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Non-Gaussian Measures in Infinite Dimensional Spaces: the Gamma-Grey Noise

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Abstract

In the context of non-Gaussian analysis, Schneider [29] introduced grey noise measures, built upon Mittag-Leffler functions; analogously, grey Brownian motion and its generalizations were constructed (see, for example, [6, 7, 9, 27]). In this paper, we construct and study a new non-Gaussian measure, by means of the incomplete-gamma function (exploiting its complete monotonicity). We label this measure Gamma-grey noise and we prove, for it, the existence of Appell system. The related generalized processes, in the infinite dimensional setting, are also defined and, through the use of the Riemann-Liouville fractional operators, the (possibly tempered) Gamma-grey Brownian motion is consequently introduced. A number of different characterizations of these processes are also provided, together with the integro-differential equation satisfied by their transition densities. They allow to model anomalous diffusions, mimicking the procedures of classical stochastic calculus.

Keywords Incomplete gamma function · Completely monotone functions · Grey noise · Hitting times · Fractional Brownian motion · Elliptically contoured measures

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1 Introduction

Non-Gaussian analysis has been introduced in the Nineties (see, for example, [2, 3, 5]), in order to extend the standard infinite-dimensional (or white noise) constructions; see also [28]. In particular, grey noise has been defined for the first time by Schneider in [29], exploit-

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ing the complete monotonicity property of the Mittag-Leffler function. Consequently, grey Brownian motion was also introduced in the same paper and studied in [30], allowing to model anomalous diffusions by mimicking the classical procedures. These models represent a family of (self-similar) stochastic processes, with stationary increments, which includes, as special cases, both standard and fractional Brownian motion.

A further generalization (generalized grey Brownian motion, hereafter ggBm) is due to [27]; it is also proved in [26] that its marginal density function is the fundamental solution of a stretched time-fractional master equation.

The ggBm, denoted by $B_\alpha^\beta := \{B_\alpha^\beta(t), t > 0\}$, for any $\alpha, \beta \in (0, 1]$, is characterized by the following n-times characteristic function: for $\xi_j \in \mathbb{R}$, $j = 1, ..., n$, and $0 \leq t_1 \leq ... \leq t_n < \infty$

$$\mathbb{E}e^{i \sum_{j=1}^n \xi_j B_\alpha^\beta(t_j)} = E_\beta \left(-\frac{1}{2} \sum_{j,k=1}^n \xi_j \xi_k \gamma_\alpha(t_j, t_k) \right), \tag{1.1}$$

where $\gamma_\alpha(t_j, t_k) := t_k^\alpha + t_j^\alpha - |t_k - t_j|^\alpha$ and $E_\beta(x)$ is the Mittag-Leffler function $E_\beta(x) := \sum_{j=0}^\infty x^j / \Gamma(\beta j + 1)$, $x \in \mathbb{R}$ (see Appendix A, for details on the Mittag-Leffler function in a more general definition).

The link with Ornstein-Uhlenbeck process is explored, by means of the stochastic calculus tools, in [6]; this is made possible by the representation of ggBm as a product of a fractional Brownian motion and an independent random variable (with distribution depending on β).

It is easy to see from (1.1) that, for $\beta = 1$, the process B_α^β reduces to fractional Brownian motion with Hurst parameter $H = \alpha/2$; for $\alpha = \beta$, it is called grey Brownian motion (see [29]); on the other hand, for $\alpha = \beta = 1$, it coincides with standard Brownian motion. A slightly different construction of the process, by means of the so-called Mittag-Leffler analysis, can be found in [14] and [15]. Finally, stochastic differential equations driven by ggBm are studied in [8].

Our aim in this paper is to define, analogously to ggBm, another class of processes that includes, as special cases both standard and fractional Brownian motion. Our starting point is a result proved in [4], i.e. that the upper incomplete gamma function $\Gamma(\rho, x) := \int_x^{+\infty} e^{-w} w^{\rho-1} dw$ is completely monotone and that the inverse Laplace transform of

$$\varphi(\eta) = \Gamma(\rho, \eta), \qquad \eta \geq 0, \tag{1.2}$$

reads

$$f_\rho(y) := \mathcal{L}^{-1} \{\varphi(\cdot); y\} = 1_{y>1} G_{1,1}^{1,0} \left[\frac{1}{y} \middle| \begin{array}{c} 2 \\ 1 + \rho \end{array} \right], \qquad \rho \in (0, 1], \tag{1.3}$$

where $G_{p,q}^{m,n} [\cdot|\ ]$ is the Meijer G-function (see (A.3), in Appendix A).

Moreover, (1.3) is a proper density function, up to the constant $1/\Gamma(\rho)$. We introduce here a tempering factor θ, for $\theta \geq 0$, i.e. we will refer to $\Gamma(\rho, \theta + \eta)$, for $\eta \geq 0$; the tempering is necessary in order to ensure finite moments to the corresponding measure. The complete monotonicity of $\Gamma(\rho, \theta + \cdot)$ easily follows. Once normalized by $\Gamma(\rho, \theta)$, it will be used to define the characteristic functional of a measure, that we will call Γ-grey measure.

In Section 2 we define the Γ-grey measure both on the finite and infinite dimensional spaces, computing its moments and discussing the existence of the Appell system [21]. These steps are necessary in order to extend the non-Gaussian analysis to the Γ-grey noise space and require some well-known preliminary results on complexification and holomorphic property in infinite dimensional spaces, that we present in the Appendix (together with some formulae on special functions).

On the Γ-grey noise space, in Section 3, we define the tempered Γ-grey Brownian motion $B_{\alpha,\rho}^{\theta} := \{B_{\alpha,\rho}^{\theta}(t), t > 0\}$, for any $\alpha, \rho \in (0, 1]$, $\theta \geq 0$, as generalized process, by means of the fractional operator $M_{-}^{\alpha/2}$, defined below (in terms of Riemann-Liouville derivative or integral, depending on the values of α). The tempering parameter θ is introduced in order to ensure finiteness of moments, while the parameter ρ (of the upper-incomplete gamma function) represents the "distance" from the white noise setting: for $\rho = 1$ (for any θ), the process $B_{\alpha,\rho}^{\theta}$ coincides with the fractional Brownian motion with Hurst parameter $H = \alpha/2$, while, if we also put $\alpha = 1$, we obtain the standard Brownian motion B. We prove that, in the n-dimensional space, the tempered Γ-grey Brownian motion can be fully characterized as a product of a fractional Brownian motion and an independent random variable, defined on $[1, +\infty)$ and with distribution depending on ρ and θ. This factorization permits us to interpret the distribution of the process as a Gaussian variance mixture and, moreover, it is suitable for path-simulating purposes.

In Section 5 we discuss the time-change representation of this process (which is valid for its one-dimensional distribution and for $\theta = 0$), i.e. the following equality in distribution

$$B_{\alpha,\rho}(t) \stackrel{d}{=} B(Y_{\rho}(t^{\alpha})), \qquad t \geq 0. \tag{1.4}$$

Here we put, for simplicity, $B_{\alpha,\rho} := B_{\alpha,\rho}^{0}$ while $Y_{\rho} := \{Y_{\rho}(t), t \geq 0\}$ is a stochastic process, independent of the Brownian motion B, taking values in $[t, +\infty)$, for any t. Moreover, we derive, in the same setting, the differential equations satisfied by its characteristic function and by its transition density. Unlike what happens in the case of the ggBm, the time-stretching parameter in (1.4) depends only on α, while does not involve ρ.

