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Distribution-Path Dependent Nonlinear SPDEs with Application to Stochastic Transport Type Equations

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Abstract
By using a regularity approximation argument, the global existence and uniqueness are derived for a class of nonlinear SPDEs depending on both the whole history and the distribution under strong enough noise. As applications, the global existence and uniqueness are proved for distribution-path dependent stochastic transport type equations, which are arising from stochastic fluid mechanics with forces depending on the history and the environment. In particular, the distribution-path dependent stochastic Camassa-Holm equation with or without Coriolis effect has a unique global solution when the noise is strong enough, whereas for the deterministic model wave-breaking may occur. This indicates that the noise may prevent blow-up almost surely.

Keywords Distribution-Path Dependent Nonlinear SPDEs · Stochastic transport type equation · Stochastic Camassa-Holm type equation

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1 Introduction

To describe the evolutions of stochastic systems depending on the history and micro environment, distribution-path dependent SDEs of the following type

$$\mathrm{d}X(t) = b(t, X_t, \mathscr{L}_{X_t})\mathrm{d}t + \sigma(t, X_t, \mathscr{L}_{X_t})\mathrm{d}W(t), \quad X(0) = X_0 \in \mathbb{R}^d, \quad t \in [0, T] \tag{1.1}$$

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have been studied intensively investigated, see for instance [1–6] and references therein. However, the existing study in the literature does not cover distribution-path dependent nonlinear SPDEs containing a singular term which is not well-defined on the state space. The main purpose of this paper is to solve a class of such SPDEs including transport type fluid models.

Nowadays there is a vast amount of literature on stochastic fluid models under random perturbation, and we do not attempt to survey it here. Instead, we recommend readers to refer to the lecture notes [7, 8] and the monographs [9, 10]. On one hand, in the real world, it is natural that random perturbations may rely on both the sample path due to inertia and averaged stochastic interactions from the environment. On the other hand, almost nothing is known if the randomness in stochastic fluid models also depends on the distribution and path of unknown variables, i.e., distribution-path dependent stochastic fluid models. For such problems, the fundamental question of well-posedness (even merely the existence) of solutions remains open. Although both linear and nonlinear (distribution-path independent) stochastic transport type equations have been intensively investigated (see for example [11–18]), there has been no study on distribution-path dependent stochastic transport type equations.

To study distribution-path dependent stochastic fluid models, we may need extend (1.1) to infinite dimensional case, i.e., assuming that X takes value in a separable Hilbert space $\mathbb{H}$. If this is the case, a singular term, which is not well-defined on $\mathbb{H}$, may occur and the existing study in the literature does not cover this case. More precisely, we consider the case that (1.1) contains one more singular drift term B taking value in a larger separable Hilbert space $\mathbb{B}$ such that $\mathbb{H} \hookrightarrow \mathbb{B}$, i.e.,

$$\begin{cases} \mathrm{d}X(t) = \left\{B(t, X(t)) + b(t, X_t, \mathscr{L}_{X_t})\right\} \mathrm{d}t + \sigma(t, X_t, \mathscr{L}_{X_t})\mathrm{d}W(t), \quad t \in [0, T], \\ X(0) = X_0 \in \mathbb{H}. \end{cases} \tag{1.2}$$

Indeed, when we consider certain stochastic fluid models in Sobolev spaces $\mathbb{H} = H^s$, if $B(t, X(t))$ involves ∇X or some derivatives of X (see Examples 1.1 and 1.2), then $B(t, X(t))$ may not be expected to be in $\mathbb{H} = H^s$. Particularly, when $B(t, X(t)) = -(X(t) \cdot \nabla)X(t)$, (1.2) reduces to the following transport type equation

$$\begin{cases} \mathrm{d}X(t) = \left\{-(X(t) \cdot \nabla)X(t) + b(t, X_t, \mathscr{L}_{X_t})\right\} \mathrm{d}t + \sigma(t, X_t, \mathscr{L}_{X_t})\mathrm{d}W(t), \ t \in [0, T], \\ X(0) = X_0 \in H^s. \end{cases} \tag{1.3}$$

We refer to Sections 1.1 and 1.2 for the precise meaning of the notations and precise setting of (1.2) and (1.3), respectively. Before going further, we would like to explain that working with the abstract framework in (1.2) entails some difficulties:

- **(a)** We assume that the coefficients B, b and σ are locally Lipschitz in X since we want to cover some stochastic fluid models in the abstract system (1.2). As a result, we do not *a priori* know that the solution exists globally in time. This brings us an essential difficulty. Indeed, the distribution, as a global object on the path space, does not exist for explosive stochastic processes whose paths are killed at the life time. Therefore, to investigate distribution dependent SDEs/SPDEs, we have to either consider the non-explosive setting or modify the "distribution" by a local notion (for example, conditional distribution given by solution does not blow up at present time).
- **(b)** We will have to localize the coefficients (by using stopping times) when we need to fix the changing Lipschitz constants since they are only locally Lipschitz in X. For instance, this happens when uniqueness is considered. Then, we will be confronted with the difficulty

that distribution can not be controlled by any local condition, again. We need to identify some appropriate topology under which the distribution can be measured locally.

(c) Because of the singular term $B(t, X)$, compared to classical case, the Itô formula is no longer available. Indeed, to estimate $\|X\|_{\mathbb{H}}^2$, to use the Itô formula in a Hilbert space (cf. [19]), the $\mathbb{H}$ inner product $\langle B(t, X), X\rangle_{\mathbb{H}}$ is required to be well-defined. But it is not because we only assume that B takes value in $\mathbb{B} \hookleftarrow \mathbb{H}$. Likewise, to apply the Itô formula under a Gelfand triplet ([20, 21]), the dual product ${}_{\mathbb{B}}\langle B(t, X), X\rangle_{\mathbb{B}^*}$ needs to be well-defined, where $\mathbb{B}^*$ is the dual space of $\mathbb{B}$ with respect to $\mathbb{H}$. Because $\mathbb{H} \hookrightarrow \mathbb{B}$, we see that $\mathbb{B}^* \hookrightarrow \mathbb{H}$. However, we do not *a priori* know that the solution X takes value in $\mathbb{B}^*$ because we only assume $X(0) \in \mathbb{H}$.

The first major goal of this paper is to establish an abstract framework for (1.2). The second goal of this work is to apply the abstract theory for (1.2) to (1.3), which gives some new results for some ideal fluid systems.

- To achieve the first goal, we introduce the precise assumptions in Section 1.1 (see Assumption (**A**)). Then, we provide our main results for (1.2) in Theorem 1.1. The key requirements for the proof are the assumption on the existence of appropriate Lipschitz-continuous and monotone regularizations for the singular term B. For difficulty (**a**), in this paper, we restrict our attention to the non-explosive case only. To this end, we assume that the noise grows fast enough (cf. $\mathbf{A}_3$), and then we will show that the blow-up of solutions can be prevented. For difficulty (**b**), we introduce a "local" Wasserstein distance (see (1.7)) and assumption ($\mathbf{A}_5$) to measure the difference of two measures, which enables us to prove uniqueness. By introducing a mollifier satisfying certain estimates (see assumption $\mathbf{A}_4$), we can overcome difficulty (**c**).
- With the general framework at hand, for nonlinear stochastic transport type equations, we are able to construct such regular approximation schemes by using mollifying operators and establishing a commutator estimate (see Lemma 4.1). From this, we can verify the assumptions introduced in Section 1.1 and obtain global existence and uniqueness of solutions in Sobolev spaces. This result is stated in Theorem 1.2. Two examples of Theorem 1.2 are given. The first one, cf. Example 1.1, is a general nonlinear stochastic transport equation, and the second one is the distribution-path dependent stochastic Camassa-Holm equation with or without Coriolis effect, cf. Example 1.2.

1.1 A General Framework

Let $\mathbb{H}, \mathbb{U}$ be two separable Hilbert spaces and let $\{W(t)\}_{t\in[0,T]}$ be a cylindrical Brownian motion on $\mathbb{U}$ with respect to a complete filtration probability space $(\Omega, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$, i.e.

$$W(t) = \sum_{i\geq 1} \beta^i(t)e_i, \quad t \in [0, T]$$

for an orthonormal basis $\{e_i\}_{i\geq 1}$ of $\mathbb{U}$ and a sequence of independent one-dimensional Brownian motions $\{\beta^i\}_{i\geq 1}$ on $(\Omega, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$. Let $\mathcal{L}_2(\mathbb{U}; \mathbb{H})$ be the space of Hilbert-Schmidt operators from $\mathbb{U}$ to $\mathbb{H}$ with Hilbert-Schmidt norm $\|\cdot\|_{\mathcal{L}_2(\mathbb{U};\mathbb{H})}$. Throughout the paper we fix the separable Hilbert space $\mathbb{U}$ and a time $T > 0$.

For a Banach space $\mathbb{M}$, we denote by $\mathscr{C}_{T,\mathbb{M}} := C([0, T]; \mathbb{M})$ the path space. We also consider the weakly continuous path space

$$\mathscr{C}^w_{T,\mathbb{M}} := \{\xi : [0, T] \to \mathbb{M} \text{ is weak continuous}\}.$$

Both $\mathscr{C}_{T,\mathbb{M}}$ and $\mathscr{C}^w_{T,\mathbb{M}}$ are Banach spaces under the uniform norm

$$\|\xi\|_{T,\mathbb{M}} := \sup_{t\in[0,T]} \|\xi(t)\|_{\mathbb{M}}.$$

Then we let $\mathscr{P}^w_{T,\mathbb{M}}$ be the sets of probability measures (with weak convergence topology) on $\mathscr{C}^w_{T,\mathbb{M}}$. Denote $\mathscr{P}_{T,\mathbb{M}} = \{\mu \in \mathscr{P}^w_{T,\mathbb{M}} : \mu(\mathscr{C}_{T,\mathbb{M}}) = 1\}$. For any $N > 0$, we let

$$\mathscr{C}^w_{T,\mathbb{M},N} = \{\xi \in \mathscr{C}^w_{T,\mathbb{M}} : \|\xi\|_{T,\mathbb{M}} \le N\}, \quad \mathscr{P}^w_{T,\mathbb{M},N} = \{\mu \in \mathscr{P}^w_{T,\mathbb{M}} : \mu(\mathscr{C}^w_{T,\mathbb{M},N}) = 1\}. \quad (1.4)$$

For any map $\xi : [0, T] \to \mathbb{M}$ and $t \in [0, T]$, the path $\pi_t(\xi)$ of ξ before time t is given by

$$\pi_t(\xi) := \xi_t : [0, T] \to \mathbb{M}, \quad \xi_t(s) := \xi(t \wedge s), \quad s \in [0, T].$$

Then the marginal distribution before time t of a probability measure $\mu \in \mathscr{P}^w_{T,\mathbb{M}}$ reads

$$\mu_t := \mu \circ \pi_t^{-1}.$$

Let $\mathscr{L}_\xi$ stand for the distribution of a random variable ξ. When more than one probability measure are considered, we denote $\mathscr{L}_\xi$ by $\mathscr{L}_{\xi|\mathbb{P}}$ to emphasize the reference probability measure $\mathbb{P}$. Throughout the paper, $\mathbf{I}$ stands for the identity mapping.

Consider the following nonlinear distribution-path dependent SPDE on $\mathbb{H}$:

$$\mathrm{d}X(t) = \left\{B(t, X(t)) + b(t, X_t, \mathscr{L}_{X_t})\right\} \mathrm{d}t + \sigma(t, X_t, \mathscr{L}_{X_t})\mathrm{d}W(t), \quad t \in [0, T],$$

where, for some separable Hilbert space $\mathbb{B}$ with $\mathbb{H} \hookrightarrow\hookrightarrow \mathbb{B}$ (" $\hookrightarrow\hookrightarrow$ " means the embedding is compact),

$$\begin{cases} B : [0, T] \times \mathbb{H} \times \Omega \to \mathbb{B}, \\ b : [0, T] \times \mathscr{C}^w_{T,\mathbb{H}} \times \mathscr{P}^w_{T,\mathbb{H}} \times \Omega \to \mathbb{H}, \\ \sigma : [0, T] \times \mathscr{C}^w_{T,\mathbb{H}} \times \mathscr{P}^w_{T,\mathbb{H}} \times \Omega \to \mathcal{L}_2(\mathbb{U}; \mathbb{H}) \end{cases} \quad (1.5)$$

are progressively measurable maps.

Definition 1.1 The (strong) solution and weak solution to (1.2) are defined as follows:

(1) A progressively measurable process $X_T := \{X(t)\}_{t\in[0,T]}$ on $\mathbb{H}$ is called a solution to (1.2), if it is continuous in $\mathbb{B}$ and $\mathbb{P}$-a.s.

$$X(t) = X(0) + \int_0^t \left\{B(s, X(s)) + b(s, X_s, \mathscr{L}_{X_s})\right\} \mathrm{d}s + \int_0^t \sigma(s, X_s, \mathscr{L}_{X_s})\mathrm{d}W(s), \; t \in [0, T],$$

where $\int_0^t \left\{B(s, X(s)) + b(s, X(s), \mathscr{L}_{X_s})\right\} \mathrm{d}s$ is the Bochner integral on $\mathbb{B}$ and the stochastic integral $\int_0^t \sigma(s, X(s), \mathscr{L}_{X_s})\mathrm{d}W(s)$ is a continuous local martingale on $\mathbb{H}$.

(2) A couple $(\tilde{X}_T, \tilde{W}_T) = (\tilde{X}(t), \tilde{W}(t))_{t\in[0,T]}$ is called a weak solution to (1.2), if there exists a complete filtration probability space $(\tilde{\Omega}, \{\tilde{\mathscr{F}}_t\}_{t\ge0}, \tilde{\mathbb{P}})$ such that $\tilde{W}_T$ is a cylindrical Brownian motion on $\mathbb{U}$ and $\tilde{X}_T$ is a solution to (1.2) for $(\tilde{W}_T, \tilde{\mathbb{P}})$ replacing $(W_T, \mathbb{P})$.

Since both $X(t)$ and $\int_0^t b(s, X_s, \mathscr{L}_{X_s})\mathrm{d}s + \int_0^t \sigma(s, X_s, \mathscr{L}_{X_s})\mathrm{d}W(s)$ are stochastic processes on $\mathbb{H}$, so is $\int_0^t B(s, X(s))\mathrm{d}s$, although $B(s, X(s))$ only takes values in $\mathbb{B}$.

To ensure the non-explosion such that the distribution is well defined, we will take a Lyapunov type condition $(\mathbf{A}_3)$ below. We write $V \in \mathscr{V}$, if $V \in C^2([0, \infty); [0, \infty))$ satisfies

$$V(0) = 0, \quad V'(r) > 0 \text{ and } V''(r) \le 0 \text{ for } r \ge 0, \; V(\infty) := \lim_{r\to\infty} V(r) = \infty.$$

Let $\mathbb{W}_{2,\mathbb{M}}(\cdot,\cdot)$ be the L^2-Wasserstein distance on $\mathscr{P}^w_{T,\mathbb{M}}$, i.e.,

$$\mathbb{W}_{2,\mathbb{M}}(\mu,\nu) := \inf_{\pi\in\mathfrak{C}(\mu,\nu)} \int_{\mathscr{C}^w_{T,\mathbb{M}}\times\mathscr{C}^w_{T,\mathbb{M}}} \|\xi-\eta\|^2_{T,\mathbb{M}}\,\pi(\mathrm{d}\xi,\mathrm{d}\eta),\quad \mu,\nu\in\mathscr{P}^w_{T,\mathbb{M}},$$

where $\mathfrak{C}(\mu,\nu)$ is the set of couplings of μ and ν. Let

$$\mathfrak{t}^\xi_n := T\wedge\inf\{t\ge 0: \|\xi(t)\|_{\mathbb{M}}\ge n\},\quad \xi\in\mathscr{C}^w_{T,\mathbb{M}}, \tag{1.6}$$

Here and in the sequel, we set $\inf\emptyset=\infty$ by convention. We remark that $\mathfrak{t}^\xi_n$ is a continuous (hence measurable) function in ξ, so that $\mathfrak{t}^X_n$ is a stopping time for an adapted random variable X on $\mathscr{C}^w_{T,\mathbb{M}}$.

For $\mu,\nu\in\mathscr{P}_{T,\mathbb{M}}$, we introduce the "local" L^2-Wasserstein distance defined by

$$\mathbb{W}_{2,\mathbb{M},N}(\mu,\nu) = \inf_{\pi\in\mathfrak{C}(\mu,\nu)} \left(\int_{\mathscr{C}_{T,\mathbb{M}}\times\mathscr{C}_{T,\mathbb{M}}} \|\xi_{t\wedge\mathfrak{t}^\xi_N\wedge\mathfrak{t}^\eta_N}-\eta_{t\wedge\mathfrak{t}^\xi_N\wedge\mathfrak{t}^\eta_N}\|^2_{T,\mathbb{M}}\,\pi(\mathrm{d}\xi,\mathrm{d}\eta)\right)^{\frac{1}{2}}. \tag{1.7}$$

We write $\mu\in\mathscr{P}^V_{T,\mathbb{H}}$ if $\mu\in\mathscr{P}_{T,\mathbb{H}}$ and

$$\|\mu\|_V := \int_{\mathscr{C}_{T,\mathbb{H}}} V(\|\xi\|^2_{T,\mathbb{H}})\mu(\mathrm{d}\xi) < \infty.$$

In general, $\|\cdot\|_V$ may not be a norm, but we use this notation for simplicity. A subset $A\subset\mathscr{P}^V_{T,\mathbb{H}}$ is called V-bounded if $\sup_{\mu\in A}\|\mu\|_V<\infty$.

