $\mathscr{N}$ is locally invariant for the SDE (1.16).

We refer to Theorem 6.3, which establishes the announced link between the invariance of submanifolds for SPDEs in Hermite Sobolev spaces and the invariance of submanifolds for finite dimensional SDEs. In particular, in some situations it turns out that locally invariance of $\mathscr{M}$ for the SPDE (1.1) is easier to prove, which is the key for providing new invariance results for finite dimensional SDEs.

One application of this connection appears in the situation, where we consider the conditions

$$b|_{\mathscr{N}} \in \Gamma(T\mathscr{N}), \tag{1.17}$$

$$\sigma^j|_{\mathscr{N}} \in \Gamma(T\mathscr{N}), \quad j \in \mathbb{N}, \tag{1.18}$$

and where we are interested in finding an additional condition ensuring that $\mathscr{N}$ is locally invariant for the SDE (1.16). In this regard, we will show that under conditions (1.17) and (1.18) the following conditions are equivalent:

(i) $\mathscr{N}$ is locally invariant for the SDE (1.16).
(ii) We have

$$\sum_{j=1}^{\infty} \big([A^j|_{\mathscr{M}}, A^j|_{\mathscr{M}}]_{\mathscr{M}} - [\bar{A}^j(A^j(\cdot), \cdot)|_{\mathscr{M}}]_{\Gamma(T\mathscr{M})} \big) = [0]_{\Gamma(T\mathscr{M})},$$

where, in accordance with (1.5), we have set

$$\bar{A}^j(y, z) := -\sum_{i=1}^{d} \langle \sigma_i^j, z \rangle \partial_i y, \quad j \in \mathbb{N}.$$

We refer to Theorem 6.5 for further details. A consequence of this result is that the conditions

$$b|_{\mathscr{N}} \in \Gamma(T\mathscr{N}),$$

$$\sigma^j|_{\mathscr{N}} \in \Gamma^*(T\mathscr{N}), \quad j \in \mathbb{N},$$

where $\Gamma^*(T\mathscr{N})$ denotes the space of all locally simultaneous vector fields on $\mathscr{N}$, are sufficient for local invariance of $\mathscr{N}$ for the SDE (1.16); see Proposition 6.6. This is a generalization of the result that an affine submanifold $\mathscr{N}$ is locally invariant if and only if we have (1.17) and (1.18). We also establish such a result in the general framework for SPDEs of the type (1.1); see Corollary 3.9.

Another application of the connection between invariance of submanifolds for SDEs and SPDEs occurs in the situation, where the submanifold $\mathscr{N}$ is given by the zeros of smooth functions. More precisely, we assume that the dimension of $\mathscr{N}$ is given by $m = d - n$, where $n < d$, and that there exist an open subset $O \subset \mathbb{R}^d$ and a mapping $f : \mathbb{R}^d \to \mathbb{R}^n$ such that

$$\mathscr{N} = \{x \in O : f(x) = 0\}.$$

Concerning the components of f we assume that $f_k \in \mathscr{S}_{q+1}(\mathbb{R}^d)$ for all $k = 1, \dots, n$. As we will show, then the following statements are equivalent:

(i) The submanifold $\mathscr{N}$ is locally invariant for the SDE (1.16).
(ii) For all $k = 1, \dots, n$ and all $x \in \mathscr{N}$ we have

$$\langle \sigma^j(x), \nabla f_k(x) \rangle = 0, \quad j \in \mathbb{N}, \tag{1.19}$$

$$\langle b(x), \nabla f_k(x) \rangle + \frac{1}{2} \mathrm{tr}\big(\sigma(x)\sigma(x)^\top \mathbf{H}_{f_k}(x)\big) = 0, \tag{1.20}$$

where $\mathbf{H}_{f_k}(x)$ denotes the Hessian matrix of f_k at point x.

For this result, we refer to Theorem 6.13. Note that the conditions (1.19) and (1.20) are similar to those in [3, 4], where invariance of closed subsets for controlled diffusions with Lipschitz coefficients has been studied.

We illustrate Theorem 6.13 with the example of the unit sphere $\mathbb{S}^{d-1}$ (Corollary 6.15 and Example 6.16) and recover an earlier result of Stroock. Furthermore, we provide an example with the unit sphere $\mathbb{S}^1$, where our results apply, but, to the best of our knowledge, none of the known results can be applied; see Example 6.17.

The paper is organized as follows: Section 2 introduces SPDEs in the framework of continuously embedded Hilbert spaces. In Section 3 we present our main result concerning invariant manifolds. Afterwards, in Section 4 we present consequences for quasi-semilinear SPDEs, which includes the particular case of semilinear SPDEs. In Section 5 we study the invariance of manifolds which are generated by orbit maps; this includes Itô type SPDEs. In Section 6 we provide the link between the invariance of submanifolds for finite dimensional SDEs and the invariance of submanifolds for SPDEs in Hermite Sobolev spaces, and provide new invariance results for finite dimensional SDEs. For convenience of the reader, in the electronic appendix [5] (which is a more detailed version of this paper) we provide the necessary background, including finite dimensional submanifolds in embedded Hilbert spaces, multi-parameter strongly continuous groups and Hermite Sobolev spaces. The proofs of some technical auxiliary results are also deferred to [5].

2 Stochastic Partial Differential Equations in Continuously Embedded Hilbert Spaces

In this section we provide the required prerequisites about SPDEs in continuously embedded Hilbert spaces.

Definition 2.1 *We call $W = (W^j)_{j \in \mathbb{N}}$ a* standard $\mathbb{R}^\infty$-Wiener process *if $(W^j)_{j \in \mathbb{N}}$ is a sequence of independent real-valued standard Wiener processes on some stochastic basis.*

For a Hilbert space H we denote by $\ell^2(H)$ the Hilbert space of all H-valued sequences $y = (y^j)_{j \in \mathbb{N}}$ such that

$$\|y\|_{\ell^2(H)} := \left(\sum_{j=1}^{\infty} \|y^j\|_H^2 \right)^{1/2} < \infty.$$

Proposition 2.2 *Let $W = (W^j)_{j \in \mathbb{N}}$ be a standard $\mathbb{R}^\infty$-Wiener process on a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$, let H be a separable Hilbert space, and let A be a predictable $\ell^2(H)$-valued process such that we have $\mathbb{P}$-almost surely*

$$\int_0^t \|A_s\|_{\ell^2(H)}^2 ds < \infty, \quad t \in \mathbb{R}_+. \tag{2.1}$$

Then the process $(\int_0^t A_s dW_s)_{t \in \mathbb{R}_+}$ given by

$$\int_0^t A_s dW_s := \sum_{j=1}^\infty \int_0^t A_s^j dW_s^j, \quad t \in \mathbb{R}_+ \tag{2.2}$$

is a well-defined H-valued continuous local martingale, and the convergence is in probability, uniformly on compact intervals.

Proof Let $T > 0$ be arbitrary. We denote by $M_T^2(H)$ the space of all H-valued square-integrable martingales $M = (M_t)_{t \in [0,T]}$, which, endowed with the norm

$$\|M\|_\infty = \mathbb{E}\bigg[\sup_{t \in [0,T]} \|M_t\|_H^2 \bigg]^{1/2}, \quad M \in M_T^2(H)$$

is a Hilbert space. Furthermore, by Doob's martingale inequality, an equivalent norm is given by

$$\|M\|_T = \mathbb{E}\big[\|M_T\|_H^2\big]^{1/2}, \quad M \in M_T^2(H).$$

Concerning the predictable process A, we first suppose that

$$\mathbb{E}\bigg[\int_0^T \|A_s\|_{\ell^2(H)}^2 ds \bigg] < \infty.$$

Then by the Itô isometry and the monotone convergence theorem we have

$$\sum_{j=1}^\infty \mathbb{E}\bigg[\bigg\| \int_0^T A_s^j dW_s^j \bigg\|_H^2 \bigg] = \sum_{j=1}^\infty \mathbb{E}\bigg[\int_0^T \|A_s^j\|_H^2 ds \bigg] = \mathbb{E}\bigg[\int_0^T \|A_s\|_{\ell^2(H)}^2 ds \bigg] < \infty,$$

and hence the series $\sum_{j=1}^\infty \int_0^T A_s^j dW_s^j$ converges in $M_T^2(H)$. The situation with a general predictable process A satisfying (2.1) follows by localization, and, by the definition of the norm $\| \cdot \|_\infty$, the convergence is in probability, uniformly on compact intervals. □

Definition 2.3 *Let G and H be two normed spaces. Then we call (G, H)* continuously embedded normed spaces *(or* normed spaces with continuous embedding*) if the following conditions are fulfilled:*

(1) *We have $G \subset H$ as sets.*
(2) *The embedding operator $\mathrm{Id} : (G, \| \cdot \|_G) \to (H, \| \cdot \|_H)$ is continuous; that is, there is a constant $K > 0$ such that*

$$\|x\|_H \leq K \|x\|_G \quad \text{for all } x \in G.$$

Definition 2.4 *Let $H_1, \dots, H_n$ be normed spaces. Then we call $(H_1, \dots, H_n)$* continuously embedded normed spaces *if for each $k = 1, \dots, n-1$ the pair (H_k, H_{k+1}) is a pair of continuously embedded normed spaces.*

Now, let (G, H) be separable Hilbert spaces with continuous embedding. Furthermore, let $L : G \to H$ and $A : G \to \ell^2(H)$ be continuous[1] mappings. Then for each $j \in \mathbb{N}$ the component $A^j : G \to H$ is continuous.

Definition 2.5 *Let $y_0 \in G$ be arbitrary. A triplet $(\mathbb{B}, W, Y)$ is called a* local martingale solution *to the SPDE (1.1) with $Y_0 = y_0$ if the following conditions are fulfilled:*

(1) *$\mathbb{B} = (\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$ is a stochastic basis; that is, a filtered probability space satisfying the usual conditions.*
(2) *W is a standard $\mathbb{R}^\infty$-Wiener process on the stochastic basis $\mathbb{B}$.*
(3) *Y is a G-valued adapted[2] process such that for some strictly positive stopping time $\tau > 0$ we have $\mathbb{P}$-almost surely*

$$\int_0^{t\wedge\tau} \big(\|L(Y_s)\|_H + \|A(Y_s)\|_{\ell^2(H)}^2\big)ds < \infty, \quad t \in \mathbb{R}_+ \tag{2.3}$$

and $\mathbb{P}$-almost surely

$$Y_{t\wedge\tau} = y_0 + \int_0^{t\wedge\tau} L(Y_s)ds + \int_0^{t\wedge\tau} A(Y_s)dW_s, \quad t \in \mathbb{R}_+, \tag{2.4}$$

where the stochastic integral is defined according to (2.2). The stopping time τ is also called the lifetime *of Y.*

If we can choose $\tau = \infty$, then $(\mathbb{B}, W, Y)$ is also called a global martingale solution *(or simply a* martingale solution*) to the SPDE (1.1) with $Y_0 = y_0$.*

Remark 2.6 *As it is apparent from the integrability condition (2.3), the stochastic integrals appearing in (2.4) are understood as stochastic integrals in the Hilbert space $(H, \|\cdot\|_H)$. Therefore, the right-hand side of (2.4) is generally H-valued, whereas the left-hand side is G-valued. This indicates that the existence of martingale solutions to the SPDE (1.1) can generally not be warranted. If there exists a martingale solution Y, then its sample paths are continuous with respect to the norm $\|\cdot\|_H$, but they do not need to be continuous with respect to the norm $\|\cdot\|_G$.*

Remark 2.7 *Let $\mathbb{B}$ be a stochastic basis. In our situation, there are two reasonable ways to define what it means that a G-valued process Y is* adapted*; namely:*

(1) *We regard Y as a process taking its values in the subspace G of the Hilbert space $(H, \|\cdot\|_H)$ and call it* adapted *if for each $t \in \mathbb{R}_+$ the mapping $Y_t : \Omega \to G$ is $\mathscr{F}_t$-$\mathscr{B}(H)_G$-measurable, where $\mathscr{B}(H)_G$ denotes the trace σ-algebra*

$$\mathscr{B}(H)_G = \{B \cap G : B \in \mathscr{B}(H)\}.$$

(2) *We regard Y as a process taking its values in the Hilbert space $(G, \|\cdot\|_G)$ and call it* adapted *if for each $t \in \mathbb{R}_+$ the mapping $Y_t : \Omega \to G$ is $\mathscr{F}_t$-$\mathscr{B}(G)$-measurable.*

However, by Kuratowski's theorem (see, for example [30, Thm. I.3.9]) we have $\mathscr{B}(G) = \mathscr{B}(H)_G$, showing that these two concepts of adaptedness are equivalent.

[1] More precisely, here and in the sequel, we call a mapping $L : G \to H$ *continuous* if $L : (G, \|\cdot\|_G) \to (H, \|\cdot\|_H)$ is continuous. The continuity of A is understood analogously.

[2] See Remark 2.7 for details about this notion.

Remark 2.8 *The SPDE (1.1) can also be realized as an SPDE driven by a trace class Wiener process, as considered, for example in* [9, 18]. *Indeed, let U be a separable Hilbert space, and let $\bar{W}$ be an U-valued Q-Wiener process for some nuclear, self-adjoint, positive definite linear operator $Q \in L_1^{++}(U)$; see, for example* [9, Def. 4.2]. *There exist an orthonormal basis $\{e_j\}_{j \in \mathbb{N}}$ of U and a sequence $(\lambda_j)_{j \in \mathbb{N}} \subset (0, \infty)$ with $\sum_{j \in \mathbb{N}} \lambda_j < \infty$ such that*

$$Qe_j = \lambda_j e_j \quad \text{for all } j \in \mathbb{N}.$$

The space $U_0 := Q^{1/2}(U)$, equipped with the inner product

$$\langle u, v \rangle_{U_0} := \langle Q^{-1/2} u, Q^{-1/2} v \rangle_U, \quad u, v \in U_0$$

is another separable Hilbert space. We fix the orthonormal basis $\{g_j\}_{j \in \mathbb{N}}$ of U_0 given by $g_j := \sqrt{\lambda_j} e_j$ for each $j \in \mathbb{N}$, and we denote by $L_2^0(H) := L_2(U_0, H)$ the space of all Hilbert-Schmidt operators from U_0 into H. Note that $L_2^0(H) \cong \ell^2(H)$, because $T \mapsto (Tg_j)_{j \in \mathbb{N}}$ is an isometric isomorphism. By [9, Prop. 4.3] *the sequence $(\bar{W}^j)_{j \in \mathbb{N}}$ defined as*

$$\bar{W}^j := \frac{1}{\sqrt{\lambda_j}} \langle \bar{W}, e_j \rangle_U, \quad j \in \mathbb{N}$$

is a sequence of independent real-valued standard Wiener processes. Hence $W = (\bar{W}^j)_{j \in \mathbb{N}}$ is a standard $\mathbb{R}^\infty$-Wiener process. As a consequence of the series representation of the stochastic integral with respect to the trace class Wiener process $\bar{W}$ (see, for example [25, Prop. 2.4.5]*), the SPDE (1.1) can be expressed as*

$$\begin{cases} dY_t = L(Y_t)dt + \bar{A}(Y_t)d\bar{W}_t \\ Y_0 = y_0 \end{cases} \tag{2.5}$$

where the continuous mapping $\bar{A} : G \to L_2^0(H)$ is given by

$$\bar{A}(y) := \sum_{j=1}^{\infty} \langle \bullet, g_j \rangle_{U_0} A^j(y), \quad y \in G,$$

and, vice versa, the SPDE (2.5) can be expressed by the SPDE (1.1), where the continuous mapping $A : G \to \ell^2(H)$ is given by

$$A(y) := (\bar{A}(y) g_j)_{j \in \mathbb{N}}, \quad y \in G.$$

Remark 2.9 *In the particular case $G = H = \mathbb{R}^d$ the SPDE (1.1) is rather an SDE, and a martingale solution $(\mathbb{B}, W, Y)$ is a weak solution. If, in this case, the continuous mappings $L : \mathbb{R}^d \to \mathbb{R}^d$ and $A : \mathbb{R}^d \to \ell^2(\mathbb{R}^d)$ satisfy the linear growth condition, then for each $y_0 \in \mathbb{R}^d$ there exists a global weak solution $(\mathbb{B}, W, Y)$ to the SDE (1.1) with $Y_0 = y_0$. Indeed, taking into account Remark 2.8, this follows from* [19, Thm. 2] *(or* [18, Thm. 3.12]*), applied with $H = H_{-1} = \mathbb{R}^m$ and $J = \mathrm{Id}_{\mathbb{R}^m}$.*

Remark 2.10 *The situation where the Wiener process W is $\mathbb{R}^r$-valued is covered by choosing $A^j \equiv 0$ for all $j > r$. If we are additionally in the situation of Remark 2.9, then the existence of global weak solutions also follows from* [21, Thms. IV.2.3 and IV.2.4].

Remark 2.11 *If there is no ambiguity, we will simply call Y a local martingale solution or a global martingale solution to the SPDE (1.1) with $Y_0 = y_0$.*

Now, let $\mathscr{M} \subset G$ be a subset. In this paper, the subset $\mathscr{M}$ will typically be a finite dimensional submanifold.

Definition 2.12 *The subset $\mathscr{M}$ is called* locally invariant *for the SPDE (1.1) if for each $y_0 \in \mathscr{M}$ there exists a local martingale solution Y to the SPDE (1.1) with $Y_0 = y_0$ and lifetime $\tau > 0$ such that $Y^\tau \in \mathscr{M}$ up to an evanescent set*[3].

Definition 2.13 *The subset $\mathscr{M}$ is called* globally invariant *(or simply* invariant*) for the SPDE (1.1) if for each $y_0 \in \mathscr{M}$ there exists a global martingale solution Y to the SPDE (1.1) with $Y_0 = y_0$ such that $Y \in \mathscr{M}$ up to an evanescent set.*

3 The General Invariance Result

In this section we provide the general invariance result. Let (G, H) be separable Hilbert spaces with continuous embedding, and consider the SPDE (1.1) with continuous mappings $L : G \to H$ and $A : G \to \ell^2(H)$. Let $\mathscr{M}$ be a finite dimensional (G, H)-submanifold of class C^2. More precisely, $\mathscr{M}$ is a finite dimensional C^2-submanifold of H such that $\mathscr{M} \subset G$ and $\tau_H \cap \mathscr{M} = \tau_G \cap \mathscr{M}$, where τ_G and τ_H denote the respective topologies. Then for each $y \in \mathscr{M}$ there exists a local parametrization $\phi : V \to U \cap \mathscr{M}$ around y, which satisfies $\phi \in C(V; G) \cap C^2(V; H)$; see [5, Prop. 3.21]. Recall that $\Gamma(T\mathscr{M})$ denotes the subspace of all vector fields on $\mathscr{M}$; that it, the space of all mappings $A : \mathscr{M} \to H$ such that $A(y) \in T_y\mathscr{M}$ for each $y \in \mathscr{M}$, where $T_y\mathscr{M}$ denotes the tangent space to $\mathscr{M}$ at y.

Definition 3.1 *Let $\phi : V \to U \cap \mathscr{M}$ be a local parametrization of $\mathscr{M}$.*

(1) *For a mapping $a : V \to \mathbb{R}^m$ we define $\phi_* a : U \cap \mathscr{M} \to H$ as*

$$(\phi_* a)(y) := D\phi(x)a(x), \quad y \in U \cap \mathscr{M},$$

where $x := \phi^{-1}(y) \in V$.

(2) *Similarly, for two mappings $a, b : V \to \mathbb{R}^m$ we define $\phi_{**}(a, b) : U \cap \mathscr{M} \to H$ as*

$$(\phi_{**}(a, b))(y) := D^2\phi(x)(a(x), b(x)), \quad y \in U \cap \mathscr{M},$$

where $x := \phi^{-1}(y) \in V$.

(3) *Setting $\mathscr{M}_U := U \cap \mathscr{M}$, for a vector field $A \in \Gamma(T\mathscr{M}_U)$ we define $\phi_*^{-1} A : V \to \mathbb{R}^m$ as*

$$(\phi_*^{-1} A)(x) := D\phi(x)^{-1} A(y), \quad x \in V,$$

where $y := \phi(x) \in U \cap \mathscr{M}$.

We denote by $A(\mathscr{M})$ be the linear space of all mappings $A : \mathscr{M} \to H$. In the following definition we consider the quotient space $A(\mathscr{M})/\Gamma(T\mathscr{M})$, and for each $A \in A(\mathscr{M})$ we denote by $[A]_{\Gamma(T\mathscr{M})}$ the corresponding equivalence class, for which any representative is of the form $A + B$ with some vector field $B \in \Gamma(T\mathscr{M})$.