2 The Gamma-Grey Noise

We define the Γ-grey noise starting from the n-dimensional Euclidean space, in analogy with the construction of the grey noise (see [29]) and the generalized grey noise (see [27]). In particular, we will follow the slightly different approach introduced by [14]. By the complete monotonicity of $\Gamma(\rho, \theta + \cdot)$, for $\theta \geq 0$ and applying the Bernstein's theorem, there exists a unique probability measure $\mu_{\rho,\theta}$ on $[0, +\infty)$ such that

$$\frac{\Gamma(\rho, \theta + \eta)}{\Gamma(\rho, \theta)} = \int_{0}^{+\infty} e^{-\eta s} d\mu_{\rho,\theta}(s), \qquad \eta, \theta \geq 0. \tag{2.1}$$

Moreover, the mapping

$$\mathbb{R}^{n} \ni \xi \to \frac{\Gamma(\rho, \theta + \frac{1}{2} \langle \xi, \xi \rangle_{euc})}{\Gamma(\rho, \theta)} \in \mathbb{R}, \tag{2.2}$$

(where $\langle \cdot, \cdot \rangle_{euc}$ denotes the Euclidean scalar product on $\mathbb{R}^{n}$) is a characteristic function.

Definition 2.1 *Let* $n \in \mathbb{N}$*,* $\rho \in (0, 1]$ *and* $\theta > 0$*. The n-dimensional* Γ*-grey measure is the unique probability measure* $v_{\rho,\theta}^{n}$ *on* $(\mathbb{R}^{n}, \mathcal{B}(\mathbb{R}^{n}))$ *that satisfies:*

$$\int_{\mathbb{R}^{n}} e^{i \langle x, \xi \rangle} dv_{\rho,\theta}^{n}(x) = \frac{\Gamma(\rho, \theta + \frac{1}{2} \langle \xi, \xi \rangle_{euc})}{\Gamma(\rho, \theta)}, \qquad \xi \in \mathbb{R}^{n}. \tag{2.3}$$

We define $\Phi_{\rho,\theta}(\xi)$ *as its characteristic function and we call* $(\mathbb{R}^{n}, \mathcal{B}(\mathbb{R}^{n}), v_{\rho,\theta}^{n})$ *the n-dimensional* Γ*-grey space.*

Remark 2.1 For $\rho = 1$ and for any θ, the measure $\nu_{\rho,\theta}^n$ reduces to the multivariate Gaussian measure (with independent components).

We now prove that the moments of $\nu_{\rho,\theta}^1$ are finite, so that we can decompose the space $L^2\left(\mathbb{R}, \nu_{\rho,\theta}^1\right)$ through the polynomials $H_n^{\rho,\theta}$. We obtain $H_n^{\rho,\theta}$ applying the Gram-Schmidt orthogonalization to the monomials x^n.

Lemma 2.1 Let $\rho \in (0, 1]$ and $\theta > 0$. The moments of $\nu_{\rho,\theta}^1$ are equal to zero, for $k = 2n+1$, $n \in \mathbb{N}$ and

$$\int_{\mathbb{R}} x^k d\nu_{\rho,\theta}^1(x) = \frac{(-1)^{n+1}(2n)!\Gamma(\rho)\theta^{\rho-n}}{n!2^n\Gamma(\rho,\theta)} E_{1,\rho+1-n}^\rho(-\theta), \qquad k = 2n, n \in \mathbb{N}. \quad (2.4)$$

The first polynomials $H_n^{\rho,\theta}$, $n = 0, 1, 2, 3$, orthogonal in $L^2\left(\mathbb{R}, \nu_{\rho,\theta}^1\right)$ and with $\deg H_n^{\rho,\theta} = n$, are given by

$$H_0^{\rho,\theta}(x) = 1 \qquad\qquad H_1^{\rho,\theta}(x) = x \qquad\qquad (2.5)$$

$$H_2^{\rho,\theta}(x) = x^2 - \frac{\theta^{\rho-1}e^{-\theta}}{\Gamma(\rho,\theta)} \qquad\qquad H_3^{\rho,\theta}(x) = x^3 - 3x(1 + (1-\rho)\theta^{-1}).$$

Proof We evaluate the derivatives of (2.3): for $n = 1$ we get

$$\frac{d}{d\xi}\Phi_{\rho,\theta}(\xi) = \frac{d}{d\xi}\frac{\Gamma(\rho, \frac{1}{2}\xi^2 + \theta)}{\Gamma(\rho,\theta)} = -\frac{\xi}{\Gamma(\rho,\theta)}e^{-(\frac{\xi^2}{2}+\theta)}(\frac{\xi^2}{2} + \theta)^{\rho-1},$$

which vanishes, for $\xi = 0$. For $n > 1$ and $l = 1, 2, \dots$ we have instead that

$$\frac{d^{l+1}}{d\xi^{l+1}}\Phi_{\rho,\theta}(\xi) = -\frac{1}{\Gamma(\rho,\theta)}\frac{d^l}{d\xi^l}\left[\xi e^{-(\frac{\xi^2}{2}+\theta)}(\frac{\xi^2}{2} + \theta)^{\rho-1}\right]$$

$$= \frac{1}{\Gamma(\rho,\theta)}\frac{d^l}{d\xi^l}\left[\xi \sum_{j=0}^\infty (-1)^{j+1}\frac{(\frac{\xi^2}{2} + \theta)^{j+\rho-1}}{j!}\right]$$

$$= \frac{1}{\Gamma(\rho,\theta)}\frac{d^l}{d\xi^l}\left[\sum_{j=0}^\infty \frac{(-1)^{j+1}}{j!}\sum_{k=0}^\infty \frac{1}{2^k}\binom{j+\rho-1}{k}\xi^{2k+1}\theta^{j+\rho-1-k}\right]$$

$$= \frac{1}{\Gamma(\rho,\theta)}\sum_{j=0}^\infty \frac{(-1)^{j+1}}{j!}\sum_{k=\lceil(l-1)/2\rceil}^\infty \frac{1}{2^k}\frac{(2k+1)!}{(2k+1-l)!}\binom{j+\rho-1}{k}\xi^{2k+1-l}\theta^{j+\rho-1-k}.$$

For $\xi = 0$ and for odd $l = 2n+1$, the term $k = (l-1)/2 = n$ is only one different from zero, so that we get:

$$\frac{d^{l+1}}{d\xi^{l+1}}\Phi_{\rho,\theta}(\xi)\bigg|_{\xi=0} = -\frac{(2n+1)!\theta^{\rho-n-1}}{n!2^n\Gamma(\rho,\theta)}\sum_{j=0}^\infty \frac{(-\theta)^j}{j!}\frac{\Gamma(j+\rho)}{\Gamma(j+\rho-n)}$$

$$= -\frac{(2n+1)!\theta^{\rho-n-1}\Gamma(\rho)}{n!2^n\Gamma(\rho,\theta)}E_{1,\rho-n}^\rho(-\theta).$$