Assumptions (A) Assume that $\mathbb{H}\hookrightarrow\hookrightarrow\mathbb{B}$ is dense, and there exists a dense subset $\mathbb{H}_0$ of $\mathbb{B}^*$, the dual space of $\mathbb{B}$ with respect to $\mathbb{H}$, such that the following conditions hold for B, b and σ in (1.5).

(A$_1$) $\|b(\cdot,0,\delta_0)\|_{\mathbb{H}}+\|\sigma(\cdot,0,\delta_0)\|_{\mathcal{L}_2(\mathbb{U};\mathbb{H})}$ is bounded on $[0,T]\times\Omega$. For any $N\ge 1$, there exists a constant $C_N>0$ such that for any $\xi,\eta\in\mathscr{C}_{T,\mathbb{H},N}$ and $\mu,\nu\in\mathscr{P}^V_{T,\mathbb{H}}$,

$$\begin{aligned}&\|b(t,\xi_t,\mu_t)-b(t,\eta_t,\nu_t)\|_{\mathbb{H}}+\|\sigma(t,\xi_t,\mu_t)-\sigma(t,\eta_t,\nu_t)\|_{\mathcal{L}_2(\mathbb{U};\mathbb{H})}\\ &\le C_N\left\{\|\xi_t-\eta_t\|_{T,\mathbb{H}}+\mathbb{W}_{2,\mathbb{B}}(\mu_t,\nu_t)\right\},\quad t\in[0,T].\end{aligned}$$

Next, for any $\psi\in\mathbb{H}_0$ and bounded sequences $\{(\xi^n,\mu^n)\}_{n\ge1}\subset\mathscr{C}_{T,\mathbb{H}}\times\mathscr{P}^V_{T,\mathbb{H}}$ satisfying $\|\xi^n-\xi\|_{T,\mathbb{B}}\to 0$ and $\mu^n\to\mu$ weakly in $\mathscr{P}_{T,\mathbb{B}}$ as $n\to\infty$, we have $\mathbb{P}$-a.s.

$$\lim_{n\to\infty}\left\{\left|{}_{\mathbb{B}}\langle b(t,\xi^n,\mu^n_t)-b(t,\xi,\mu_t),\psi\rangle_{\mathbb{B}^*}\right|+\|\{\sigma(t,\xi^n,\mu^n_t)-\sigma(t,\xi,\mu_t)\}^*\psi\|_{\mathbb{U}}\right\}=0,$$

and for any $N\ge1$ there exists a constant $\tilde{C}_N>0$ such that

$$\sup_{t\in[0,T],\eta\in\mathscr{C}_{T,\mathbb{B},N}}\left\{\|b(t,\eta,\mu^n_t)\|_{\mathbb{B}}+\|\sigma(t,\eta,\mu^n_t)\|_{\mathcal{L}_2(\mathbb{U};\mathbb{B})}\right\}\le\tilde{C}_N.$$

(A$_2$) There exist constants $\{C_N, C_{n,N}>0: n,N\ge1\}$ and a sequence of progressively measurable maps

$$B_n:[0,T]\times\mathbb{H}\times\Omega\to\mathbb{H},\quad n\ge1$$

such that for all $n,N\ge1$,

$$\sup_{t\in[0,T],\|x\|_{\mathbb{H}}\le N}\left(\|B(t,x)\|_{\mathbb{B}}+\|B_n(t,x)\|_{\mathbb{B}}\right)\le C_N,$$

$$\sup_{t\in[0,T],\|x\|_{\mathbb{H}}\vee\|y\|_{\mathbb{H}}\le N}\left\{\|B_n(t,x)\|_{\mathbb{H}}+1_{\{x\neq y\}}\frac{\|B_n(t,x)-B_n(t,y)\|_{\mathbb{H}}}{\|x-y\|}\right\}\le C_{n,N}.$$

Moreover, for any bounded sequence $\{\xi^n\}_{n\ge1}$ in $\mathscr{C}^w_{T,\mathbb{H}}$ with $\|\xi^n-\xi\|_{T,\mathbb{B}}\to 0$ as $n\to\infty$, we have

$$\lim_{n\to\infty}\int_0^T \left|{}_{\mathbb{B}}\langle B_n(t,\xi^n(t))-B(t,\xi(t)),\psi\rangle_{\mathbb{B}^*}\right|\,\mathrm{d}t=0,\quad \psi\in\mathbb{H}_0.$$

($\mathbf{A}_3$) There exist $V\in\mathscr{V}$ and constants $K_1,K_2>0$ such that for any $\mu\in\mathscr{P}_{T,\mathbb{H}}$, $t\in[0,T]$, $\xi\in\mathscr{C}_{T,\mathbb{H}}$ and $n\ge1$,

$$\begin{aligned}&V'(\|\xi(t)\|^2_{\mathbb{H}})\left\{2\langle B_n(t,\xi(t))+b(t,\xi_t,\mu_t),\xi(t)\rangle_{\mathbb{H}}+\|\sigma(t,\xi_t,\mu_t)\|^2_{\mathcal{L}_2(\mathbb{U};\mathbb{H})}\right\}\\&+2V''(\|\xi(t)\|^2_{\mathbb{H}})\|\sigma(t,\xi_t,\mu_t)^*\xi(t)\|^2_{\mathbb{U}}\le K_1-K_2\frac{\{V'(\|\xi(t)\|^2_{\mathbb{H}})\|\sigma(t,\xi_t,\mu_t)^*\xi(t)\|_{\mathbb{U}}\}^2}{1+V(\|\xi(t)\|^2_{\mathbb{H}})}.\end{aligned}$$

($\mathbf{A}_4$) There exists a sequence of continuous linear operators $\{T_n\}_{n\ge1}$ from $\mathbb{B}$ to $\mathbb{H}$ with

$$\|T_nx\|_{\mathbb{H}}\le\|x\|_{\mathbb{H}},\quad \lim_{n\to\infty}\|T_nx-x\|_{\mathbb{H}}=0,\quad x\in\mathbb{H}, \tag{1.8}$$

such that for any $N\ge1$, there exists a constant $C_N>0$ such that

$$\sup_{\|x\|_{\mathbb{H}}\le N,n\ge1}|\langle T_nB(t,x),T_nx\rangle_{\mathbb{H}}|\le C_N. \tag{1.9}$$

($\mathbf{A}_5$) There exist constants $K,\varepsilon>0$ and an increasing map $C_\cdot:\mathbb{N}\to(0,\infty)$ such that for any $N\ge1$, $\xi,\eta\in\mathscr{C}^w_{T,\mathbb{H},N}$ and $\mu,\nu\in\mathscr{P}^w_{T,\mathbb{H}}$,

$$\langle B(t,\xi(t))-B(t,\eta(t)),\xi(t)-\eta(t)\rangle_{\mathbb{B}}\le C_N\|\xi(t)-\eta(t)\|^2_{\mathbb{B}},\quad t\in[0,T],$$

and

$$\begin{aligned}&\|b(t,\xi_t,\mu_t)-b(t,\eta_t,\nu_t)\|_{\mathbb{B}}+\|\sigma(t,\xi_t,\mu_t)-\sigma(t,\eta_t,\nu_t)\|_{\mathcal{L}_2(\mathbb{U};\mathbb{B})}\\&\le C_N\left\{\|\xi_t-\eta_t\|_{T,\mathbb{B}}+\mathbb{W}_{2,\mathbb{B},N}(\mu_t,\nu_t)+K\mathrm{e}^{-\varepsilon C_N}\left(1\wedge\mathbb{W}_{2,\mathbb{B}}(\mu_t,\nu_t)\right)\right\},\quad t\in[0,T].\end{aligned}$$

Theorem 1.1 *Let $X_0\in L^2(\Omega\to\mathbb{H},\mathscr{F}_0,\mathbb{P})$.*

(i) *Assume ($\mathbf{A}_1$)–($\mathbf{A}_3$). Then (1.2) has a weak solution $(\tilde{X}_T,\tilde{W}_T)$ such that $\mathscr{L}_{\tilde{X}(0)|\tilde{\mathbb{P}}}=\mathscr{L}_{X_0|\mathbb{P}}$ and*

$$\tilde{\mathbb{E}}\left[V(\|\tilde{X}_T\|^2_{T,\mathbb{H}})\right]\le 2K_1T+1+\frac{64}{K_2}\left(K_1T+\tilde{\mathbb{E}}[V(\|\tilde{X}(0)\|^2_{\mathbb{H}})]\right)<\infty.$$

(ii) *If ($\mathbf{A}_4$) holds, then the weak solution is continuous in $\mathbb{H}$.*
(iii) *If ($\mathbf{A}_5$) holds, then (1.2) has a unique solution with initial value X_0.*

Now we give some remarks regarding the proof of Theorem 1.1 and Assumption (**A**).

Remark 1.1 Except for the difficulties (**a**), (**b**) and (**c**), we will be confronted with one additional technical obstacle. Indeed, we notice that the singular term B is in general not monotone in the sense of [22] (see also [21]). Therefore, even coming back to the distribution-path independent case, the Galerkin approximation under a Gelfand triple developed for quasi-linear SPDEs does not work for the present model. To overcome this obstacle, we will take a different regularization argument. The proof of Theorem 1.1 includes two main steps:

Step 1: **Regular case.** We first establish the solvability of the regular case, i.e., $B = 0$ (see Proposition 2.1). In this step, we need $(\mathbf{A}_1)$ as $(\mathbf{A}_1)$ describes the local Lipschitz continuity of the regular coefficients $b(t, \xi, \mu)$ and $\sigma(t, \xi, \mu)$ in (ξ, μ) under the metric induced by $\|\cdot\|_{\mathbb{H}}$ and $\mathbb{W}_{2,\mathbb{B}}$. We also note that, in finite dimensional space, all norms are equivalent and hence $\mathbb{H}$ reduces to $\mathbb{B}$ and the compact embedding $\mathbb{H} \hookrightarrow\hookrightarrow \mathbb{B}$ is no longer needed. Recalling the difficulty (**a**) mentioned before, we restrict our attention to the non-explosive case. Hence we need the assumption $(\mathbf{A}_3)$, which is a Lyapunov type condition ensuring the global existence of the solution. Furthermore, $(\mathbf{A}_5)$ means that the dependence on the distribution of the coefficients is asymptotically determined by the distribution of local paths, and it will be used to prove the pathwise uniqueness. Actually, $(\mathbf{A}_5)$ is proposed to overcome the difficulty (**b**).

Step 2: **Singular case.** We will propose a regularization argument to establish existence and uniqueness to (1.2). Therefore, in $(\mathbf{A}_2)$, we assume that the singular term $B \in \mathbb{B}$ can be approximated by a regular term $B_n \in \mathbb{H}$ with certain nice properties. The result in **Step 1** guarantees that the approximation problem (see (3.1), where B in (1.2) is replaced by B_n) can be uniquely solved on $[0, T]$ for any given $T > 0$, and we refer to Proposition 2.1. Then, we use the martingale approach to pass limit to the original problem (1.2), where we need the continuity of the coefficient in μ under the weak topology (see $\mathbf{A}_1$). Precisely speaking, by Prokhorov's theorem and Skorokhod's theorem, we can get almost sure convergence of the approximation solutions relative to a new probability space. Then, by the martingale representation theorem, we can identify the limit of the stochastic integral. Finally, we establish the uniqueness, which together with the Yamada-Watanabe type result gives the existence and uniqueness of a pathwise solution. As mentioned before, Itô's formula cannot be applied directly to $\|X(t)\|^2_{\mathbb{H}}$ (see difficulty (**c**)). Hence, it is not obvious to obtain the time continuity of the solution in $\mathbb{H}$. We need to mollify the equation first by using some mollifiers. Therefore, $(\mathbf{A}_4)$ provides certain properties of such mollifiers.

1.2 Distribution-Path Dependent Stochastic Transport Type Equations

Let $d \geq 1$ and $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$ be the d-dimensional torus. Let Δ be the Laplacian operator on $\mathbb{T}^d$, and let i denote the imaginary unit. Then $\{\mathrm{e}^{\mathrm{i}\langle k,\cdot\rangle}\}_{k\in\mathbb{Z}^d}$ consists of an eigenbasis of the Laplacian Δ in the complex L^2-space of the normalized volume measure $\mu(\mathrm{d}x) := (2\pi)^{-d}\mathrm{d}x$ on $\mathbb{T}^d$:

$$\Delta \mathrm{e}^{\mathrm{i}\langle k,\cdot\rangle} = -|k|^2\mathrm{e}^{\mathrm{i}\langle k,\cdot\rangle}, \ \ k \in \mathbb{Z}^d.$$

For a function $f \in L^2(\mu)$, its Fourier transform is given by

$$\widehat{f}(y) := \mathcal{F}(f)(y) = \int_{\mathbb{T}^d} f\mathrm{e}^{\mathrm{i}\langle y,\cdot\rangle}\mathrm{d}\mu, \ \ y \in \mathbb{R}^d.$$

It is well known that

$$\|f\|^2_{L^2(\mu)} = \sum_{k\in\mathbb{Z}^d} |\widehat{f}(k)|^2, \ \ f \in L^2(\mu), \tag{1.10}$$

and

$$\sum_{m\in\mathbb{Z}^d} \widehat{g}(k-m)\widehat{f}(m) = \widehat{fg}(k), \ \ k \in \mathbb{Z}^d, \ \ f, g \in L^4(\mu). \tag{1.11}$$

By the spectral representation, for any $s \geq 0$, we have

$$D^s f := (\mathbf{I} - \Delta)^{\frac{s}{2}} f = \sum_{k \in \mathbb{Z}^d} (1 + |k|^2)^{\frac{s}{2}} \widehat{f}(k) \mathrm{e}^{\mathrm{i}\langle k, \cdot \rangle}, \quad k \in \mathbb{Z}^d,$$

$$f \in \mathscr{D}(D^s) := \left\{ f \in L^2(\mu) : \|D^s f\|_{L^2(\mu)}^2 = \sum_{k \in \mathbb{Z}^d} (1 + |k|^2)^s |\widehat{f}(k)|^2 < \infty \right\}.$$

Then

$$H^s := \{ f = (f_1, \cdots, f_d) : f_i \in \mathscr{D}(D^s), 1 \leq i \leq d \}$$

is a separable Hilbert space with inner product

$$\langle f, g \rangle_{H^s} := \sum_{i=1}^d \langle D^s f_i, D^s g_i \rangle_{L^2(\mu)} = \sum_{k \in \mathbb{Z}^d} (1 + |k|^2)^s \langle \widehat{f}(k), \widehat{g}(k) \rangle_{\mathbb{R}^d}.$$

Now, we recall the stochastic transport SPDE (1.3) on H^s:

$$\mathrm{d}X(t) = \left\{ -(X(t) \cdot \nabla) X(t) + b(t, X_t, \mathscr{L}_{X_t}) \right\} \mathrm{d}t + \sigma(t, X_t, \mathscr{L}_{X_t}) \mathrm{d}W(t), \quad t \in [0, T],$$

where $W(t)$ is the cylindrical Brownian motion, and

$$b : [0, T] \times \mathscr{C}^w_{T, H^s} \times \mathscr{P}^w_{T, H^s} \times \Omega \to H^s, \quad \sigma : [0, T] \times \mathscr{C}^w_{T, H^s} \times \mathscr{P}^w_{T, H^s} \times \Omega \to \mathcal{L}_2(\mathbb{U}; H^s)$$

are measurable.

To apply Theorem 1.1, we make the following assumptions on b and σ.

Assumptions (B) Let $d \geq 1$, $V \in \mathscr{V}$, $s > \frac{d}{2} + 2$, $s' = s - 1$. We assume that the following conditions hold for $\mathbb{H} = H^s$ and $\mathbb{B} = H^{s'}$.