Definition 3.2 *Let $A, B \in \Gamma(T\mathscr{M})$ be two vector fields on $\mathscr{M}$. We define the mapping*

$$[A, B]_{\mathscr{M}} \in A(\mathscr{M})/\Gamma(T\mathscr{M})$$

as follows. For each local parametrization $\phi : V \to U \cap \mathscr{M}$ a local representative of $[A, B]_{\mathscr{M}}$ on $U \cap \mathscr{M}$ is given by

$$\phi_{**}(\phi_*^{-1} A|_{U \cap \mathscr{M}}, \phi_*^{-1} B|_{U \cap \mathscr{M}}),$$

where we recall the notation from Definition 3.1.

[3] A random set $A \subset \Omega \times \mathbb{R}_+$ is called *evanescent* if the set $\{\omega \in \Omega : (\omega, t) \in A$ for some $t \in \mathbb{R}_+\}$ is a $\mathbb{P}$-nullset, cf. [23, 1.1.10].

Remark 3.3 *Note that, according to* [5, Lemma 3.19], *the Definition 3.2 of* $[A, B]_{\mathscr{M}}$ *does not depend on the choice of the parametrization.*

Now, we are ready to state our main result.

Theorem 3.4 *The following statements are equivalent:*

(i) *The submanifold* $\mathscr{M}$ *is locally invariant for the SPDE (1.1).*
(ii) *We have*

$$A^j|_{\mathscr{M}} \in \Gamma(T\mathscr{M}), \quad j \in \mathbb{N}, \tag{3.1}$$

$$[L|_{\mathscr{M}}]_{\Gamma(T\mathscr{M})} - \frac{1}{2}\sum_{j=1}^{\infty}[A^j|_{\mathscr{M}}, A^j|_{\mathscr{M}}]_{\mathscr{M}} = [0]_{\Gamma(T\mathscr{M})}. \tag{3.2}$$

(iii) *The mappings*

$$A|_{\mathscr{M}} : (\mathscr{M}, \|\cdot\|_H) \to (\ell^2(H), \|\cdot\|_{\ell^2(H)}), \tag{3.3}$$

$$L|_{\mathscr{M}} : (\mathscr{M}, \|\cdot\|_H) \to (H, \|\cdot\|_H) \tag{3.4}$$

are continuous, and for each $y_0 \in \mathscr{M}$ *there exists a local martingale solution* Y *to the SPDE (1.1) with* $Y_0 = y_0$ *and lifetime* τ *such that* $Y^\tau \in \mathscr{M}$ *up to an evanescent set and the sample paths of* Y^τ *are continuous with respect to* $\|\cdot\|_G$*.*

Proposition 3.5 *Suppose that the submanifold* $\mathscr{M}$ *is locally invariant for the SPDE (1.1). If the submanifold* $\mathscr{M}$ *has one chart with a global parametrization* $\phi : V \to \mathscr{M}$*, and the open set* V *is globally invariant for the* $\mathbb{R}^m$*-valued SDE*

$$\begin{cases} dX_t = \ell(X_t)dt + a(X_t)dW_t \\ X_0 = x_0, \end{cases}$$

where the continuous mappings $\ell : V \to \mathbb{R}^m$ *and* $a : V \to \ell^2(\mathbb{R}^m)$ *are the unique solutions of the equations*

$$A^j|_{\mathscr{M}} = \phi_* a^j, \quad j \in \mathbb{N}, \tag{3.5}$$

$$L|_{\mathscr{M}} = \phi_*\ell + \frac{1}{2}\sum_{j=1}^{\infty}\phi_{**}(a^j, a^j), \tag{3.6}$$

then the submanifold $\mathscr{M}$ *is globally invariant for the SPDE (1.1).*

Remark 3.6 *In view of Proposition 3.5 and subsequent results (such as Proposition 3.11) we point out that the unique mappings* $\ell : V \to \mathbb{R}^m$ *and* $a : V \to \ell^2(\mathbb{R}^m)$ *which solve the Eqs. 3.5 and 3.6 are automatically continuous. This is a consequence of* [5, Lemma 4.13].

Remark 3.7 *Choosing* $G = H = \mathbb{R}^d$*, we see that Theorem 3.4 and Proposition 3.5 cover the well-known situation of finite dimensional SDEs.*

Before we provide the proofs of Theorem 3.4 and Proposition 3.5, let us state some consequences of these results. Consider the conditions

$$L|_{\mathscr{M}} \in \Gamma(T\mathscr{M}), \tag{3.7}$$

$$A^j|_{\mathscr{M}} \in \Gamma(T\mathscr{M}), \quad j \in \mathbb{N}. \tag{3.8}$$

We are interested in finding an additional condition which ensures such that $\mathscr{M}$ is locally invariant for the SPDE (1.1).

Proposition 3.8 *Suppose that conditions (3.7) and (3.8) is fulfilled. Then the following statements are equivalent:*

(i) *$\mathscr{M}$ is locally invariant for the SPDE (1.1).*
(ii) *We have*

$$\sum_{j=1}^{\infty} [A^j|_{\mathscr{M}}, A^j|_{\mathscr{M}}]_{\mathscr{M}} = [0]_{\Gamma(T\mathscr{M})}.$$

Proof This is a consequence of Theorem 3.4. □

We say that the submanifold $\mathscr{M}$ is *affine* if for any local parametrization $\phi : V \to U \cap \mathscr{M}$ we have $D^2\phi = 0$.

Corollary 3.9 *Suppose the submanifold $\mathscr{M}$ is affine. Then the following statements are equivalent:*

(i) *$\mathscr{M}$ is locally invariant for the SPDE (1.1).*
(ii) *We have (3.7) and (3.8).*

Proof This is a consequence of Theorem 3.4 and Proposition 3.8. □

Remark 3.10 *Consider the situation $G = H$ and $A^j \in C^1(H)$ for all $j \in \mathbb{N}$. If $\sum_{j=1}^{\infty} DA^j(y)A^j(y)$ converges for each $y \in H$, and the mapping $\sum_{j=1}^{\infty} DA^j \cdot A^j$ is continuous, then we can rewrite the SPDE (1.1) in Stratonovich form as*

$$\begin{cases} dY_t = K(Y_t)dt + A(Y_t) \circ dW_t \\ Y_0 = y_0, \end{cases}$$

where $K : H \to H$ is given by

$$K = L - \frac{1}{2}\sum_{j=1}^{\infty} DA^j \cdot A^j.$$

If we have (3.1), then by the decomposition [5, Prop. 3.25, eqn. (3.2)] *we have*

$$[K|_{\mathscr{M}}]_{\Gamma(T\mathscr{M})} = [L|_{\mathscr{M}}]_{\Gamma(T\mathscr{M})} - \frac{1}{2}\sum_{j=1}^{\infty} [A^j|_{\mathscr{M}}, A^j|_{\mathscr{M}}]_{\mathscr{M}},$$

and hence condition (3.2) is equivalent to

$$K|_{\mathscr{M}} \in \Gamma(T\mathscr{M}).$$

We will present a corresponding result for continuously embedded Hilbert spaces with an additional intermediate space later on; see Theorem 3.15 below.

We can express the statement of Theorem 3.4 in local coordinates as follows.

Proposition 3.11 *The following statements are equivalent:*

(i) *The submanifold $\mathscr{M}$ is locally invariant for the SPDE (1.1).*

(ii) *For each local parametrization $\phi : V \to U \cap \mathscr{M}$ there are continuous mappings $\ell : V \to \mathbb{R}^m$ and $a : V \to \ell^2(\mathbb{R}^m)$ which are the unique solutions of the equations*

$$A^j|_{U\cap\mathscr{M}} = \phi_* a^j, \quad j \in \mathbb{N}, \tag{3.9}$$

$$L|_{U\cap\mathscr{M}} = \phi_* \ell + \frac{1}{2}\sum_{j=1}^{\infty} \phi_{**}(a^j, a^j). \tag{3.10}$$

(iii) *For each $y \in \mathscr{M}$ there exist a local parametrization $\phi : V \to U \cap \mathscr{M}$ around y and continuous mappings $\ell : V \to \mathbb{R}^m$ and $a : V \to \ell^2(\mathbb{R}^m)$ which are the unique solutions of the Eqs. 3.9 and 3.10.*

Proof This is an immediate consequence of Theorem 3.4. □

In the following two results we assume that the submanifold $\mathscr{M}$ is induced $(\psi, \mathscr{N})$, where $\mathscr{N}$ is an m-dimensional C^2-submanifold of $\mathbb{R}^d$, and $\psi \in C^2(\mathbb{R}^d; H)$ is a C^2-immersion on $\mathscr{N}$ such that $\psi|_{\mathscr{N}} : \mathscr{N} \to \psi(\mathscr{N})$ is a homeomorphism; see [5, Def. 3.32].

Theorem 3.12 *The following statements are equivalent:*

(i) *The submanifold $\mathscr{M}$ is locally invariant for the SPDE (1.1).*
(ii) *The submanifold $\mathscr{N}$ is locally invariant for the SDE*

$$\begin{cases} dX_t = b(X_t)dt + \sigma(X_t)dW_t \\ X_0 = x_0, \end{cases} \tag{3.11}$$

where the continuous mappings[4] *$b : \mathscr{N} \to \mathbb{R}^d$ and $\sigma : \mathscr{N} \to \ell^2(\mathbb{R}^d)$ are the unique solutions of the equations*

$$A^j|_{\mathscr{M}} = \psi_* \sigma^j, \quad j \in \mathbb{N}, \tag{3.12}$$

$$L|_{\mathscr{M}} = \psi_* b + \frac{1}{2}\sum_{j=1}^{\infty} \psi_{**}(\sigma^j, \sigma^j). \tag{3.13}$$

Proof (i) ⇒ (ii): Let $y \in \mathscr{M}$ be arbitrary, and let $\varphi : V \to W \cap \mathscr{N}$ be a local parametrization around $x := \psi^{-1}(y) \in \mathscr{N}$. By [5, Lemma 3.31] there exists an open neighborhood $U \subset H$ of y such that $\phi := \psi \circ \varphi : V \to U \cap \mathscr{M}$ is a local parametrization around y. Furthermore, by Proposition 3.11 there are continuous mappings $\ell : V \to \mathbb{R}^m$ and $a : V \to \ell^2(\mathbb{R}^m)$ which are the unique solutions of the Eqs. 3.9 and 3.10. We define the continuous mappings $b : W \cap \mathscr{N} \to \mathbb{R}^d$ and $\sigma : W \cap \mathscr{N} \to \ell^2(\mathbb{R}^d)$ as

$$\sigma^j := \varphi_* a^j, \quad j \in \mathbb{N},$$

$$b := \varphi_* \ell + \frac{1}{2}\sum_{j=1}^{\infty} \varphi_{**}(a^j, a^j).$$

Since $y \in \mathscr{M}$ was arbitrary, by Proposition 3.11 we deduce that the submanifold $\mathscr{N}$ is locally invariant for the SDE (3.11). Furthermore, by [5, Lemma 3.35] we obtain

$$A^j|_{U\cap\mathscr{M}} = \phi_* a^j = \psi_* \varphi_* a^j = \psi_* \sigma^j, \quad j \in \mathbb{N}$$

[4] If the SDE (3.11) is locally invariant, then it suffices to specify the coefficients b and σ on the submanifold $\mathscr{N}$.

as well as

$$\begin{aligned} L|_{U\cap\mathscr{M}} &= \phi_*\ell + \frac{1}{2}\sum_{j=1}^{\infty}\phi_{**}(a^j, a^j) \\ &= \psi_*\varphi_*\ell + \frac{1}{2}\sum_{j=1}^{\infty}\big(\psi_{**}(\varphi_* a^j, \varphi_* a^j) + \psi_*\varphi_{**}(a^j, a^j)\big) \\ &= \psi_* b + \frac{1}{2}\sum_{j=1}^{\infty}\psi_{**}(\sigma^j, \sigma^j). \end{aligned}$$

Since the element $y \in \mathscr{M}$ was arbitrary, this procedure provides us with continuous mappings $b : \mathscr{N} \to \mathbb{R}^d$ and $\sigma : \mathscr{N} \to \ell^2(\mathbb{R}^d)$ which are the unique solutions of the Eqs. 3.12 and 3.13.

(ii) $\Rightarrow$ (i): Let $y \in \mathscr{M}$ be arbitrary, and let $\varphi : V \to W \cap \mathscr{N}$ be a local parametrization around $x := \psi^{-1}(y) \in \mathscr{N}$. By [5, Lemma 3.31] there exists an open neighborhood $U \subset H$ of y such that $\phi := \psi \circ \varphi : V \to U \cap \mathscr{M}$ is a local parametrization around y. Since $\mathscr{N}$ is locally invariant for the SDE (3.11), by Proposition 3.11 there are continuous mappings $\ell : V \to \mathbb{R}^m$ and $a : V \to \ell^2(\mathbb{R}^m)$ which are the unique solutions of the equations

$$\begin{aligned} \sigma^j|_{W\cap\mathscr{N}} &= \varphi_* a^j, \quad j \in \mathbb{N}, \\ b|_{W\cap\mathscr{N}} &= \varphi_*\ell + \frac{1}{2}\sum_{j=1}^{\infty}\varphi_{**}(a^j, a^j). \end{aligned}$$

By [5, Lemma 3.35] we obtain

$$A^j|_{U\cap\mathscr{M}} = \psi_*\sigma^j|_{W\cap\mathscr{N}} = \psi_*\varphi_* a^j = \phi_* a^j, \quad j \in \mathbb{N}$$

as well as

$$\begin{aligned} L|_{U\cap\mathscr{M}} &= \psi_* b|_{W\cap\mathscr{N}} + \frac{1}{2}\sum_{j=1}^{\infty}\psi_{**}(\sigma^j|_{W\cap\mathscr{N}}, \sigma^j|_{W\cap\mathscr{N}}) \\ &= \psi_*\varphi_*\ell + \frac{1}{2}\sum_{j=1}^{\infty}\big(\psi_{**}(\varphi_* a^j, \varphi_* a^j) + \psi_*\varphi_{**}(a^j, a^j)\big) \\ &= \phi_*\ell + \frac{1}{2}\sum_{j=1}^{\infty}\phi_{**}(a^j, a^j). \end{aligned}$$

Therefore, by Proposition 3.11 the submanifold $\mathscr{M}$ is locally invariant for the SPDE (1.1). □

For the next result, recall that the submanifold $\mathscr{M}$ has one chart if $\mathscr{N}$ has one chart; see [5, Lemma 3.33].

Proposition 3.13 *If the submanifold $\mathscr{M}$ is locally invariant for the SPDE (1.1) and the submanifold $\mathscr{N}$ has one chart with a global parametrization $\varphi : V \to \mathscr{N}$, then for continuous mappings $\ell : V \to \mathbb{R}^m$ and $a : V \to \ell^2(\mathbb{R}^m)$ the following statements are equivalent:*

(i) *$\ell : V \to \mathbb{R}^m$ and $a : V \to \ell^2(\mathbb{R}^m)$ are the unique solutions of the Eqs. 3.5 and 3.6.*
(ii) *$\ell : V \to \mathbb{R}^m$ and $a : V \to \ell^2(\mathbb{R}^m)$ are the unique solutions of the equations*

$$\sigma^j = \varphi_* a^j, \quad j \in \mathbb{N}, \tag{3.14}$$

$$b = \varphi_* \ell + \frac{1}{2} \sum_{j=1}^{\infty} \varphi_{**}(a^j, a^j), \tag{3.15}$$

where the continuous mappings $b : \mathscr{N} \to \mathbb{R}^d$ and $\sigma : \mathscr{N} \to \ell^2(\mathbb{R}^d)$ are the unique solutions of the Eqs. 3.12 and 3.13.

If any of the previous two conditions is fulfilled and the open set V is globally invariant for the $\mathbb{R}^m$-valued SDE

$$\begin{cases} d\Xi_t = \ell(\Xi_t)dt + a(\Xi_t)dW_t \\ \Xi_0 = \xi_0, \end{cases}$$

then the submanifold $\mathscr{M}$ is globally invariant for the SPDE (1.1), and the submanifold $\mathscr{N}$ is globally invariant for the SPDE (3.11).

Proof By [5, Lemma 3.33] the submanifold $\mathscr{M}$ has one chart with global parametrization $\phi := \psi \circ \varphi : V \to \mathscr{M}$.
(i) $\Rightarrow$ (ii): Taking into account [5, Lemma 3.35], by (3.12) and (3.5) we obtain

$$\sigma^j = \psi_*^{-1} \psi_* \sigma^j = \psi_*^{-1} A^j|_{\mathscr{M}} = \psi_*^{-1} \phi_* a^j = \psi_*^{-1} \psi_* \varphi_* a^j = \varphi_* a^j, \quad j \in \mathbb{N},$$

and by (3.13) and (3.6) we obtain

$$\begin{aligned} b - \frac{1}{2} \sum_{j=1}^{\infty} \varphi_{**}(a^j, a^j) &= \psi_*^{-1} \psi_* \left(b - \frac{1}{2} \sum_{j=1}^{\infty} \varphi_{**}(a^j, a^j) \right) \\ &= \psi_*^{-1} \left(L|_{\mathscr{M}} - \frac{1}{2} \sum_{j=1}^{\infty} \big(\psi_{**}(\varphi_* a^j, \varphi_* a^j) + \psi_* \varphi_{**}(a^j, a^j) \big) \right) \\ &= \psi_*^{-1} \left(L|_{\mathscr{M}} - \frac{1}{2} \sum_{j=1}^{\infty} \phi_{**}(a^j, a^j) \right) = \psi_*^{-1} \phi_* \ell = \psi_*^{-1} \psi_* \varphi_* \ell = \varphi_* \ell. \end{aligned}$$

(ii) $\Rightarrow$ (i): Taking into account [5, Lemma 3.35], by (3.12) and (3.14) we obtain

$$A^j|_{\mathscr{M}} = \psi_* \sigma^j = \psi_* \varphi_* a^j = \phi_* a^j, \quad j \in \mathbb{N},$$

and by (3.13) and (3.15) we obtain

$$\begin{aligned} L|_{\mathscr{M}} &= \psi_* b + \frac{1}{2} \sum_{j=1}^{\infty} \psi_{**}(\sigma^j, \sigma^j) = \psi_* b + \frac{1}{2} \sum_{j=1}^{\infty} \psi_{**}(\varphi_* a^j, \varphi_* a^j) \\ &= \psi_* b + \frac{1}{2} \sum_{j=1}^{\infty} \big(\phi_{**}(a^j, a^j) - \psi_* \varphi_{**}(a^j, a^j) \big) \\ &= \psi_* \left(b - \frac{1}{2} \sum_{j=1}^{\infty} \varphi_{**}(a^j, a^j) \right) + \frac{1}{2} \sum_{j=1}^{\infty} \phi_{**}(a^j, a^j) \\ &= \psi_* \varphi_* \ell + \frac{1}{2} \sum_{j=1}^{\infty} \phi_{**}(a^j, a^j) = \phi_* \ell + \frac{1}{2} \sum_{j=1}^{\infty} \phi_{**}(a^j, a^j). \end{aligned}$$

The additional statement is a consequence of Proposition 3.5. □

Now, we approach the proofs of Theorem 3.4 and Proposition 3.5. Consider the $\mathbb{R}^m$-valued SDE

$$\begin{cases} dX_t = \ell(X_t)dt + a(X_t)dW_t \\ X_0 = x_0 \end{cases} \tag{3.16}$$

with continuous mappings $\ell : V \to \mathbb{R}^m$ and $a : V \to \ell^2(\mathbb{R}^m)$, where $V \subset \mathbb{R}^m$ is an open subset.