Thus we obtain formula (2.4) and the first even moments read

$$\int_{\mathbb{R}} x^2 d\nu_{\rho,\theta}^1(x) = \frac{\theta^{\rho-1}e^{-\theta}}{\Gamma(\rho,\theta)} \qquad\qquad (2.6)$$

$$\int_{\mathbb{R}} x^4 d\nu_{\rho,\theta}^1(x), = \frac{3e^{-\theta}}{\Gamma(\rho,\theta)}\left[\theta^{\rho-1} + (1-\rho)\theta^{\rho-2}\right].$$

The polynomials in (2.5) follow from (2.6), by solving the following equations, for $H_k^{\rho,\theta}(x)$, $k = 0, 1, ...,$

$$\mathbb{E}_{\nu_{\rho,\theta}^1} \left[\left(a_0 + a_1 X + ... + X^r \right) X^k \right] = 0,$$

for $r = 0, 1, ...k$. □

Remark 2.2 For $\rho = 1$ and for any θ, formula (2.4) reduces to the k-th moment (for $k = 2n$) of a Gaussian random variable with variance 1:

$$\int_{\mathbb{R}} x^{2n} d\nu_{\rho,\theta}^1(x) = \frac{(2n)!}{n!2^n} \frac{(-1)^{n+1}\theta^{1-n}}{e^{-\theta}} E_{1,2-n}^1(-\theta) = \frac{(2n)!}{n!2^n}$$

where we use the fact that $\theta^{\rho-n} E_{1,\rho+1-n}^\rho(-\theta) = \frac{d^{n-1}}{d\theta^{n-1}} \left[\theta^{\rho-1} E_{1,\rho}^\rho(-\theta) \right]$.

Correspondingly, for $\rho = 1$ and for any θ, $H_n^{\rho,\theta}$, given in (2.5), for $n = 0, 1, 2, 3$, reduce to the first four Hermite polynomials.

We can now extend the n-dimensional Γ-grey measure to the infinite dimensional space $\mathcal{S}'(\mathbb{R})$, dual of the space of Schwartz functions $\mathcal{S}(\mathbb{R})$ (respectively $\mathcal{S}'$ and $\mathcal{S}$, hereinafter).

Recalling that $\mathcal{S} \subset L^2(\mathbb{R}, dx) \subset \mathcal{S}'$ is a nuclear triple, we can define the measure $\nu_{\rho,\theta}$ on $(\mathcal{S}', \sigma^*)$ via the Bochner-Minlos theorem, where σ^* is the σ-algebra generated by the cylinders [16].

In analogy to the above definition of $\nu_{\rho,\theta}^n$ in $\mathbb{R}^n$, we have the following:

Definition 2.2 For $\rho \in (0, 1]$, $\theta > 0$, the Γ-grey measure $\nu_{\rho,\theta}$ is the unique probability measure fulfilling

$$\int_{\mathcal{S}'} e^{i\langle x,\xi \rangle} d\nu_{\rho,\theta}(x) = \frac{\Gamma(\rho, \theta + \frac{1}{2} \langle \xi, \xi \rangle)}{\Gamma(\rho, \theta)}, \quad \xi \in \mathcal{S}. \tag{2.7}$$

We call $(\mathcal{S}', \sigma^*, \nu_{\rho,\theta})$ the Γ-grey noise space.

Remark 2.3 For $\rho = 1$ it reduces to the Gaussian white noise measure $\nu := \nu_{1,\theta}$, for any θ.

The moments and the covariance of generalized stochastic processes on $(\mathcal{S}', \sigma^*, \nu_{\rho,\theta})$ can be obtained by considering those of the one-dimensional measure, given in Lemma 2.1.

Corollary 2.1 Let $\rho \in (0, 1]$ and $\theta > 0$. Let $\xi, \eta \in \mathcal{S}$ and $n \in \mathbb{N}$, then $\int_{\mathcal{S}'} \langle x, \xi \rangle^{2n+1} d\nu_{\rho,\theta}(x) = 0$ and

$$\int_{\mathcal{S}'(\mathbb{R})} \langle x, \xi \rangle^{2n} d\nu_{\rho,\theta}(x) = \frac{(-1)^{n+1}(2n)!\Gamma(\rho)\theta^{\rho-n} \langle \xi, \xi \rangle^n}{n!2^n\Gamma(\rho, \theta)} E_{1,\rho+1-n}^\rho(-\theta). \tag{2.8}$$

Moreover,

$$\mathbb{E}_{\nu_{\rho,\theta}}(\langle \omega, \xi_1 \rangle \langle \omega, \xi_2 \rangle) = \frac{\theta^{\rho-1}e^{-\theta}}{\Gamma(\rho, \theta)} \langle \xi_1, \xi_2 \rangle, \tag{2.9}$$

for $\xi_1, \xi_2 \in \mathcal{S}$ and $\omega \in \mathcal{S}'$. Moreover, $\|\langle \cdot, \xi \rangle\|_{L^2(\nu_{\rho,\theta})}^2 = \theta^{\rho-1}e^{-\theta}\|\xi\|^2 / \Gamma(\rho, \theta)$.

Proof Since the moments are easily obtained from Lemma 2.1, we just compute the covariance as

$$\mathbb{E}(\langle \omega, \xi_1 \rangle \langle \omega, \xi_2 \rangle) = i^{-2} D_{a_1,a_2} \left(\frac{\Gamma(\rho, \theta + \frac{1}{2} \|a_1 \xi_1 + a_2 \xi_2\|^2)}{\Gamma(\rho, \theta)} \right) \Bigg|_{a_1 = a_2 = 0} \qquad \text{for } \omega \in \mathcal{S}',$$

where D_{a_1,a_2} is the derivative w.r.t. a_1 and a_2.

We can write $\|a_1 \xi_1 + a_2 \xi_2\|^2 = \langle a_1 \xi_1 + a_2 \xi_2, a_1 \xi_1 + a_2 \xi_2 \rangle$ and thanks to the bi-linearity we have $a_1^2 \|\xi_1\|^2 + 2 a_1 a_2 \langle \xi_1, \xi_2 \rangle + a_2^2 \|\xi_2\|^2 =: F(a_1, a_2)$. Hence,

$$D_{a_1,a_2} \left(\frac{\Gamma(\rho, \theta + \frac{1}{2} \|a_1 \xi_1 + a_2 \xi_2\|^2)}{\Gamma(\rho, \theta)} \right) = \frac{1}{\Gamma(\rho, \theta)} D_{a_1,a_2} \left(\Gamma(\rho, \theta + \frac{1}{2} F(a_1, a_2)) \right)$$

$$= -\frac{1}{\Gamma(\rho, \theta)} D_{a_1} \left((\theta + \frac{1}{2} F(a_1, a_2))^{\rho-1} e^{-(\theta + \frac{1}{2} F(a_1, a_2))} (a_1 \langle \xi_1, \xi_2 \rangle + a_2 \|\xi_2\|^2) \right)$$

$$= -\frac{1}{\Gamma(\rho, \theta)} \left((\rho-1)(\theta + \frac{1}{2} F(a_1, a_2))^{\rho-2} e^{-(\theta + \frac{1}{2} F(a_1, a_2))} (a_1 \|\xi_1\|^2 + a_2 \langle \xi_1, \xi_2 \rangle)(a_1 \langle \xi_1, \xi_2 \rangle + a_2 \|\xi_2\|^2) \right)$$

$$+ (\theta + \frac{1}{2} F(a_1, a_2))^{\rho-1} e^{-(\theta + \frac{1}{2} F(a_1, a_2))} (-(a_1 \|\xi_1\|^2 + a_2 \langle \xi_1, \xi_2 \rangle))(a_1 \langle \xi_1, \xi_2 \rangle + a_2 \|\xi_2\|^2)$$

$$+ (\theta + \frac{1}{2} F(a_1, a_2))^{\rho-1} e^{-(\theta + \frac{1}{2} F(a_1, a_2))} \langle \xi_1, \xi_2 \rangle \Big),$$

which, taking $a_1 = a_2 = 0$ and multiplying by $i^{-2} = -1$, coincides with (2.9). $\qquad \square$