- ($\mathbf{B}_1$) Conditions in ($\mathbf{A}_1$) hold.
- ($\mathbf{B}_2$) There exist constants $K_1, K_2 > 0$ such that for any $\mu \in \mathscr{P}_{T, \mathbb{H}}$, $t \in [0, T]$, $\xi \in \mathscr{C}_{T, \mathbb{H}}$ and $n \geq 1$,

$$V'(\|\xi(t)\|_{\mathbb{H}}^2) \left\{ 2K_0 \|\xi(t)\|_{\mathbb{B}} \|\xi(t)\|_{\mathbb{H}}^2 + 2\langle b(t, \xi_t, \mu_t), \xi(t) \rangle_{\mathbb{H}} + \|\sigma(t, \xi_t, \mu_t)\|_{\mathcal{L}_2(\mathbb{U}; \mathbb{H})}^2 \right\}$$
$$+ 2V''(\|\xi(t)\|_{\mathbb{H}}^2) \|\sigma(t, \xi_t, \mu_t)^* \xi(t)\|_{\mathbb{U}}^2 \leq K_1 - K_2 \frac{\{V'(\|\xi(t)\|_{\mathbb{H}}^2) \|\sigma(t, \xi_t, \mu_t)^* \xi(t)\|_{\mathbb{U}}\}^2}{1 + V(\|\xi(t)\|_{\mathbb{H}}^2)}.$$

- ($\mathbf{B}_3$) There exist constants $K, \varepsilon > 0$ and an increasing map $C_\cdot : \mathbb{N} \to (0, \infty)$ such that for any $N \geq 1$, $\xi, \eta \in \mathscr{C}^w_{T, \mathbb{H}, N}$ and $\mu, \nu \in \mathscr{P}^w_{T, \mathbb{H}}$,

$$\|b(t, \xi_t, \mu_t) - b(t, \eta_t, \nu_t)\|_{\mathbb{B}} + \|\sigma(t, \xi_t, \mu_t) - \sigma(t, \eta_t, \nu_t)\|_{\mathcal{L}_2(\mathbb{U}; \mathbb{B})}$$
$$\leq C_N \left\{ \|\xi_t - \eta_t\|_{T, \mathbb{B}} + \mathbb{W}_{2, \mathbb{B}, N}(\mu_t, \nu_t) + K \mathrm{e}^{-\varepsilon C_N} \left(1 \wedge \mathbb{W}_{2, \mathbb{B}}(\mu_t, \nu_t) \right) \right\}, \quad t \in [0, T].$$

Then we have the following result:

Theorem 1.2 *Assume $s > \frac{d}{2} + 2$, ($\boldsymbol{B}_1$) and ($\boldsymbol{B}_2$). For any $X_0 \in L^2(\Omega \to H^s, \mathscr{F}_0, \mathbb{P})$, (1.3) has a weak solution $(\tilde{X}_T, \tilde{W}_T)$ such that $\mathscr{L}_{\tilde{X}(0)|\tilde{\mathbb{P}}} = \mathscr{L}_{X_0|\mathbb{P}}$, $\tilde{X}_T$ is continuous in H^s and*

$$\tilde{\mathbb{E}} \left[V(\|\tilde{X}_T\|_{T, H^s}^2) \right] \leq 2K_1 T + 1 + \frac{64}{K_2} \left(K_1 T + \tilde{\mathbb{E}}[V(\|\tilde{X}(0)\|_{H^s})] \right).$$

If, moreover, ($\boldsymbol{B}_3$) holds, then (1.3) has a unique solution.

Below we give some remarks concerning Theorem 1.2.

Remark 1.2 We first notice that (1.3) does not contain the viscous term $\Delta X(t)$, which provides additional regularization effect to make the problem of existence easier, see [23, Chapter 5]. Besides the existence and uniqueness, it is interesting to clarify the effect of noise on the properties of solutions. We notice that existing results on the regularization effects by noises for transport type equations are mainly for linear equations or for linear growing noises, see for instance [11–13] for linear transport equations, and [24–28] for linear noise. For nonlinear equations with nonlinear noise, there are examples with positive answers showing that noises can be used to regularize singularities caused by nonlinearity. For example, for the stochastic 2D Euler equations, coalescence of vortices may disappear [29]. But there are also counterexamples such as the fact that noise does not prevent shock formation in the Burgers' equation, see [8, 15]. Therefore, for nonlinear SPDEs, what kind of nonlinear noise can prevent blow-up is a question worthwhile to study. In the current work, the main idea is to use the stochastic part of the equation to avoid any blow-up phenomena that could arise under the presence of the singular drift. Hence we use the Lyapunov type condition ($\mathbf{B}_2$) to measure how strong the noise term needs to be (see also [30, Theorem III.4.1] for the finite dimensional case and [31] for the stochastic nonlinear beam equations). In this way, the noise effect given by the large enough noise is macroscopic and it is different from many previous works, where small noise can also bring regularization effect, see for example [13, 29]. Here we remark that the noise structure in [13, 29] are transport noise in the Stratonovich sense. *A priori*, it is not clear how to interpret the noise term in (1.3). In this work, our main interest is mainly mathematical and we believe that searching for nonlinear noise such that blow-up can be prevented is important because it helps us understand the regularizing mechanisms of noise. This in turn brings us one further step closer to finding the correct and physical noise which provides such regularization.

Remark 1.3 We remark here that there is a gap between the index $s > \frac{d}{2} + 2$ in Theorem 1.2 and the critical value $s > \frac{d}{2} + 1$ such that $H^s \hookrightarrow W^{1,\infty}$. Formally speaking, on one hand, because the transport term $(u \cdot \nabla)u$ loses one order of regularity, we have to consider uniqueness in $H^{s'}$ with $s' \leq s - 1$, i.e., we ask $\mathbb{B} = H^{s'}$ in ($\mathbf{B}_3$). One the other hand, since $\langle (u \cdot \nabla)u, u\rangle_{H^s} \leq c_s \|u\|_{W^{1,\infty}} \|u\|_{H^s}^2$ for smooth u, to verify ($\mathbf{B}_2$), we have to pick $s' \leq s - 1$ such that $\mathbb{B} = H^{s'} \hookrightarrow W^{1,\infty}$. Therefore we have to require $s - 1 > \frac{d}{2} + 1$. However, if we only consider local solutions in H^s without assuming ($\mathbf{B}_2$) (as is explained before, in this case the distribution has to be modified), then $s > \frac{d}{2} + 1$ will be enough.

To conclude this section, we present below two examples to illustrate Theorem 1.2.

Example 1.1 Let $s, s' = s - 1$ be in Assumption (B), $\mathbb{U} = H^s$ and $\mu(F) = \int F \mathrm{d}\mu$ for $F \in L^1(\mu)$. Let

$$b(t, \xi, \mu) = h(t, \|\xi\|_{H^{s'}}, \mu(F_b))\xi(t),$$
$$\sigma(t, \xi, \mu) = \beta(1 + \|\xi\|_{T,H^{s'}})^{\alpha} \langle \xi(t), \cdot \rangle_{H^s} x_0 + \sigma_0(t, \|\xi\|_{H^{s'}}, \mu(F_\sigma)),$$

where $\alpha, \beta > 0$ are constants to be determined, and

(1) $x_0 \in H^s$ with $\|x_0\|_{H^s} = 1$ is a fixed element;
(2) $F_b, F_\sigma : \mathscr{C}_{T,H^{s'}} \to \mathbb{R}^m$ are bounded and Lipschtiz continuous for some $m \geq 1$;
(3) $h(t, \cdot, \cdot) : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ is locally Lipschtiz continuous uniformly in $t \in [0, T]$ such that

$$\sup_{(t,z) \in [0,T] \times \mathbb{R}^m, |x| \leq r} |h(t, x, z)| \leq c(1 + r^{2\alpha}), \quad r \geq 0$$

holds for some constant $c > 0$;

(4) $\sigma_0(t,\cdot,\cdot) : \mathbb{R} \times \mathbb{R}^m \to \mathcal{L}_2(H^s; H^s)$ is bounded and locally Lipschtiz continuous uniformly in $t \in [0, T]$.

If $\alpha \geq \frac{1}{2}$ and β is large enough, then for any probability measure μ_0 on H^s with $\mu_0(\|\cdot\|^2_{H^s}) < \infty$, (1.2) has a weak solution $(\tilde{X}_T, \tilde{W}_T)$ with $\mathscr{L}_{\tilde{X}(0)|\tilde{\mathbb{P}}} = \mu_0$, which is continuous in H^s and satisfies

$$\tilde{\mathbb{E}}\left[\log(1 + \|\tilde{X}_T\|^2_{T,H^s})\right] < \infty.$$

In particular, if $m = 1$ and $F_b(\xi) = F_\sigma(\xi) = \|\xi\|_{T,H^{s'}} \wedge R$ for some constant $R > 0$, then for any $X(0) \in L^2(\Omega \to H^s, \mathscr{F}_0, \mathbb{P})$, (1.2) has a unique solution, which is continuous in H^s and satisfies

$$\mathbb{E}\left[\log(1 + \|X_T\|^2_{T,H^s})\right] < \infty.$$

Proof Let $\alpha \geq \frac{1}{2}$, and take $V(r) = \log(1+r) \in \mathscr{V}$. By Theorem 1.2, we only need to verify conditions $(\mathbf{A}_1)$, $(\mathbf{B}_2)$ with $\mathbb{H} = \mathbb{U} = H^s$, $\mathbb{B} = H^{s'}$, $\mathbb{H}_0 = H^{s+1}$ and large enough $\beta > 0$, and finally prove $(\mathbf{B}_3)$ with $m = 1$ and $F_b(\xi) = F_\sigma(\xi) = \|\xi\|_{T,H^s} \wedge R$.

To begin with, it is easy to see that the weak convergence in $\mathscr{P}_{T,\mathbb{B}}$ is equivalent to that in the metric

$$\mathbb{W}_{1,\mathbb{B}}(\mu,\nu) := \inf_{\pi \in \mathfrak{C}(\mu,\nu)} \int_{\mathscr{C}_{T,\mathbb{B}} \times \mathscr{C}_{T,\mathbb{B}}} (1 \wedge \|\xi - \eta\|_{T,\mathbb{B}}) \pi(\mathrm{d}\xi, \mathrm{d}\eta).$$

Then (1)-(4) and $\mathbb{H} \hookrightarrow \mathbb{B}$ imply that for any $N \geq 1$ there exists a constant $C_N > 0$ such that for all $\eta \in H^{s+1}$,

$$\|b(t,\xi,\mu) - b(t,\eta,\nu)\|_{\mathbb{H}} + \|\sigma(t,\xi,\mu) - \sigma(t,\eta,\nu)\|_{\mathcal{L}_2(\mathbb{U};\mathbb{H})} \leq C_N \left(\|\xi - \eta\|_{T,\mathbb{H}} + \mathbb{W}_{1,\mathbb{B}}(\mu,\nu)\right).$$

Therefore, $(\mathbf{A}_1)$ holds.

Next, let $C = \sup_{(t,r,z) \in [0,T] \times [0,\infty) \times \mathbb{R}^m} \|\sigma_0(t,r,z)\|^2_{\mathcal{L}_2(\mathbb{U};\mathbb{H})}$. We have

$$\begin{aligned}
&V'(\|\xi(t)\|^2_{\mathbb{H}}) \left\{2K_0 \|\xi(t)\|_{\mathbb{B}} \|\xi(t)\|^2_{\mathbb{H}} + 2\langle b(t,\xi_t,\mu_t), \xi(t)\rangle_{\mathbb{H}} + \|\sigma(t,\xi_t,\mu_t)\|^2_{\mathcal{L}_2(\mathbb{U};\mathbb{H})}\right\} \\
&\leq \frac{2K_0 \|\xi(t)\|_{\mathbb{B}} \|\xi(t)\|^2_{\mathbb{H}} + \frac{5\beta^2}{4}(1 + \|\xi_t\|^\alpha_{T,\mathbb{B}})^2 \|\xi(t)\|^2_{\mathbb{H}} + 5C}{1 + \|\xi(t)\|^2_{\mathbb{H}}} \\
&\leq \frac{\|\xi(t)\|^2_{\mathbb{H}}}{1 + \|\xi(t)\|^2_{\mathbb{H}}} \left\{C_1(1 + \|\xi_t\|^{2\alpha}_{T,\mathbb{B}}) + \frac{5\beta^2}{4}(1 + \|\xi_t\|^\alpha_{T,\mathbb{B}})^2\right\}
\end{aligned}$$

for some constant $C_1 > 0$, and on the other hand,

$$\begin{aligned}
2V''(\|\xi(t)\|^2_{\mathbb{H}}) \|\sigma(t,\xi_t,\mu_t)^* \xi(t)\|^2_{\mathbb{U}} &\leq -\frac{2\|\xi(t)\|^4_{\mathbb{H}}}{(1 + \|\xi(t)\|^2_{\mathbb{H}})^2} \left\{\frac{3\beta^2}{4}(1 + \|\xi_t\|^\alpha_{T,\mathbb{B}})^2 - 4C\right\} \\
\frac{\{V'(\|\xi(t)\|^2_{\mathbb{H}}) \|\sigma(t,\xi_t,\mu_t)^* \xi(t)\|_{\mathbb{U}}\}^2}{1 + V(\|\xi(t)\|^2_{\mathbb{H}})} &\leq \frac{\|\xi(t)\|^4_{\mathbb{H}}}{(1 + \|\xi(t)\|^2_{\mathbb{H}})^2} \left\{\beta^2(1 + \|\xi_t\|^\alpha_{T,\mathbb{B}})^2 + 2C\right\}.
\end{aligned}$$

Therefore, when $\beta > 2\sqrt{C_1}$, $(\mathbf{B}_2)$ holds for some constants $K_1, K_2 > 0$.

Finally, let $m = 1$, $F_b(\xi) = F_\sigma(\xi) = \|\xi\|_{T,\mathbb{B}} \wedge R$. It suffices to verify $(\mathbf{B}_3)$ for $N \geq R$. In this case, by the formulation of b, σ and conditions (1)-(4), for any $N \geq R$, there exists a constant $C_N > 0$ such that

$$\begin{aligned}
&\|b(t,\xi,\mu) - b(t,\eta,\nu)\|_{\mathbb{B}} + \|\sigma(t,\xi,\mu) - \sigma(t,\eta,\nu)\|_{\mathcal{L}_2(\mathbb{U};\mathbb{B})} \\
&\leq C_N \left(\|\xi - \eta\|_{T,\mathbb{B}} + |\mu_t(\|\cdot\|_{T,\mathbb{B}} \wedge R) - \nu_t(\|\cdot\|_{T,\mathbb{B}} \wedge R)|\right).
\end{aligned} \tag{1.12}$$

Recall (1.6) and then denote

$$\|\xi-\eta\|_{\mathrm{t}_N}=\sup_{t\in[0,T\wedge \mathrm{t}_N^{\xi}\wedge \mathrm{t}_N^{\eta}]}\|\xi(t)-\eta(t)\|_{\mathbb{B}}.$$

When $N\geq R$ we have

$$\begin{cases}\big|\|\xi_t\|_{T,\mathbb{B}}\wedge R-\|\xi_t\|_{T,\mathbb{B}}\wedge R\big|\leq \|\xi_t-\eta_t\|_{T,\mathbb{B}}=\|\xi-\eta\|_{\mathrm{t}_N}, & \text{if } \mathrm{t}_N^{\xi}\wedge \mathrm{t}_N^{\eta}>t,\\ \big|\|\xi_t\|_{T,\mathbb{B}}\wedge R-\|\xi_t\|_{T,\mathbb{B}}\wedge R\big|=R-\|\eta_t\|_{T,\mathbb{B}}\wedge R\leq \|\xi-\eta\|_{\mathrm{t}_N}, & \text{if } \mathrm{t}_N^{\xi}\leq t,\ \mathrm{t}_N^{\eta}>t\\ \big|\|\xi_t\|_{T,\mathbb{B}}\wedge R-\|\xi_t\|_{T,\mathbb{B}}\wedge R\big|=R-\|\xi_t\|_{T,\mathbb{B}}\wedge R\leq \|\xi-\eta\|_{\mathrm{t}_N}, & \text{if } \mathrm{t}_N^{\xi}>t,\ \mathrm{t}_N^{\eta}\leq t\\ \big|\|\xi_t\|_{T,\mathbb{B}}\wedge R-\|\xi_t\|_{T,\mathbb{B}}\wedge R\big|=0\leq \|\xi-\eta\|_{\mathrm{t}_N}, & \text{if } \mathrm{t}_N^{\xi}\vee \mathrm{t}_N^{\eta}\leq t.\end{cases}$$

Consequently,

$$|\mu_t(\|\cdot\|_{T,\mathbb{B}}\wedge R)-\nu_t(\|\cdot\|_{T,\mathbb{B}}\wedge R)|\leq \inf_{\pi\in\mathfrak{C}(\mu_t,\nu_t)}\int_{\mathscr{C}_{T,\mathbb{B}}\times\mathscr{C}_{T,\mathbb{B}}}\|\xi-\eta\|_{\mathrm{t}_N}\mathrm{d}\pi\leq \mathbb{W}_{2,\mathbb{B},N}(\mu_t,\nu_t),$$

so that (1.12) implies $(\mathbf{B}_3)$ for $K=0$. □

Example 1.2 Now we consider a family of stochastic models which are more physical relevant. Let s,s' be in assumption (B) with $d=1$ and $\mathbb{U}=H^s$. We focus on the following PDE

$$\partial_t u+u\partial_x u+(\mathbf{I}-\partial_{xx}^2)^{-1}\partial_x\left(a_0u+a_1u^2+a_2(\partial_x u)^2+a_3u^3+a_4u^4\right)=0, \tag{1.13}$$

where a_i $(i=0,1,2,3,4)$ are some constants. Before we consider its stochastic versions, we briefly recall some background of (1.13). Due to the abundance of literature on (1.13), here we only mention a few related results. If $a_1=1$, $a_2=\frac{1}{2}$ and $a_0=a_3=a_4=0$, (1.13) becomes the Camassa-Holm equation

$$\partial_t u+u\partial_x u+(\mathbf{I}-\partial_{xx}^2)^{-1}\partial_x\left(u^2+\frac{1}{2}(\partial_x u)^2\right)=0. \tag{1.14}$$

Equation (1.14) models the unidirectional propagation of shallow water waves over a flat bottom and it appeared initially in the context of hereditary symmetries studied by Fuchssteiner and Fokas [32] as a bi-Hamiltonian generalization of KdV equation. Later, Camassa and Holm [33] derived it by approximating directly in the Hamiltonian for Euler equations in the shallow water regime. It is well known that (1.14) exhibits both phenomena of (peaked) soliton interaction and wave-breaking. When $a_1=\frac{b}{2}$, $a_2=\frac{3-b}{2}$ with $b\in\mathbb{R}$ and $a_0=a_3=a_4=0$, (1.13) reduces to the so-called b-family equations, cf. [34, 35],

$$\partial_t u+u\partial_x u+(\mathbf{1}-\partial_{xx}^2)^{-1}\partial_x\left(\frac{b}{2}u^2+\frac{3-b}{2}(\partial_x u)^2\right)=0.$$

When $a_0\in\mathbb{R}$, $a_1=1$, $a_2=\frac{1}{2}$ and $a_3=a_4=0$, (1.13) is a dispersive evolution equation derived by Dullin et al. in [36] as a model governing planar solutions to Euler's equations in the shallow-water regime. Finally, when a_i $(i=0,1,2,3,4)$ are suitably chosen, (1.13) becomes the recently derived rotation Camassa-Holm equation describing the motion of the fluid with the Coriolis effect from the incompressible shallow water in the equatorial region, cf. [37, equation (4.9)]. In this case, $a_3\neq 0$ and $a_4\neq 0$ so that the equation has a cubic and quartic nonlinearities.