Lemma 3.14 *V is a C^∞-submanifold of $\mathbb{R}^m$, which is locally invariant for the SDE (3.16).*

Proof It is obvious that V is a C^∞-submanifold of $\mathbb{R}^m$. Let $x_0 \in V$ be arbitrary. Since V is open, there exists a compact, convex neighborhood $K \subset V$ of x_0. Let $P_K : \mathbb{R}^m \to K$ be the orthogonal projection on K. We consider the SDE

$$\begin{cases} d\bar{X}_t = \bar{\ell}(\bar{X}_t)dt + \bar{a}(\bar{X}_t)dW_t \\ \bar{X}_0 = x_0, \end{cases} \tag{3.17}$$

where the coefficients are given by

$$\bar{\ell} := \ell \circ P_K : \mathbb{R}^m \to \mathbb{R}^m,$$
$$\bar{a} := a \circ P_K : \mathbb{R}^m \to \ell^2(\mathbb{R}^m).$$

Note that $\bar{\ell}$ and $\bar{a}$ are continuous and bounded. Hence, by Remark 2.9 there exists a global weak solution $(\mathbb{B}, W, \bar{X})$ to the SDE (3.17) with $\bar{X}_0 = x_0$. Now, we define the positive stopping time $\tau > 0$ as

$$\tau := \inf\{t \in \mathbb{R}_+ : \bar{X}_t \notin K\}.$$

Setting $X := \bar{X}^\tau$, we have $X^\tau \in K \subset V$, and, since $\ell|_K = \bar{\ell}|_K$ and $a|_K = \bar{a}|_K$, the triplet $(\mathbb{B}, W, X)$ is a local weak solution to the SDE (3.16) with $X_0 = x_0$ and lifetime τ. □

Now, we are ready to provide the proof of Theorem 3.4.

Proof of Theorem 3.4 (i) ⇒ (ii): Let $y_0 \in \mathscr{M}$ be arbitrary. According to [5, Prop. 3.14] there exist a local parametrization $\phi : V \to U \cap \mathscr{M}$ around y_0 and a bounded linear operator $\psi \in L(H, \mathbb{R}^m)$ such that $\phi^{-1} = \psi|_{U \cap \mathscr{M}}$ and we have

$$D\psi(y)|_{T_y\mathscr{M}} = D\phi(x)^{-1} \quad \text{for all } y \in U \cap \mathscr{M},$$

where $x := \psi(y) \in V$. By [5, Prop. 3.21] we have $\phi \in C(V; G) \cap C^2(V; H)$. Now, let $y \in U \cap \mathscr{M}$ be arbitrary, and set $x := \psi(y) \in V$. Since the submanifold $\mathscr{M}$ is locally invariant for the SPDE (1.1), there exist a positive stopping time $\tau > 0$ and a local martingale solution Y to (1.1) with $Y_0 = y$ and lifetime τ such that $Y^\tau \in \mathscr{M}$ up to an evanescent set. Since U is an open subset of H and the sample paths of Y are continuous with respect to $\| \cdot \|_H$, we may assume that $Y^\tau \in U \cap \mathscr{M}$ up to an evanescent set. Now, we define the continuous $\mathbb{R}^m$-valued process $X := \psi(Y)$. Then we have $X^\tau \in V$, and since ψ is linear, the process X is a local weak solution to the SDE

$$\begin{cases} dX_t = (\psi_* L)(X_t)dt + (\psi_* A)(X_t)dW_t \\ X_0 = x \end{cases}$$

with lifetime τ. The sample paths of $Y^\tau = \phi(X^\tau)$ are continuous with respect to $\|\cdot\|_G$, because $\phi \in C(V;G)$. Since also $\phi \in C^2(V;H)$, by Itô's formula (see [14, Thm. 2.3.1]) we obtain that the process Y is a local martingale solution to the SPDE

$$\begin{cases} dY_t = \big((\phi_* \psi_* L)(Y_t) + \frac{1}{2}\sum_{j=1}^\infty \phi_{**}(\psi_* A^j, \psi_* A^j)(Y_t)\big)dt + (\phi_*\psi_* A)(Y_t)dW_t \\ Y_0 = y \end{cases}$$

with lifetime τ. On the other hand, the process Y is a local martingale solution to the original SPDE (1.1) with $Y_0 = y$ and lifetime τ. We set $\mathscr{M}_U := U \cap \mathscr{M}$. By [5, Lemmas 4.13 and 4.14] the mappings

$$\phi_*\psi_* A|_{\mathscr{M}_U} : (\mathscr{M}_U, \|\cdot\|_G) \to (\ell^2(H), \|\cdot\|_{\ell^2(H)}),$$

$$\phi_*\psi_* L|_{\mathscr{M}_U} + \frac{1}{2}\sum_{j=1}^\infty \phi_{**}(\psi_* A^j|_{\mathscr{M}_U}, \psi_* A^j|_{\mathscr{M}_U}) : (\mathscr{M}_U, \|\cdot\|_G) \to (H, \|\cdot\|_H)$$

are continuous. Therefore, and since the sample paths of Y^τ are continuous with respect to $\|\cdot\|_G$, we may apply [5, Lemma 4.18], which gives us

$$A^j|_{\mathscr{M}_U} = \phi_*\psi_* A^j|_{\mathscr{M}_U}, \quad j \in \mathbb{N},$$

$$L|_{\mathscr{M}_U} = \phi_*\psi_* L|_{\mathscr{M}_U} + \frac{1}{2}\sum_{j=1}^\infty \phi_{**}(\psi_* A^j|_{\mathscr{M}_U}, \psi_* A^j|_{\mathscr{M}_U}).$$

Therefore, by [5, Prop. 4.15] we deduce that

$$A^j|_{\mathscr{M}_U} \in \Gamma(T\mathscr{M}_U), \quad j \in \mathbb{N}.$$

Furthermore, using [5, Prop. 4.16] we obtain

$$L|_{\mathscr{M}_U} - \frac{1}{2}\sum_{j=1}^\infty \phi_{**}(\psi_* A^j|_{\mathscr{M}_U}, \psi_* A^j|_{\mathscr{M}_U})$$
$$= \phi_*\psi_*\Bigg(L|_{\mathscr{M}_U} - \frac{1}{2}\sum_{j=1}^\infty \phi_{**}(\psi_* A^j|_{\mathscr{M}_U}, \psi_* A^j|_{\mathscr{M}_U})\Bigg).$$

Therefore, by [5, Prop. 4.15] we deduce that

$$L|_{\mathscr{M}_U} - \frac{1}{2}\sum_{j=1}^\infty \phi_{**}(\phi_*^{-1} A^j|_{\mathscr{M}_U}, \phi_*^{-1} A^j|_{\mathscr{M}_U}) \in \Gamma(T\mathscr{M}_U).$$

Since the point $y_0 \in \mathscr{M}$ chosen at the beginning of this proof was arbitrary, we deduce (3.1) and (3.2).

(ii) $\Rightarrow$ (iii): Let $y_0 \in \mathscr{M}$ be arbitrary, and let $\phi : V \to U \cap \mathscr{M}$ be an arbitrary local parametrization around y_0. By [5, Prop. 3.21] we have $\phi \in C(V;G) \cap C^2(V;H)$. We set $x_0 := \phi^{-1}(y_0) \in V$. By [5, Lemma 4.13] the mappings

$$a := \phi_*^{-1} A|_{\mathscr{M}_U} : V \to \ell^2(\mathbb{R}^m)$$

$$\ell := \phi_*^{-1}\Bigg(L|_{\mathscr{M}_U} - \frac{1}{2}\sum_{j=1}^\infty \phi_{**}(a^j, a^j)\Bigg) : V \to \mathbb{R}^m$$

are continuous. Therefore, by Lemma 3.14 the open set V is locally invariant for the SDE (3.16). Hence, there exist a stopping time $\tau > 0$ and a local weak solution X to (3.16) with

$X_0 = x_0$ and lifetime τ such that $X^\tau \in V$ up to an evanescent set. We define the $\mathscr{M}$-valued process $Y := \phi(X)$. Then we have $Y^\tau \in U \cap \mathscr{M}$. Furthermore, since $\phi \in C(V; G)$, the sample paths of Y^τ are continuous with respect to $\| \cdot \|_G$. Taking into account [5, Lemma 4.13], the mapping

$$A|_{\mathscr{M}_U} = \phi_* \phi_*^{-1} A|_{\mathscr{M}_U} = \phi_* a : (\mathscr{M}_U, \| \cdot \|_H) \to (\ell^2(H), \| \cdot \|_{\ell^2(H)}) \tag{3.18}$$

is continuous. Furthermore, taking into account [5, Lemma 4.13] we have

$$L|_{\mathscr{M}_U} - \frac{1}{2} \sum_{j=1}^{\infty} \phi_{**}(a^j, a^j) = \phi_* \phi_*^{-1} \left(L|_{\mathscr{M}_U} - \frac{1}{2} \sum_{j=1}^{\infty} \phi_{**}(a^j, a^j) \right) = \phi_* \ell,$$

and hence, by [5, Lemma 4.13] the mapping

$$L|_{\mathscr{M}_U} = \phi_* \ell + \frac{1}{2} \sum_{j=1}^{\infty} \phi_{**}(a^j, a^j) : (\mathscr{M}_U, \| \cdot \|_H) \to (H, \| \cdot \|_H) \tag{3.19}$$

is continuous. Moreover, by Itô's formula (see [14, Thm. 2.3.1]) and relations (3.18), (3.19) we obtain that Y^τ is a local martingale solution to the SPDE

$$\begin{aligned} dY_t &= \left((\phi_* \ell)(Y_t) + \frac{1}{2} \sum_{j=1}^{\infty} \phi_{**}(a^j, a^j)(Y_t) \right) dt + (\phi_* a)(Y_t) dW_t \\ &= L(Y_t) dt + A(Y_t) dW_t, \end{aligned}$$

which is just the original SPDE (1.1), with $Y_0 = y_0$ and lifetime τ. This proves that $\mathscr{M}$ is locally invariant for the SPDE (1.1).
(iii) $\Rightarrow$ (i): This implication is obvious. □

Proof of Proposition 3.5 This follows from inspecting the proof of the implication (ii) $\Rightarrow$ (iii) from Theorem 3.4. □

Now, let H_0 be another separable Hilbert space such that (G, H_0, H) are continuously embedded, and suppose that $\mathscr{M}$ is a (G, H_0, H)-submanifold of class C^2.

Theorem 3.15 *Suppose that for each $j \in \mathbb{N}$ we have $A^j \in C(G; H_0)$ with an extension $A^j \in C^1(H_0; H)$, and that for each $y \in \mathscr{M}$ the series $\sum_{j=1}^{\infty} DA^j(y) A^j(y)$ converges in H. Then the following statements are equivalent:*

(i) *The submanifold $\mathscr{M}$ is locally invariant for the SPDE (1.1).*
(ii) *We have*

$$A^j|_{\mathscr{M}} \in \Gamma(T\mathscr{M}), \quad j \in \mathbb{N}, \tag{3.20}$$

$$L|_{\mathscr{M}} - \frac{1}{2} \sum_{j=1}^{\infty} DA^j \cdot A^j|_{\mathscr{M}} \in \Gamma(T\mathscr{M}). \tag{3.21}$$

(iii) *The mappings (3.3) and (3.4) are continuous, and for each $y_0 \in \mathscr{M}$ there exists a local martingale solution Y to the SPDE (1.1) with $Y_0 = y_0$ and lifetime τ such that $Y^\tau \in \mathscr{M}$ up to an evanescent set and the sample paths of Y^τ are continuous with respect to $\| \cdot \|_G$.*

Proof By the decomposition [5, Prop. 3.25, eqn. (3.2)] we have

$$[A^j|_{\mathscr{M}}, A^j|_{\mathscr{M}}]_{\mathscr{M}} = [DA^j \cdot A^j]_{\Gamma(T\mathscr{M})}, \quad j \in \mathbb{N}.$$

Hence, the result is a consequence of Theorem 3.4. □

In the next result we present sufficient conditions for local invariance under the assumption that the volatilities A^j, $j \in \mathbb{N}$ have a quasi-linear structure. Recall that for any $z \in \mathscr{M}$ the space $\Gamma_z(T\mathscr{M})$ denotes the space of all local vector fields on $\mathscr{M}$ around z; see [5, Def. 3.10].

Theorem 3.16 *We suppose that for each $j \in \mathbb{N}$ there exists a continuous mapping $\bar{A}^j : G \times G \to H_0$ such that*

$$A^j(y) = \bar{A}^j(y, y), \quad y \in G$$

having a continuous extension $\bar{A}^j : H_0 \times G \to H$ such that $\bar{A}^j_z := \bar{A}^j(\cdot, z)$ belongs to $L(H_0, H)$ for each $z \in G$. Furthermore, we assume that for each $y \in \mathscr{M}$ the series $\sum_{j=1}^{\infty} \bar{A}^j(A^j(y), y)$ converges in H, and that

$$\bar{A}^j_z|_{\mathscr{M}} \in \Gamma_z(T\mathscr{M}), \quad z \in \mathscr{M}, \quad j \in \mathbb{N}, \tag{3.22}$$

$$L|_{\mathscr{M}} - \frac{1}{2}\sum_{j=1}^{\infty} \bar{A}^j(A^j(\cdot), \cdot)|_{\mathscr{M}} \in \Gamma(T\mathscr{M}). \tag{3.23}$$

Then mappings (3.3) and (3.4) are continuous, and for each $y_0 \in \mathscr{M}$ there exists a local martingale solution Y to the SPDE (1.1) with $Y_0 = y_0$ and lifetime τ such that $Y^\tau \in \mathscr{M}$ up to an evanescent set and the sample paths of Y^τ are continuous with respect to $\| \cdot \|_G$. In particular, the submanifold $\mathscr{M}$ is locally invariant for the SPDE (1.1).

Proof Note that condition (3.22) implies (3.1). Furthermore, using the decomposition [5, Prop. 3.25, eqn. (3.4)] we obtain

$$[A^j|_{\mathscr{M}}, A^j|_{\mathscr{M}}]_{\mathscr{M}} = [\bar{A}^j(A^j(\cdot), \cdot)]_{\Gamma(T\mathscr{M})}, \quad j \in \mathbb{N},$$

and hence, condition (3.23) is equivalent to (3.2). Consequently, applying Theorem 3.4 completes the proof. □

Remark 3.17 *Suppose that conditions (3.22) and (3.23) from Theorem 3.16 are fulfilled such that $\bar{A}^j$ even has an extension $\bar{A}^j \in C^1(H_0 \times H_0; H)$ for each $j \in \mathbb{N}$. Then the submanifold $\mathscr{M}$ is locally invariant for the SPDE (1.1), and the mapping $A^j \in C(G; H_0)$ has an extension $A^j \in C^1(H_0; H)$ for each $j \in \mathbb{N}$. If for each $y \in \mathscr{M}$ the series $\sum_{j=1}^{\infty} DA^j(y)A^j(y)$ converges in H, then by Theorem 3.15 the invariance condition (3.21) is satisfied as well. The vector fields in (3.21) and (3.23) do not, in general, coincide. Using* [5, Prop. 3.25], *we can determine their difference by using local coordinates. Namely, if $\phi : V \to U \cap \mathscr{M}$ is a local parametrization, then by the decomposition* [5, Prop. 3.25, eqn. (3.6)] *we have*

$$\left(L - \frac{1}{2}\sum_{j=1}^{\infty} \bar{A}^j(A^j(\cdot), \cdot) \right)\Big|_{\mathscr{M}_U} - \left(L - \frac{1}{2}\sum_{j=1}^{\infty} DA^j \cdot A^j \right)\Big|_{\mathscr{M}_U} = \frac{1}{2}\sum_{j=1}^{\infty} \phi_*(D_2\bar{a}^j \cdot a^j),$$

where the notation is analogous to that in [5, Prop. 3.25].

We conclude this section by indicating a result analogous to Theorem 3.4 for deterministic PDEs of the kind

$$\begin{cases} dY_t = K(Y_t)dt \\ Y_0 = y_0 \end{cases} \tag{3.24}$$

with a continuous mapping $K : G \to H$. Here G and H may be Banach spaces, and $\mathscr{M}$ only needs to be a (G, H)-submanifold of class C^1. The proof of following result is similar to that of Theorem 3.4; indeed the arguments are even simpler.

Theorem 3.18 *The following statements are equivalent:*

(i) *The submanifold $\mathscr{M}$ is locally invariant for the PDE (3.24).*
(ii) *We have $K|_{\mathscr{M}} \in \Gamma(T\mathscr{M})$.*
(iii) *The mapping $K|_{\mathscr{M}} : (\mathscr{M}, \|\cdot\|_H) \to (H, \|\cdot\|_H)$ is continuous, and for each $y_0 \in \mathscr{M}$ there exists a local solution $Y : [0,T] \to G$ to the PDE (3.24) with $Y_0 = y_0$ for some deterministic time $T > 0$ such that $Y \in \mathscr{M}$ and Y is continuous with respect to $\|\cdot\|_G$.*

4 Quasi-Semilinear Stochastic Partial Differential Equations

In this section we investigate invariance of submanifolds for quasi-semilinear SPDEs. It is organized as follows: In Section 4.1 we treat the general situation, and in Section 4.2 we draw consequences for semilinear SPDEs.

4.1 The General Situation

Let (G, H) be separable Hilbert spaces with continuous embedding, and let $L : G \to H$ and $A : G \to \ell^2(H)$ be continuous mappings. Throughout this section, we assume that the following assumption is satisfied.

Assumption 4.1 (Quasi-semilinearity) *We suppose that the following conditions are fulfilled:*

(1) *G is a dense subspace of H.*
(2) *There exist a continuous mapping $\bar{L} : G \times H \to H$ and a continuous mapping $\alpha : H \to H$ such that*

$$L(y) = \bar{L}(y, y) + \alpha(y), \quad y \in G,$$

and for each $z \in H$ the mapping

$$\bar{L}_z := \bar{L}(\cdot, z) : G \to H$$

extends to a closed operator $\bar{L}_z : H \supset D(\bar{L}_z) \to H$.
(3) *There exist a continuous mapping $\bar{A} : G \times H \to \ell^2(H)$ and a continuous mapping $\sigma : H \to \ell^2(H)$ such that*

$$A(y) = \bar{A}(y, y) + \sigma(y), \quad y \in G,$$

and for each $z \in H$ and each $j \in \mathbb{N}$ the mapping

$$\bar{A}_z^j := \bar{A}^j(\cdot, z) : G \to H$$

extends to a closed operator $\bar{A}_z^j : H \supset D(\bar{A}_z^j) \to H$.
(4) *For each $z \in H$ we have*

$$G = D(\bar{L}_z) \cap \left(\bigcap_{j=1}^{\infty} D(\bar{A}_z^j) \right). \tag{4.1}$$

(5) *There is a dense subspace $H_0 \subset H$ such that for each $z \in H$ we have*

$$H_0 \subset D(\bar{L}_z^*) \cap \left(\bigcap_{j=1}^{\infty} D(\bar{A}_z^{j,*}) \right),$$

and for each $\zeta \in H_0$ *we have* $\bar{A}_z^* \zeta := (\bar{A}_z^{j,*} \zeta)_{j \in \mathbb{N}} \in \ell^2(H)$*, and the mappings*

$$H \to H, \quad z \mapsto \bar{L}_z^* \zeta, \tag{4.2}$$

$$H \to \ell^2(H), \quad z \mapsto \bar{A}_z^* \zeta \tag{4.3}$$

are continuous.

In view of condition (5), recall that for a densely defined operator $A : H \supset D(A) \to H$ the adjoint operator $A^* : H \supset D(A^*) \to H$ is defined on the subspace

$$D(A^*) := \{z \in H : \xi \mapsto \langle A\xi, z \rangle_H \text{ is continuous on } D(A)\}, \tag{4.4}$$

and that it is characterized by the property

$$\langle Ay, z \rangle_H = \langle y, A^* z \rangle_H \quad \text{for all } y \in D(A) \text{ and } z \in D(A^*). \tag{4.5}$$

Proposition 4.2 *[36, Thm. 13.12] Let* $A : H \supset D(A) \to H$ *be densely defined and closed. Then* A^* *is densely defined and we have* $A = A^{**}$*.*

If Assumption 4.1 is fulfilled, then we also call the SPDE (1.1) a quasi-semilinear SPDE.