We now want to prove that the Γ-grey measure $\nu_{\rho,\theta}$ belongs to the class of measures for which the Appell systems exist. The latter are bi-orthogonal polynomials which replace the Wick-ordered polynomials of Gaussian analysis and have been proved to be fundamental tools in the non-Gaussian context. To this aim, it is sufficient to prove the following conditions are satisfied (see [17] for details):

C1 For $\rho \in (0, 1]$ and $\theta > 0$, $\nu_{\rho,\theta}$ has an analytic Laplace transform in a neighborhood of zero, i.e. the following mapping is holomorphic in a neighborhood $\mathcal{U} \subset \mathcal{S}_{\mathbb{C}}$ of zero:

$$\mathcal{S}_{\mathbb{C}} \ni \phi \mapsto \ell_\nu(\phi) := \int_{\mathcal{S}'} \exp \langle x, \phi \rangle d\nu_{\rho,\theta}(x) \in \mathbb{C}$$

C2 For $\rho \in (0, 1]$ and $\theta > 0$, $\nu_{\rho,\theta}(\mathcal{U}) > 0$ for any non-empty open subset $\mathcal{U} \subset \mathcal{S}'$.

As far as C1 is concerned, we recall in Appendix B some definitions and well-known results on holomorphic property; on the basis of the latter, we show that for $\rho \in (0, 1]$ and $\theta > 0$ the measure $\nu_{\rho,\theta}$ admits a Laplace transform defined only on a subset of $\mathcal{S}_{\mathbb{C}}$ but it is holomorphic on that subset and it is positive on non-empty, open subsets.

First we show that $\ell_\nu(\xi)$ is well-defined on a subset of $\mathcal{S}$.

Lemma 2.2 Let $\rho \in (0, 1)$, $\theta > 0$ and $\lambda \in \mathbb{R}/\{0\}$, then the exponential function $\mathcal{S}' \ni x \mapsto e^{|\lambda \langle x, \phi \rangle|}$ is integrable and

$$\int_{\mathcal{S}'} e^{\lambda \langle x, \phi \rangle} d\nu_{\rho,\theta}(x) = \frac{\Gamma(\rho, \theta - \frac{\lambda^2}{2} \langle \phi, \phi \rangle)}{\Gamma(\rho, \theta)}, \qquad \text{for } \phi \in B_{\sqrt{2\theta/\lambda^2}}(0). \tag{2.10}$$

Proof For $\lambda \in \mathbb{R}/\{0\}$ we start by proving the integrability. We can define the monotone increasing sequence $g_N(\cdot) := \sum_{n=0}^{N} \frac{1}{n!} |\langle \cdot, \lambda\phi \rangle|^n$. We divide the elements of g_N into odd and even terms,

$$g_N(\cdot) = \sum_{n=0}^{\lfloor N/2 \rfloor} \frac{1}{(2n)!} |\langle \cdot, \lambda\phi \rangle|^{2n} + \sum_{n=0}^{\lceil N/2 \rceil - 1} \frac{1}{(2n+1)!} |\langle \cdot, \lambda\phi \rangle|^{2n+1}$$

and we apply the integral to each term. For the even terms we get:

$$\frac{1}{(2n)!} \int_{S'} |\langle x, \lambda\phi \rangle|^{2n} dv_{\rho,\theta}(x) = \frac{\Gamma(\rho)}{\Gamma(\rho,\theta)} \frac{(-1)^{n+1} \theta^{\rho-n} E^{\rho}_{1,\rho+1-n}(-\theta)}{n! 2^n} \langle \lambda\phi, \lambda\phi \rangle^n.$$

By considering that $\theta^{\rho-n} E^{\rho}_{1,\rho+1-n}(-\theta) = \frac{d^{n-1}}{d\theta^{n-1}} [\theta^{\rho-1} E^{\rho}_{1,\rho}(-\theta)]$ (see (1.9.6) in [20]) and $E^{\rho}_{1,\rho}(-\theta) = \frac{1}{\Gamma(\rho)} e^{-\theta}$, we get

$$\theta^{\rho-n} E^{\rho}_{1,\rho+1-n}(-\theta) = \frac{1}{\Gamma(\rho)} \frac{d^{n-1}}{d\theta^{n-1}} [\theta^{\rho-1} e^{-\theta}] = \frac{1}{\Gamma(\rho)} \frac{\partial^{n-1}}{\partial\theta^{n-1}} \left[-\frac{\partial}{\partial\theta} \Gamma(\rho,\theta) \right]$$

$$= -\frac{1}{\Gamma(\rho)} \frac{\partial^n}{\partial\theta^n} \Gamma(\rho,\theta) = -\frac{1}{\Gamma(\rho)} \frac{\partial^n}{\partial x^n} \Gamma(\rho,\theta+x) \Big|_{x=0}$$

Hence, we have that

$$(-1)^{n+1} \theta^{\rho-n} E^{\rho}_{1,\rho+1-n}(-\theta) = \frac{(-1)^n}{\Gamma(\rho)} \frac{\partial^n}{\partial x^n} \Gamma(\rho,\theta+x) \Big|_{x=0},$$

so that each even term is equal to:

$$\frac{1}{(2n)!} \int_{S'} |\langle x, \lambda\phi \rangle|^{2n} dv_{\rho,\theta}(x) = \frac{1}{\Gamma(\rho,\theta)} \frac{\frac{\partial^n}{\partial x^n} \Gamma(\rho,\theta+x) \Big|_{x=0}}{n! 2^n} (-\langle \lambda\phi, \lambda\phi \rangle)^n =: E(n).$$