For this family of PDEs, if distribution-path dependent noise is involved, we consider

$$\mathrm{d}u+[u\partial_x u+G(u)]\,\mathrm{d}t=\sigma(t,u_t,\mathscr{L}_{u_t})\mathrm{d}W(t). \tag{1.15}$$

In (1.15),

$$G(u)=(\mathbf{I}-\partial_{xx}^2)^{-1}\partial_x\left(a_0u+a_1u^2+a_2(\partial_xu)^2+a_3u^3+a_4u^4\right),$$
$$\sigma(t,u,\mu)=\beta(1+\|u\|_{T,H^{s'}})^{\alpha}\langle u(t),\cdot\rangle_{H^s}\cdot v+\sigma_0(t,\|u\|_{H^{s'}},\mu(F_\sigma)),$$

where $v\in H^s$ is a fixed element such that $\|v\|_{H^s}=1$ and σ_0 satisfies condition **(4)** with $m=1$ as in Example 1.1. It is easy to show that there is a constant $C>0$ such that

$$\|G(u)\|_{H^s}\le C\left(|a_0|+(|a_1|+|a_2|)\|u\|_{W^{1,\infty}}+|a_3|\|u\|_{W^{1,\infty}}^2+|a_4|\|u\|_{W^{1,\infty}}^3\right)\|u\|_{H^s},$$

and

$$\|G(u)-G(v)\|_{H^{s'}}\le C\left[|a_0|+(|a_1|+|a_2|)I_s(u,v)+|a_3|I_s^2(u,v)+|a_4|I_s^3(u,v)\right]\|u-v\|_{H^s},$$

where $I_s(u,v)=\|u\|_{H^s}+\|v\|_{H^s}$. Since $H^{s'}\hookrightarrow W^{1,\infty}$, $G(\cdot)$ satisfies the the estimates for drift part as in ($\mathbf{B}_1$) and ($\mathbf{B}_3$). Going along the lines as in the proof of Example 1.1 (see also the proofs of Theorems 1.1 and 1.2) with minor modification, we can see that if $\beta>1$ is large enough and

$$\begin{cases}\alpha\ge 3/2, & \text{if } a_4\neq 0,\ a_0,a_1,a_2,a_3\in\mathbb{R}\ \text{(with Coriolis effect)},\\ \alpha\ge 1, & \text{if } a_4=0,\ a_3\neq 0,\ a_0,a_1,a_2\in\mathbb{R},\\ \alpha\ge 1/2, & \text{if } a_3=a_4=0,\ a_1\neq 0,\ a_2\neq 0,\ a_0\in\mathbb{R}\ \text{(without Coriolis effect)},\end{cases}$$

then for any $u(0)\in L^2(\Omega\to H^s,\mathscr{F}_0,\mathbb{P})$, (1.15) has a unique solution with continuous path in H^s and

$$\mathbb{E}\left[\log(1+\|u_T\|_{T,H^s}^2)\right]<\infty.$$

Therefore, in contrast to the deterministic case where wave-breaking phenomenon may occur in finite time, see [38–40], the blow-up is prevented when the growth of the noise coefficient in (1.15) is faster enough. For other Camassa-Holm type equations with random noise, we refer to [17, 41–45] and the references therein.

The remainder of the paper is organized as follows. In Section 2, we consider the regular case where $B=0$. Then we prove Theorems 1.1 and 1.2 in Section 3 and Section 4 respectively.

2 Regular Case: $B=0$

We consider the following distribution-path dependent SPDE:

$$\mathrm{d}X(t)=b(t,X_t,\mathscr{L}_{X_t})\mathrm{d}t+\sigma(t,X_t,\mathscr{L}_{X_t})\mathrm{d}W(t),\quad X(0)=X_0,\quad t\in[0,T]. \tag{2.1}$$

Recall (1.6),

$$\mathbb{W}_{2,\mathbb{M}}(\mu,\nu):=\inf_{\pi\in\mathfrak{C}(\mu,\nu)}\left(\int_{\mathscr{C}_{T,\mathbb{M}}^w\times\mathscr{C}_{T,\mathbb{M}}^w}\|\xi-\eta\|_{T,\mathbb{M}}^2\,\pi(\mathrm{d}\xi,\mathrm{d}\eta)\right)^{\frac12},$$

and

$$\mathbb{W}_{2,\mathbb{M},N}(\mu,\nu)=\inf_{\pi\in\mathfrak{C}(\mu,\nu)}\left(\int_{\mathscr{C}_{T,\mathbb{M}}\times\mathscr{C}_{T,\mathbb{M}}}\|\xi_{t\wedge\mathfrak{t}_N^\xi\wedge\mathfrak{t}_N^\eta}-\eta_{t\wedge\mathfrak{t}_N^\xi\wedge\mathfrak{t}_N^\eta}\|_{T,\mathbb{M}}^2\,\pi(\mathrm{d}\xi,\mathrm{d}\eta)\right)^{\frac12}.$$

Then Assumption (**A**) for $B=0$ implies the following assumption (**C**):

Assumptions (C) With the same notation as in (1.4), we assume the following, for some Hilbert space $\mathbb{B}$ with dense and compact embedding $\mathbb{H} \hookrightarrow\hookrightarrow \mathbb{B}$:

($\mathbf{C}_1$) For any $N \geq 1$, there exists a constant $C_N > 0$ such that for any $\xi, \eta \in \mathscr{C}_{T,\mathbb{H},N}$ and $\mu, \nu \in \mathscr{P}^V_{T,\mathbb{H}}$, we have that $\mathbb{P}$-a.s. for $t \in [0, T]$,

$$\|b(t, \xi_t, \mu_t)\|_{\mathbb{H}} + \|\sigma(t, \xi_t, \mu_t)\|_{\mathcal{L}_2(\mathbb{U};\mathbb{H})} \leq C_N,$$

$$\|b(t, \xi_t, \mu_t) - b(t, \eta_t, \nu_t)\|_{\mathbb{H}} + \|\sigma(t, \xi_t, \mu_t) - \sigma(t, \eta_t, \nu_t)\|_{\mathcal{L}_2(\mathbb{U};\mathbb{H})} \leq C_N \left\{\|\xi_t - \eta_t\|_{T,\mathbb{H}} + \mathbb{W}_{2,\mathbb{B}}(\mu_t, \nu_t)\right\}.$$

($\mathbf{C}_2$) There exists a dense subset $\mathbb{H}_0 \subset \mathbb{H}$ such that for any bounded sequence $\{(\xi^n, \mu^n)\}_{n\geq 1} \subset \mathscr{C}_{T,\mathbb{H}} \times \mathscr{P}^V_{T,\mathbb{H}}$ with $\|\xi^n - \xi\|_{T,\mathbb{H}} \to 0$ and $\mu^n \to \mu$ weakly in $\mathscr{P}_{T,\mathbb{B}}$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \left\{\left|\langle b(t, \xi^n, \mu^n_t) - b(t, \xi, \mu_t), \psi\rangle_{\mathbb{H}}\right| + \|\{\sigma(t, \xi^n, \mu^n_t) - \sigma(t, \xi, \mu_t)\}^* \psi\|_{\mathbb{U}}\right\} = 0, \quad \psi \in \mathbb{H}_0.$$

($\mathbf{C}_3$) There exist constants $K_1, K_2 > 0$ such that for any $\mu \in \mathscr{P}_{T,\mathbb{H}}$, $t \in [0, T]$ and $\xi \in \mathscr{C}_{T,H}$,

$$V'(\|\xi(t)\|^2_{\mathbb{H}}) \left\{2\langle b(t, \xi_t, \mu_t), \xi(t)\rangle_{\mathbb{H}} + \|\sigma(t, \xi_t, \mu_t)\|^2_{\mathcal{L}_2(\mathbb{U};\mathbb{H})}\right\} + 2V''(\|\xi(t)\|^2_{\mathbb{H}})\|\sigma(t, \xi_t, \mu_t)^*\xi(t)\|^2_{\mathbb{U}} \leq K_1 - K_2 \frac{\{V'(\|\xi(t)\|^2_{\mathbb{H}})\|\sigma(t, \xi_t, \mu_t)^*\xi(t)\|_{\mathbb{U}}\}^2}{1 + V(\|\xi(t)\|^2_{\mathbb{H}})}.$$

($\mathbf{C}_4$) There exist constants $K, \varepsilon > 0$, an increasing map $C_\cdot : \mathbb{N} \to (0, \infty)$, such that for any $\xi, \eta \in \mathscr{C}_{T,\mathbb{H},N}$ and $\mu, \nu \in \mathscr{P}^w_{T,\mathbb{H}}$,

$$\|b(t, \xi_t, \mu_t) - b(t, \eta_t, \nu_t)\|_{\mathbb{B}} + \|\sigma(t, \xi_t, \mu_t - \sigma(t, \eta_t, \nu_t)\|_{\mathcal{L}_2(\mathbb{U};\mathbb{B})} \leq C_N \left\{\|\xi_t - \eta_t\|_{\mathbb{B}} + \mathbb{W}_{2,\mathbb{B},N}(\mu_t, \nu_t) + K\mathrm{e}^{-\varepsilon C_N}\left(1 \wedge \mathbb{W}_{2,\mathbb{B}}(\mu_t, \nu_t)\right)\right\}, \quad t \in [0, T].$$

The main result of this section is the following.

Proposition 2.1 *Assume ($\mathbf{C}_1$)–($\mathbf{C}_3$). If $X_0 \in L^2(\Omega \to \mathbb{H}, \mathscr{F}_0, \mathbb{P})$, then (2.1) has a solution $X \in \mathscr{C}_{T,\mathbb{H}}$ satisfying*

$$\mathbb{E}\left[V(\|X_T\|^2_{T,\mathbb{H}})\right] \leq 2K_1T + 1 + \frac{64}{K_2}\left(K_1T + \mathbb{E}\left[V(\|X_0\|^2_{\mathbb{H}})\right]\right) < \infty. \tag{2.2}$$

Moreover, if ($\mathbf{C}_4$) holds, then the solution is unique.

To prove this result, we first consider the global monotone situation, and then extend to the local case.

Lemma 2.2 *Let $b(t, \xi, \mu)$ and $\sigma(t, \xi, \mu)$ be continuous in $(\xi, \mu) \in \mathscr{C}_{T,\mathbb{H}} \times \mathscr{P}_{T,\mathbb{H}}$. Assume that there exists a positive random variable γ with $\mathbb{E}[\gamma] < \infty$ and a constant $K > 0$ such that for any $\mathscr{C}_{T,\mathbb{H}}$-valued random variables ξ and η with $\xi(0) = \eta(0)$, we have $\mathbb{P}$-a.s. that for all $t \in [0, T]$, $\mu, \nu \in \mathscr{P}^w_{T,\mathbb{H}}$,*

$$\begin{cases} 2\langle b(t, \xi_t, \mu_t), \xi(t)\rangle_{\mathbb{H}} + \|\sigma(t, \xi_t, \mu_t)\|^2_{\mathcal{L}_2(\mathbb{U};\mathbb{H})} \leq K\left\{\gamma + \|\xi_t\|^2_{T,\mathbb{H}} + \mu_t(\|\cdot\|^2_{T,\mathbb{H}})\right\}, \\ 2\langle b(t, \xi_t, \mu_t) - b(t, \eta_t, \nu_t), \xi(t) - \eta(t)\rangle_{\mathbb{H}} \leq K\left\{\|\xi_t - \eta_t\|^2_{T,\mathbb{H}} + \mathbb{W}_{2,\mathbb{H}}(\mu_t, \nu_t)^2\right\}, \\ \|\sigma(t, \xi_t, \mu_t) - \sigma(t, \eta_t, \nu_t)\|^2_{\mathcal{L}_2(\mathbb{U};\mathbb{H})} \leq K\left\{\|\xi_t - \eta_t\|^2_{T,\mathbb{H}} + \mathbb{W}^2_{2,\mathbb{H}}(\mu_t, \nu_t)^2\right\}. \end{cases} \tag{2.3}$$

Then for any $X_0 \in L^2(\Omega \to \mathbb{H}, \mathscr{F}_0, \mathbb{P})$, (2.1) has a unique solution which is continuous in $\mathbb{H}$.

Proof By (2.3), the uniqueness follows from Itô's formula and Grönwall's inequality. Below we only prove the existence by using the procedure as in [5].

Let $X^0(t) \equiv X_0$ and $\mu_t^{(0)} \equiv \mathscr{L}_{X_t^0}$. We construct the following iteration scheme:

$$\begin{cases} \mathrm{d}X^{(n)}(t) = b(s, X_t^{(n)}, \mu_t^{(n-1)})\,\mathrm{d}t + \sigma(s, X_t^{(n)}, \mu_t^{(n-1)})\,\mathrm{d}W(t), \quad t \in [0, T], \\ X^{(n)}(0) = X_0 \in \mathbb{H}, \quad \mu_t^{(n-1)} = \mathscr{L}_{X_t^{(n-1)}}, \quad n \geq 1. \end{cases} \tag{2.4}$$

By (2.3) and induction, we can construct a sequence of continuous adapted processes $\{X_T^{(n)}\}_{n\geq 1}$ on $\mathscr{C}_{T,\mathbb{H}}$ with $\sup_{n\geq 1} \mathbb{E}[\|X_T^{(n)}\|_{T,\mathbb{H}}^2] < \infty$. Below we prove that $\{X_T^{(n)}\}_{n\geq 1}$ is a Cauchy sequence in $L^2(\Omega \to \mathscr{C}_{T,\mathbb{H}}; \mathbb{P})$, and hence has a limit X_T in this space as $n \to \infty$, so that due to (2.3) and the continuity of $b(t, \xi, \mu)$ and $\sigma(t, \xi, \mu)$ in (ξ, μ), we may let $n \to \infty$ in (2.4) for $t \in [0, T]$ to conclude that X_T is a solution to (2.1).

By (2.3) and Itô's formula, for $Z^{(n)}(t) := X^{(n)}(t) - X^{(n-1)}(t)$,

$$\|Z^{(n)}(t)\|_{\mathbb{H}}^2 \leq K \int_0^t \left\{\|Z_s^{(n)}\|_{T,\mathbb{H}}^2 + \mathbb{E}\|Z_s^{(n-1)}\|_{T,\mathbb{H}}^2\right\} \mathrm{d}s + M(t)$$

where

$$M(t) := 2\int_0^t \left\langle Z^{(n)}(s), \{\sigma(s, X_s^{(n)}, \mu_s^{(n-1)}) - \sigma(s, X_s^{(n-1)}, \mu_s^{(n-2)})\}\mathrm{d}W(s)\right\rangle_{\mathbb{H}}.$$

Then for $\lambda > 0$,

$$\begin{aligned} &\mathrm{e}^{-\lambda t}\mathbb{E}\|Z_t^{(n)}\|_{T,\mathbb{H}}^2 \\ &\leq K\mathrm{e}^{-\lambda t} \int_0^t \left\{\mathbb{E}\|Z_s^{(n)}\|_{T,\mathbb{H}}^2 + \mathbb{E}\|Z_s^{(n-1)}\|_{T,\mathbb{H}}^2\right\} \mathrm{d}s + \mathrm{e}^{-\lambda t}\mathbb{E}\left(\sup_{0\leq s\leq t} M(s)\right) \\ &=: I^{(1)}(t) + I^{(2)}(t), \quad t \in [0, T]. \end{aligned} \tag{2.5}$$

We observe that

$$\begin{aligned} I^{(1)}(t) &= K\int_0^t \mathrm{e}^{-\lambda(t-s)} \left\{\mathrm{e}^{-\lambda s}\mathbb{E}\|Z_s^{(n)}\|_{T,\mathbb{H}}^2 + \mathrm{e}^{-\lambda s}\mathbb{E}\|Z_s^{(n-1)}\|_{T,\mathbb{H}}^2\right\} \mathrm{d}s \\ &\leq \frac{K}{\lambda} \sup_{0\leq s\leq t} \left(\mathrm{e}^{-\lambda s}\mathbb{E}\|Z_s^{(n)}\|_{T,\mathbb{H}}^2\right) + \frac{K}{\lambda} \sup_{0\leq s\leq t} \left(\mathrm{e}^{-\lambda s}\mathbb{E}\|Z_s^{(n-1)}\|_{T,\mathbb{H}}^2\right). \end{aligned} \tag{2.6}$$