Definition 4.3 *Let* $y_0 \in H$ *be arbitrary. A triplet* $(\mathbb{B}, W, Y)$ *is called a* local analytically weak martingale solution *to the SPDE (1.1) with* $Y_0 = y_0$ *if the following conditions are fulfilled:*

1. $\mathbb{B} = (\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$ *is a stochastic basis; that is, a filtered probability space satisfying the usual conditions.*
2. W *is a standard* $\mathbb{R}^\infty$*-Wiener process on the stochastic basis* $\mathbb{B}$*.*
3. Y *is an* H*-valued adapted, continuous process such that, for some strictly positive stopping time* $\tau > 0$*, for each* $\zeta \in H_0$ *we have* $\mathbb{P}$*-almost surely*

$$\begin{aligned} \int_0^{t \wedge \tau} \bigg(& \big| \langle \bar{L}_{Y_s}^* \zeta, Y_s \rangle_H + \langle \zeta, \alpha(Y_s) \rangle_H \big| \\ & + \big\| \langle \bar{A}_{Y_s}^* \zeta, Y_s \rangle_H + \langle \zeta, \sigma(Y_s) \rangle_H \big\|_{\ell^2(H)}^2 \bigg) ds < \infty, \quad t \in \mathbb{R}_+ \end{aligned} \tag{4.6}$$

and $\mathbb{P}$*-almost surely*

$$\begin{aligned} \langle \zeta, Y_{t \wedge \tau} \rangle_H = \langle \zeta, y_0 \rangle_H & + \int_0^{t \wedge \tau} \big(\langle \bar{L}_{Y_s}^* \zeta, Y_s \rangle_H + \langle \zeta, \alpha(Y_s) \rangle_H \big) ds \\ & + \int_0^{t \wedge \tau} \big(\langle \bar{A}_{Y_s}^* \zeta, Y_s \rangle_H + \langle \zeta, \sigma(Y_s) \rangle_H \big) dW_s, \quad t \in \mathbb{R}_+, \end{aligned} \tag{4.7}$$

where for each $y \in H$ *we agree on the notation*

$$\begin{aligned} \langle \bar{A}_y^* \zeta, y \rangle_H &:= \big(\langle \bar{A}_y^{j,*} \zeta, y \rangle_H \big)_{j \in \mathbb{N}} \in \ell^2(H), \\ \langle \zeta, \sigma(y) \rangle_H &:= \big(\langle \zeta, \sigma^j(y) \rangle_H \big)_{j \in \mathbb{N}} \in \ell^2(H). \end{aligned}$$

The stopping time τ *is also called the* lifetime *of* Y*.*

If we can choose $\tau = \infty$*, then* $(\mathbb{B}, W, Y)$ *is also called a* global analytically weak martingale solution *(or simply an* analytically weak martingale solution*) to the SPDE (1.1) with* $Y_0 = y_0$*.*

Remark 4.4 *Note that the integrands in (4.6) and (4.7) are continuous and adapted by virtue of the continuity of the mappings (4.2) and (4.3).*

Remark 4.5 *If there is no ambiguity, we will simply call Y a local analytically weak martingale solution or a global analytically weak martingale solution to the SPDE (1.1) with $Y_0 = y_0$.*

Let $\mathscr{M}$ be a finite dimensional C^2-submanifold of H.

Definition 4.6 *The submanifold $\mathscr{M}$ is called* weakly locally invariant *for the SPDE (1.1) if for each $y_0 \in \mathscr{M}$ there exists a local analytically weak martingale solution Y to the SPDE (1.1) with $Y_0 = y_0$ and lifetime $\tau > 0$ such that $Y^\tau \in \mathscr{M}$ up to an evanescent set.*

Definition 4.7 *The submanifold $\mathscr{M}$ is called* weakly globally invariant *(or simply* weakly invariant*) for the SPDE (1.1) if for each $y_0 \in \mathscr{M}$ there exists a global analytically weak martingale solution Y to the SPDE (1.1) with $Y_0 = y_0$ such that $Y \in \mathscr{M}$ up to an evanescent set.*

Remark 4.8 *If $\mathscr{M}$ is locally invariant (or globally invariant) for the SPDE (1.1), then $\mathscr{M}$ is also weakly locally invariant (or weakly globally invariant) for the SPDE (1.1).*

Theorem 4.9 *Suppose that Assumption 4.1 is fulfilled. Then the following statements are equivalent:*

(i) *The submanifold $\mathscr{M}$ is weakly locally invariant for the SPDE (1.1).*
(ii) *We have $\mathscr{M} \subset G$, the submanifold $\mathscr{M}$ is locally invariant for the SPDE (1.1), and the mappings*

$$L|_{\mathscr{M}} : (\mathscr{M}, \|\cdot\|_H) \to (H, \|\cdot\|_H), \tag{4.8}$$

$$A|_{\mathscr{M}} : (\mathscr{M}, \|\cdot\|_H) \to (\ell^2(H), \|\cdot\|_{\ell^2(H)}) \tag{4.9}$$

are continuous.

Proof (ii) $\Rightarrow$ (i): See Remark 4.8.
(i) $\Rightarrow$ (ii): Let $y_0 \in \mathscr{M}$ be arbitrary. Since H_0 is dense in H, by [5, Prop. 3.14] there exist a local parametrization $\phi : V \to U \cap \mathscr{M}$ around y_0 and a bounded linear operator $\psi \in L(H, \mathbb{R}^m)$ of the form $\psi = \langle \zeta, \cdot \rangle_H$ with $\zeta_1, \ldots, \zeta_m \in H_0$ such that we have $\phi^{-1} = \psi|_{U \cap \mathscr{M}}$. Now, let $y \in U \cap \mathscr{M}$ be arbitrary, and set $x := \psi(y) \in V$. Since the submanifold $\mathscr{M}$ is weakly locally invariant for the SPDE (1.1), there exist a positive stopping time $\tau > 0$ and a local analytically weak martingale solution Y to (1.1) with $Y_0 = y$ and lifetime τ such that $Y^\tau \in \mathscr{M}$ up to an evanescent set. Since U is an open subset of H and the sample paths of Y are continuous, we may assume that $Y^\tau \in U \cap \mathscr{M}$ up to an evanescent set. Now, we define the continuous $\mathbb{R}^m$-valued process $X := \psi(Y)$. Then we have $X^\tau \in V$, and since $\zeta_1, \ldots, \zeta_m \in H_0$, the process X is a local strong solution to the SDE

$$\begin{cases} dX_t = L_\zeta(X_t)dt + A_\zeta(X_t)dW_t \\ X_0 = x \end{cases}$$

with lifetime τ, where $L_\zeta : V \to \mathbb{R}^m$ and $A_\zeta : V \to \ell^2(\mathbb{R}^m)$ are given by

$$L_\zeta(z) := \langle \bar{L}^*_{\phi(z)}\zeta, \phi(z)\rangle_H + \langle \zeta, \alpha(\phi(z))\rangle_H, \tag{4.10}$$

$$A^j_\zeta(z) := \langle \bar{A}^{j,*}_{\phi(z)}\zeta, \phi(z)\rangle_H + \langle \zeta, \sigma^j(\phi(z))\rangle_H, \quad j \in \mathbb{N}. \tag{4.11}$$

Note that the mappings L_ζ and A_ζ are continuous by virtue of the continuity of the mappings (4.2) and (4.3). Since $\phi \in C^2(V; H)$, by Itô's formula (see [14, Thm. 2.3.1]) we obtain that the process Y is a local solution to the SPDE

$$\begin{cases} dY_t = \big((\phi_* L_\zeta)(Y_t) + \frac{1}{2}\sum_{j=1}^\infty \phi_{**}(A_\zeta^j, A_\zeta^j)(Y_t)\big)dt + (\phi_* A_\zeta)(Y_t)dW_t \\ Y_0 = y \end{cases} \tag{4.12}$$

with lifetime τ, where we recall the notation from [5, Def. 3.12]. Let $\xi \in H_0$ be arbitrary. Then we have

$$\langle \xi, Y_{t\wedge\tau}\rangle_H = \langle \xi, y\rangle_H + \int_0^{t\wedge\tau} \left\langle \xi, (\phi_* L_\zeta)(Y_s) + \frac{1}{2}\sum_{j=1}^\infty \phi_{**}(A_\zeta^j, A_\zeta^j)(Y_s)\right\rangle_H ds$$
$$+ \int_0^{t\wedge\tau} \langle \xi, (\phi_* A_\zeta)(Y_s)\rangle_H dW_s, \quad t \in \mathbb{R}_+$$

On the other hand, the process Y is a local analytically weak martingale solution to the original SPDE (1.1) with $Y_0 = y$ and lifetime τ. Therefore, we have

$$\langle \xi, Y_{t\wedge\tau}\rangle_H = \langle \xi, y\rangle_H + \int_0^{t\wedge\tau} \big(\langle \bar{L}_{Y_s}^* \xi, Y_s\rangle_H + \langle \xi, \alpha(Y_s)\rangle_H\big)ds$$
$$+ \int_0^{t\wedge\tau} \big(\langle \bar{A}_{Y_s}^* \xi, Y_s\rangle_H + \langle \xi, \sigma(Y_s)\rangle_H\big)dW_s, \quad t \in \mathbb{R}_+.$$

Thus, taking into account [5, Lemma 4.13] and the continuity of the mappings (4.2) and (4.3), we have

$$\langle \bar{L}_y^* \xi, y\rangle_H = \left\langle \xi, (\phi_* L_\zeta)(y) + \frac{1}{2}\sum_{j=1}^\infty \phi_{**}(A_\zeta^j, A_\zeta^j)(y) - \alpha(y)\right\rangle_H, \tag{4.13}$$

$$\langle (\bar{A}_y^j)^* \xi, y\rangle_H = \langle \xi, (\phi_* A_\zeta^j)(y) - \sigma^j(y)\rangle_H, \quad j \in \mathbb{N}. \tag{4.14}$$

Taking into account Proposition 4.2 and (4.4), we have

$$D(\bar{L}_y) = D(\bar{L}_y^{**}) = \{z \in H : \xi \mapsto \langle \bar{L}_y^* \xi, z\rangle_H \text{ is continuous on } D(\bar{L}_y^*)\}$$

as well as

$$D(\bar{A}_y^j) = D((\bar{A}_y^j)^{**}) = \{z \in H : \xi \mapsto \langle (\bar{A}_y^j)^* \xi, z\rangle_H \text{ is continuous on } D((\bar{A}_y^j)^*)\}$$

for all $j \in \mathbb{N}$. This proves $y \in D(\bar{L}_y)$ and $y \in D(\bar{A}_y^j)$ for all $j \in \mathbb{N}$. Taking into account (4.1), we deduce that $y \in G$. Consequently, we have $\mathscr{M} \subset G$. By (4.10) and (4.11) we obtain

$$L_\zeta(x) = \langle \zeta, \bar{L}_{\phi(x)}\phi(x)\rangle_H + \langle \zeta, \alpha(\phi(x))\rangle_H = \langle \zeta, L(\phi(x))\rangle_H,$$
$$A_\zeta^j(x) = \langle \zeta, \bar{A}_{\phi(x)}^j \phi(x)\rangle_H + \langle \zeta, \sigma^j(\phi(x))\rangle_H = \langle \zeta, A^j(\phi(x))\rangle_H, \quad j \in \mathbb{N}$$

for each $x \in V$. Furthermore, from (4.13) and (4.14) we obtain

$$\langle \xi, \bar{L}_y y\rangle_H = \left\langle \xi, (\phi_* L_\zeta)(y) + \frac{1}{2}\sum_{j=1}^\infty \phi_{**}(A_\zeta^j, A_\zeta^j)(y) - \alpha(y)\right\rangle_H,$$
$$\langle \xi, \bar{A}_y^j y\rangle_H = \langle \xi, (\phi_* A_\zeta^j)(y) - \sigma^j(y)\rangle_H, \quad j \in \mathbb{N}$$

for all $\xi \in H_0$ and all $y \in U \cap \mathscr{M}$. Since H_0 is dense in H, we obtain

$$L(y) = \bar{L}(y, y) + \alpha(y) = (\phi_* L_\zeta)(y) + \frac{1}{2}\sum_{j=1}^\infty \phi_{**}(A_\zeta^j, A_\zeta^j)(y),$$

$$A^j(y) = \bar{A}^j(y, y) + \sigma^j(y) = (\phi_* A_\zeta^j)(y), \quad j \in \mathbb{N}$$

for all $y \in U \cap \mathscr{M}$. Since $y_0 \in \mathscr{M}$ at the beginning of the proof was chosen arbitrary, by [5, Lemma 4.13] we deduce that the mappings (4.8) and (4.9) are continuous. Furthermore, by taking into account (4.12), we see that Y is local strong solution to the SPDE (1.1) with $Y_0 = y_0$, proving that $\mathscr{M}$ is locally invariant for the SPDE (1.1). □

4.2 Semilinear Stochastic Partial Differential Equations

In this section we present consequences of our previous findings for semilinear SPDEs of the form

$$\begin{cases} dY_t = (BY_t + \alpha(Y_t))dt + \sigma(Y_t)dW_t \\ Y_0 = y_0. \end{cases} \tag{4.15}$$

Such equations have been studied, for example, in [9, 18, 25, 31]. Here the state space H is a separable Hilbert space, and $B : H \supset D(B) \to H$ is a densely defined, closed operator. Moreover $\alpha : H \to H$ and $\sigma : H \to \ell^2(H)$ are continuous mappings. We endow $G := D(B)$ with the graph norm

$$\|y\|_G := \sqrt{\|y\|_H^2 + \|By\|_H^2}, \quad y \in G. \tag{4.16}$$

By [5, Prop A.7], the pair (G, H) consists of separable Hilbert spaces with continuous embedding.

Remark 4.10 *Note that the semilinear SPDE (4.15) is of the type (1.1) with $L = B + \alpha$ and $A = \sigma$. Furthermore, note that Assumption 4.1 is fulfilled with*

$$\bar{L}(y, z) = B(y) \quad \text{for all } y \in G \text{ and } z \in H,$$
$$\bar{A} = 0,$$

and $H_0 = D(B^*)$. The concept of a local martingale solution (or a global martingale solution) from Definition 2.5 is just the concept of a local strong solution (or a global strong solution) for the semilinear SPDE (4.15) in the sense of martingale solutions. Accordingly, the concept of a local analytically weak martingale solution (or a global analytically weak martingale solution) from Definition 4.3 is just the concept of a local weak solution (or a global weak solution) for the semilinear SPDE (4.15) in the sense of martingale solutions.

Remark 4.11 *If B generates a C_0-semigroup on H, then we can also consider mild solutions. However, this is not required for our upcoming results.*

Let $\mathscr{M}$ be a finite dimensional C^2-submanifold of H. Invariant manifolds of weak solutions to semilinear SPDEs have been studied, for example, in [13, 29]; see also [15] for the case of jump-diffusions and submanifolds with boundary.

Lemma 4.12 *The following statements are equivalent:*

(i) *$\mathscr{M}$ is a finite dimensional (G, H)-submanifold of class C^2*
(ii) *$\mathscr{M} \subset G$ and the restriction $B|_{\mathscr{M}} : (\mathscr{M}, \|\cdot\|_H) \to (H, \|\cdot\|_H)$ is continuous.*

Proof This is an immediate consequence of [5, Prop. 3.37]. □

Proposition 4.13 *For a finite dimensional C^2-submanifold $\mathscr{M}$ of H the following statements are equivalent:*

(i) *The submanifold $\mathscr{M}$ is weakly locally invariant for the semilinear SPDE (4.15).*
(ii) *$\mathscr{M}$ is a (G, H)-submanifold of class C^2, which is locally invariant for the semilinear SPDE (4.15).*

Proof (i) $\Rightarrow$ (ii): By Theorem 4.9 we have $\mathscr{M} \subset G$, the submanifold $\mathscr{M}$ is locally invariant for the semilinear SPDE (4.15), and the restriction $B|_{\mathscr{M}} : (\mathscr{M}, \| \cdot \|_H) \to (H, \| \cdot \|_H)$ is continuous. Moreover, by Lemma 4.12 the submanifold $\mathscr{M}$ is a (G, H)-submanifold of class C^2.
(ii) $\Rightarrow$ (i): This implication is obvious. $\square$

Theorem 4.14 *Let $\mathscr{M}$ be a finite dimensional C^2-submanifold of H. Then the following statements are equivalent:*

(i) *The submanifold $\mathscr{M}$ is weakly locally invariant for the semilinear SPDE (4.15).*
(ii) *$\mathscr{M}$ is a (G, H)-submanifold of class C^2, and we have*

$$\sigma^j|_{\mathscr{M}} \in \Gamma(T\mathscr{M}), \quad j \in \mathbb{N}, \tag{4.17}$$

$$[(B+\alpha)|_{\mathscr{M}}]_{\Gamma(T\mathscr{M})} - \frac{1}{2}\sum_{j=1}^{\infty}[\sigma^j|_{\mathscr{M}}, \sigma^j|_{\mathscr{M}}]_{\mathscr{M}} = [0]_{\Gamma(T\mathscr{M})}. \tag{4.18}$$

(iii) *$\mathscr{M}$ is a (G, H)-submanifold of class C^2, the mapping $B|_{\mathscr{M}} : (\mathscr{M}, \|\cdot\|_H) \to (H, \|\cdot\|_H)$ is continuous, and for each $y_0 \in \mathscr{M}$ there exists a local martingale solution Y to the SPDE (1.1) with $Y_0 = y_0$ and lifetime τ such that $Y^\tau \in \mathscr{M}$ up to an evanescent set and the sample paths of Y^τ are continuous with respect to the graph norm $\| \cdot \|_G$.*

Proof This is a consequence of Proposition 4.13 and Theorem 3.4. $\square$

Remark 4.15 *If we even have $\sigma^j \in C^1(H)$ for all $j \in \mathbb{N}$, and for each $y \in \mathscr{M}$ the series $\sum_{j=1}^{\infty} D\sigma^j(y)\sigma^j(y)$ converges in H, then conditions (i)–(iii) are equivalent to the following:*

(iv) *$\mathscr{M}$ is a (G, H)-submanifold of class C^2, and we have (4.17) as well as*

$$B|_{\mathscr{M}} + \alpha|_{\mathscr{M}} - \frac{1}{2}\sum_{j=1}^{\infty} D\sigma^j \cdot \sigma^j|_{\mathscr{M}} \in \Gamma(T\mathscr{M}).$$

This is a consequence of the decomposition [5, Prop. 3.25, eqn. (3.2)].

Remark 4.16 *Let $k \in \mathbb{N}$ and $l \in \mathbb{N}_0$ be arbitrary, let $\mathscr{M}$ be a C^k-submanifold of H and assume that $\sigma^j \in C^l(H)$ for all $j \in \mathbb{N}$. Then k is the degree of smoothness of the submanifold, and l is the degree of smoothness of the volatilities. In the literature, the following situations have been considered:*

(1) *In* [13] *it is assumed that $k = 2$ and $l = 1$.*
(2) *In* [29] *(which uses the support theorem from* [28]*) it is assumed that $k = 1$ and $l = 1$.*
(3) *Here, in Theorem 4.14 we assume that $k = 2$ and $l = 0$.*

Summing up these degrees of smoothness, we see that in our result we have also achieved $k + l = 2$.

5 Invariant Manifolds Generated by Orbit Maps

In this section we investigate invariance of submanifolds generated by orbit maps. It is organized as follows: In Section 5.1 we investigate the structure of the coefficients of the SPDE in case of invariance of such a submanifold, and in Section 5.2 we treat the structure of invariant submanifolds for SPDEs with such coefficients. In Section 5.3 we apply our findings to SPDEs in Hermite Sobolev spaces.

5.1 Coefficients given by Generators of Group Actions

Let (G, H_0, H) be separable Hilbert spaces with continuous embeddings. We consider the SPDE (1.1) with continuous mappings $L : G \to H$ and $A : G \to \ell^2(H)$. Let $d \in \mathbb{N}$ be a positive integer, and let $T = (T(t))_{t \in \mathbb{R}^d}$ be a multi-parameter C_0-group on H such that $T|_G$ is a multi-parameter C_0-group on G, and $T|_{H_0}$ is a multi-parameter C_0-group on H_0. We denote by $B = (B_1, \dots, B_d)$ the generator of T; see [5, App. A] for further details. We assume that $H_0 \subset D(B)$ and $G \subset D(B^2)$. Furthermore, we assume that $B_i|_{H_0} \in L(H_0, H)$ and $B_i|_G \in L(G, H_0)$ for each $i = 1, \dots, d$. Let $y_0 \in G$ be arbitrary, and denote by $\psi \in C^2(\mathbb{R}^d; H)$ the orbit map given by $\psi(t) := T(t)y_0$ for each $t \in \mathbb{R}^d$. Let $\mathscr{N}$ be an m-dimensional C^2-submanifold of $\mathbb{R}^d$ for some $m \leq d$, and let $\mathscr{M}$ be an m-dimensional (G, H_0, H)-submanifold of class C^2, which is induced by $(\psi, \mathscr{N})$; see [5, Def. 3.32]. Recall that this requires that $\psi|_{\mathscr{N}} : \mathscr{N} \to \psi(\mathscr{N})$ is a homeomorphism, and that ψ is a C^2-immersion on $\mathscr{N}$.