We can estimate the odd terms using the Cauchy-Schwarz inequality on $L^2(S', \sigma^*, v_{\rho,\theta})$ and the inequality $st \le 1/2(s^2 + t^2)$, for $s, t \in \mathbb{R}$:

$$\frac{1}{(2n+1)!} \int_{S'} |\langle x, \lambda\phi \rangle|^{2n+1} dv_{\rho,\theta}(x)$$

$$= \frac{1}{(2n+1)!} \int_{S'} |\langle x, \lambda\phi \rangle|^{n+1} |\langle x, \lambda\phi \rangle|^n dv_{\rho,\theta}(x)$$

$$\le \frac{1}{(2n+1)!} \left(\int_{S'} |\langle x, \lambda\phi \rangle|^{2n+2} dv_{\rho,\theta}(x) \right)^{1/2} \left(\int_{S'} |\langle x, \lambda\phi \rangle|^{2n} dv_{\rho,\theta}(x) \right)^{1/2}$$

$$\le \frac{1}{(2n+1)!} \left(\frac{1}{2} \int_{S'} |\langle x, \lambda\phi \rangle|^{2n+2} dv_{\rho,\theta}(x) + \frac{1}{2} \int_{S'} |\langle x, \lambda\phi \rangle|^{2n} dv_{\rho,\theta}(x) \right)$$

$$\le \frac{1}{(2n+1)!} \left(\frac{(2n+2)!}{(n+1)! 2^{n+2} \Gamma(\rho,\theta)} \frac{\partial^{n+1}}{\partial x^{n+1}} \Gamma(\rho,\theta+x) \Big|_{x=0} (-\langle \lambda\phi, \lambda\phi \rangle)^{n+1} \right)$$

$$+\frac{1}{(2n+1)!}\left(\frac{(2n)!}{n!2^{n+1}\Gamma(\rho,\theta)}\frac{\partial^n}{\partial x^n}\Gamma(\rho,\theta+x)\bigg|_{x=0}(-\langle\lambda\phi,\lambda\phi\rangle)^n\right)$$

$$=\frac{1}{\Gamma(\rho,\theta)}\frac{1}{n!2^{n+1}}\frac{\partial^{n+1}}{\partial x^{n+1}}\Gamma(\rho,\theta+x)\bigg|_{x=0}(-\langle\lambda\phi,\lambda\phi\rangle)^{n+1}$$

$$+\frac{1}{(2n+1)}\frac{1}{\Gamma(\rho,\theta)}\frac{1}{n!2^{n+1}}\frac{\partial^n}{\partial x^n}\Gamma(\rho,\theta+x)\bigg|_{x=0}(-\langle\lambda\phi,\lambda\phi\rangle)^n$$

$$=:O'(n)+O''(n).$$

Thus, by integrating g_N, we get that

$$\int_{\mathcal{S}'}g_N(x)d\nu_{\rho,\theta}(x)\leq\sum_{n=0}^{\lfloor N/2\rfloor}E(n)+\sum_{n=0}^{\lceil N/2\rceil-1}O'(n)+\sum_{n=0}^{\lceil N/2\rceil-1}O''(n).$$

We have that the sum of the even terms $E(n)$ converges to $\frac{1}{\Gamma(\rho,\theta)}\Gamma(\rho,\theta-\lambda^2\langle\phi,\phi\rangle/2)$ if $\langle\phi,\phi\rangle=\|\phi\|^2<2\theta/\lambda^2$, as the Taylor expansion for $\Gamma(\rho,\eta)$ holds for $\rho\in(0,1)$, if $\eta>0$. For the odd terms $O'(n)$ we have that:

$$\sum_{n=0}^{\lceil N/2\rceil-1}O'(n)=\frac{1}{\Gamma(\rho,\theta)}\sum_{n=0}^{\lceil N/2\rceil-1}\frac{n+1}{2^{n+1}}\frac{\frac{\partial^{n+1}}{\partial x^{n+1}}\Gamma(\rho,\theta+x)\bigg|_{x=0}}{(n+1)!}(-\langle\lambda\phi,\lambda\phi\rangle)^{n+1}$$

$$\leq\frac{1}{2\Gamma(\rho,\theta)}\sum_{m=1}^{\lceil N/2\rceil}\frac{\frac{\partial^m}{\partial x^m}\Gamma(\rho,\theta+x)\bigg|_{x=0}}{m!}(-\langle\lambda\phi,\lambda\phi\rangle)^m,$$

where the last sum converges. On the other hand the sum of the odd terms $O''(n)$ converges since

$$\sum_{n=0}^{\lceil N/2\rceil-1}O''(n)=\sum_{n=0}^{\lceil N/2\rceil-1}\frac{1}{(2n+1)2^{n+1}}\frac{1}{\Gamma(\rho,\theta)}\frac{\frac{\partial^n}{\partial x^n}\Gamma(\rho,\theta+x)\bigg|_{x=0}}{n!}(-\langle\lambda\phi,\lambda\phi\rangle)^n$$

$$<\frac{1}{\Gamma(\rho,\theta)}\sum_{n=0}^{\lceil N/2\rceil-1}\frac{\frac{\partial^n}{\partial x^n}\Gamma(\rho,\theta+x)\bigg|_{x=0}}{n!}(-\langle\lambda\phi,\lambda\phi\rangle)^n\xrightarrow[N\to\infty]{}\frac{1}{\Gamma(\rho,\theta)}\Gamma(\rho,\theta-\lambda^2\langle\phi,\phi\rangle).$$

Therefore, by applying the monotone convergence theorem (as each term is positive), we get, for $\phi\in B_{\sqrt{2\theta/\lambda^2}}(0)$, that:

$$\int_{\mathcal{S}'}e^{\langle x,\lambda\phi\rangle}d\nu_{\rho,\theta}(x)=\lim_{N\to\infty}\int_{\mathcal{S}'}g_N(x)d\nu_{\rho,\theta}(x)$$

$$=\frac{1}{\Gamma(\rho,\theta)}\sum_{n\geq0}\frac{\frac{\partial^n}{\partial x^n}\Gamma(\rho,\theta+x)\bigg|_{x=0}}{n!}(-1/2)^n\langle\lambda\phi,\lambda\phi\rangle^n,$$

which coincides with (2.10). □

Now, we prove that $\ell_\nu(\xi)$ is holomorphic on some neighborhood of 0 in $\mathcal{S}_{\mathbb{C}}$ for $\rho\in(0,1)$ and $\theta>0$. Hence we have that $\ell_\nu(\xi)$ is holomorphic on $\mathcal{U}_\theta:=B_{\sqrt{2\theta}}(0)\oplus i\mathcal{S}=\{\xi_1+i\xi_2|\xi_1\in B_{\sqrt{2\theta}}(0)$ and $\xi_2\in\mathcal{S}\}$.

Theorem 2.1 *Let $\rho \in (0, 1)$ and $\theta > 0$, then the function*

$$\mathcal{S}_{\mathbb{C}} \supset \mathcal{U}_\theta \ni \xi \mapsto \int_{\mathcal{S}'} e^{\langle x, \xi \rangle} dv_{\rho,\theta}(x)$$

is holomorphic from $\mathcal{U}_\theta$ to $\mathbb{C}$.

Proof We show that it is bounded on $\mathcal{U}_\theta$. Let $\xi \in \mathcal{U}_\theta$, then we have that

$$|\ell_\nu(\xi)| \le \int_{\mathcal{S}'} |e^{\langle x, \xi \rangle}| dv_{\rho,\theta}(x)$$

Noting that $|e^{\langle x, \xi \rangle}| = |e^{\langle x, \xi_1 \rangle}||e^{-i\langle x, \xi_2 \rangle}| = |e^{\langle x, \xi_1 \rangle}| = e^{\langle x, \xi_1 \rangle}$, for $\xi = \xi_1 + i\xi_2$, we get

$$\int_{\mathcal{S}'} |e^{\langle x, \xi \rangle}| dv_{\rho,\theta}(x) = \int_{\mathcal{S}'} e^{\langle x, \xi_1 \rangle} dv_{\rho,\theta}(x) = \frac{\Gamma(\rho, \theta - 1/2\|\xi_1\|^2)}{\Gamma(\rho, \theta)} < \infty,$$

by using Lemma 2.2, for the second equality.