By BDG's inequality, for some constants $c_1, c_2 > 0$, we have

$$\begin{aligned} &I^{(2)}(t) \\ &\leq c_1\mathrm{e}^{-\lambda t}\mathbb{E}\left(\int_0^t \|Z^{(n)}(s)\|_{\mathbb{H}}^2 \left\{\|Z_s^{(n)}\|_{T,\mathbb{H}}^2 + \mathbb{E}\|Z_s^{(n-1)}\|_{T,\mathbb{H}}^2\right\} \mathrm{d}s\right)^{\frac{1}{2}} \\ &\leq c_1\mathrm{e}^{-\lambda t}\left(\mathbb{E}\|Z_t^{(n)}\|_{T,\mathbb{H}}^2 \int_0^t \left\{\mathbb{E}\|Z_s^{(n)}\|_{T,\mathbb{H}}^2 + \mathbb{E}\|Z_s^{(n-1)}\|_{T,\mathbb{H}}^2\right\} \mathrm{d}s\right)^{\frac{1}{2}} \end{aligned}$$

$$\le \frac{1}{2}\mathrm{e}^{-\lambda t}\mathbb{E}\|Z_t^{(n)}\|_{T,\mathbb{H}}^2 + c_2\int_0^t \mathrm{e}^{-\lambda(t-s)}\left\{\mathrm{e}^{-\lambda s}\mathbb{E}\|Z_s^{(n)}\|_{T,\mathbb{H}}^2 + \mathrm{e}^{-\lambda s}\mathbb{E}\|Z_s^{(n-1)}\|_{T,\mathbb{H}}^2\right\}\mathrm{d}s$$
$$\le \frac{1}{2}\mathrm{e}^{-\lambda t}\mathbb{E}\|Z_t^{(n)}\|_{T,\mathbb{H}}^2 + \frac{c_2}{\lambda}\sup_{0\le s\le t}\left(\mathrm{e}^{-\lambda s}\mathbb{E}\|Z_s^{(n)}\|_{T,\mathbb{H}}^2\right)$$
$$+\frac{c_2}{\lambda}\sup_{0\le s\le t}\left(\mathrm{e}^{-\lambda s}\mathbb{E}\|Z_s^{(n-1)}\|_{T,\mathbb{H}}^2\right). \tag{2.7}$$

Substituting (2.6) and (2.7) into (2.5) yields that for $t \in [0, T]$,

$$\mathrm{e}^{-\lambda t}\mathbb{E}\|Z_t^{(n)}\|_{T,\mathbb{H}}^2$$
$$\le \frac{2(K+c_2)}{\lambda}\sup_{0\le s\le t}\left(\mathrm{e}^{-\lambda s}\mathbb{E}\|Z_s^{(n)}\|_{T,\mathbb{H}}^2\right) + \frac{2(K+c_2)}{\lambda}\sup_{0\le s\le t}\left(\mathrm{e}^{-\lambda s}\mathbb{E}\|Z_s^{(n-1)}\|_{T,\mathbb{H}}^2\right),$$

which implies

$$\sup_{0\le s\le T}\left(\mathrm{e}^{-\lambda s}\mathbb{E}\|Z_s^{(n)}\|_{T,\mathbb{H}}^2\right)$$
$$\le \frac{2(K+c_2)}{\lambda}\left(\sup_{0\le s\le T}\left(\mathrm{e}^{-\lambda s}\mathbb{E}\|Z_s^{(n)}\|_{T,\mathbb{H}}^2\right) + \sup_{0\le s\le T}\left(\mathrm{e}^{-\lambda s}\mathbb{E}\|Z_s^{(n-1)}\|_{T,\mathbb{H}}^2\right)\right).$$

Taking $\lambda = 6(K + c_2)$, we arrive at

$$\sup_{0\le s\le T}\left(\mathrm{e}^{-\lambda s}\mathbb{E}\|Z_s^{(n)}\|_{T,\mathbb{H}}^2\right) \le \frac{2(K+c_2)}{\lambda-2(K+c_2)}\sup_{0\le s\le T}\left(\mathrm{e}^{-\lambda s}\mathbb{E}\|Z_s^{(n-1)}\|_{T,\mathbb{H}}^2\right)$$
$$= \frac{1}{2}\sup_{0\le s\le T}\left(\mathrm{e}^{-\lambda s}\mathbb{E}\|Z_s^{(n-1)}\|_{T,\mathbb{H}}^2\right).$$

Hence, for any $n \ge 2$ we have

$$\sup_{0\le s\le T}\left(\mathrm{e}^{-\lambda s}\mathbb{E}\|Z_s^{(n)}\|_{T,\mathbb{H}}^2\right) \le \frac{1}{2^{n-1}}\sup_{0\le s\le T}\left(\mathrm{e}^{-\lambda s}\mathbb{E}\|Z_s^{(1)}\|_{T,\mathbb{H}}^2\right).$$

Therefore, $\{X_T^{(n)}\}_{n\ge 1}$ is a Cauchy sequence as desired. □

Lemma 2.3 *Assume* ($\mathbf{C}_1$)*–*($\mathbf{C}_3$)*. For any* $T > 0$, $X(0) \in L^2(\Omega \to \mathbb{H}, \mathscr{F}_0, \mathbb{P})$*, and any* $\mu \in \mathscr{P}_{T,\mathbb{H}}^V$*, the SPDE*

$$\mathrm{d}X^\mu(t) = b(t, X_t^\mu, \mu_t)\mathrm{d}t + \sigma(t, X_t^\mu, \mu_t)\mathrm{d}W(t), \quad X^\mu(0) = X_0$$

has a unique solution X_T^μ *satisfying*

$$\mathbb{E}\left[V(\|X_T^\mu\|_{T,\mathbb{H}}^2)\right] \le 2K_1T + 1 + \frac{64}{K_2}\left(K_1T + \mathbb{E}[V(\|X_0\|_{\mathbb{H}}^2)]\right). \tag{2.8}$$

Proof By ($\mathbf{C}_1$), we see that this equation has a unique solution up to the life time τ. Now we prove that $\tau > T$ (i.e. the solution is non-explosive) and (2.8). To this end, with the convention $\inf \emptyset = \infty$ we set

$$\tau_n = \inf\{t \ge 0 : \|X^\mu(t)\|_{\mathbb{H}}^2 \ge n\}, \quad n \ge 1,$$

$$H(t) := \frac{\{V'(\|X^\mu(t)\|_{\mathbb{H}}^2)\|\sigma(t, X_t^\mu, \mu_t)^* X(t)\|_{\mathbb{U}}\}^2}{1 + V(\|X^\mu(t)\|_{\mathbb{H}}^2)}, \quad t \in [0, T].$$

By ($\mathbf{C}_3$) and Itô's formula, we obtain

$$\mathrm{d}V(\|X^\mu(t)\|_{\mathbb{H}}^2) \le \{K_1 - K_2 H(t)\} + 2V'(\|X^\mu(t)\|_{\mathbb{H}}^2)\langle X^\mu(t), \sigma(t, X_t^\mu, \mu_t)\mathrm{d}W(t)\rangle_{\mathbb{H}}. \quad (2.9)$$

This gives rise to

$$\mathbb{E}[V(\|X^\mu(T\wedge\tau_n)\|_{\mathbb{H}}^2)] + K_2\mathbb{E}\int_0^{T\wedge\tau_n} H(t)\mathrm{d}t \le K_1 T + \mathbb{E}[V(\|X_0\|_{\mathbb{H}}^2)] =: C, \quad n \ge 1. \quad (2.10)$$

Then

$$V(n)\mathbb{P}(\tau_n \le T) \le \mathbb{E}[V(\|X^\mu(T\wedge\tau_n)\|_{\mathbb{H}}^2)] \le C, \quad n \ge 1,$$

so that by $\tau \ge \tau_n$ we obtain $\mathbb{P}(\tau \le T) \le \frac{C}{V(n)} \to 0$ as $n \to \infty$. Thus, $\mathbb{P}(\tau > T) = 1$. Moreover, by (2.9) and BDG inequality, we obtain that for all $n \ge 1$,

$$\begin{aligned}
\mathbb{E}\Big[V(\|X_{T\wedge\tau_n}^\mu\|_{T,\mathbb{H}}^2)\Big] &\le K_1 T + 8\mathbb{E}\bigg(\int_0^{T\wedge\tau_n}\{V'(\|X^\mu(t)\|_{\mathbb{H}}^2)\}^2\|\sigma^*(t, X_t^\mu, \mu_t)X^\mu(t)\|_{\mathbb{U}}^2\mathrm{d}t\bigg)^{\frac{1}{2}} \\
&= K_1 T + 8\mathbb{E}\bigg(\Big(1 + V(\|X_{T\wedge\tau_n}^\mu\|_{T,\mathbb{H}}^2)\Big)\int_0^{T\wedge\tau_n} H(t)\mathrm{d}t\bigg)^{\frac{1}{2}} \\
&\le K_1 T + \frac{1}{2}\mathbb{E}\Big[\Big(1 + V(\|X_{T\wedge\tau_n}^\mu\|_{T,\mathbb{H}}^2)\Big)\Big] + 32\mathbb{E}\int_0^T H(t)\mathrm{d}t.
\end{aligned}$$

Combining this with (2.10), we arrive at

$$\mathbb{E}\Big[V(\|X_{T\wedge\tau_n}^\mu\|_{T,\mathbb{H}}^2)\Big] \le 2K_1 T + 1 + 64\frac{C}{K_2} =: \delta, \quad n \ge 1. \quad (2.11)$$

As C does not depend on n, letting $n \to \infty$ and noting (2.10) give rise to (2.8). □

Now we are in the position o prove Proposition 2.1.

Proof of Proposition 2.1 The estimate (2.2) is implied by Lemma 2.3 with $\mu_t = \mathscr{L}_{X_t}$ once existence has been established. So, it remains to prove the existence and uniqueness. The key point is to apply Lemma 2.2 with a localization argument. For the case that the target problem is finite-dimensional and path independent, we refer to [46, Theorem 1.1].

(a) Existence. To construct a solution using Lemma 2.2, we make a localized approximation of b and σ as follows. Let $\mathfrak{t}_n^\xi$ be defined in (1.6) for $\mathbb{M} = \mathbb{H}$, and let

$$\phi_n(\xi)(t) := \xi(t \wedge \mathfrak{t}_n^\xi), \quad \xi \in \mathscr{C}_{T,\mathbb{H}},\ n \ge 1,\ t \in [0, T],$$

so that $\phi_n(\xi)$ is continuous (hence measurable) in $\xi \in \mathscr{C}_{T,\mathbb{H}}$. For all $t \in [0, T]$, $\xi \in \mathscr{C}_{T,\mathbb{H}}$, $\mu \in \mathscr{P}_{T,\mathbb{H}}$ and $n \ge 1$, define

$$b^n(t, \xi, \mu) = b(t, \phi_n(\xi), \mu \circ \phi_n^{-1}), \quad \sigma^n(t, \xi, \mu) = \sigma(t, \phi_n(\xi), \mu \circ \phi_n^{-1}).$$

By ($\mathbf{C}_1$), we see that for each $n \ge 1$, b^n and σ^n satisfy (2.3) for $\gamma = 1$ and some constant K depending on n. Therefore, by Lemma 2.2, the equation

$$X^n(t) = X(0) + \int_0^t b^n(s, X_s^n, \mathscr{L}_{X_s^n})\mathrm{d}s + \int_0^t \sigma^n(s, X_s^n, \mathscr{L}_{X_s^n})\mathrm{d}W(s) \quad (2.12)$$

has a unique solution on $[0, T]$. By the definition of ϕ_n, we have

$$\phi_n(X_s^n) = X_{s\wedge\tau^n}^n \ \text{ with } \ \tau^n := \inf\{t \ge 0 : \|X^n(t)\|_{\mathbb{H}} \ge n\}, \quad s \in [0, T],\ n \ge 1.$$

Moreover, for any measurable set $A \subset \mathscr{C}_{T,\mathbb{H}}$, we have

$$\big\{(\mathscr{L}_{X_s^n}) \circ \phi_n^{-1}\big\}(A) = \mathbb{P}\big(X_s^n \in \phi_n^{-1}(A)\big) = \mathbb{P}\big(\phi_n(X_s^n) \in A\big) = \mathscr{L}_{\phi_n(X_s^n)}(A) = \mathscr{L}_{X^n_{s\wedge\tau^n}}(A),$$

so that (2.12) reduces to

$$X^n(t) = X(0) + \int_0^t b(s, X^n_{s\wedge\tau^n}, \mathscr{L}_{X^n_{s\wedge\tau^n}})\mathrm{d}s + \int_0^t \sigma(s, X^n_{s\wedge\tau^n}, \mathscr{L}_{X^n_{s\wedge\tau^n}})\mathrm{d}W(s). \tag{2.13}$$

So, by ($\mathbf{C}_3$) and applying Itô's formula to $V(\|X^n(t)\|^2_{\mathbb{H}})$ up to time $T \wedge \tau^n$, as in (2.11), we derive

$$\mathbb{E}\big[V(\|X^n_{T\wedge\tau_n}\|^2_{T,\mathbb{H}})\big] \le \delta, \quad n \ge 1. \tag{2.14}$$

Consequently, the stopping times

$$\tau_N^n := \inf\{t \ge 0 : \|X_t^n\|_{T,\mathbb{H}} \ge N\}, \quad n \ge N \ge 1$$

satisfy

$$\mathbb{P}(\tau_N^n < T) \le \frac{\delta}{V(N^2)}, \quad n \ge N \ge 1. \tag{2.15}$$

Next, by ($\mathbf{C}_1$) and (2.12), we find a constant $C_N > 0$ such that for any $n \ge N$,

$$\mathbb{E}\left[\sup_{s,t\in[0,T],|t-s|\le\varepsilon} \|X^n(t\wedge\tau_N^n) - X^n(s\wedge\tau_N^n)\|_{\mathbb{H}}\right] \le C_N\varepsilon^{\frac{1}{3}}, \quad s \le t, \ \varepsilon \in (0,T). \tag{2.16}$$

Indeed, for any $l \ge 1$, by ($\mathbf{C}_1$), (2.12) and BDG inequality, there exists a constant $C_{N,l} > 0$ such that

$$\mathbb{E}\left[\sup_{t\in[s,(s+\varepsilon)\wedge T]} \|X^n(t\wedge\tau_N^n) - X^n(s\wedge\tau_N^n)\|^{2l}_{\mathbb{H}}\right] \le C_{N,l}\varepsilon^l, \quad n \ge N, \ s \in [0, T-\varepsilon].$$

Let $k \in \mathbb{N}$ such that $k\varepsilon \in [T, T+\varepsilon)$. We find some constant $c(l) > 0$ such that for all $n \ge N$,

$$\begin{aligned}&\mathbb{E}\left[\sup_{s,t\in[0,T],|t-s|\le\varepsilon} \|X^n(t\wedge\tau_N^n) - X^n(s\wedge\tau_N^n)\|^{2l}_{\mathbb{H}}\right]\\ &\le c(l)\sum_{i=1}^k \mathbb{E}\left[\sup_{t\in[(i-1)\varepsilon,\,(i\varepsilon)\wedge T]} \|X^n(t\wedge\tau_N^n) - X^n(\{(i-1)\varepsilon\}\wedge\tau_N^n)\|^{2l}_{\mathbb{H}}\right] \le C_{N,l}(T+\varepsilon)\varepsilon^{l-1}.\end{aligned}$$

Therefore, by Jensen's inequality, we obtain

$$\mathbb{E}\left[\sup_{s,t\in[0,T],|t-s|\le\varepsilon} \|X^n(t\wedge\tau_N^n) - X^n(s\wedge\tau_N^n)\|_{\mathbb{H}}\right] \le \big\{C_{N,l}(T+\varepsilon)\big\}^{\frac{1}{2l}}\varepsilon^{\frac{1}{2}-\frac{1}{2l}}, \quad n \ge N.$$

Taking $l \ge 1$ such that $\frac{1}{2} - \frac{1}{2l} \ge \frac{1}{3}$, we obtain (2.16). Particularly, (2.16) holds true for $n = N$. In this case, $\tau_N^n = \tau_n^n = \tau^n$. Due to this and (2.14), and noting that embedding $\mathbb{H} \hookrightarrow \mathbb{B}$ is compact, we deduce from the Arzelá-Ascoli type theorem for measures that $\{\mu^n := \mathscr{L}_{X^n_{T\wedge\tau^n}}\}_{n\ge1}$ is tight in $\mathscr{P}_{T,\mathbb{B}}$. By the Prokhorov theorem, for some subsequence $\{n_k\}_{k\ge1}$ we have $\mu^{n_k} \to \mu$ weakly in $\mathscr{P}_{T,\mathbb{B}}$ as $k \to \infty$. Notice that $\phi_n(\xi) = \xi$ for $\xi \in \mathscr{C}_{T,\mathbb{H},n}$ and define

$$\tau_N^{k,j} := \tau_N^{n_k} \wedge \tau_N^{n_j}.$$

Then we find

$$\phi_{n_i}(X^{n_j}_{t\wedge\tau_N^{k,l}}) = X^{n_j}_{t\wedge\tau_N^{k,l}}, \quad i, j \in \{k, l\},$$

and

$$\lim_{k\to\infty}\lim_{l\to\infty}\mu^{n_k}\circ\phi_{n_l}^{-1}=\mu \text{ weakly in } \mathscr{P}_{T,\mathbb{B}}.$$