Remark 5.1 *For a multi-dimensional sequence $\sigma = (\sigma_1, \dots, \sigma_d) \in \ell^2(\mathbb{R}^d) \cong \ell^2(\mathbb{R})^{\times d}$ we denote by $\sigma\sigma^\top \in \mathbb{R}^{d \times d}$ the matrix with elements $(\sigma\sigma^\top)_{ik} := \langle \sigma_i, \sigma_k \rangle_{\ell^2(\mathbb{R})}$ for all $i, k = 1, \dots, d$. If there is an index $r \in \mathbb{N}$ such that $\sigma^j = 0$ for all $j > r$, then we may regard the sequence σ as a matrix $\sigma \in \mathbb{R}^{d \times r}$, and $\sigma\sigma^\top$ is just the usual matrix multiplication with the transpose matrix.*

Theorem 5.2 *The following statements are equivalent:*

(i) *The submanifold $\mathscr{M}$ is locally invariant for the SPDE (1.1).*
(ii) *The submanifold $\mathscr{N}$ is locally invariant for the $\mathbb{R}^d$-valued SDE*

$$\begin{cases} dX_t = \bar{b}(X_t)dt + \bar{\sigma}(X_t)dW_t \\ X_0 = x_0, \end{cases} \tag{5.1}$$

where the continuous mappings $\bar{\sigma} : \mathscr{N} \to \ell^2(\mathbb{R}^d)$ and $\bar{b} : \mathscr{N} \to \mathbb{R}^d$ are the unique solutions of the equations

$$L|_{\mathscr{M}} = \frac{1}{2} \sum_{i,j=1}^d (\bar{\sigma}\bar{\sigma}^\top)_{ij} \circ \psi^{-1}|_{\mathscr{M}} \, B_{ij}|_{\mathscr{M}} + \sum_{i=1}^d \bar{b}_i \circ \psi^{-1}|_{\mathscr{M}} \, B_i|_{\mathscr{M}}, \tag{5.2}$$

$$A^j|_{\mathscr{M}} = \sum_{i=1}^d \bar{\sigma}_i^j \circ \psi^{-1}|_{\mathscr{M}} \, B_i|_{\mathscr{M}}, \quad j \in \mathbb{N}. \tag{5.3}$$

Proof Let $y \in \mathscr{M}$ be arbitrary, and set $x := \psi^{-1}(y) \in \mathscr{N}$. By [5, Prop. A.11] for $j \in \mathbb{N}$ we have

$$(\psi_* \bar{\sigma}^j)(y) = D\psi(x)\bar{\sigma}^j(x) = \sum_{i=1}^d B_i \psi(x) \bar{\sigma}_i^j(x) = \sum_{i=1}^d \bar{\sigma}_i^j(\psi^{-1}(y)) B_i y$$

as well as

$$
\begin{aligned}
(\psi_* \bar{b})(y) + \frac{1}{2} \sum_{j=1}^{\infty} \psi_{**}(\bar{\sigma}^j, \bar{\sigma}^j)(y) &= D\psi(x)\bar{b}(x) + \frac{1}{2} \sum_{j=1}^{\infty} D^2\psi(x)(\bar{\sigma}^j(x), \bar{\sigma}^j(x)) \\
&= \sum_{i=1}^{d} B_i \psi(x) \bar{b}_i(x) + \frac{1}{2} \sum_{j=1}^{\infty} \sum_{i,k=1}^{d} B_{ik} \psi(x) \bar{\sigma}_i^j(x) \bar{\sigma}_k^j(x) \\
&= \sum_{i=1}^{d} \bar{b}_i(\psi^{-1}(y)) B_i y + \frac{1}{2} \sum_{i,k=1}^{d} \bar{\sigma}(\psi^{-1}(y)) \bar{\sigma}(\psi^{-1}(y))^{\top} B_{ik} y.
\end{aligned}
$$

Therefore, applying Theorem 3.12 concludes the proof. □

Proposition 5.3 *Suppose that the following conditions are fulfilled:*

(1) *The submanifold $\mathscr{M}$ is locally invariant for the SPDE (1.1).*
(2) *The submanifold $\mathscr{N}$ has one chart with a global parametrization $\varphi : V \to \mathscr{N}$.*
(3) *The open set V is globally invariant for the $\mathbb{R}^m$-valued SDE*

$$
\begin{cases} d\Xi_t = \ell(\Xi_t)dt + a(\Xi_t)dW_t \\ \Xi_0 = \xi_0, \end{cases} \tag{5.4}
$$

whose coefficients $a : V \to \ell^2(\mathbb{R}^m)$ and $\ell : V \to \mathbb{R}^m$ are the unique solutions of the equations

$$
\bar{\sigma}^j = \varphi_* a^j, \quad j \in \mathbb{N}, \tag{5.5}
$$

$$
\bar{b} = \varphi_* \ell + \frac{1}{2} \sum_{j=1}^{\infty} \varphi_{**}(a^j, a^j), \tag{5.6}
$$

where the continuous mappings $\bar{\sigma} : \mathscr{N} \to \ell^2(\mathbb{R}^d)$ and $\bar{b} : \mathscr{N} \to \mathbb{R}^d$ are the unique solutions of the Eqs. 5.2 and 5.3

Then the submanifold $\mathscr{M}$ is globally invariant for the SPDE (1.1), and the submanifold $\mathscr{N}$ is globally invariant for the SDE (5.1).

Proof This is a consequence of Proposition 3.13. □

Remark 5.4 *Examples of submanifolds $\mathscr{M}$ as in Theorem 5.2 are obtained from* [5, Ex. 3.38] *with $k = 2$ and choosing $G = D(B^2)$ as well as $H_0 = D(B)$. Moreover, regarding Proposition 5.3, recall that the submanifold $\mathscr{M}$ has one chart if $\mathscr{N}$ has one chart; see* [5, Lemma 3.33].

5.2 The Structure of Invariant Submanifolds

In the previous we have considered invariant submanifolds which are induced by $(\psi, \mathscr{N})$, and shown that the coefficients of the SPDE (1.1) must be of the form (5.2) and (5.3). In this section, we will show that for such coefficients an invariant submanifold must, subject to appropriate regularity conditions, necessarily be an induced submanifold.

Let $T = (T(t))_{t \in \mathbb{R}^d}$ be a multi-parameter C_0-group on H as in Section 5.1. Furthermore, let $\mathscr{M}$ be an m-dimensional (G, H_0, H)-submanifold of class C^2 for some $m \leq d$, which is locally invariant for the SPDE (1.1). Suppose that for each $j = 1, \ldots, m$ we have $A^j \in$

$C(G; H_0)$ with an extension $A^j \in C^1(H_0; H)$. Let $y_0 \in \mathscr{M}$ be arbitrary. By [5, Prop. 3.24] there exists a local parametrization $\phi : V \to U \cap \mathscr{M}$ around y_0 such that

$$\phi \in C(V; G) \cap C^1(V; H_0) \cap C^2(V; H).$$

We assume there exists a mapping $\Lambda : V \to \mathbb{R}^{m \times d}$ of class C^1 such that

$$A(y) = \Lambda(x) B(y), \quad y \in U \cap \mathscr{M}, \tag{5.7}$$

where $x := \phi^{-1}(y) \in V$, and where we use the notations $A = (A^1, \dots, A^m)$ and $B = (B_1, \dots, B_d)$. Then the volatilities $A^1, \dots, A^m$ are locally of the form (5.3). We assume that

$$\dim \operatorname{lin}\{A^1 y, \dots, A^m y\} = m \quad \text{for each } y \in U \cap \mathscr{M}.$$

By Theorem 3.4 we have $A^1, \dots, A^m \in \Gamma(T\mathscr{M})$, and hence

$$T_y \mathscr{M} = \operatorname{lin}\{A^1 y, \dots, A^m y\} \quad \text{for each } y \in U \cap \mathscr{M}. \tag{5.8}$$

Lemma 5.5 *There exists a mapping $\Gamma : V \to \mathbb{R}^{m \times m}$ of class C^1 such that*

$$\nabla \phi(x) = \Gamma(x) A \phi(x), \quad x \in V. \tag{5.9}$$

Proof Let $x \in V$ be arbitrary, and set $y := \phi(x) \in U \cap \mathscr{M}$. Noting (5.8), the two sets

$$\{\partial_1 \phi(x), \dots, \partial_m \phi(x)\} \quad \text{and} \quad \{A^1 \phi(x), \dots, A^m \phi(x)\}$$

are bases of $T_y \mathscr{M}$. Hence, there is a unique matrix $\Gamma(x) \in \mathbb{R}^{m \times m}$ such that $\nabla \phi(x) = \Gamma(x) A \phi(x)$. This gives us a mapping $\Gamma : V \to \mathbb{R}^{m \times m}$ satisfying (5.9). The mapping $\nabla \phi : V \to H$ is of class C^1 because $\phi \in C^2(V; H)$. Furthermore, the mapping $A\phi$ is of class C^1 because $\phi \in C^1(V; H_0)$ and $A^j \in C^1(H_0; H)$ for each $j = 1, \dots, m$. Consequently, the mapping Γ is of class C^1, which concludes the proof. □

Now, we consider the product $\Phi := \Gamma \cdot \Lambda : V \to \mathbb{R}^{m \times d}$, which is again of class C^1. Furthermore, we set $x_0 := \phi^{-1}(y_0) \in V$. Recall that $\psi \in C^2(\mathbb{R}^d; H)$ denotes the orbit map given by $\psi(t) := T(t) y_0$ for each $t \in \mathbb{R}^d$.

Theorem 5.6 *Suppose that Φ has a primitive and satisfies* $\operatorname{rk} \Phi(x_0) = m$. *Then there exist an m-dimensional C^2-submanifold $\mathscr{N}$ of $\mathbb{R}^d$ and an open neighborhood $U_0 \subset U$ of y_0 such that the submanifold $U_0 \cap \mathscr{M}$ is induced by $(\psi, \mathscr{N})$.*

Proof We may assume that the open set V is a connected neighborhood of x_0. By (5.7) and (5.9) the mapping $\phi \in C^2(V; H)$ is a $D(B)$-valued solution to the PDE

$$\begin{cases} \nabla \phi(x) = \Phi(x) B \phi(x), & x \in V, \\ \phi(x_0) = y_0. \end{cases}$$

By assumption the mapping Φ has a primitive $\varphi : V \to \mathbb{R}^d$. We may assume that $\varphi(x_0) = 0$. Thus, by [5, Prop. A.12] we obtain $\phi = \psi \circ \varphi$. Since $\nabla \varphi = \Phi$ and $\operatorname{rk} \Phi(x_0) = m$, the mapping φ is a C^2-immersion at x_0. Hence, by [5, Lemma 3.30] there exists an open neighborhood $V_0 \subset V$ of zero such that $\varphi|_{V_0} : V_0 \to \varphi(V_0)$ is a homeomorphism and $\varphi|_{V_0}$ is a C^2-immersion. Moreover, by [5, Lemma 3.31] the set $\mathscr{N} := \varphi(V_0)$ is an m-dimensional C^2-submanifold of $\mathbb{R}^d$. Since $\phi : V \to U \cap \mathscr{M}$ is a homeomorphism, there exists an open neighborhood $U_0 \subset U$ of y_0 such that $\phi(V_0) = U_0 \cap \mathscr{M}$, and hence $U_0 \cap \mathscr{M} = \psi(\mathscr{N})$. Note that $\psi|_{\mathscr{N}} : \mathscr{N} \to \psi(\mathscr{N})$ is a homeomorphism, because $\phi|_{V_0} : V_0 \to \psi(\mathscr{N})$ and

$\varphi|_{V_0} : V_0 \to \mathscr{N}$ are homeomorphisms. Furthermore, by the chain rule, for each $x \in \mathscr{N}$ we have

$$D\psi(x)|_{T_x\mathscr{N}} = D\phi(\xi)D\varphi(\xi)^{-1} \in L(T_x\mathscr{N}, H),$$

where $\xi := \varphi^{-1}(x) \in V_0$, showing that ψ is a C^2-immersion on $\mathscr{N}$. □

Remark 5.7 *We may assume that the open set V is a simply connected neighborhood of x_0. Then Φ has a primitive if and only if*

$$\frac{\partial \Phi_{ik}}{\partial x_j} = \frac{\partial \Phi_{jk}}{\partial x_i} \quad \text{for all } i, j = 1, \dots, m \text{ and } k = 1, \dots, d.$$

5.3 Invariant Submanifolds in Hermite Sobolev Spaces

In this section we will apply our findings from Section 5.1 in order to construct examples of invariant submanifolds in Hermite Sobolev spaces; see [5, App. B] for further details about Hermite Sobolev spaces. Let $p \in \mathbb{R}$ be arbitrary and set $G := \mathscr{S}_{p+1}(\mathbb{R}^d)$, $H_0 := \mathscr{S}_{p+\frac{1}{2}}(\mathbb{R}^d)$ and $H := \mathscr{S}_p(\mathbb{R}^d)$. Furthermore, let $\tau = (\tau_x)_{x\in\mathbb{R}^d}$ be the translation group. We recall from [34] that for every $q \in \mathbb{R}$ the space $\mathscr{S}_q(\mathbb{R}^d)$ is invariant under the translation group. Let $b \in \mathscr{S}_{-(p+1)}(\mathbb{R}^d; \mathbb{R}^d)$ and $\sigma \in \ell^2(\mathscr{S}_{-(p+1)}(\mathbb{R}^d; \mathbb{R}^d))$ be given, where for any $q \in \mathbb{R}$ we agree on the notation

$$\mathscr{S}_q(\mathbb{R}^d; \mathbb{R}^d) := \mathscr{S}_q(\mathbb{R}^d)^{\times d},$$

which, endowed with the norm

$$\|f\|_{q,d} := \left(\sum_{i=1}^d \|f_i\|_q^2 \right)^{1/2}, \quad f \in \mathscr{S}_q(\mathbb{R}^d; \mathbb{R}^d),$$

is also a separable Hilbert space. Furthermore, the norm on $\ell^2(\mathscr{S}_q(\mathbb{R}^d; \mathbb{R}^d))$ will be denoted by $\| \cdot \|_{q,\ell^2}$. We define the coefficients $L : G \to H$ and $A^j : G \to H_0$ for $j \in \mathbb{N}$ of the SPDE (1.1) as

$$L(y) := \frac{1}{2} \sum_{i,j=1}^d (\langle \sigma, y\rangle \langle \sigma, y\rangle^\top)_{ij} \partial_{ij}^2 y - \sum_{i=1}^d \langle b_i, y\rangle \partial_i y, \tag{5.10}$$

$$A^j(y) := - \sum_{i=1}^d \langle \sigma_i^j, y\rangle \partial_i y, \quad j \in \mathbb{N}, \tag{5.11}$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pair on $\mathscr{S}_{-(p+1)}(\mathbb{R}^d) \times \mathscr{S}_{p+1}(\mathbb{R}^d)$; see [5, Lemma B.3] and also [5, Rem. 3.4]. Furthermore $\langle \sigma, y\rangle \in \ell^2(\mathbb{R}^d)$ is given by $\langle \sigma, y\rangle := (\langle \sigma^j, y\rangle)_{j\in\mathbb{N}}$, where for $c \in \mathscr{S}_{-(p+1)}(\mathbb{R}^d; \mathbb{R}^d)$ we define $\langle c, y\rangle \in \mathbb{R}^d$ as $\langle c, y\rangle := (\langle c_i, y\rangle)_{i=1,\dots,d}$. Recalling the notation introduced in Remark 5.1, it is obvious that $L : G \to H$ is continuous. Furthermore, according to [5, Lemma 6.8] the sequence $A := (A^j)_{j\in\mathbb{N}}$ provides a continuous mapping $A : G \to \ell^2(H_0)$.

Remark 5.8 *Note that the mapping $A : G \to \ell^2(H_0)$ generally does not satisfy the smoothness assumption imposed in Theorem 3.15, where it is required that for every $j \in \mathbb{N}$ the mapping $A^j \in C(G; H_0)$ admits an extension $A^j \in C^1(H_0; H)$. Indeed, for this we would need that for all $i = 1, \dots, d$ and all $j \in \mathbb{N}$ the continuous linear functional $\langle \sigma_i^j, \cdot \rangle : G \to \mathbb{R}$*

admits a continuous extension $\langle \sigma^j_d, \cdot \rangle : H_0 \to \mathbb{R}$*, and this is only true if we make the stronger assumption* $\sigma \in \ell^2(\mathscr{S}_{-(p+\frac{1}{2})}(\mathbb{R}^d; \mathbb{R}^d))$.

Let $\Phi \in G$ be arbitrary, and denote by $\psi \in C^2(\mathbb{R}^d; H)$ the orbit map given by $\psi(x) = \tau_x \Phi$ for each $x \in \mathbb{R}^d$. Due to the results from [5, Sec. 3.2] we are in the mathematical setting of Section 5.1. In particular, by [5, Prop. 3.42] we have $H_0 \subset D(-\partial)$ and $G \subset D((-\partial)^2)$. Let $\mathscr{N}$ be an m-dimensional C^2-submanifold of $\mathbb{R}^d$, and let $\mathscr{M}$ be an m-dimensional (G, H_0, H)-submanifold of class C^2, which is induced by $(\psi, \mathscr{N})$. Recall that this requires that $\psi|_{\mathscr{N}} : \mathscr{N} \to \psi(\mathscr{N})$ is a homeomorphism, and that ψ is a C^2-immersion on $\mathscr{N}$.

Theorem 5.9 *The following statements are equivalent:*

(i) *The submanifold* $\mathscr{M}$ *is locally invariant for the SPDE (1.1).*
(ii) *The submanifold* $\mathscr{N}$ *is locally invariant for the* $\mathbb{R}^d$*-valued SDE*

$$\begin{cases} dX_t = \bar{b}(X_t)dt + \bar{\sigma}(X_t)dW_t \\ X_0 = x_0, \end{cases} \tag{5.12}$$

where the continuous mappings $\bar{\sigma} : \mathbb{R}^d \to \ell^2(\mathbb{R}^d)$ *and* $\bar{b} : \mathbb{R}^d \to \mathbb{R}^d$ *are defined as*

$$\bar{\sigma}^j := \langle \sigma^j, \psi(\cdot) \rangle, \quad j \in \mathbb{N}, \tag{5.13}$$

$$\bar{b} := \langle b, \psi(\cdot) \rangle. \tag{5.14}$$

Proof Noting the definitions (5.11) and (5.10), this is a consequence of Theorem 5.2. □

Proposition 5.10 *Suppose that the following conditions are fulfilled:*

(1) *The submanifold* $\mathscr{M}$ *is locally invariant for the SPDE (1.1).*
(2) *The submanifold* $\mathscr{N}$ *has one chart with a global parametrization* $\varphi : V \to \mathscr{N}$.
(3) *The open set* V *is globally invariant for the* $\mathbb{R}^m$*-valued SDE (5.4), whose coefficients* $a : V \to \ell^2(\mathbb{R}^m)$ *and* $\ell : V \to \mathbb{R}^m$ *are the unique solutions of the equations*

$$\bar{\sigma}^j|_{\mathscr{N}} = \varphi_* a^j, \quad j \in \mathbb{N},$$

$$\bar{b}|_{\mathscr{N}} = \varphi_* \ell + \frac{1}{2} \sum_{j=1}^{\infty} \varphi_{**}(a^j, a^j),$$

where the continuous mappings $\bar{\sigma} : \mathbb{R}^d \to \ell^2(\mathbb{R}^d)$ *and* $\bar{b} : \mathbb{R}^d \to \mathbb{R}^d$ *are given by (5.13) and (5.14)*

Then the submanifold $\mathscr{M}$ *is globally invariant for the SPDE (1.1), and the submanifold* $\mathscr{N}$ *is globally invariant for the SDE (5.12).*

Proof This is a consequence of Proposition 5.3. □

Now, we will construct some examples of induced submanifolds which are invariant for the SPDE (1.1) with coefficients given by (5.10) and (5.11). Recall that $b \in \mathscr{S}_{-(p+1)}(\mathbb{R}^d; \mathbb{R}^d)$ and $\sigma \in \ell^2(\mathscr{S}_{-(p+1)}(\mathbb{R}^d; \mathbb{R}^d))$, and that $\langle \cdot, \cdot \rangle$ denotes the dual pair on $\mathscr{S}_{-(p+1)}(\mathbb{R}^d) \times \mathscr{S}_{p+1}(\mathbb{R}^d)$. In each of the upcoming examples, we will impose conditions on the choice of p. Also recall that $\mathbb{R}^m \times \{0\} \subset \mathbb{R}^d$ denotes the subspace $\mathbb{R}^m \times \{0\} = \text{lin}\{e_1, \ldots, e_m\}$, where $e_1, \ldots, e_m \in \mathbb{R}^d$ are the first m unit vectors. The following examples of invariant submanifolds are consequences of Theorem 5.9, Proposition 5.10 and [5, Ex. 3.48] with $k = 2$. For the first example, recall that every finite signed measure μ on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ may be regarded as a distribution $\mu \in \mathscr{S}_p(\mathbb{R}^d)$ for each $p < -\frac{d}{4}$; see [5, Lemma B.13].