Now we show that, for $\xi = \xi_1 + i\xi_2 \in \mathcal{U}_\theta$, $\eta = \eta_1 + i\eta_2 \in \mathcal{S}_{\mathbb{C}}$ and $z \in B_r(0)$ where $0 < r < \frac{\sqrt{2\theta} - \|\xi_1\|}{3(\|\eta_1\| + \|\eta_2\|)}$, the function $\mathbb{C} \supset B_r(0) \ni z \mapsto \ell_\nu(\xi + z\eta) =: f(z) \in \mathbb{C}$ is continous. The radius length is such that, for all $\lambda \in B_r(0)$, we have $\xi + \lambda\eta \in \mathcal{U}_\theta$.

We take $\{z_n\}_{n \in \mathbb{N}} \subset B_r(0)$ such that $z_n \to z$, for $n \to \infty$. Denoting by $(\cdot)_1$ the real part of a function in $\mathcal{S}_{\mathbb{C}}$, we have that

$$\begin{aligned}
|f(z) - f(z_n)| &\le \int_{\mathcal{S}'} |e^{\langle x, \xi + z\eta \rangle} - e^{\langle x, \xi + z_n\eta \rangle}| dv_{\rho,\theta}(x) \\
&\le \int_{\mathcal{S}'} |e^{\langle x, \xi \rangle}||e^{\langle x, z\eta \rangle} - e^{\langle x, z_n\eta \rangle}| dv_{\rho,\theta}(x) \\
&\le \int_{\mathcal{S}'} e^{\langle x, \xi_1 \rangle + \langle x, (z_n\eta)_1 \rangle + ((z-z_n)\eta)_1 \rangle} dv_{\rho,\theta}(x)
\end{aligned}$$

We note that $|e^{\langle x, z\eta \rangle} - e^{\langle x, z_n\eta \rangle}| = |e^{\langle x, z_n\eta \rangle}||e^{\langle x, (z-z_n)\eta \rangle} - 1|$. Moreover, for sufficiently large n, we have that $|e^{\langle x, (z-z_n)\eta \rangle} - 1| \le e^{\langle x, ((z-z_n)\eta)_1 \rangle}$. Since $|z_n| + |z - z_n| < 3r$ for each n, we can ensure that $\xi + 3r\eta \in \mathcal{U}_\theta$, so that $e^{\langle x, (\xi + 3r\eta)_1 \rangle} \in L^1(\mathcal{S}', \sigma^*, v_{\rho,\theta})$. Hence, we can apply the dominated convergence theorem to gain the continuity of f in $z \in B_r(0)$, as follows

$$\begin{aligned}
\lim_{n \to \infty} |f(z) - f(z_n)| &\le \lim_{n \to \infty} \int_{\mathcal{S}'} |e^{\langle x, \xi_1 \rangle}||e^{\langle x, z\xi_2 \rangle} - e^{\langle x, z_n\xi_2 \rangle}| dv_{\rho,\theta}(x) \\
&= \int_{\mathcal{S}'} \lim_{n \to \infty} |e^{\langle x, \xi_1 \rangle}||e^{z\langle x, \xi_2 \rangle} - e^{z_n\langle x, \xi_2 \rangle}| dv_{\rho,\theta}(x) = 0.
\end{aligned}$$

Now we apply the Morera's theorem to show that $f(z)$ is holomorphic, which means that $\ell_\nu(\xi)$ is G-holomorphic on $\mathcal{U}_\theta$ (see Definition B.3 in Appendix). Let γ be a closed and bounded curve in $B_r(0) \subset \mathbb{C}$, since γ is compact and $\int_{\mathcal{S}'} e^{\langle x, \xi + z\eta \rangle} dv_{\rho,\theta}(x) < \infty$, we can use the Fubini theorem to get:

$$\int_\gamma \int_{\mathcal{S}'} e^{\langle x, \xi + z\eta \rangle} dv_{\rho,\theta}(x) dz = \int_{\mathcal{S}'} \int_\gamma e^{\langle x, \xi + z\eta \rangle} dz dv_{\rho,\theta}(x) = 0$$

as the exponential function is holomorphic. By the Morera's theorem and by Lemma B.1 in Appendix B, we have that ℓ_ν is holomorphic on $\mathcal{U}_\theta$. □

Remark 2.4 It is easy to check that $B_{r_1}(0) \oplus i B_{r_2}(0)$ is an open set in the topology induced by $\langle \cdot, \cdot \rangle_{\mathcal{H}_\mathbb{C}}$, as follows: let us define the projections of an element of $\mathcal{H}_\mathbb{C}$ as

$$\pi_1 : \mathcal{H}_\mathbb{C} \to \mathcal{H} : \xi_1 + i\xi_2 \mapsto \xi_1$$

and

$$\pi_2 : \mathcal{H}_\mathbb{C} \to \mathcal{H} : \xi_1 + i\xi_2 \mapsto \xi_2.$$

Let $x \in B_{r_1}(0) \oplus i B_{r_2}(0)$, then we have $\pi_1(x) \in B_{r_1}(0)$ and $\pi_2(x) \in B_{r_2}(0)$. The sets $B_{r_1}(0)$ and $B_{r_2}(0)$ are open in the topology of $\mathcal{H}$; then $\exists \epsilon_1, \epsilon_2 > 0$ such that $B_{\epsilon_1}(\pi_1(x)) \subset B_{r_1}(0)$ and $B_{\epsilon_2}(\pi_2(x)) \subset B_{r_2}(0)$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$, then we have that $B_\epsilon^\mathbb{C}(x) \subset B_{r_1}(0) \oplus i B_{r_2}(0)$.

In order to verify that C2 is satisfied by $\nu_{\rho,\theta}$, we prove that, for $\rho \in (0, 1)$ and $\theta > 0$, they are always strictly positive on non-empty, open subsets, by resorting to their representation as mixture of Gaussian measures.

Theorem 2.2 *For any open, non-empty set $U \subset \mathcal{S}'$ and for any $\rho \in (0, 1)$, $\theta > 0$, we have that $\nu_{\rho,\theta}(U) > 0$.*

Proof By applying Theorem 4.5 in [14], it is sufficient to prove that $\nu_{\rho,\theta}$ is an elliptically contoured measure, i.e. if we denote by μ^s the centered Gaussian measure on $\mathcal{S}'$ with variance $s > 0$, the following holds:

$$\nu_{\rho,\theta} = \int_0^\infty \mu^s d\mu_{\rho,\theta}(s), \tag{2.11}$$

where $\mu_{\rho,\theta}$ is the measure defined on $(0, \infty)$ by (2.1). The identity in (2.11) can be checked by considering that

$$\int_{\mathcal{S}'} e^{i\langle x, \xi \rangle} d\mu^s(x) = \exp\left\{ -\frac{s}{2}\langle \xi, \xi \rangle \right\}, \quad \xi \in \mathcal{S}$$

and thus, by (2.1),

$$\int_0^\infty \exp\left\{ -\frac{s}{2}\langle \xi, \xi \rangle \right\} d\mu_{\rho,\theta}(s) = \frac{\Gamma(\rho, \theta + \frac{1}{2}\langle \xi, \xi \rangle)}{\Gamma(\rho, \theta)}, \tag{2.12}$$

which coincides with $\int_{\mathcal{S}'} e^{i\langle x, \xi \rangle} d\nu_{\rho,\theta}(x)$. □

Remark 2.5 For $\rho = 1$, C1 and C2 are satisfied because $\nu_{1,\theta}$ is Gaussian, for each $\theta \geq 0$.