Indeed, by $\lim_{n\to\infty}\phi_n=\mathbf{I}$ and $\mathscr{L}_{X^{n_k}_{T\wedge\tau^{n_k}}}=\mu^{n_k}\to\mu$ weakly in $\mathscr{P}_{T,\mathbb{B}}$, we have that for all $F\in C_b(\mathscr{C}_{T,\mathbb{B}})$,

$$\begin{aligned}\lim_{k\to\infty}\lim_{l\to\infty}\int_{\mathscr{C}_{T,\mathbb{B}}}F(\xi)\{\mu^{n_k}\circ\phi_{n_l}^{-1}\}(\mathrm{d}\xi)&=\lim_{k\to\infty}\lim_{l\to\infty}\int_{\mathscr{C}_{T,\mathbb{B}}}F(\phi_{n_l}(\xi))\mu^{n_k}(\mathrm{d}\xi)\\&=\lim_{k\to\infty}\int_{\mathscr{C}_{T,\mathbb{B}}}F(\xi)\mu^{n_k}(\mathrm{d}\xi)=\int_{\mathscr{C}_{T,\mathbb{B}}}F(\xi)\mu(\mathrm{d}\xi).\end{aligned}$$

From these properties, (2.14) and ($\mathbf{C}_1$), we find a family of constants $\{\varepsilon_{k,l}:k,l\geq 1\}$ with $\varepsilon_{k,l}\to 0$ as $k,l\to\infty$ such that

$$\begin{aligned}&\left\|b\left(t,X^{n_k}_{t\wedge\tau^{k,l}_N},\mu^{n_k}_t\right)-b\left(t,X^{n_l}_{t\wedge\tau^{k,l}_N},\mu^{n_l}_t\right)\right\|_{\mathbb{H}}\\&=\left\|b\left(t,X^{n_k}_{t\wedge\tau^{k,l}_N},\mu^{n_k}_t\circ\phi_{n_l}^{-1}\right)-b\left(t,X^{n_l}_{t\wedge\tau^{k,l}_N},\mu^{n_k}_t\circ\phi_{n_l}^{-1}\right)\right\|_{\mathbb{H}}\\&\quad+\left\|b\left(t,X^{n_l}_{t\wedge\tau^{k,l}_N},\mu^{n_k}_t\circ\phi_{n_l}^{-1}\right)-b\left(t,X^{n_l}_{t\wedge\tau^{k,l}_N},\mu^{n_l}_t\circ\phi_{n_l}^{-1}\right)\right\|_{\mathbb{H}}\\&\leq C_N\left\|X^{n_k}_{s\wedge\tau^{k,l}_N}-X^{n_l}_{s\wedge\tau^{k,l}_N}\right\|_{T,\mathbb{H}}+C_N\varepsilon_{k,l},\quad l\geq k\geq N\ \ \mathbb{P}\text{-a.s.}\end{aligned}\tag{2.17}$$

Similarly, we also have

$$\begin{aligned}&\left\|\sigma(t,X^{n_k}_{t\wedge\tau^{k,l}_N},\mu^{n_k}_t)-\sigma(t,X^{n_k}_{t\wedge\tau^{k,l}_N},\mu^{n_l}_t)\right\|_{\mathcal{L}_2(\mathbb{U};\mathbb{H})}\\&\leq C_N\left\|X^{n_k}_{s\wedge\tau^{k,l}_N}-X^{n_l}_{s\wedge\tau^{k,l}_N}\right\|_{T,\mathbb{H}}+C_N\varepsilon_{k,l},\quad l\geq k\geq N\ \ \mathbb{P}\text{-a.s.}\end{aligned}\tag{2.18}$$

By (2.17), (2.18), ($\mathbf{C}_1$), and applying BDG inequality, we find a constant $C_N>0$ such that for $t\in[0,T]$ and $l\geq k\geq N$,

$$\mathbb{E}\left[\left\|X^{n_k}_{t\wedge\tau^{k,l}_N}-X^{n_l}_{t\wedge\tau^{k,l}_N}\right\|^2_{T,\mathbb{H}}\right]\leq C_N^2\int_0^T\mathbb{E}\left[\left\|X^{n_k}_{s\wedge\tau^{k,l}_N}-X^{n_l}_{s\wedge\tau^{k,l}_N}\right\|^2_{T,\mathbb{H}}\right]\mathrm{d}s+C_N^2\varepsilon_{k,l}^2T.$$

Applying Grönwall's inequality with noting that $\varepsilon_{k,l}\to 0$ as $k,l\to\infty$, we derive

$$\lim_{k\to\infty}\sup_{l\geq k}\mathbb{E}\left[\left\|X^{n_k}_{T\wedge\tau^{k,l}_N}-X^{n_l}_{T\wedge\tau^{k,l}_N}\right\|^2_{T,\mathbb{H}}\right]\leq C_N^2\lim_{k\to\infty}\sup_{l\geq k}\varepsilon^2_{k,l}T\mathrm{e}^{C_N^2T}=0.\tag{2.19}$$

Then we infer from (2.15) that for any $\epsilon>0$,

$$\begin{aligned}&\mathbb{P}\left(\|X^{n_k}_T-X^{n_l}_T\|_{T,\mathbb{H}}>\epsilon\right)\\&\leq\mathbb{P}(\tau^{n_k}_N\leq T)+\mathbb{P}(\tau^{n_l}_N\leq T)+\mathbb{P}\left(\|X^{n_k}_{T\wedge\tau^{k,l}_N}-X^{n_l}_{T\wedge\tau^{k,l}_N}\|_{T,\mathbb{H}}>\epsilon\right)\\&\leq\frac{2\delta}{V(N^2)}+\mathbb{P}\left(\|X^{n_k}_{T\wedge\tau^{k,l}_N}-X^{n_l}_{T\wedge\tau^{k,l}_N}\|_{T,\mathbb{H}}>\epsilon\right),\quad l\geq k\geq N.\end{aligned}$$

Combining this with (2.19), we obtain

$$\lim_{k\to\infty}\sup_{l\geq k}\mathbb{P}\left(\|X^{n_k}_T-X^{n_l}_T\|_{T,\mathbb{H}}>\epsilon\right)\leq\frac{2\delta}{V(N^2)},\quad N\geq 1,\ \epsilon>0.$$

Letting $N \to \infty$, we conclude that $X_T^{n_k}$ converges in probability to some $\mathscr{C}_{T,\mathbb{H}}$-valued random variable X_T. Since for each $n \ge 1$, X_T^n is adapted, so is X_T. Therefore, up to a subsequence $\{\tilde{n}_k\}_{k\ge1}$, we have $\mathbb{P}$-a.s.

$$\lim_{n\to\infty} \|X_T^{\tilde{n}_k} - X_T\|_{T,\mathbb{H}} = 0.$$

In particular, $\mathscr{L}_{X_T^{\tilde{n}_k}} \to \mathscr{L}_{X_T}$ weakly in $\mathscr{P}_{T,\mathbb{H}}$, and

$$\tau_N' := \inf\left\{\sup_{k\ge1} \|X^{\tilde{n}_k}(t)\|_{\mathbb{H}} \ge N\right\} \uparrow \infty \text{ as } N \uparrow \infty. \tag{2.20}$$

Indeed, let $\tau_\infty' := \lim_{N\to\infty} \tau_N'$. Since $X_T^{n_k}$ converges in probability to X_T, we may take a subsequence such that

$$\mathbb{P}\Big(\|X_T^{n_k} - X_T\|_{T,\mathbb{H}} > 1\Big) \le 2^{-k}, \quad k \ge 1.$$

Since $\|X_T\|_{T,\mathbb{H}} < \infty$ $\mathbb{P}$-a.s., we find

$$\{\tau_\infty' < \infty\} \subset \bigcup_{l\ge k} \{\|X_T^{n_k} - X_T\|_{T,\mathbb{H}} > 1\}$$

and hence

$$\mathbb{P}\left(\tau_\infty' < \infty\right) \le \sum_{l\ge k} \mathbb{P}\Big(\|X_T^{n_k} - X_T\|_{T,\mathbb{H}} > 1\Big) \le \sum_{l\ge k} 2^{-l} = 2^{-(k-1)}.$$

Letting $k \to \infty$, we have $\mathbb{P}\left(\tau_\infty' < \infty\right) = 0$.

Since $\mu^{\tilde{n}_k} \to \mu$ weakly in $\mathscr{P}_{T,\mathbb{B}} \supset \mathscr{P}_{T,\mathbb{H}}$, as is proved above, we have $\mathscr{L}_{X_T} = \mu$. Combining this with ($\mathbf{C}_1$), ($\mathbf{C}_2$) and (2.14), we may let $k \to \infty$ in (2.12) (equivalently, (2.13)) for $n = \tilde{n}_k$ to conclude that X_T satisfies

$$\begin{aligned}
&\langle X(t \wedge \tau_N'), \psi\rangle_{\mathbb{H}} - \langle X(0), \psi\rangle_{\mathbb{H}} \\
&= \int_0^{t\wedge\tau_N'} \langle b(s, X_s, \mu_s), \psi\rangle_{\mathbb{H}} \mathrm{d}s + \int_0^{t\wedge\tau_N'} \langle \sigma(s, X_s, \mu_s)\mathrm{d}W(s), \psi\rangle_{\mathbb{H}}, \quad t \in [0,T],\ N \ge 1,\ \psi \in \mathbb{H}_0.
\end{aligned}$$

Since $\mathbb{H}_0$ is dense in $\mathbb{H}$ and $\tau_N' \uparrow \infty$ as $N \uparrow \infty$, this implies that X_T solves (2.1).

(b) Uniqueness. If C_N is bounded, by letting $N \to \infty$ in ($\mathbf{C}_4$) we find a global Lipschitz condition on the coefficients which, as is well known, implies the pathwise uniqueness. So, below we assume $C_N \to \infty$ as $N \to \infty$.

(b1) We first prove the pathwise uniqueness up to a time $t_0 \in (0, T]$. Let X_T and Y_T be two solutions with $X(0) = Y(0)$. As explained after (1.6), t_n^X and t_n^Y are stopping times. Let

$$\tau_n = \mathrm{t}_n^X \wedge \mathrm{t}_n^Y = T \wedge \inf\{t \ge 0 : \|X(t)\|_{\mathbb{H}} \vee \|Y(t)\|_{\mathbb{H}} \ge n\}, \quad n \ge 1. \tag{2.21}$$

Then $Z_T = X_T - Y_T$ satisfies

$$\begin{aligned}
Z(t \wedge \tau_n) = &\int_0^{t\wedge\tau_n} \big(b(t, X_t, \mathscr{L}_{X_t}) - b(t, Y_t, \mathscr{L}_{Y_t})\big)\,\mathrm{d}t \\
&+ \int_0^{t\wedge\tau_n} \big(\sigma(t, X_t, \mathscr{L}_{X_t}) - \sigma(t, Y_t, \mathscr{L}_{Y_t})\big)\,\mathrm{d}W(t)
\end{aligned}$$

By Itô's formula and BDG's inequality, there exist constants $c_1, c_2 > 0$ such that

$$\begin{aligned}
&\mathbb{E}\|Z_{\tau_n\wedge s}\|_{T,\mathbb{B}}^2\\
&\le c_1\mathbb{E}\int_0^{\tau_n\wedge s}\|b(t,X_t,\mathscr{L}_{X_t})-b(t,Y_t,\mathscr{L}_{Y_t})\|_{\mathbb{B}}\|Z(t)\|_{\mathbb{B}}\mathrm{d}t\\
&\quad+c_1\mathbb{E}\left(\int_0^{\tau_n\wedge s}\|\sigma(t,X_t,\mathscr{L}_{X_t})-\sigma(t,Y_t,\mathscr{L}_{Y_t})\|_{\mathcal{L}_2(\mathbb{U};\mathbb{B})}^2\|Z(t)\|_{\mathbb{B}}^2\mathrm{d}t\right)^{\frac{1}{2}}\\
&\quad+c_1\mathbb{E}\int_0^{\tau_n\wedge s}\left\|\left(\sigma(t,X_t,\mathscr{L}_{X_t})-\sigma(t,Y_t,\mathscr{L}_{Y_t})\right)\right\|_{\mathcal{L}_2(\mathbb{U};\mathbb{B})}^2\mathrm{d}t\\
&\le\frac{1}{2}\mathbb{E}\|Z_{\tau_n\wedge s}\|_{T,\mathbb{B}}^2+c_2\mathbb{E}\int_0^{\tau_n\wedge s}\|b(t,X_t,\mathscr{L}_{X_t})-b(t,Y_t,\mathscr{L}_{Y_t})\|_{\mathbb{B}}^2\mathrm{d}t\\
&\quad+c_2\mathbb{E}\int_0^{\tau_n\wedge s}\left\|\left(\sigma(t,X_t,\mathscr{L}_{X_t})-\sigma(t,Y_t,\mathscr{L}_{Y_t})\right)\right\|_{\mathcal{L}_2(\mathbb{U};\mathbb{B})}^2\mathrm{d}t,\quad s\in[0,T].
\end{aligned}\tag{2.22}$$

Since $\pi_t := \mathscr{L}_{(X_t,Y_t)} \in \mathfrak{C}(\mathscr{L}_{X_t},\mathscr{L}_{Y_t})$ is a probability measure on $\mathscr{C}_{T,\mathbb{B}}\times\mathscr{C}_{T,\mathbb{B}}$, for the function

$$F(\xi,\eta):=\|\xi_{t\wedge \mathrm{t}_n^\xi\wedge \mathrm{t}_n^\eta}-\eta_{t\wedge \mathrm{t}_n^\xi\wedge \mathrm{t}_n^\eta}\|_{T,\mathbb{B}}^2=\sup_{s\in[0,t\wedge \mathrm{t}_n^\xi\wedge \mathrm{t}_n^\eta]}\|\xi(s)-\eta(s)\|_{\mathbb{B}}^2,\quad \xi,\eta\in\mathscr{C}_{T,\mathbb{B}},$$

we have

$$\mathbb{E}\|X_{\tau_n\wedge t}-Y_{\tau_n\wedge t}\|_{T,\mathbb{B}}^2=\mathbb{E}F(X_t,Y_t)=\int_{\mathscr{C}_{T,\mathbb{B}}\times\mathscr{C}_{T,\mathbb{B}}}F(\xi,\eta)\pi_t(\mathrm{d}\xi,\mathrm{d}\eta).$$

Combining this with the definition of $\mathbb{W}_{2,\mathbb{B},n}$ (see (1.7)), we obtain

$$\mathbb{W}_{2,\mathbb{B},n}(\mathscr{L}_{X_t},\mathscr{L}_{Y_t})^2\le\int_{\mathscr{C}_{T,\mathbb{B}}\times\mathscr{C}_{T,\mathbb{B}}}F(\xi,\eta)\,\pi_t(\mathrm{d}\xi,\mathrm{d}\eta)=\mathbb{E}\|X_{\tau_n\wedge t}-Y_{\tau_n\wedge t}\|_{T,\mathbb{B}}^2.$$

So, by ($\mathbf{C}_4$), we have

$$\begin{aligned}
&\mathbb{E}\int_0^{\tau_n\wedge s}\left\{\|b(t,X_t,\mathscr{L}_{X_t})-b(t,Y_t,\mathscr{L}_{Y_t})\|_{\mathbb{B}}^2+\left\|\left(\sigma(t,X_t,\mathscr{L}_{X_t})-\sigma(t,Y_t,\mathscr{L}_{Y_t})\right)\right\|_{\mathcal{L}_2(\mathbb{U};\mathbb{B})}^2\right\}\mathrm{d}t\\
&\le C_n\mathbb{E}\int_0^{\tau_n\wedge s}\left[\|X_t-Y_t\|_{T,\mathbb{B}}^2+\mathbb{W}_{2,\mathbb{B},n}(\mathscr{L}_{X_t},\mathscr{L}_{Y_t})^2+C_0\mathrm{e}^{-C_n\varepsilon}\right]\mathrm{d}t\\
&\le C_n\int_0^s\left[\mathbb{E}\|Z_{\tau_n\wedge s}\|_{T,\mathbb{B}}^2+\mathbb{W}_{2,\mathbb{B},n}(\mathscr{L}_{X_t},\mathscr{L}_{Y_t})^2+C_0\mathrm{e}^{-C_n\varepsilon}\right]\mathrm{d}t\\
&\le 2C_n\int_0^s\mathbb{E}\|Z_{\tau_n\wedge s}\|_{T,\mathbb{B}}^2\mathrm{d}t+C_nC_0\mathrm{e}^{-C_n\varepsilon},
\end{aligned}$$

which together with (2.22) yields

$$\mathbb{E}\left[\|Z_{\tau_n\wedge s}\|_{T,\mathbb{B}}^2\right]\le CC_n\int_0^s\left\{\mathbb{E}\|Z_{\tau_n\wedge t}\|_{\mathbb{B}}^2+C_0\mathrm{e}^{-\varepsilon C_n}\right\}\mathrm{d}t,\quad n\ge 1\tag{2.23}$$

for some constant $C>0$. Applying Fatou's lemma and Grönwall's inequality, we derive

$$\mathbb{E}\|Z_s\|_{T,\mathbb{B}}^2\le\liminf_{n\to\infty}\mathbb{E}\left[\|Z_{\tau_n\wedge s}\|_{T,\mathbb{B}}^2\right]\le sCC_0\liminf_{n\to\infty}C_n\mathrm{e}^{-C_n(\varepsilon-Cs)}=0,\quad T\ge s\in(0,\varepsilon/C).$$

This implies the pathwise uniqueness up to time $t_0:=\{\varepsilon/C\}\wedge T$.