Example 5.11 (Distributions given by measures) *The collection of Dirac measures*

$$\mathscr{M} = \{\delta_x : x \in \mathbb{R}^d\}$$

is the prime example of an invariant submanifold. It is known that $\delta_x \in \mathscr{S}_p(\mathbb{R}^d)$ *if and only if* $p < -\frac{d}{4}$*; see* [35]. *We generalize the preceding result as follows. We choose* $p \in \mathbb{R}$ *such that* $p + 1 < -\frac{d}{4}$, *and let* $\Phi = \mu \in G$ *be a finite signed measure on* $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ *with compact support such that* $\mu(\mathbb{R}^d) \neq 0$. *Furthermore, setting* $\mathscr{N} := \mathbb{R}^m \times \{0\}$ *we assume that for all* $x \in \mathscr{N}$ *we have*

$$\langle b, \tau_x \mu \rangle \in \mathscr{N}, \tag{5.15}$$

$$\langle \sigma^j, \tau_x \mu \rangle \in \mathscr{N}, \quad j \in \mathbb{N}. \tag{5.16}$$

Then the set

$$\mathscr{M} := \psi(\mathscr{N}) = \{\tau_x \mu : x \in \mathscr{N}\}$$

is an m*-dimensional* (G, H_0, H)*-submanifold of class* C^2 *with one chart, which is globally invariant for the SPDE (1.1). The global invariance follows from Remark 2.9, because the coefficients* $a : \mathbb{R}^m \to \ell^2(\mathbb{R}^m)$ *and* $\ell : \mathbb{R}^m \to \mathbb{R}^m$ *are bounded by virtue of* [5, Lemma B.13].

For the next example, recall that every polynomial $f : \mathbb{R}^d \to \mathbb{R}$ in several variables with $\deg(f) = n$ for some $n \in \mathbb{N}_0$ may be regarded as a distribution $f \in \mathscr{S}_p(\mathbb{R}^d)$ for each $p < -\frac{d}{4} - \frac{n}{2}$; see [5, Lemma B.14].

Example 5.12 (Distributions given by polynomials) *We choose* $p \in \mathbb{R}$ *such that* $p + 1 < -\frac{d}{4} - \frac{n}{2}$ *for some* $n \in \mathbb{N}$ *such that* $m \leq n \leq d$, *and let* $\Phi = f \in G$ *be the polynomial* $f : \mathbb{R}^d \to \mathbb{R}$ *given by* $f(x) = x_1 \cdot \ldots \cdot x_n$. *Furthermore, setting* $\mathscr{N} := \mathbb{R}^m \times \{0\}$ *we assume that for all* $x \in \mathscr{N}$ *we have*

$$\langle b, \tau_x f \rangle \in \mathscr{N}, \tag{5.17}$$

$$\langle \sigma^j, \tau_x f \rangle \in \mathscr{N}, \quad j \in \mathbb{N}. \tag{5.18}$$

Then the set

$$\mathscr{M} := \psi(\mathscr{N}) = \{\tau_x f : x \in \mathscr{N}\}$$

is an m*-dimensional* (G, H_0, H)*-submanifold of class* C^2 *with one chart, which is locally invariant for the SPDE (1.1). If* $m = n = 1$, *which means that* $\mathscr{N} = \mathbb{R} \times \{0\}$ *and* $f(x) = x_1$, *then* $\mathscr{M}$ *is even globally invariant for the SPDE (1.1). Taking into account Remark 2.9, this follows from* [5, Lemma B.15], *which ensures that the coefficients* $a : \mathbb{R}^m \to \ell^2(\mathbb{R}^m)$ *and* $\ell : \mathbb{R}^m \to \mathbb{R}^m$ *satisfy the linear growth condition.*

For the next result, recall that $\mathscr{S}_p(\mathbb{R}^d) \subset C_0^1(\mathbb{R}^d)$ for each $p > \frac{d}{4} + \frac{1}{2}$. This is a consequence of the Sobolev embedding theorem for Hermite Sobolev spaces; see [5, Thm. B.19].

Example 5.13 (Distributions given by C^1-functions) *We choose* $p \in \mathbb{R}$ *such that* $p + 1 > \frac{d}{4} + \frac{1}{2}$, *and let* $\Phi = \varphi \in G$ *be arbitrary. Setting* $\mathscr{N} := \mathbb{R}^m \times \{0\}$, *we assume there are* $z_1, \ldots, z_m \in \mathbb{R}^d$ *such that the matrix* $(\partial_i \varphi(z_j))_{i,j=1,\ldots,m} \in \mathbb{R}^{m \times m}$ *is invertible, and we assume that for all* $x \in \mathscr{N}$ *we have*

$$\langle b, \tau_x \varphi \rangle \in \mathscr{N}, \tag{5.19}$$

$$\langle \sigma^j, \tau_x \varphi \rangle \in \mathscr{N}, \quad j \in \mathbb{N}. \tag{5.20}$$

Then the set

$$\mathscr{M} := \psi(\mathscr{N}) = \{\tau_x \varphi : x \in \mathscr{N}\}$$

is an m-dimensional (G, H_0, H)-submanifold of class C^2 with one chart, which is locally invariant for the SPDE (1.1). Note that the invertibility of the matrix $(\partial_i \varphi(z_j))_{i,j=1,\dots,m}$ is required in order to ensure that ψ is an immersion on $\mathscr{N}$; see [5, Prop. 3.47]. *If $b_i \in L^2(\mathbb{R}^d)$ for $i = 1, \dots, d$ and $\sigma_i^j \in L^2(\mathbb{R}^d)$ for $i = 1, \dots, d$ and $j \in \mathbb{N}$, then $\mathscr{M}$ is even globally invariant for the SPDE (1.1). This follows from Remark 2.9, because, recalling that $L^2(\mathbb{R}^d) = \mathscr{S}_0(\mathbb{R}^d)$, by* [5, Lemma B.16] *the coefficients $a : \mathbb{R}^m \to \ell^2(\mathbb{R}^m)$ and $\ell : \mathbb{R}^m \to \mathbb{R}^m$ are bounded.*

Remark 5.14 *Note that in each of the previous examples we have considered the submanifold $\mathscr{N} := \mathbb{R}^m \times \{0\}$, which ensures that in any case the assumptions from [5, Ex. 3.48] concerning $\mathscr{N}$ are fulfilled. Since the submanifold $\mathscr{N}$ is a linear space, in any case the respective conditions (5.15)–(5.16), (5.17)–(5.18) or (5.19)–(5.20) ensures that $\mathscr{N}$ is locally invariant for the SDE (5.12); see Corollary 3.9. Of course, we can also consider other choices of the submanifold $\mathscr{N}$ such that the assumptions from* [5, Ex. 3.48] *are fulfilled. In particular, noting Theorem 3.4, in the situation of Example 5.11 we can choose any m-dimensional C^2-submanifold $\mathscr{N}$ of $\mathbb{R}^d$ such that*

$$\bar{\sigma}^j|_{\mathscr{N}} \in \Gamma(T\mathscr{N}), \quad j \in \mathbb{N},$$
$$[\bar{b}|_{\mathscr{N}}]_{\Gamma(T\mathscr{N})} - \frac{1}{2}\sum_{j=1}^{\infty}[\bar{\sigma}^j|_{\mathscr{N}}, \bar{\sigma}^j|_{\mathscr{N}}]_{\mathscr{N}} = [0]_{\Gamma(T\mathscr{N})},$$

where the continuous mappings $\bar{\sigma} : \mathbb{R}^d \to \ell^2(\mathbb{R}^d)$ and $\bar{b} : \mathbb{R}^d \to \mathbb{R}^d$ are defined as

$$\bar{\sigma}^j(x) := \langle \sigma^j, \tau_x \mu \rangle, \quad j \in \mathbb{N},$$
$$\bar{b}(x) := \langle b, \tau_x \mu \rangle$$

for each $x \in \mathbb{R}^d$.

Remark 5.15 *Consider the particular situation $m = d$, $\mathscr{N} = \mathbb{R}^d$ and $\Phi = \delta_0$, which is covered by Example 5.11. Then, by* [5, Lemma B.12] *the invariant submanifold is given by*

$$\mathscr{M} = \{\delta_x : x \in \mathbb{R}^d\},$$

and the coefficients of the SDE (5.12) are simply given by $\bar{b} = b$ and $\bar{\sigma} = \sigma$.

Remark 5.16 *Note that the findings of this section are in accordance with* [32, Lemma 3.6], *where it was shown that solutions to the SPDE (1.1) with coefficients (5.10) and (5.11) can be realized locally as $Y_t = \tau_{X_t}\Phi$ with an $\mathbb{R}^d$-valued Itô process X.*

6 Interplay between SPDEs and Finite Dimensional SDEs

In this section we illustrate how our findings from the previous Section 5.3 can be used in order to study stochastic invariance for finite dimensional diffusions. Consider the $\mathbb{R}^d$-valued SDE

$$\begin{cases} dX_t = b(X_t)dt + \sigma(X_t)dW_t \\ X_0 = x_0 \end{cases} \tag{6.1}$$

with measurable mappings $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \ell^2(\mathbb{R}^d)$. We assume that for some $q > \frac{d}{4}$ we have $b \in \mathscr{S}_q(\mathbb{R}^d; \mathbb{R}^d)$ and $\sigma \in \ell^2(\mathscr{S}_q(\mathbb{R}^d; \mathbb{R}^d))$.

Remark 6.1 *Note that sufficient conditions for the assumption that the components of b and σ belong to $\mathscr{S}_q(\mathbb{R}^d)$ are provided by* [5, Prop. B.21 and Cor. B.22].

Lemma 6.2 *The mappings $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \ell^2(\mathbb{R}^d)$ are continuous and bounded.*

Proof By the Sobolev embedding theorem for Hermite Sobolev spaces ([5, Thm. B.19]) the mapping $b : \mathbb{R}^d \to \mathbb{R}^d$ is continuous and bounded, and for each $j \in \mathbb{N}$ the mapping $\sigma^j : \mathbb{R}^d \to \mathbb{R}^d$ is continuous and bounded. Let $x \in \mathbb{R}^d$ and $j \in \mathbb{N}$ be arbitrary. By [5, Thm. B.19] we have

$$\|\sigma^j(x)\| \leq C\|\sigma^j\|_{q,d}$$

with a universal constant $C > 0$. Therefore, by Lebesgue's dominated convergence theorem the claim follows. □

Consequently, by Remark 2.9 for each $x_0 \in \mathbb{R}^d$ there exists a global weak solution X to the SDE (6.1) with $X_0 = x_0$. Let $\mathscr{N}$ be an m-dimensional C^2-submanifold of $\mathbb{R}^d$ for some $m \leq d$. Taking into account Remark 5.15, our idea is to link invariance of the submanifold $\mathscr{N}$ for the SDE (6.1) with invariance of the submanifold $\mathscr{M}$ for the SPDE (1.1) in Hermite Sobolev spaces, where $\mathscr{M}$ is defined in (6.2) below. For this purpose, we set $p := -(q+1)$. Then we have $q = -(p+1)$ as well as $p + 1 < -\frac{d}{4}$, and hence, we can consider the SPDE (1.1) with coefficients (5.10) and (5.11) in the framework of the previous Section 5.3 with $G = \mathscr{S}_{-q}(\mathbb{R}^d)$, $H_0 = \mathscr{S}_{-(q+\frac{1}{2})}(\mathbb{R}^d)$, $H = \mathscr{S}_{-(q+1)}(\mathbb{R}^d)$ and $\Phi = \delta_0$. As pointed out in Remark 5.15, then the coefficients of the SDE (5.12) are simply given by $\bar{b} = b$ and $\bar{\sigma} = \sigma$, and hence, the SDE (6.1) from this section coincides with the SDE (5.12). By [5, Lemma B.12], the orbit map $\psi \in C^2(\mathbb{R}^d; H)$ is given by $\psi(x) = \delta_x$ for each $x \in \mathbb{R}^d$. Therefore, by [5, Ex. 3.48] with $k = 2$ the set

$$\mathscr{M} := \psi(\mathscr{N}) = \{\delta_x : x \in \mathscr{N}\} \tag{6.2}$$

is a d-dimensional (G, H_0, H)-submanifold of class C^2, which is induced by $(\psi, \mathscr{N})$. The following result shows how local invariance of the submanifold $\mathscr{N}$ for the SDE (6.1) is connected with local invariance of the submanifold $\mathscr{M}$ for the SPDE (1.1).

Theorem 6.3 *The following statements are equivalent:*

(i) *The submanifold $\mathscr{M}$ is locally invariant for the SPDE (1.1).*
(ii) *The submanifold $\mathscr{N}$ is locally invariant for the SDE (6.1).*

Proof Taking into account Remark 5.15, this is a consequence of Theorem 5.9. □

Proposition 6.4 *Suppose that the submanifold $\mathscr{N}$ is locally invariant for the SDE (6.1). Then the following statements are true:*

(1) *If the submanifold $\mathscr{N}$ has one chart with a global parametrization $\varphi : V \to \mathscr{N}$, and the open set V is globally invariant for the $\mathbb{R}^m$-valued SDE (5.4), whose coefficients $a : V \to \ell^2(\mathbb{R}^m)$ and $\ell : V \to \mathbb{R}^m$ are the unique solutions of the equations*

$$\sigma^j|_{\mathscr{N}} = \varphi_* a^j, \quad j \in \mathbb{N},$$

$$b|_{\mathscr{N}} = \varphi_* \ell + \frac{1}{2}\sum_{j=1}^{\infty} \varphi_{**}(a^j, a^j),$$

then the submanifold $\mathscr{M}$ is globally invariant for the SPDE (1.1), and the submanifold $\mathscr{N}$ is globally invariant for the SDE (6.1).

(2) *If the submanifold $\mathscr{N}$ is closed as a subset of $\mathbb{R}^d$, then it is globally invariant for the SDE (6.1).*

Proof The first statement is a consequence of Proposition 5.10. In the situation of the second statement, let $x_0 \in \mathscr{N}$ be arbitrary, and let X be a global weak solution to the SDE (6.1) with $X_0 = x_0$. We define the stopping time

$$\tau := \inf\{t \in \mathbb{R}_+ : X_t \notin \mathscr{N}\},$$

and, since $\mathscr{N}$ is closed as a subset of $\mathbb{R}^d$, arguing by contradiction we can show that $\mathbb{P}(\tau = \infty) = 1$; see, for example, the proof of [15, Thm. 2.8]. □

Consequently, when we are interested in proving local invariance of the submanifold $\mathscr{N}$ for the SDE (6.1), we can alternatively show local invariance of the submanifold $\mathscr{M}$ for the SPDE (1.1), which turns out to be simpler in certain situations. We illustrate this procedure in the upcoming two subsections, which are organized as follows: In Section 6.1 we treat the invariance of submanifolds for coefficients given by vector fields, and in Section 6.2 we investigate the invariance of submanifolds given by the zeros of smooth functions.

6.1 Coefficients given by Vector Fields

For the following results, consider the conditions

$$b|_{\mathscr{N}} \in \Gamma(T\mathscr{N}), \tag{6.3}$$

$$\sigma^j|_{\mathscr{N}} \in \Gamma(T\mathscr{N}), \quad j \in \mathbb{N}. \tag{6.4}$$

We are interested in finding an additional condition ensuring that $\mathscr{N}$ is locally invariant for the SDE (6.1). In the general framework of Section 3, such a condition is provided by Proposition 3.8. In the present situation, we will establish another equivalent condition by using the connection to the SPDE (1.1). For each $j \in \mathbb{N}$ we define $\bar{A}^j : H_0 \times G \to H$ as

$$\bar{A}^j(y, z) := -\sum_{i=1}^d \langle \sigma_i^j, z \rangle \partial_i y, \quad (y, z) \in H_0 \times G.$$

Then we have

$$A^j(y) = \bar{A}^j(y, y) \quad \text{for all } y \in G \text{ and } j \in \mathbb{N}.$$

Concerning the notation used in Eqs. 6.5 and 6.6 below, we refer to Definition 3.2. By virtue of [5, Lemma 6.8] the series $\sum_{j=1}^\infty \bar{A}^j(A^j(y), y)$, which appears in (6.6), is well-defined for each $y \in G$.

Theorem 6.5 *Suppose that conditions (6.3) and (6.4) are fulfilled. Then the following statements are equivalent:*

(i) *$\mathscr{N}$ is locally invariant for the SDE (6.1).*

(ii) *We have*

$$\sum_{j=1}^\infty [\sigma^j|_{\mathscr{N}}, \sigma^j|_{\mathscr{N}}]_{\mathscr{N}} = [0]_{\Gamma(T\mathscr{N})}. \tag{6.5}$$

(iii) *We have*

$$\sum_{j=1}^{\infty} \big([A^j|_{\mathscr{M}}, A^j|_{\mathscr{M}}]_{\mathscr{M}} - [\bar{A}^j(A^j(\cdot), \cdot)|_{\mathscr{M}}]_{\Gamma(T\mathscr{M})} \big) = [0]_{\Gamma(T\mathscr{M})}. \tag{6.6}$$

Proof (i) ⇔ (ii): This equivalence is a consequence of Proposition 3.8.
(i) ⇔ (iii): By [5, Lemmas 7.5 and 3.36] we have

$$A^j|_{\mathscr{M}} \in \Gamma(T\mathscr{M}), \quad j \in \mathbb{N},$$
$$L|_{\mathscr{M}} - \frac{1}{2} \sum_{j=1}^{\infty} \bar{A}^j(A^j(\cdot), \cdot)|_{\mathscr{M}} \in \Gamma(T\mathscr{M}).$$

The latter relation shows that

$$[L|_{\mathscr{M}}]_{\Gamma(T\mathscr{M})} - \frac{1}{2} \sum_{j=1}^{\infty} [A^j|_{\mathscr{M}}, A^j|_{\mathscr{M}}]_{\mathscr{M}}$$
$$= \frac{1}{2} \sum_{j=1}^{\infty} \big([A^j|_{\mathscr{M}}, A^j|_{\mathscr{M}}]_{\mathscr{M}} - [\bar{A}^j(A^j(\cdot), \cdot)|_{\mathscr{M}}]_{\Gamma(T\mathscr{M})} \big),$$

and thus, the stated equivalence is a consequence of Theorem 6.3 and Theorem 3.4. □

If the submanifold $\mathscr{N}$ is affine, then it is locally invariant for the SDE (6.1) if and only if we have (6.3) and (6.4). This is a consequence of Corollary 3.9. More generally, we have the following result. Recall that $\Gamma^*(T\mathscr{N})$ denotes the space of all locally simultaneous vector fields on $\mathscr{N}$; see [5, Def. 3.11].

Proposition 6.6 *Suppose that*

$$b|_{\mathscr{N}} \in \Gamma(T\mathscr{N}), \tag{6.7}$$
$$\sigma^j|_{\mathscr{N}} \in \Gamma^*(T\mathscr{N}), \quad j \in \mathbb{N}. \tag{6.8}$$

Then the submanifold $\mathscr{N}$ is locally invariant for the SDE (6.1).