3 The Tempered Gamma-Grey Brownian Motion as Generalized Stochastic Process

We can now consider the fractional operator $M_-^{\alpha/2}$ defined, for any $f \in \mathcal{S}$, as

$$M_-^{\alpha/2} f := \begin{cases} \sqrt{C_\alpha} D_-^{(1-\alpha)/2} f, & \alpha \in (0, 1) \\ f, & \alpha = 1 \\ \sqrt{C_\alpha} I_-^{(\alpha-1)/2} f, & \alpha \in (1, 2) \end{cases},$$

where $C_\alpha = \Gamma(\alpha + 1) \sin(\pi\alpha/2)$ (see [15] togheter with [25], p.9) and

$$D_-^\beta f(x) := -\frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \int_x^\infty f(t)(t-x)^{-\beta} dt, \quad x \in \mathbb{R}, \ \beta \in (0, 1),$$

is the Riemann-Liouville fractional derivative and

$$I_-^\beta f(x) := \frac{1}{\Gamma(\beta)} \int_x^\infty f(t)(t-x)^{\beta-1}dt, \quad x \in \mathbb{R}, \ \beta \in (0,1),$$

is the Riemann-Liouville fractional integral.

We extend the dual pairing $\langle \cdot, \cdot \rangle$ to $\mathcal{S}'(\mathbb{R}) \times L^2(\mathbb{R}, dx)$ and by considering that $M_-^{\alpha/2} 1_{[0,t)} \in L^2(\mathbb{R}, dx)$, where $1_{[a,b)}$ is the indicator function of $[a,b]$, we introduce the tempered Γ-grey Brownian motion (hereafter Γ-GBM) as follows:

Definition 3.1 *Let $\alpha \in (0,2)$, $\rho \in (0,1]$ and $\theta > 0$. The tempered Γ-GBM is defined on the probability space $\left(\mathcal{S}'(\mathbb{R}), \sigma^*, \nu_{\rho,\theta}\right)$ as the generalized process*

$$B_{\alpha,\rho}^\theta(t,\omega) := \left\langle \omega, M_-^{\alpha/2} 1_{[0,t)} \right\rangle, \quad t \geq 0, \ \omega \in \mathcal{S}'(\mathbb{R}). \tag{3.1}$$

We notice that for each t, $B_{\alpha,\rho}^\theta(t,\cdot) \in L^2(\nu_{\rho,\theta})$.

Remark 3.1 For $\rho = 1$, $\alpha = 1$ and for each θ, we have that $B_{1,1}^\theta$ is a Brownian Motion, indeed for each t, $B_t(\omega) = \langle \omega, 1_{[0,t)} \rangle \in L^2(\nu)$ where ν is Gaussian.

In order to study the continuity of this process, we recall the following relationship obtained in [15]:

$$\left\langle M_-^{\alpha/2} \xi, M_-^{\alpha/2} \eta \right\rangle_{L^2(\mathbb{R},dx)} = C_\alpha \int_{\mathbb{R}} |x|^{1-\alpha} \widehat{\xi}(x) \widehat{\eta}(x) dx, \quad \xi, \eta \in \mathcal{S}(\mathbb{R}), \tag{3.2}$$

where $\widehat{f}(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\langle \omega, x \rangle} f(\omega) d\omega$ denotes the Fourier transform of $f(\cdot)$ (see also [25], for details).

It is proved in [15] that (3.2) holds not only on $\mathcal{S}$, but also for indicator functions and that

$$\left\langle M_-^{\alpha/2} 1_{[0,t)}, M_-^{\alpha/2} 1_{[0,s)} \right\rangle_{L^2(\mathbb{R},dx)} = \frac{1}{2}(s^\alpha + t^\alpha - |t-s|^\alpha).$$

Similarly to what was done in [15] for the ggBm, it is easy to prove the following result.

Theorem 3.1 *For $\alpha \in (0,2)$, $\rho \in (0,1]$ and $\theta > 0$, the tempered Γ-GBM has a γ-Hölder continuous version with $\gamma < \alpha/2$.*

Proof In order to apply the Kolmogorov's continuity theorem, we only need to show that

$$\mathbb{E}_{\nu_{\rho,\theta}}\left(\left(B_{\alpha,\rho}^\theta(t) - B_{\alpha,\rho}^\theta(s)\right)^{2n}\right) \leq K|t-s|^{q+1}, \tag{3.3}$$

for some $q > 0$ and $s, t \geq 0$. By definition and by recalling (2.8), we have that, for $s < t$,

$$\mathbb{E}_{\nu_{\rho,\theta}}\left(\left|B_{\alpha,\rho}^\theta(t) - B_{\alpha,\rho}^\theta(s)\right|^{2n}\right)$$

$$= \int_{\mathcal{S}'(\mathbb{R})} \left\langle \omega, M_-^{\alpha/2} 1_{[s,t)} \right\rangle^{2n} d\nu_{\rho,\theta}(\omega)$$

$$= \frac{(-1)^{n+1}(2n)!\Gamma(\rho)\theta^{\rho-n}}{n!2^n \Gamma(\rho,\theta)} E_{1,\rho+1-n}^\rho(-\theta) \left\langle M_-^{\alpha/2} 1_{[s,t)}, M_-^{\alpha/2} 1_{[s,t)} \right\rangle^n$$

.

We now prove that

$$K_{\theta,\rho}^n := \frac{(-1)^{n+1}(2n)!\Gamma(\rho)\theta^{\rho-n}}{n!2^n \Gamma(\rho,\theta)} E_{1,\rho+1-n}^\rho(-\theta)$$

is positive, for any n, by considering ([A.6](#)), as follows

$$(-1)^{n+1}\theta^{\rho-n}E^{\rho}_{1,\rho+1-n}(-\theta) = (-1)^{n+1}\frac{d^{n-1}}{d\theta^{n-1}}\left[\theta^{\rho-1}E^{\rho}_{1,\rho}(-\theta)\right]$$

$$= (-1)^{n+1}\sum_{j=0}^{n-1}\binom{n-1}{j}\frac{d^j}{d\theta^j}\left[\theta^{\rho-1}\right]\frac{d^{n-1-j}}{d\theta^{n-1-j}}\frac{e^{-\theta}}{\Gamma(\rho)}$$

$$= \frac{1}{\Gamma(\rho)}\sum_{j=0}^{n-1}\binom{n-1}{j}\left\{(-1)^j\frac{d^j}{d\theta^j}\left[\theta^{\rho-1}\right]\right\}\left\{(-1)^{n-1-j}\frac{d^{n-1-j}}{d\theta^{n-1-j}}e^{-\theta}\right\} \geq 0.$$

In the last step we resorted to the complete monotonicity of both the Prabhakar function (in the special case $\beta = \gamma = \rho < 1$) and of $\theta^{\rho-1}$, for $\rho < 1$.