(b2) If $t_0 = T$, then the proof is finished. Otherwise, since $Z_{t_0} = 0$, (2.23) implies

$$\mathbb{E}\left[\|Z_{\tau_n\wedge s}\|_{T,\mathbb{B}}^2\right] \le CC_n \int_{t_0}^{s} \mathbb{E}\|Z_{\tau_n\wedge t}\|_{\mathbb{B}}^2 \mathrm{d}t + sC_0\mathrm{e}^{-\varepsilon C_n},\ \ n\ge 1,\ s\in[t_0,T].$$

Using Fatou's lemma and Grönwall's inequality as before, we arrive at

$$\begin{aligned}\mathbb{E}\|Z_s\|_{T,\mathbb{B}}^2 &\le \liminf_{n\to\infty} \mathbb{E}\left[\|Z_{\tau_n\wedge s}\|_{T,\mathbb{B}}^2\right]\\ &\le sCC_0 \liminf_{n\to\infty} C_n \mathrm{e}^{-C_n(\varepsilon - C(s-t_0))} = 0,\ \ T\ge s\in(t_0, t_0+\varepsilon/C).\end{aligned}$$

Thus, the uniqueness holds up to time $(2t_0)\wedge T$. Repeating the procedure for finite many times, we prove the uniqueness up to time T. The proof of Proposition 2.1 is completed.

3 Proof of Theorem 1.1

Proof of (i) in Theorem 1.1. For each $n\ge 1$, let

$$b_n(t,\xi,\mu) := B_n(t,\xi(t)) + b(t,\xi_t,\mu_t),\ \ (t,\xi,\mu)\in[0,T]\times\mathscr{C}_{T,\mathbb{H}}\times\mathscr{P}_{T,\mathbb{H}}.$$

Obviously, $(\mathbf{A}_1)$–$(\mathbf{A}_3)$ imply $(\mathbf{C}_1)$–$(\mathbf{C}_3)$ for (b_n,σ) replacing (b,σ). Thus, by Proposition 2.1, there exists a continuous adapted process $X^n(t)$ on $\mathbb{H}$ such that

$$\begin{aligned}X^n(t) = X(0) &+ \int_0^t \left\{B_n(s,X^n(s)) + b(s,X_s^n,\mathscr{L}_{X_s^n})\right\}\mathrm{d}s\\ &+\int_0^t \sigma(s,X_s^n,\mathscr{L}_{X_s^n})\mathrm{d}W(s),\ \ t\in[0,T],\end{aligned}\tag{3.1}$$

and

$$\mathbb{E}\left[V(\|X_T^n\|_{T,\mathbb{H}}^2)\right] \le \delta = 2K_1T + 1 + \frac{64}{K_2}\left(K_1T + \mathbb{E}[V(\|X_0\|_{\mathbb{H}}^2)]\right),\ \ n\ge 1.\tag{3.2}$$

As a result, by convenient abuse of notation, the stopping times

$$\tau_N^n := \inf\{t\ge 0: \|X_t^n\|_{T,\mathbb{H}} \ge N\},\ \ n,N\ge 1$$

satisfy

$$\mathbb{P}(\tau_N^n < T) \le \frac{\delta}{V(N^2)},\ \ n,N\ge 1.\tag{3.3}$$

Next, similarly to (2.16), by $(\mathbf{A}_1)$, the first inequality in $(\mathbf{A}_2)$, (3.1) and noting that $\|\cdot\|_{\mathbb{B}} \le c\|\cdot\|_{\mathbb{H}}$ for some constant $c>0$, we find a constant $C_N>0$ such that

$$\mathbb{E}\left[\sup_{s,t\in[0,T],|t-s|\le\varepsilon} \|X^n(t\wedge\tau_N^n) - X^n(s\wedge\tau_N^n)\|_{\mathbb{B}}\right] \le C_N\varepsilon^{\frac{1}{3}},\ \ s\le t,\ \varepsilon\in(0,T).\tag{3.4}$$

Now, combining (3.4) with (3.3), we arrive at

$$\begin{aligned}&\mathbb{E}\left[\sup_{s,t\le T,|s-t|\le\varepsilon}(1\wedge\|X^n(s)-X^n(t)\|_{\mathbb{B}})\right]\\ &\le \mathbb{P}(\tau_N^n\le T) + \mathbb{E}\left[\sup_{s,t\le T\wedge\tau_N^n,|s-t|\le\varepsilon}(1\wedge\|X^n(s)-X^n(t)\|_{\mathbb{B}})\right]\\ &\le \frac{\delta}{V(N^2)} + C_N\varepsilon^{\frac{1}{3}},\ \ n,N\ge 1,\varepsilon>0.\end{aligned}$$

Since $V(N)\uparrow\infty$ as $N\uparrow\infty$, we obtain

$$\mathbb{E}\left[\sup_{s,t\le T,|s-t|\le\varepsilon}\left(1\wedge\|X^n(s)-X^n(t)\|_{\mathbb{B}}\right)\right]\le\inf_{N>0}\left\{\frac{\delta}{V(N^2)}+C_N\varepsilon^{\frac{1}{3}}\right\}\downarrow 0 \text{ as } \varepsilon\downarrow 0.$$

Due to this and (3.2), one can use the Arzelá-Ascoli theorem for measures to find that $\{\mu^n:=\mathscr{L}_{X_T^n}\}_{n\ge1}$ is tight in $\mathscr{P}_{T,\mathbb{B}}$, so is $\{\Lambda^n:=\mathscr{L}_{(X_T^n,Y_T^n,W_T)}\}_{n\ge1}$, where W_T is a continuous process on a separable Hilbert space $\tilde{\mathbb{U}}$ such that the embedding $\mathbb{U}\subset\tilde{\mathbb{U}}$ is Hilbert-Schmidt, and

$$Y^n(t):=\int_0^t\sigma(s,X_s^n,\mu_s^n)\mathrm{d}W(s),\quad t\in[0,T]$$

is a continuous process on $\mathbb{B}$. By the Prokhorov theorem, there exists a subsequence $\{n_k\}_{k\ge1}$ such that $\mu^{(n_k)}\to\mu$ weakly in $\mathscr{P}_{T,\mathbb{B}}$, and $\Lambda^{n_k}\to\Lambda$ weakly in the probability space on $\mathscr{P}(\mathscr{C}^2_{T,\mathbb{B}}\times\tilde{\mathbb{U}})$. Then the Skorokhod theorem guarantees that there exists a complete filtration probability space $(\tilde\Omega,\{\tilde{\mathscr{F}}_t\}_{t\ge0},\tilde{\mathbb{P}})$ and a sequence $(\tilde X_T^{n_k},\tilde Y_T^{n_k},\tilde W_T^{n_k})$ such that $\Lambda^{n_k}=\mathscr{L}_{(\tilde X_T^{n_k},\tilde Y_T^{n_k},\tilde W_T^{n_k})|\tilde{\mathbb{P}}}$ and

$$\lim_{k\to\infty}\left(\|\tilde X_T^{n_k}-\tilde X_T\|_{T,\mathbb{B}}+\|\tilde Y_T^{n_k}-\tilde Y_T\|_{T,\mathbb{B}}\right)=0 \tag{3.5}$$

holds for some continuous adapted process $(\tilde X_T,\tilde Y_T)$ on $\mathbb{B}$. Since the embedding $\mathbb{H}\hookrightarrow\mathbb{B}$ is continuous, there exist continuous maps $\pi_m:\mathbb{B}\to\mathbb{H},\ m\ge1$ such that

$$\|\pi_m x\|_{\mathbb{H}}\le\|x\|_{\mathbb{H}},\quad \lim_{m\to\infty}\|\pi_m x\|_{\mathbb{H}}=\|x\|_{\mathbb{H}},\quad x\in\mathbb{B},$$

where $\|x\|_{\mathbb{H}}:=\infty$ if $x\notin\mathbb{H}$. Recalling $\mathscr{L}_{\tilde X_T^{n_k}|\tilde{\mathbb{P}}}=\mathscr{L}_{X_T^{n_k}|\mathbb{P}}$, $\tilde X_T^{n_k}\to\tilde X_T$ in $\mathscr{C}_{T,\mathbb{B}}$ as $k\to\infty$, (3.2) and Fatou's lemma, one has

$$\begin{aligned}\tilde{\mathbb{E}}\left[V(\|\tilde X_T\|^2_{T,\mathbb{H}})\right]&\le\tilde{\mathbb{E}}\left[\lim_{m\to\infty}V(\|\pi_m\tilde X_T\|^2_{T,\mathbb{H}})\right]\le\liminf_{m\to\infty}\tilde{\mathbb{E}}\left[V(\|\pi_m\tilde X_T\|^2_{T,\mathbb{H}})\right]\\&=\liminf_{m\to\infty}\liminf_{k\to\infty}\tilde{\mathbb{E}}\left[V(\|\pi_m\tilde X_T^{n_k}\|^2_{T,\mathbb{H}})\right]\le\delta<\infty.\end{aligned} \tag{3.6}$$

Similar to (2.20), we can infer from $\mathscr{L}_{\tilde X_T^{n_k}|\tilde{\mathbb{P}}}=\mathscr{L}_{X_T^{n_k}|\mathbb{P}}$, (3.2) and (3.6) that $\tilde{\mathbb{P}}$-a.s.,

$$\tilde\tau_N:=\inf\left\{t\ge0:\sup_{k\ge1}\|\tilde X^{n_k}(t)\|_{\mathbb{H}}\ge N\right\}\uparrow\infty \text{ as } N\uparrow\infty. \tag{3.7}$$

Since $\tilde Y_T^{n_k}$ is a continuous local martingale on $\mathbb{B}$ with quadratic variational process

$$\langle\tilde Y^{n_k}\rangle(t)=\int_0^t(\sigma^*\sigma)\left(s,\tilde X_s^{n_k},\mu_s^{n_k}\right)\mathrm{d}s,\quad t\in[0,T],$$

we deduce from (3.2), (3.5), (3.7) and ($\mathbf{A}_1$) that $\tilde Y_T$ is a continuous local martingale on $\mathbb{B}$ with quadratic variational process

$$\langle\tilde Y\rangle(t)=\int_0^t(\sigma^*\sigma)\left(s,\tilde X_s,\mathscr{L}_{\tilde X_s|\tilde{\mathbb{P}}}\right)\mathrm{d}s,\quad t\in[0,T].$$

By the martingale representation theorem, there exists a cylindrical Brownian motion $\tilde W(t)$ on $\mathbb{U}$ under $\tilde{\mathbb{P}}$ such that

$$\tilde Y(t)=\int_0^t\sigma\left(s,\tilde X_s,\mathscr{L}_{\tilde X_s|\tilde{\mathbb{P}}}\right)\mathrm{d}\tilde W(s),\quad t\in[0,T]. \tag{3.8}$$

Moreover, it follows from (3.1) and $\mathscr{L}_{(\tilde{X}_T^{n_k}, \tilde{W}_T^{n_k})|\tilde{\mathbb{P}}} = \mathscr{L}_{(X_T^{n_k}, W_T)|\mathbb{P}}$ that $\tilde{\mathbb{P}}$-a.s.,

$$\begin{aligned}&\tilde{X}^{n_k}(t) - \tilde{X}^{n_k}(0)\\ &= \int_0^t \left\{B_{n_k}\left(s, \tilde{X}^{n_k}(s)\right) + b\left(s, \tilde{X}_s^{n_k}, \mu_s^{n_k}\right)\right\} \mathrm{d}s + \tilde{Y}^{n_k}(t), \quad t \in [0, T], \quad k \geq 1.\end{aligned}$$

So, for any $N, k \geq 1$ and $t \in [0, T]$,

$$\tilde{X}^{n_k}(t \wedge \tilde{\tau}_N) = \tilde{X}^{n_k}(0) + \int_0^{t\wedge\tilde{\tau}_N} \left\{B_{n_k}\left(s, \tilde{X}^{n_k}(s)\right) + b\left(s, \tilde{X}_s^{n_k}, \mu_s^{n_k}\right)\right\} \mathrm{d}s + \tilde{Y}^{n_k}(t \wedge \tilde{\tau}_N).$$

Summarizing this, $(\mathbf{A}_1)$, $(\mathbf{A}_2)$, (3.2), (3.5) and (3.8), and then letting $k \to \infty$, we derive

$$\begin{aligned}{}_{\mathbb{B}}\left\langle \tilde{X}(t\wedge\tilde{\tau}_N), \psi\right\rangle_{\mathbb{B}^*} =& {}_{\mathbb{B}}\left\langle \tilde{X}(0), \psi\right\rangle_{\mathbb{B}^*} + \int_0^{t\wedge\tilde{\tau}_N} \left\{{}_{\mathbb{B}}\left\langle B(s, \tilde{X}) + b\left(s, \tilde{X}_s, \mathscr{L}_{\tilde{X}_s|\tilde{\mathbb{P}}}\right), \psi\right\rangle_{\mathbb{B}^*}\right\} \mathrm{d}s\\ &+ {}_{\mathbb{B}}\left\langle \int_0^{t\wedge\tilde{\tau}_N} \sigma\left(s, \tilde{X}_s, \mathscr{L}_{\tilde{X}_s|\tilde{\mathbb{P}}}\right) \mathrm{d}\tilde{W}(s), \psi\right\rangle_{\mathbb{B}^*}, \quad \psi \in \mathbb{H}_0.\end{aligned}$$

It is easy to see that $(\mathbf{A}_1)$, $(\mathbf{A}_2)$ and (3.6) imply that for some constant $\tilde{C}_N > 0$,

$$\sup_{s\in[0,T\wedge\tilde{\tau}_N]} \|\sigma(s, \tilde{X}_s, \mathscr{L}_{\tilde{X}_s|\tilde{\mathbb{P}}})\|_{\mathcal{L}_2(\mathbb{U};\mathbb{H})} \leq \tilde{C}_N,$$

which means $\int_0^{t\wedge\tilde{\tau}_N} \sigma(s, \tilde{X}_s, \mathscr{L}_{\tilde{X}_s|\tilde{\mathbb{P}}})\mathrm{d}\tilde{W}(s)$ is an adapted continuous process on $\mathbb{H} \subset \mathbb{B}$. Similarly, by $(\mathbf{A}_1)$, $(\mathbf{A}_2)$ and (3.6),

$$\int_0^{t\wedge\tilde{\tau}_N} \{B(s, \tilde{X}) + b(s, \tilde{X}_s, \mathscr{L}_{\tilde{X}_s|\tilde{\mathbb{P}}})\}\mathrm{d}s$$

is a continuous process on $\mathbb{B}$ as well. On account of (3.6) and (3.7), we identify that $(\tilde{X}_T, \tilde{W}_T)$ is a weak solution to (1.2).

Proof of (ii) in Theorem 1.1. Now, assume $(\mathbf{A}_4)$. We aim to prove the continuity of $\tilde{X}(t)$ in $\mathbb{H}$. Since $X(t)$ is an adapted continuous process on $\mathbb{B}$, and hence weak continuous in $\mathbb{H}$, it suffices to prove the continuity of $[0, T] \ni t \mapsto \|\tilde{X}(t)\|_{\mathbb{H}}$. By (3.7), we only need to prove the continuity up to time $\tilde{\tau}_N$ for each $N \geq 1$, where τ_N is given in (3.7). If $\tilde{X} \in \mathbb{H}$, then $B(t, \tilde{X}) \in \mathbb{B}$ and $\langle B(t, \tilde{X}), \tilde{X}\rangle_{\mathbb{H}}$ does not make sense, therefore we can not use the Itô formula to $\|\tilde{X}\|_{\mathbb{H}}^2$ directly. To overcome this difficulty, we consider $\|T_m\tilde{X}\|_{\mathbb{H}}^2$ firstly, where T_m is the operator as in $(\mathbf{A}_4)$. Applying T_m to (1.2) with noting $(\mathbf{A}_4)$, we see that

$$\begin{aligned}T_m\tilde{X}(t\wedge\tilde{\tau}_N) = T_m(\tilde{X}(0)) &+ \int_0^{t\wedge\tilde{\tau}_N} T_m\left\{B(r, \tilde{X}(r)) + b(r, \tilde{X}_r, \mathscr{L}_{\tilde{X}_r|\tilde{\mathbb{P}}})\right\}\mathrm{d}r\\ &+ \int_0^{t\wedge\tilde{\tau}_N} T_m\sigma(r, \tilde{X}_r, \mathscr{L}_{\tilde{X}_r|\tilde{\mathbb{P}}})\mathrm{d}W(r), \quad t \in [0, T]\end{aligned}$$

is an L^p-semimartingale on $\mathbb{H}$ for any $p \in [1, \infty)$.

Combining this with $(\mathbf{A}_1)$, $(\mathbf{A}_4)$ and the Itô's formula, we find a constant $C_N > 0$ such that for $m \geq 1$,

$$\tilde{\mathbb{E}}\left[\left(\|T_m\tilde{X}(t\wedge\tilde{\tau}_N)\|_{\mathbb{H}}^2 - \|T_m\tilde{X}(s\wedge\tilde{\tau}_N)\|_{\mathbb{H}}^2\right)^4\right] \leq C_N(t-s)^2, \quad [s, t] \subset [0, T], \ t - s < 1.$$

Since $\|T_m x - x\|_{\mathbb{H}} \to 0$ as $m \to \infty$ holds for $x \in \mathbb{H}$ and $\tilde{X}(t)$ takes values in $\mathbb{H}$, Fatou's lemma implies

$$\tilde{\mathbb{E}}\left[\left(\|\tilde{X}(t \wedge \tilde{\tau}_N)\|_{\mathbb{H}}^2 - \|\tilde{X}(s \wedge \tilde{\tau}_N)\|_{\mathbb{H}}^2\right)^4\right] \le C_N (t-s)^2, \quad [s,t] \subset [0,T], \ t-s<1.$$

Therefore, Kolmogorov's continuity theorem ensures the continuity of $t \mapsto \|\tilde{X}(t \wedge \tilde{\tau}_N)\|_{\mathbb{H}}$ as desired.