Proof By [5, Lemmas 7.5 and 3.36] we have

$$\bar{A}_z^j|_{\mathscr{M}} \in \Gamma_z(T\mathscr{M}), \quad z \in \mathscr{M}, \quad j \in \mathbb{N}, \tag{6.9}$$
$$L|_{\mathscr{M}} - \frac{1}{2} \sum_{j=1}^{\infty} \bar{A}^j(A^j(\cdot), \cdot)|_{\mathscr{M}} \in \Gamma(T\mathscr{M}), \tag{6.10}$$

where $\Gamma_z(T\mathscr{M})$ denotes the space of all local vector fields on $\mathscr{M}$ around z; see [5, Def. 3.10]. Using the decomposition [5, Prop. 3.25, eqn. (3.4)], we obtain

$$[A^j|_{\mathscr{M}}, A^j|_{\mathscr{M}}]_{\mathscr{M}} = [\bar{A}^j(A^j(\cdot), \cdot)]_{\Gamma(T\mathscr{M})}, \quad j \in \mathbb{N}.$$

Therefore, condition (6.6) is fulfilled, and hence, by Theorem 6.5 the submanifold $\mathscr{N}$ is locally invariant for the SDE (6.1). □

Remark 6.7 *Once we have established (6.9) and (6.10), alternatively we can also use Theorem 3.16 and Theorem 6.3 in order to conclude the proof of Proposition 6.6.*

Remark 6.8 *Consider the $\mathbb{R}^d$-valued Stratonovich SDE*

$$\begin{cases} dX_t = c(X_t)dt + \sigma(X_t) \circ dW_t \\ X_0 = x_0 \end{cases} \tag{6.11}$$

with a continuous mapping $c : \mathbb{R}^d \to \mathbb{R}^d$. It is well-known that the submanifold $\mathscr{N}$ is locally invariant for the Stratonovich SDE (6.11) if and only if

$$c|_{\mathscr{N}} \in \Gamma(T\mathscr{N}),$$
$$\sigma^j|_{\mathscr{N}} \in \Gamma(T\mathscr{N}), \quad j \in \mathbb{N},$$

see, for example [27, Cor. 1.ii]. *In Proposition 6.6 we present similar conditions, namely (6.7) and (6.8), which are sufficient for local invariance of the submanifold $\mathscr{N}$ for the Itô SDE (6.1).*

For the following results we will assume that even $\sigma \in \ell^2(\mathscr{S}_{q+\frac{1}{2}}(\mathbb{R}^d; \mathbb{R}^d))$. Note that $p + \frac{1}{2} = -(q + \frac{1}{2})$ and $q + \frac{1}{2} = -(p + \frac{1}{2})$, which shows that

$$H_0 = \mathscr{S}_{p+\frac{1}{2}}(\mathbb{R}^d) = \mathscr{S}_{-(q+\frac{1}{2})}(\mathbb{R}^d).$$

According to [5, Lemmas 7.10 and 7.11], for each $j \in \mathbb{N}$ the mappings $A^j : H_0 \to H$ and $\sigma^j : \mathbb{R}^d \to \mathbb{R}^d$ are of class C^1, and the series $\sum_{j=1}^\infty DA^j \cdot A^j$ and $\sum_{j=1}^\infty D\sigma^j \cdot \sigma^j$ are convergent.

Remark 6.9 *If $\sigma \in \ell^2(\mathscr{S}_{q+\frac{1}{2}}(\mathbb{R}^d; \mathbb{R}^d))$, then the Itô SDE (6.1) can equivalently be expressed by the Stratonovich SDE (6.11), where the continuous mapping $c : \mathbb{R}^d \to \mathbb{R}^d$ is given by*

$$c = b - \frac{1}{2} \sum_{j=1}^\infty D\sigma^j \cdot \sigma^j. \tag{6.12}$$

Proposition 6.10 *Suppose that $\sigma \in \ell^2(\mathscr{S}_{q+\frac{1}{2}}(\mathbb{R}^d; \mathbb{R}^d))$. Then we have the decomposition*

$$\sum_{j=1}^\infty DA^j \cdot A^j|_{\mathscr{M}} = \sum_{j=1}^\infty \bar{A}^j(A^j(\cdot), \cdot)|_{\mathscr{M}} + \psi_* \bigg(\sum_{j=1}^\infty D\sigma^j \cdot \sigma^j|_{\mathscr{N}} \bigg). \tag{6.13}$$

Proof Let $j \in \mathbb{N}$ be arbitrary. By the Leibniz rule we have

$$DA^j(y)z = \bar{A}^j(y, z) + \bar{A}^j(z, y), \quad y, z \in G,$$

and hence

$$DA^j(y)A^j(y) = \bar{A}^j(y, A^j(y)) + \bar{A}^j(A^j(y), y), \quad y \in G.$$

Now, let $y \in \mathscr{M}$ be arbitrary. Then we have $y = \delta_x$, where $x := \psi^{-1}(y) \in \mathscr{N}$. Therefore, by duality we obtain

$$\begin{aligned} \bar{A}^j(y, A^j(y)) &= -\sum_{i=1}^d \langle \sigma_i^j, A^j(y) \rangle \partial_i y = \sum_{i=1}^d \sum_{k=1}^d \langle \sigma_i^j, \langle \sigma_k^j, y \rangle \partial_k y \rangle \partial_i y \\ &= -\sum_{i=1}^d \sum_{k=1}^d \langle \partial_k \sigma_i^j, y \rangle \langle \sigma_k^j, y \rangle \partial_i y = -\sum_{i=1}^d \sum_{k=1}^d \partial_k \sigma_i^j(x) \sigma_k^j(x) \partial_i y \\ &= -\sum_{i=1}^d \langle e_i, D\sigma^j(x) \sigma^j(x) \rangle \partial_i y. \end{aligned}$$

Therefore, for all $y \in \mathscr{M}$ we obtain

$$\sum_{j=1}^{\infty} DA^j(y)A^j(y) = \sum_{j=1}^{\infty} \bar{A}^j(A^j(y), y) - \sum_{i=1}^{d} \left\langle e_i, \sum_{j=1}^{\infty} D\sigma^j(x)\sigma^j(x) \right\rangle \partial_i y.$$

Consequently, using [5, Prop. 3.43] completes the proof. □

Proposition 6.11 *Suppose that $\sigma \in \ell^2(\mathscr{S}_{q+\frac{1}{2}}(\mathbb{R}^d; \mathbb{R}^d))$. If conditions (6.3) and (6.4) are fulfilled, then the following statements are equivalent:*

(i) *The submanifold $\mathscr{N}$ is locally invariant for the SDE (6.1).*
(ii) *We have $\sum_{j=1}^{\infty} D\sigma^j \cdot \sigma^j|_{\mathscr{N}} \in \Gamma(T\mathscr{N})$.*
(iii) *We have $c|_{\mathscr{N}} \in \Gamma(T\mathscr{N})$, where the continuous mapping $c : \mathbb{R}^d \to \mathbb{R}^d$ is given by (6.12).*

If any of the previous conditions is fulfilled, then we have

$$L|_{\mathscr{M}} - \frac{1}{2} \sum_{j=1}^{\infty} \bar{A}^j(A^j(\cdot), \cdot)|_{\mathscr{M}} = \psi_* b|_{\mathscr{N}} \in \Gamma(T\mathscr{M}),$$

$$L|_{\mathscr{M}} - \frac{1}{2} \sum_{j=1}^{\infty} DA^j \cdot A^j|_{\mathscr{M}} = \psi_* c|_{\mathscr{N}} \in \Gamma(T\mathscr{M}),$$

and the difference is given by

$$\left(L|_{\mathscr{M}} - \frac{1}{2} \sum_{j=1}^{\infty} \bar{A}^j(A^j(\cdot), \cdot)|_{\mathscr{M}} \right) - \left(L|_{\mathscr{M}} - \frac{1}{2} \sum_{j=1}^{\infty} DA^j \cdot A^j|_{\mathscr{M}} \right)$$
$$= \frac{1}{2} \psi_* \left(\sum_{j=1}^{\infty} D\sigma^j \cdot \sigma^j|_{\mathscr{N}} \right) \in \Gamma(T\mathscr{M}).$$

Proof Noting [5, Lemma 7.10], the equivalences (i) ⇔ (ii) ⇔ (iii) are a consequence of Theorem 3.15. The additional statements follow from [5, Lemma 7.5] and the decomposition (6.13) from Proposition 6.10. □

Consequently, we see the following connection between the coefficients of the SDE (6.1) and the associated SPDE (1.1). The vector field in (3.23) corresponds to the drift b, and the vector field in (3.21) corresponds to the Stratonovich corrected drift c. Furthermore, we have computed the difference between these two vector fields, which in the general situation has been determined in Remark 3.17.

6.2 Submanifolds given by Zeros of Smooth Functions

Now, let $\mathscr{N}$ be a $(d-n)$-dimensional C^2-submanifold of $\mathbb{R}^d$ for some $n \in \mathbb{N}$ such that $n < d$. We assume there exist an open subset $O \subset \mathbb{R}^d$ and a mapping $f : \mathbb{R}^d \to \mathbb{R}^n$ such that

$$\mathscr{N} = \{x \in O : f(x) = 0\}. \tag{6.14}$$

Concerning the components of f we assume that $f_k \in \mathscr{S}_{q+1}(\mathbb{R}^d)$ for all $k = 1, \dots, n$. Recalling that $q > \frac{d}{4}$, by the Sobolev embedding theorem for Hermite Sobolev spaces (see

[5, Thm. B.19]) we have $f \in C^2(\mathbb{R}^d; \mathbb{R}^n)$. We also assume that $Df(x)\mathbb{R}^d = \mathbb{R}^n$ for all $x \in \mathscr{N}$.

Remark 6.12 *Note that the structure (6.14) of the submanifold does not mean a severe restriction. Indeed, it is well-known that for every $x_0 \in \mathscr{M}$ there are an open neighborhood $O \subset \mathbb{R}^d$ of x_0 and a mapping $f \in C^2(O; \mathbb{R}^n)$ such that $Df(x)\mathbb{R}^d = \mathbb{R}^n$ for all $x \in O$ and*

$$O \cap \mathscr{N} = \{x \in O : f(x) = 0\}.$$

For what follows, for a function $g \in C^1(\mathbb{R}^d; \mathbb{R})$ we denote by $\nabla g(x)$ the gradient at some point $x \in \mathbb{R}^d$, and for a function $g \in C^2(\mathbb{R}^d; \mathbb{R})$ we denote by $\mathbf{H}_g(x)$ the Hessian matrix at some point $x \in \mathbb{R}^d$.

Theorem 6.13 *The following statements are equivalent:*

(i) *The submanifold $\mathscr{N}$ is locally invariant for the SDE (6.1).*
(ii) *For all $k = 1, \dots, n$ and all $x \in \mathscr{N}$ we have*

$$\langle \sigma^j(x), \nabla f_k(x) \rangle = 0, \quad j \in \mathbb{N}, \tag{6.15}$$

$$\langle b(x), \nabla f_k(x) \rangle + \frac{1}{2} \operatorname{tr}\big(\sigma(x)\sigma(x)^\top \mathbf{H}_{f_k}(x)\big) = 0. \tag{6.16}$$

Before we provide the proof of Theorem 6.13, let us state some consequences.

Proposition 6.14 *Conditions (6.3) and (6.4) are satisfied if and only if for all $k = 1, \dots, n$ and all $x \in \mathscr{N}$ we have*

$$\langle \sigma^j(x), \nabla f_k(x) \rangle = 0, \quad j \in \mathbb{N},$$
$$\langle b(x), \nabla f_k(x) \rangle = 0,$$

and in this case, the following statements are equivalent:

(i) *$\mathscr{N}$ is locally invariant for the SDE (6.1).*
(ii) *For all $k = 1, \dots, n$ and all $x \in \mathscr{N}$ we have*

$$\operatorname{tr}\big(\sigma(x)\sigma(x)^\top \mathbf{H}_{f_k}(x)\big) = 0.$$

Proof The first equivalence follows from [5, Lemma 3.17], and in this case, the equivalence (i) $\Leftrightarrow$ (ii) is a consequence of Theorem 6.13. □

Corollary 6.15 (Unit sphere) *Let $d \geq 2$ be arbitrary, and consider the unit sphere $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$. Then the following statements are equivalent:*

(i) *$\mathbb{S}^{d-1}$ is globally invariant for the SDE (6.1).*
(ii) *$\mathbb{S}^{d-1}$ is locally invariant for the SDE (6.1).*
(iii) *For each $x \in \mathbb{S}^{d-1}$ we have*

$$\langle \sigma^j(x), x \rangle = 0, \quad j \in \mathbb{N}, \tag{6.17}$$

$$\langle b(x), x \rangle + \frac{1}{2} \operatorname{tr}\big(\sigma(x)\sigma(x)^\top\big) = 0. \tag{6.18}$$

Proof (i) $\Leftrightarrow$ (ii): Since $\mathbb{S}^{d-1}$ is a closed subset of $\mathbb{R}^d$, this equivalence follows from Proposition 6.4.

(ii) $\Leftrightarrow$ (iii): By [5, Lemma B.1] there exists a function $f \in \mathscr{S}_q(\mathbb{R}^d)$ such that

$$f(x) = \|x\|^2 - 1, \quad x \in O,$$

where $O \subset \mathbb{R}^d$ denotes the open set $O = \{x \in \mathbb{R}^d : \|x\| < 2\}$. Furthermore, the unit sphere $\mathbb{S}^{d-1}$ is a $(d-1)$-dimensional submanifold having the representation

$$\mathbb{S}^{d-1} = \{x \in O : f(x) = 0\}.$$

For each $x \in O$ we obtain

$$\nabla f(x) = 2x \quad \text{and} \quad \mathbf{H}_f(x) = 2\,\mathrm{Id},$$

which in particular shows that $Df(x)\mathbb{R}^d = \mathbb{R}$ for all $x \in \mathbb{S}^{d-1}$. Therefore, applying Theorem 6.13 completes the proof. $\square$

Example 6.16 (Stroock's representation of spherical Brownian motion) *Let $\mathbb{S}^{d-1}$ be the unit sphere in $\mathbb{R}^d$, and consider the $\mathbb{R}^d$-valued Stratonovich SDE*

$$\begin{cases} dX_t = (\mathrm{Id} - X_t X_t^\top) \circ dW_t \\ X_0 = x_0 \end{cases} \tag{6.19}$$

with an $\mathbb{R}^d$-valued Wiener process W; see [20, Example 3.3.2]. *With our notation, the volatilities $\sigma^1, \dots, \sigma^d : \mathbb{R}^d \to \mathbb{R}^d$ are given by*

$$\sigma^j(x) = (\delta_{ij} - x_i x_j)_{i=1,\dots,d} = e_j - x_j x, \quad j = 1, \dots, d.$$

Let us compute the corresponding Itô dynamics. For this purpose, let $x \in \mathbb{R}^d$ be arbitrary. Then we have

$$\partial_i \sigma^j(x) = -\delta_{ij} x - x_j e_i, \quad i, j = 1, \dots, d,$$

and hence, for each $j = 1, \dots, d$ we obtain

$$\begin{aligned} D\sigma^j(x)\sigma^j(x) &= \sum_{i=1}^d \sigma_{ij}(x)\partial_i \sigma^j(x) = -\sum_{i=1}^d (\delta_{ij} - x_i x_j)(\delta_{ij} x + x_j e_i) \\ &= -\sum_{i=1}^d (\delta_{ij} x + \delta_{ij} x_j e_i - x_i x_j \delta_{ij} x - x_i x_j^2 e_i) \\ &= -x - x_j e_j + x_j^2 x + x_j^2 \sum_{i=1}^d x_i e_i = -x - x_j e_j + 2x_j^2 x. \end{aligned}$$

Therefore, we have

$$\sum_{j=1}^d D\sigma^j(x)\sigma^j(x) = -dx - x + 2\|x\|^2 x = -(d + 1 - 2\|x\|^2)x.$$

In particular, for $x \in \mathbb{S}^{d-1}$ we obtain

$$\frac{1}{2} \sum_{j=1}^d D\sigma^j(x)\sigma^j(x) = -\frac{d-1}{2} x.$$

Therefore, we may alternatively consider the $\mathbb{R}^d$-valued Itô SDE

$$\begin{cases} dX_t = -\frac{d-1}{2} X_t dt + (\mathrm{Id} - X_t X_t^\top) dW_t \\ X_0 = x_0, \end{cases} \tag{6.20}$$

cf., for example, equation (2.1) in [26]. Using Corollary 6.15, we will show that the unit sphere $\mathbb{S}^{d-1}$ is globally invariant for the SDE (6.20). First, note that the SDE (6.20) is of the form (6.1). Let $O \subset \mathbb{R}^d$ be the open set $O = \{x \in \mathbb{R}^d : \|x\| < 2\}$. By virtue of [5, Lemma B.1] *there exist $b_i \in \mathscr{S}_q(\mathbb{R}^d)$, $i = 1, \dots, d$ such that*

$$b(x) = -\frac{d-1}{2} x, \quad x \in O,$$

where $b = (b_i)_{i=1,\dots,d}$, and there exist $\sigma_{ij} \in \mathscr{S}_q(\mathbb{R}^d)$, $i, j = 1, \dots, d$ such that

$$\sigma(x) = \mathrm{Id} - xx^\top, \quad x \in O,$$

where $\sigma = (\sigma_{ij})_{i,j=1,\dots,d}$. Hence, we may assume that the coefficients $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ of the SDE (6.1) are given by these mappings with components from $\mathscr{S}_q(\mathbb{R}^d)$. Now, let $x \in \mathbb{S}^{d-1}$ be arbitrary. Since the matrix $\sigma(x)$ is symmetric, taking into account the identification $\mathbb{R}^d \cong \mathbb{R}^{d \times 1}$ we have

$$\sigma(x)^\top x = \sigma(x)x = (\mathrm{Id} - xx^\top)x = x - xx^\top x = x(1 - x^\top x) = x(1 - \|x\|^2) = 0.$$

Furthermore, since $x^\top x = \|x\|^2 = 1$, we obtain

$$\begin{aligned} \sigma(x)\sigma(x)^\top &= \sigma(x)^2 = (\mathrm{Id} - xx^\top)^2 = \mathrm{Id} - 2xx^\top + xx^\top xx^\top \\ &= \mathrm{Id} - xx^\top = \sigma(x). \end{aligned}$$

Therefore, we have

$$\mathrm{tr}\big(\sigma(x)\sigma(x)^\top\big) = \mathrm{tr}\big(\mathrm{Id} - xx^\top\big) = d - \|x\|^2 = d - 1,$$

and hence

$$\langle b(x), x \rangle + \frac{1}{2}\mathrm{tr}\big(\sigma(x)\sigma(x)^\top\big) = -\frac{d-1}{2} + \frac{d-1}{2} = 0.$$

Consequently, by Corollary 6.15 the unit sphere $\mathbb{S}^{d-1}$ is globally invariant for the SDE (6.20).

Example 6.17 *Let $\mathbb{S}^1$ be the unit sphere in $\mathbb{R}^2$, and consider the $\mathbb{R}^2$-valued SDE (6.1) driven by a one-dimensional Wiener process W, where the coefficients $b, \sigma : \mathbb{R}^2 \to \mathbb{R}^2$ are given by*

$$b(x) := -\frac{1}{2}\lambda(x)^2 x, \tag{6.21}$$

$$\sigma(x) := \lambda(x)(-x_2, x_1)^\top \tag{6.22}$$

with an arbitrary continuous function $\lambda : \mathbb{R}^2 \to \mathbb{R}$. Then for each $x \in \mathbb{R}^2$ we have

$$\sigma(x)\sigma(x)^\top = \lambda(x)^2 \begin{pmatrix} x_2^2 & -x_1 x_2 \\ -x_1 x_2 & x_1^2 \end{pmatrix},$$

and hence

$$\mathrm{tr}\big(\sigma(x)\sigma(x)^\top\big) = \lambda(x)^2 \|x\|^2. \tag{6.23}$$

Therefore, the conditions (6.17) and (6.18) from Corollary 6.15 are fulfilled. As in the previous example, let $O \subset \mathbb{R}^2$ be the open set $O = \{x \in \mathbb{R}^2 : \|x\| < 2\}$. If there are functions $\tilde{b}_i, \tilde{\sigma}_i \in \mathscr{S}_q(\mathbb{R}^d)$, $i = 1, 2$ such that $b_i|_O = \tilde{b}_i|_O$ and $\sigma_i|_O = \tilde{\sigma}_i|_O$ for each $i = 1, 2$, then we can apply Corollary 6.15 and obtain that the unit sphere $\mathbb{S}^1$ is invariant for the SDE (6.1). Otherwise, we can use Theorem 3.4 as follows. Setting $\mathscr{N} := \mathbb{S}^1$, for each $x_0 \in \mathscr{N}$ a local parametrization of $\mathscr{N}$ around x_0 is given by

$$\phi : V \to U \cap \mathscr{N}, \quad \phi(t) = (\cos(t), \sin(t))^\top,$$

where $V \subset \mathbb{R}$ is a bounded, open interval. Note that

$$\phi'(t) = (-\sin(t), \cos(t))^\top, \quad t \in V,$$
$$\phi''(t) = -(\cos(t), \sin(t))^\top = -\phi(t), \quad t \in V.$$

Therefore, we have

$$T_x \mathscr{N} = \operatorname{lin}\{(-x_2, x_1)^\top\}, \quad x \in \mathscr{N},$$

and hence $\sigma|_{\mathscr{N}} \in \Gamma(T\mathscr{N})$. Now, let $x \in U \cap \mathscr{N}$ be arbitrary and set $t := \phi^{-1}(x) \in V$. Then we have

$$D\phi(t)^{-1}\sigma(x) = \lambda(x),$$

which implies

$$\frac{1}{2} D^2\phi(t)(D\phi(t)^{-1}\sigma(x), D\phi(t)^{-1}\sigma(x)) = -\frac{1}{2}\lambda(x)^2 x = b(x).$$

Therefore, we obtain

$$[b|_{\mathscr{N}}]_{\Gamma(T\mathscr{N})} - \frac{1}{2}[\sigma|_{\mathscr{N}}, \sigma|_{\mathscr{N}}]_{\mathscr{N}} = [0]_{\Gamma(T\mathscr{N})}.$$

Consequently, by Theorem 3.4 with $H = G = \mathbb{R}^2$ as well as $L = b$ and $A = \sigma$ the unit sphere $\mathbb{S}^1$ is invariant for the SDE (6.1).