Proof of (iii) in Theorem 1.1. By (i) in Theorem 1.1, (1.2) has a weak solution. Moreover, for any fixed $\mu \in \mathscr{P}^w_{T,\mathbb{H}}$, it is easy to deduce from $(\mathbf{A}_1)$, $(\mathbf{A}_2)$, $(\mathbf{A}_3)$ and $(\mathbf{A}_5)$ that the distribution independent SPDE

$$\mathrm{d}X^\mu(t) = \left\{B(t, X^\mu(t)) + b(t, X^\mu_t, \mu_t)\right\}\mathrm{d}t + \sigma(t, X^\mu_t, \mu_t)\mathrm{d}W_t, \quad X^\mu(0) = X_0$$

has a unique solution. So, by a Yamada-Watanabe type principle (see for instance [47, Lemma 3.4] and [48]), it remains to prove the pathwise uniqueness.

As is explained in step **(b2)** in the proof of Proposition 2.1, we assume that $C_N \to \infty$ as $N \to \infty$ and it suffices to prove the pathwise uniqueness up to a time $t_0 > 0$ independent of the initial value $X(0)$. Let τ_n be defined by (2.21). As is shown in **(b1)** in the proof of Proposition 2.1, it follows from $(\mathbf{A}_5)$, Itô's formula and BDG inequality that there is a constant $K_0 > 1$ such that

$$\mathbb{E}\left[\|Z_{\tau_n \wedge s}\|^2_{T,\mathbb{B}}\right] \le K_0 C_n \int_0^s \left(\mathbb{E}\left[\|Z_{\tau_n \wedge r}\|^2_{T,\mathbb{B}}\right] + \mathrm{e}^{-\varepsilon C_n}\right)\mathrm{d}r, \quad s \in [0,T], n \ge 1.$$

By Fatou's lemma and Grönwall's inequality, this implies

$$\mathbb{E}\left[\|Z_s\|^2_{T,\mathbb{B}}\right] \le \liminf_{n\to\infty} \mathbb{E}\left[\|Z_{\tau_n \wedge s}\|^2_{T,\mathbb{B}}\right] \le \liminf_{n\to\infty} s K_0 \mathrm{e}^{K_0 C_n s - \varepsilon C_n} = 0$$

provided $s < t_0 := \varepsilon / K_0$. Therefore pathwise uniqueness holds up to time t_0, and hence the proof is finished. □

4 Proof of Theorem 1.2

It suffices to verify conditions in Theorem 1.1 for suitable choices of $\mathbb{H}$, $\mathbb{B}$, B_n, J_n and T_n. Let $j(x)$ be a Schwartz function such that $0 \le \widehat{j}(\xi) \le 1$ for all the $\xi \in \mathbb{R}^d$ and $\widehat{j}(\xi) = 1$ for any $|\xi| \le 1$. For any $n \ge 1$ and $f \in H^0 := L^2(\mathbb{T}^d \to \mathbb{R}^d; \mu)$, we define

$$J_n f := j_n * f, \quad j_n(x) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}^d} \widehat{j}(k/n)\, \mathrm{e}^{\mathrm{i}\langle k, \cdot\rangle}, \tag{4.1}$$

and

$$T_n f := (\mathbf{I} - n^{-2}\Delta)^{-1} f = \sum_{k \in \mathbb{Z}^d} \left(1 + n^{-2}|k|^2\right)^{-1} \widehat{f}(k)\, \mathrm{e}^{\mathrm{i}\langle k, \cdot\rangle}. \tag{4.2}$$

It can be shown that for any $s \ge 0$, $f, g \in H^s$ and $n \ge 1$, cf. [26, 27],

$$D^s J_n = J_n D^s, \quad D^s T_n = T_n D^s, \tag{4.3}$$

$$\langle J_n f, g\rangle_{H^s} = \langle f, J_n g\rangle_{H^s}, \quad \langle T_n f, g\rangle_{H^s} = \langle f, T_n g\rangle_{H^s},, \tag{4.4}$$

$$\|J_n f\|_{H^s} \vee \|T_n f\|_{H^s} \le \|f\|_{H^s}, \ \|\nabla J_n f\|_{H^s} \vee \|\nabla T_n f\|_{H^s} \lesssim n\|f\|_{H^s}, \tag{4.5}$$

where for two sequences of positive numbers $\{a_n, b_n\}_{n\geq 1}$, $a_n \lesssim b_n$ means that $a_n \leq cb_n$ holds for some constant $c > 0$ and all $n \geq 1$. Moreover, we write $a_n = \mathrm{o}(b_n)$ if $\lim_{n\to\infty} b_n^{-1} a_n = 0$. Then

$$\|X - J_n X\|_{H^r} = \mathrm{o}(n^{r-s}), \ \ 0 \leq r \leq s, \ X \in H^s. \tag{4.6}$$

To verify conditions in Theorem 1.1, we need more properties of J_n, T_n and D^s. In general, the commutator for two operators P, Q is given by

$$[P, Q] := PQ - QP.$$

Lemma 4.1 *There exists a constant $C > 0$ such that for all $f \in L^2(\mathbb{T}^d \to \mathbb{R}^d; \mu)$ and $g \in W^{1,\infty}(\mathbb{T}^d \to \mathbb{R}^d; \mu)$,*

$$\|[T_n, (g \cdot \nabla)] f\|_{L^2(\mu)} \leq C \|\nabla g\|_{L^\infty} \|f\|_{L^2(\mu)}, \ \ n \geq 1.$$

Proof It is worth noticing that in one-dimensional case, the above commutator estimate has been established for a different mollifier on the whole space, see [49]. In current setting, periodicity is required and the mollifier is different, so we present also the proof here.

Let ∂_l denote the l-th partial derivative in $\mathbb{R}^d$. Since $[T_n, \partial_l] = 0$ for $l \in \{1, 2, \cdots, d\}$, we have

$$\begin{aligned}\|[T_n, (g \cdot \nabla)] f\|_{L^2(\mu)}^2 =& \sum_{j=1}^d \Big\| \sum_{l=1}^d T_n \left(g_l \partial_l f_j\right) - \sum_{l=1}^d g_l \partial_l \left(T_n f_j\right) \Big\|_{L^2(\mu)}^2 \\ \leq& d \sum_{j,l=1}^d \left\|T_n \left(g_l \partial_l f_j\right) - g_l T_n \left(\partial_l f_j\right)\right\|_{L^2(\mu)}^2 = d \sum_{j,l=1}^d \Big\|[T_n, g_l] \partial_l f_j\Big\|_{L^2(\mu)}^2.\end{aligned}$$

Hence, it suffices to find a constant $c > 0$ such that

$$\|[T_n, g] \partial_l f\|_{L^2(\mu)}^2 \leq c \|\nabla g\|_{L^\infty}^2 \|f\|_{L^2(\mu)}^2, \ \ f, g \in C^1(\mathbb{T}^d), \ 1 \leq l \leq d, \ n \geq 1. \tag{4.7}$$

Noting that

$$\frac{1}{1 + \frac{1}{n^2}|k|^2} - \frac{1}{1 + \frac{1}{n^2}|m|^2} = \frac{\langle m - k, m + k\rangle}{n^2(1 + \frac{1}{n^2}|k|^2)(1 + \frac{1}{n^2}|m|^2)} = \sum_{j=1}^d \frac{(m_j - k_j)(m_j + k_j)}{n^2(1 + \frac{1}{n^2}|k|^2)(1 + \frac{1}{n^2}|m|^2)},$$

by $T_n = (I - \frac{1}{n^2}\Delta)^{-1}$, (1.10), and (1.11), we find a constant $c > 0$ such that

$$\begin{aligned}&\|[T_n, g] \partial_l f\|_{L^2(\mu)}^2 = \sum_{k\in\mathbb{Z}^d} \left|(1 + n^{-2}|k|^2)^{-1} \mathcal{F}(g \partial_l f)(k) - \mathcal{F}(g T_n \partial_l f)(k)\right|^2 \\ &= \sum_{k\in\mathbb{Z}^d} \left| \left(\frac{m_l}{1 + \frac{1}{n^2}|k|^2} - \frac{m_l}{1 + \frac{1}{n^2}|m|^2} \right) \sum_{m\in\mathbb{Z}^d} \widehat{g}(k - m) \widehat{f}(m) \right|^2 \\ &= \sum_{k\in\mathbb{Z}^d} \left| \sum_{j=1}^d \sum_{m\in\mathbb{Z}^d} \widehat{\partial_j g}(k - m) \left\{ \frac{\mathcal{F}(T_n \partial_l \partial_j f)(m)}{n^2(1 + \frac{1}{n^2}|k|^2)} + \frac{\mathrm{i} k_j \mathcal{F}(T_n \partial_l f)(m)}{n^2(1 + \frac{1}{n^2}|k|^2)} \right\} \right|^2 \\ &= \sum_{k\in\mathbb{Z}^d} \left| \sum_{j=1}^d \left\{ \frac{\mathcal{F}\left((\partial_j g) T_n \partial_l \partial_j f\right)(k)}{n^2(1 + \frac{1}{n^2}|k|^2)} + \frac{\mathrm{i} k_j \mathcal{F}\left((\partial_j g) T_n \partial_l f\right)(k)}{n^2(1 + \frac{1}{n^2}|k|^2)} \right\} \right|^2 \\ &\leq 2d \sum_{j=1}^d \left\{ \frac{1}{n^4} \left\|(\partial_j g) T_n \partial_l \partial_j f\right\|_{L^2(\mu)}^2 + \frac{1}{n^2} \left\|(\partial_j g) T_n \partial_l f\right\|_{L^2(\mu)}^2 \right\} \leq c \|\nabla g\|_{L^\infty}^2 \|f\|_{L^2(\mu)}^2,\end{aligned}$$

where the last step is due to the fact that

$$\frac{1}{n^4}\|T_n\partial_l\partial_j f\|^2_{L^2(\mu)} + \frac{1}{n^2}\|T_n\partial_l f\|^2_{L^2(\mu)} \le C\|f\|^2_{L^2(\mu)}, \quad n \ge 1$$

holds for some constant $C > 0$. Then we obtain (4.7) and hence finish the proof. □

We also need the following lemma on the commutator estimates for D^s.

Lemma 4.2 *([50]). Let $p, p_2, p_3 \in (1,\infty)$ and $p_1, p_4 \in (1,\infty]$ such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Then for any $s > 0$, there exists a constant $C > 0$ such that

$$\|\left[D^s, f\right]g\|_{L^p(\mu)} \le C(\|\nabla f\|_{L^{p_1}(\mu)}\|D^{s-1}g\|_{L^{p_2}(\mu)} + \|D^s f\|_{L^{p_3}(\mu)}\|g\|_{L^{p_4}(\mu)})$$

holds for all $f, g \in H^s \cap W^{1,\infty}(\mathbb{T}^d \to \mathbb{R}^d; \mu)$.

We are now ready to prove Theorem 1.2. Let s, s' be given in Assumption (B). Take $\mathbb{H} = H^s$, $\mathbb{B} = H^{s'}$, $\mathbb{H}_0 = C^\infty(\mathbb{T}^d; \mathbb{R}^d)$, and let J_n and T_n be given in (4.1) and (4.2), respectively. Take

$$B(t, X) = B(X) = -(X\cdot\nabla)X, \quad B_n(t, X) = B_n(X) = J_n B(J_n X), \quad t \ge 0, \ X \in H^s.$$

Obviously, $(\mathbf{A}_1)$ follows from $(\mathbf{B}_1)$. So, it remains to verify $(\mathbf{A}_2)$, $(\mathbf{A}_3)$, $(\mathbf{A}_4)$ and $(\mathbf{A}_5)$.

Verifying $(\mathbf{A}_2)$. By (4.5), we have

$$\|B_n(t, X)\|_{H^s} \le \|(J_n X\cdot\nabla)J_n X\|_{H^s} \le \|J_n X\|_{H^s}\|\nabla J_n X\|_{H^s} \le n\,\|X\|^2_{H^s},$$

and

$$\begin{aligned}\|B_n(t, X) - B_n(t, Y)\|_{H^s} &\le \|(J_n X\cdot\nabla)J_n X - (J_n Y\cdot\nabla)J_n Y\|_{H^s}\\ &\le \|X\|_{H^s}\|\nabla(J_n X - J_n Y)\|_{H^s} + \|X - Y\|_{H^s}\|\nabla J_n Y\|_{H^s}\\ &\lesssim n\,(\|X\|_{H^s} + \|Y\|_{H^s})\|X - Y\|_{H^s}.\end{aligned}$$

Finally, by identifying $H^{s'}$ and $(H^{s'})^*$ via the Riesz isomorphism, then $(\mathbf{A}_2)$ follows from the above estimates and (4.6).

Verifying $(\mathbf{A}_3)$. It follows from Lemma 4.2, integration by parts, $H^{s'} \hookrightarrow W^{1,\infty}$, (4.3) and (4.5) that for some $C = C_s > 0$,

$$\begin{aligned}&|\langle B_n(X), X\rangle_{H^s}|\\ &\le \left|\left\langle\left[D^s, (J_n X\cdot\nabla)J_n X\right], D^s J_n X\right\rangle_{L^2(\mu)}\right| + \left|\left\langle(J_n X\cdot\nabla)D^s J_n X, D^s J_n X\right\rangle_{L^2(\mu)}\right|\\ &\le C_s\|J_n X\|_{H^s}\|\nabla J_n X\|_{L^\infty}\|J_n X\|_{H^s} + \|\nabla J_n X\|_{L^\infty}\|J_n X\|^2_{H^s}\\ &\le (C_s + 1)\|X\|_{H^{s'}}\|X\|^2_{H^s}, \quad X \in H^s.\end{aligned}$$

Then above estimate and $(\mathbf{B}_2)$ yields $(\mathbf{A}_3)$.

Verifying $(\mathbf{A}_4)$. Let T_n be defined in (4.2). It is easy to see that (1.8) is satisfied. So, to verify $(\mathbf{A}_4)$ it remains to check (1.9). By (4.3), (4.4), (4.5), Lemma 4.2, integration by parts, Lemma 4.1, and $H^s \hookrightarrow W^{1,\infty}$, we find constants $c_1, c_2, c_3 > 0$ such that

$$\begin{aligned}
&\left|\langle T_n\{(X\cdot\nabla)X\}, T_nX\rangle_{H^s}\right| \\
&= \left|\left\langle \left[D^s, (X\cdot\nabla)X\right], D^sT_n^2X\right\rangle_{L^2(\mu)} + \left\langle T_n\{(X\cdot\nabla)D^sX\}, D^sT_nX\right\rangle_{L^2(\mu)}\right| \\
&\le \left|\left\langle \left[D^s, (X\cdot\nabla)X\right], D^sT_n^2X\right\rangle_{L^2(\mu)}\right| + \left|\left\langle [T_n, (X\cdot\nabla)]D^sX, D^sT_nX\right\rangle_{L^2(\mu)}\right| \\
&\quad + \left|\left\langle (X\cdot\nabla)D^sT_nX, D^sT_nX\right\rangle_{L^2(\mu)}\right| \\
&\le c_1\|X\|_{H^s}\|\nabla X\|_{L^\infty}\|T_n^2X\|_{H^s} + c_2\|\nabla X\|_{L^\infty}\|X\|_{H^s}\|T_nX\|_{H^s} \\
&\le c_3\|X\|_{H^s}^3, \quad X \in H^s.
\end{aligned}$$

Therefore, (1.9) holds.

Verifying ($\mathbf{A}_5$). By ($\mathbf{B}_3$), for any $N \ge 1$ it suffices to find a constant $C_N > 0$ such that

$$\langle B(t,X) - B(t,Y), X - Y\rangle_{H^{s'}} \le C_N\|X - Y\|_{H^{s'}}^2, \quad X, Y \in \mathscr{C}_{T,H^s,N}.$$

Let $Z = X - Y$. By $H^s \hookrightarrow H^{s'} \hookrightarrow W^{1,\infty}$ and Lemma 4.2, we find constants $c_1, c_2 > 0$ such that

$$\begin{aligned}
&\langle B(t,X) - B(t,Y), X - Y\rangle_{H^{s'}} \\
=& - \langle (Z\cdot\nabla)X, Z\rangle_{H^{s'}} - \langle (Y\cdot\nabla)Z, Z\rangle_{H^{s'}} \\
\le& c_1\|X\|_{H^s}\|Z\|_{H^{s'}}^2 + \left|\left\langle D^{s'}\left((Y\cdot\nabla)Z\right), D^{s'}Z\right\rangle_{L^2(\mu)}\right| \\
\le& c_1\|X\|_{H^s}\|Z\|_{H^{s'}}^2 + c_2\|D^{s'}Y\|_{L^2(\mu)}\|\nabla Z\|_{L^\infty(\mu)}\|Z\|_{H^{s'}} + c_2\|\nabla Y\|_{L^\infty}\|Z\|_{H^{s'}}^2 \\
\le& c_1\|X\|_{H^s}\|Z\|_{H^{s'}}^2 + c_2\|Y\|_{H^s}\|Z\|_{H^{s'}}^2,
\end{aligned}$$

which is the desired estimate.

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Declarations

Conflict of Interest The authors declare that they have no conflict of interest and data sharing is not applicable to this article since no datasets were generated or analyzed during the current study.

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