Now, we construct a function λ such that none of the known results from the literature can be applied. Namely, we define

$$\lambda : \mathbb{R}^2 \to \mathbb{R}, \quad \lambda(x) := |\arg(x)|^{\frac{1}{4}},$$

where $\arg : \mathbb{R}^2 \to (-\pi, \pi]$ denotes the argument function. Then λ is continuous, which implies the continuity of b and σ. Moreover, the following statements are true:

(1) *Neither $b|_{\mathbb{S}^1}$ nor $\sigma|_{\mathbb{S}^1}$ are locally Lipschitz. This already excludes an application of most of the known results.*
(2) *The function $\sigma|_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{R}^2$ cannot be extended to a C^1-function on a neighborhood of $\mathbb{S}^1$. As a consequence, results with the Stratonovich correction term $\frac{1}{2}D\sigma \cdot \sigma$ (as, e.g., in [10]) cannot be applied.*
(3) *The function $\sigma\sigma^\top|_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{R}^{2\times 2}$ cannot be extended to a C^1-function on a neighborhood of $\mathbb{S}^1$. As a consequence, the results from [1, 2] cannot be applied.*

In order to prove these statements, let us define

$$\varphi : (-\pi/2, \pi/2) \to \mathbb{S}^1, \quad \varphi(t) := (\cos t, \sin t). \tag{6.24}$$

Due to the identity

$$\arg(x) = \arctan(x_2/x_1) \quad \textit{for each } x \in \mathbb{R}^2 \textit{ with } x_1 > 0 \tag{6.25}$$

we obtain

$$\lambda(\varphi(t)) = |\arg(\varphi(t))|^{\frac{1}{4}} = |t|^{\frac{1}{4}} \quad \textit{for all } t \in (-\pi/2, \pi/2).$$

Taking into account the definitions (6.21) and (6.22), we obtain

$$\|b(\varphi(t))\| = \frac{1}{2}|t|^{\frac{1}{2}} \quad \textit{for all } t \in (-\pi/2, \pi/2),$$
$$\|\sigma(\varphi(t))\| = |t|^{\frac{1}{4}} \quad \textit{for all } t \in (-\pi/2, \pi/2),$$

showing that neither $b|_{\mathbb{S}^1}$ nor $\sigma|_{\mathbb{S}^1}$ can be locally Lipschitz. For the proof of the second statement, suppose, on the contrary, there are an open neighborhood $O \subset \mathbb{R}^2$ with $\mathbb{S}^1 \subset O$ and an extension of $\sigma|_{\mathbb{S}^1}$ which is of class $C^1(O)$. For convenience of notation, let us denote this extension by $\sigma : O \to \mathbb{R}^2$. Then the norm

$$\rho : O \to \mathbb{R}, \quad \rho(x) := \|\sigma(x)\|^2 \tag{6.26}$$

is also of class C^1. Therefore, the function $\rho \circ \varphi : (-\pi/2, \pi/2) \to \mathbb{R}$ is of class C^1 as well, where φ was defined in (6.24). However, by (6.25) we obtain

$$\rho(\varphi(t)) = \lambda(\varphi(t))^2 = |\arg(\varphi(t))|^{\frac{1}{2}} = |t|^{\frac{1}{2}} \quad \textit{for all } t \in (-\pi/2, \pi/2), \tag{6.27}$$

which is a contradiction. For the proof of the third statement, suppose, on the contrary, there are an open neighborhood $O \subset \mathbb{R}^2$ with $\mathbb{S}^1 \subset O$ and an extension of $\sigma\sigma^\top|_{\mathbb{S}^1}$ which is of class $C^1(O)$. For convenience of notation, let us denote this extension by $\sigma\sigma^\top : O \to \mathbb{R}^{2\times 2}$. Then the trace

$$\ell : O \to \mathbb{R}, \quad \ell(x) := \operatorname{tr}\big(\sigma(x)\sigma(x)^\top\big)$$

is also of class C^1. Noting that $\ell = \rho$, where ρ as defined in (6.26), the identity (6.27) provides the desired contradiction.

Remark 6.18 *Suppose that the submanifold $\mathcal{N}$ is globally invariant for the SDE (6.1), and that its complement $\mathbb{R}^d \setminus \mathcal{N}$ consists of two connected components $\mathcal{N}_1$ and $\mathcal{N}_2$. Then the two sets $\mathcal{N}_1 \cup \mathcal{N}$ and $\mathcal{N}_2 \cup \mathcal{N}$ are also globally invariant for the SDE (6.1), and the submanifold $\mathcal{N}$ is an absorbing set in the sense that for each $y_0 \in \mathbb{R}^d$ we have $Y \in \mathcal{N}$ up to an evanescent set on $[\![\tau, \infty[\![$, where Y denotes any weak solution to the SDE (6.1) with $Y_0 = y_0$, and τ denotes the stopping time $\tau := \inf\{t \in \mathbb{R}_+ : Y_t \in \mathcal{N}\}$. Some examples for the submanifold $\mathcal{N}$ are as follows:*

- *Let $\mathcal{N}$ be a $(d-1)$-dimensional affine hyperplane. Then there are $\eta \in \mathbb{R}^d$ and $b \in \mathbb{R}$ such that*

  $$\mathcal{N} = \{x \in \mathbb{R}^d : \langle x, \eta \rangle = b\}.$$

  By Corollary 3.9 and Proposition 6.4 the affine hyperplane $\mathcal{N}$ is globally invariant for the SDE (6.1) if and only if conditions (6.3) and (6.4) are fulfilled. Its complement $\mathbb{R}^d \setminus \mathcal{N}$ consists of the two connected components

  $$\mathcal{N}_1 = \{x \in \mathbb{R}^d : \langle x, \eta \rangle < b\} \quad \textit{and} \quad \mathcal{N}_2 = \{x \in \mathbb{R}^d : \langle x, \eta \rangle > b\}.$$

- *Let $\mathscr{N} = \mathbb{S}^{d-1}$ be the unit sphere in $\mathbb{R}^d$. By Corollary 6.15 the unit sphere $\mathscr{N}$ is globally invariant for the SDE (6.1) if and and only if conditions (6.18) and (6.17) are fulfilled for each $x \in \mathscr{N}$. Its complement $\mathbb{R}^d \setminus \mathscr{N}$ consists of the two connected components*

$$\mathscr{N}_1 = \{x \in \mathbb{R}^d : \|x\| < 1\} \quad \text{and} \quad \mathscr{N}_2 = \{x \in \mathbb{R}^d : \|x\| > 1\}.$$

- *More generally, let $\mathscr{N}$ be a $(d-1)$-dimensional submanifold of $\mathbb{R}^d$ which is compact and connected. By the Jordan-Brouwer separation theorem its complement $\mathbb{R}^d \setminus \mathscr{N}$ consists of two connected components $\mathscr{N}_1$ and $\mathscr{N}_2$.*

Now, we approach the proof of Theorem 6.13. Recall that $\psi \in C^2(\mathbb{R}^d; H)$ denotes the orbit map $\psi = \xi_\Phi$ with $\Phi = \delta_0$. Thus, we have $\psi(x) = \delta_x$ for all $x \in \mathbb{R}^d$, and by [5, Prop. 3.44] the mapping ψ is a C^2-immersion, and $\psi : \mathbb{R}^d \to \psi(\mathbb{R}^d)$ is a homeomorphism. By [5, Ex. 3.48] the set

$$\mathscr{K} := \psi(O) = \{\delta_x : x \in O\} \tag{6.28}$$

is a d-dimensional (G, H_0, H)-submanifold of class C^2 with one chart. Furthermore, by [5, Ex. 3.48] the set

$$\mathscr{M} := \psi(\mathscr{N}) = \{\delta_x : x \in \mathscr{N}\} \tag{6.29}$$

is a $(d-n)$-dimensional (G, H_0, H)-submanifold of class C^2, which is induced by $(\psi, \mathscr{N})$, and obviously we have $\mathscr{M} \subset \mathscr{K}$.

Lemma 6.19 *The submanifold $\mathscr{K}$ is locally invariant for the SPDE (1.1).*

Proof Since the open subset O is locally invariant for the SDE (6.1), this is an immediate consequence of Theorem 6.3. □

Now, we are ready to provide the proof of Theorem 6.13. We define the operator $\mathscr{L} : C^2(\mathbb{R}^d) \to C(\mathbb{R}^d)$ as

$$(\mathscr{L}g)(x) := \langle b(x), \nabla g(x)\rangle + \frac{1}{2}\mathrm{tr}\big(\sigma(x)\sigma(x)^\top \mathbf{H}_g(x)\big), \quad x \in \mathbb{R}^d,$$

and for each $j \in \mathbb{N}$ we define the operator $\mathscr{A}^j : C^1(\mathbb{R}^d) \to C(\mathbb{R}^d)$ as

$$(\mathscr{A}^j g)(x) := \langle \sigma^j(x), \nabla g(x)\rangle, \quad x \in \mathbb{R}^d.$$

Proof of Theorem 6.13 (i) $\Rightarrow$ (ii): Let $x \in \mathscr{N}$ be arbitrary. There exist a global weak solution X to the SDE (3.16) with $X_0 = x$ and a positive stopping time $\tau > 0$ such that $X^\tau \in \mathscr{N}$ up to an evanescent set. Let $k = 1, \ldots, n$ be arbitrary. By Itô's formula (see [14, Thm. 2.3.1]) we have $\mathbb{P}$-almost surely

$$f_k(X_{t\wedge\tau}) = f_k(x) + \int_0^{t\wedge\tau} (\mathscr{L} f_k)(X_s)ds + \int_0^{t\wedge\tau} (\mathscr{A} f_k)(X_s)dW_s, \quad t \in \mathbb{R}_+.$$

where the continuous mapping $\mathscr{A} f_k : \mathbb{R}^d \to \ell^2(\mathbb{R}^d)$ is given by $\mathscr{A} f_k = (\mathscr{A}^j f_k)_{j\in\mathbb{N}}$. Noting that $f_k(X^\tau) = 0$, we deduce (6.15) and (6.16).
(ii) $\Rightarrow$ (i): Our strategy is to prove that the submanifold $\mathscr{M}$ defined in (6.29) is locally invariant for the SPDE (1.1) with coefficients (5.10) and (5.11), and then to apply Theorem 6.3 in order

to deduce that the submanifold $\mathscr{N}$ is locally invariant for the SDE (6.1). First, note that for all $y \in \mathscr{M}$ and all $k = 1, \ldots, n$ we have

$$\langle f_k, A^j(y) \rangle = 0, \quad j \in \mathbb{N}, \tag{6.30}$$

$$\langle f_k, L(y) \rangle = 0. \tag{6.31}$$

Indeed, let $y \in \mathscr{M}$ be arbitrary. Setting $x := \psi^{-1}(y) \in \mathscr{N}$, we have $y = \delta_x$. Thus, taking into account the definitions (5.10) and (5.11) of the coefficients, by duality for all $k = 1, \ldots, n$ we obtain

$$\begin{aligned} \langle f_k, A^j(y) \rangle &= -\sum_{i=1}^d \langle \sigma_{ij}, y \rangle \langle f_k, \partial_i y \rangle = \sum_{i=1}^d \langle \sigma_{ij}, y \rangle \langle \partial_i f_k, y \rangle \\ &= \sum_{i=1}^d \sigma_{ij}(x) \partial_i f_k(x) = \mathscr{A}^j f_k(x) = 0, \quad j \in \mathbb{N} \end{aligned}$$

as well as

$$\begin{aligned} \langle f_k, L(y) \rangle &= \frac{1}{2} \sum_{i,j=1}^d (\langle \sigma, y \rangle \langle \sigma, y \rangle^\top)_{ij} \langle f_k, \partial_{ij}^2 y \rangle - \sum_{i=1}^d \langle b_i, y \rangle \langle f_k, \partial_i y \rangle \\ &= \frac{1}{2} \sum_{i,j=1}^d (\sigma(x)\sigma(x)^\top)_{ij} \partial_{ij}^2 f_k(x) + \sum_{i=1}^d b_i(x) \partial_i f_k(x) = \mathscr{L} f_k(x) = 0. \end{aligned}$$

Now, let $y \in \mathscr{M}$ be arbitrary. Setting $x := \psi^{-1}(y) \in \mathscr{N}$, we have $y = \delta_x$. Let $\varphi : V \to W \cap \mathscr{N}$ be a local parametrization around $x := \psi^{-1}(y) \in \mathscr{N}$ with $W \subset O$. By [5, Lemma 3.31] there exists an open neighborhood $U \subset H$ of y such that $\phi := \psi \circ \varphi : V \to U \cap \mathscr{M}$ is a local parametrization around y. Hence, the mapping $\psi|_{W \cap \mathscr{N}} : W \cap \mathscr{N} \to U \cap \mathscr{M}$ is a homeomorphism, and noting (6.28) the mapping $\psi|_O : O \to \mathscr{K}$ is a homeomorphism. Since $W \subset O$, it follows that the mapping $\psi|_W : W \to U \cap \mathscr{K}$ is a local parametrization of $\mathscr{K}$ around y. By Lemma 6.19 the submanifold $\mathscr{K}$ is locally invariant for the SPDE (1.1). Therefore, by Proposition 3.11 there are continuous mappings $\bar{b} : W \to \mathbb{R}^d$ and $\bar{\sigma} : W \to \ell^2(\mathbb{R}^d)$ which are the unique solutions of the equations

$$\begin{aligned} A^j|_{U \cap \mathscr{K}} &= \psi_* \bar{\sigma}^j, \quad j \in \mathbb{N}, \\ L|_{U \cap \mathscr{K}} &= \psi_* \bar{b} + \frac{1}{2} \sum_{j=1}^\infty \psi_{**}(\bar{\sigma}^j, \bar{\sigma}^j). \end{aligned}$$

In particular, we have

$$A^j|_{\mathscr{M}} \in \Gamma(T\mathscr{K}_U), \quad j \in \mathbb{N}, \tag{6.32}$$

where $\mathscr{K}_U := U \cap \mathscr{K}$. From these equations, it follows that

$$A^j|_{U \cap \mathscr{M}} = \psi_* \bar{\sigma}^j|_{W \cap \mathscr{N}}, \quad j \in \mathbb{N}, \tag{6.33}$$

$$L|_{U \cap \mathscr{M}} = \psi_* \bar{b}|_{W \cap \mathscr{N}} + \frac{1}{2} \sum_{j=1}^\infty \psi_{**}(\bar{\sigma}^j|_{W \cap \mathscr{N}}, \bar{\sigma}^j|_{W \cap \mathscr{N}}). \tag{6.34}$$

Let $j \in \mathbb{N}$ be arbitrary. Noting (6.30) and (6.32), by [5, Lemma 7.21] we obtain

$$A^j|_{\mathscr{M}} \in \Gamma(T\mathscr{M}_U),$$

where $\mathscr{M}_U := U \cap \mathscr{M}$. Therefore, taking into account (6.33), by [5, Lemma 3.34] we deduce that

$$\bar{\sigma}^j|_{W \cap \mathscr{N}} \in \Gamma(T\mathscr{N}_W),$$

where $\mathscr{N}_W := W \cap \mathscr{N}$. Hence there is a continuous mappings $a : V \to \ell^2(\mathbb{R}^m)$ whose components are the unique solutions to the equations

$$\bar{\sigma}^j|_{W \cap \mathscr{N}} = \varphi_* a^j, \quad j \in \mathbb{N}. \tag{6.35}$$

Taking into account [5, Lemma 3.35], by (6.33) and (6.35) we obtain

$$A^j|_{U \cap \mathscr{M}} = \psi_* \bar{\sigma}^j|_{W \cap \mathscr{N}} = \psi_* \varphi_* a^j = \phi_* a^j, \quad j \in \mathbb{N}.$$

Furthermore, taking into account [5, Lemma 3.35], by (6.34) and (6.35) we have

$$\begin{aligned} L|_{U \cap \mathscr{M}} &= \psi_* \bar{b}|_{W \cap \mathscr{N}} + \frac{1}{2} \sum_{j=1}^{\infty} \psi_{**}(\varphi_* a^j, \varphi_* a^j) \\ &= \psi_* \bar{b}|_{W \cap \mathscr{N}} + \frac{1}{2} \sum_{j=1}^{\infty} \big(\phi_{**}(a^j, a^j) - \psi_* \varphi_{**}(a^j, a^j) \big) \\ &= \psi_* \bigg(\bar{b}|_{W \cap \mathscr{N}} - \frac{1}{2} \sum_{j=1}^{\infty} \varphi_{**}(a^j, a^j) \bigg) + \frac{1}{2} \sum_{j=1}^{\infty} \phi_{**}(a^j, a^j). \end{aligned} \tag{6.36}$$

Taking into account [5, Lemma 7.21], we have $\phi(V) \subset \bigcap_{k=1}^n \ker(\langle f_k, \cdot \rangle)$, and hence

$$(\phi_{**}(a^j, a^j))(U \cap \mathscr{M}) \subset \bigcap_{k=1}^n \ker(\langle f_k, \cdot \rangle) \quad \text{for all } j \in \mathbb{N}.$$

Thus, noting (6.31), by [5, Lemma 7.21] we obtain

$$L|_{U \cap \mathscr{M}} - \frac{1}{2} \sum_{j=1}^{\infty} \phi_{**}(a^j, a^j) \in \Gamma(T\mathscr{M}_U).$$

Therefore, by [5, Lemma 3.34] we deduce that

$$\bar{b}|_{W \cap \mathscr{N}} - \frac{1}{2} \sum_{j=1}^{\infty} \varphi_{**}(a^j, a^j) \in \Gamma(T\mathscr{N}_W).$$

Hence, there is a continuous mapping $\ell : V \to \mathbb{R}^m$ which is the unique solution to the equation

$$\bar{b}|_{W \cap \mathscr{N}} - \frac{1}{2} \sum_{j=1}^{\infty} \varphi_{**}(a^j, a^j) = \varphi_* \ell. \tag{6.37}$$

Therefore, using [5, Lemma 3.35], by (6.36) and (6.37) we obtain

$$L|_{U \cap \mathscr{M}} - \frac{1}{2} \sum_{j=1}^{\infty} \phi_{**}(a^j, a^j) = \psi_* \bigg(\bar{b}|_{W \cap \mathscr{N}} - \frac{1}{2} \sum_{j=1}^{\infty} \varphi_{**}(a^j, a^j) \bigg) = \psi_* \varphi_* \ell = \phi_* \ell.$$

Now, by Proposition 3.11 we deduce that the submanifold $\mathscr{M}$ is locally invariant for the SPDE (1.1). Consequently, by Theorem 6.3 it follows that the submanifold $\mathscr{N}$ is locally invariant for the SDE (6.1). $\square$

Remark 6.20 *As the proof of Theorem 6.13 reveals, the conditions $b \in \mathscr{S}_q(\mathbb{R}^d; \mathbb{R}^d)$ and $\sigma \in \ell^2(\mathscr{S}_q(\mathbb{R}^d; \mathbb{R}^d))$ are only required for the implication (ii) $\Rightarrow$ (i), whereas for the implication (i) $\Rightarrow$ (ii) we merely need that $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \ell^2(\mathbb{R}^d)$ are continuous.*

Acknowledgements We thank the reviewers for their comments and suggestions.

Funding Open Access funding enabled and organized by Projekt DEAL. Rajeev Bhaskaran gratefully acknowledges financial support from the Science and Engineering Research Board (SERB), India – vide grant CRG/2019/002594. Stefan Tappe gratefully acknowledges financial support from the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – project number 444121509, and from the Deutsche Mathematiker-Vereinigung (DMV, German Mathematical Society) – Fachgruppe Stochastik.

Data Availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflicts of interest The authors declare that they have no conflict of interest.

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