We now apply Proposition 3.8 of [[15](#)], which shows that $\left\langle M^{\alpha/2}_- 1_{[s,t)}, M^{\alpha/2}_- 1_{[s,t)}\right\rangle^n = (t-s)^{\alpha n}$, so that ([3.3](#)) holds for $q = \alpha n - 1 > 0$. The case $s > t$ can be treated analogously, so that the sufficient condition of the Kolmogorov's continuity theorem is satisfied and the Hölder-continuity parameter is $\gamma < \frac{q+1}{2n} = \frac{\alpha}{2}$.   □

Remark 3.2 The previous result agrees with the well-known γ-Hölder continuity of the fractional Brownian motion with $\gamma < H$.

4 Finite-Dimensional Characterization of the Tempered Gamma-Grey Brownian Motion

This section is devoted to the finite dimensional characterization of the generalized process $B^{\theta}_{\rho,\alpha}$. We recall that, in order to overcome the lack of moments, we introduced the tempering factor θ; we give the following definition of the process in the Euclidean space, in terms of its n-times characteristic function thanks to the σ^* algebra.

Definition 4.1 Let $\alpha \in (0,2)$, $\rho \in (0,1]$ and $\theta \geq 0$. Let, for any $\xi_k \in \mathbb{R}$, $k = 1, ..., n$ and $n \in \mathbb{N}$,

$$\Phi^{\theta}_{\alpha,\rho}(\xi_1, ..., \xi_n; t_1, ..., t_n) = \frac{\Gamma\left(\rho, \theta + \frac{1}{2}\sum_{j,k=1}^{n}\xi_j\xi_k\gamma_\alpha(t_j, t_k)\right)}{\Gamma(\rho, \theta)}, \qquad (4.1)$$

where $\gamma_\alpha(t_j, t_k) = t_k^\alpha + t_j^\alpha - |t_k - t_j|^\alpha$ and $0 \leq t_1 \leq ... \leq t_n < \infty$. Then, the process with characteristic function ([4.1](#)), will be denoted (as its infinite-dimensional counterpart) as $B^{\theta}_{\alpha,\rho} := \left\{B^{\theta}_{\alpha,\rho}(t), t \geq 0\right\}$.

Thanks to the next result, we can express the tempered Γ-GBM as a product of a random variable and a fractional Brownian motion, under the assumption that they are mutually independent.

Theorem 4.1 For $\alpha \in (0,2)$, $\rho \in (0,1]$ and $\theta > 0$, the following equality of all the finite-dimensional distribution (denoted by $\overset{f.d.d.}{=}$) holds

$$B^{\theta}_{\alpha,\rho}(t) \overset{f.d.d.}{=} \sqrt{Y^{\rho}_{\theta}}\,B^{\alpha/2}(t), \qquad t \geq 0, \qquad (4.2)$$

where $B^{\alpha/2} := \{B^{\alpha/2}(t), t \geq 0\}$ is the fractional Brownian motion with Hurst-parameter $H = \alpha/2$, for $\alpha \in (0,2)$ and Y^{ρ}_{θ} is the r.v. with density

$$l^{\theta}_{\rho}(y) = \frac{1}{\Gamma(\rho,\theta)\Gamma(1-\rho)}\frac{e^{-\theta y}}{y(y-1)^\rho}1_{y>1}, \qquad \rho \in (0,1), \theta \geq 0. \qquad (4.3)$$

independent from $B^{\alpha/2}$.

Proof We have that

$$\mathbb{E}\exp\left\{i\sqrt{Y_\theta^\rho}\sum_{k=1}^n \xi_k B^{\alpha/2}(t_k)\right\}$$

$$= \mathbb{E}\left[\mathbb{E}\left(\exp\left\{i\sqrt{Y_\theta^\rho}\sum_{k=1}^n \xi_k B^{\alpha/2}(t_k)\right\}\bigg|\, Y_\theta^\rho\right)\right]$$

$$= \mathbb{E}\left[\exp\left\{-\frac{1}{2}Y_\theta^\rho\sum_{j,k=1}^n \xi_j\xi_k\gamma_\alpha(t_j,t_k)\right\}\right]$$

which coincides with (4.1). This can be proved by taking into account that

$$\tilde{l}_\rho^\theta(\eta) = \frac{1}{\Gamma(\rho,\theta)\Gamma(1-\rho)}\int_1^{+\infty} e^{-(\theta+\eta)y}(y-1)^{-\rho}y^{-1}dy$$

$$= \frac{e^{-(\theta+\eta)}}{\Gamma(\rho,\theta)\Gamma(1-\rho)}\int_0^{+\infty} e^{-(\theta+\eta)\omega}\omega^{-\rho}(1+\omega)^{-1}d\omega$$

$$= [\text{by } (1.6.25) \text{ in } [20]]$$

$$= \frac{e^{-(\theta+\eta)}}{\Gamma(\rho,\theta)}\Psi(1-\rho, 1-\rho; \theta+\eta),$$

where $\Psi(a,b;\cdot)$ is the confluent Tricomi hypergeometric function, together with the well-known relationship $\Psi(a,a;x) = e^x\Gamma(1-a;x)$, for $a > 0$. □

For any $n \in \mathbb{N}$, the joint probability density function of $B_{\alpha,\rho}^\theta$ is therefore given by

$$f_{B_{\alpha,\rho}^\theta}(\mathbf{x}, \Sigma_\alpha) = \frac{(2\pi)^{-n/2}}{\sqrt{\det \Sigma_\alpha}}\int_0^{+\infty}\tau^{-n/2}\exp\left\{-\frac{\mathbf{x}^T\Sigma_\alpha^{-1}\mathbf{x}}{2\tau}\right\}l_\rho^\theta(\tau)d\tau, \qquad (4.4)$$

where $\Sigma_\alpha := (\gamma_\alpha(t_j,t_k))_{j,k=1}^n$ and $\mathbf{x} \in \mathbb{R}^n$.

Remark 4.1 It is easy to check that, in the special case where $\rho = 1$ and for any θ, formula (4.1) reduces to the characteristic function of the fractional Brownian motion with $H = \alpha/2$, and thus, by adding the condition $\alpha = 1$, we obtain the Brownian motion.

Remark 4.2 We note that the density (4.3) coincides with $l_\rho^\theta(y) = \exp(-\theta y)f_\rho(y)$, where $f_\rho(y)$ is given in (1.3), as can be easily checked by considering property P2 in Appendix A and (2.9.6) in [19].

Theorem 4.2 *Let $\alpha \in (0,2)$, $\rho \in (0,1)$ and $\theta > 0$. The k-th order moment of the tempered Γ-Grey Brownian Motion is given by*

$$\mathbb{E}\left[B_{\alpha,\rho}^\theta(t)^k\right] = \begin{cases} 0, & k = 2n+1 \\ \frac{(2n)!t^{\alpha n}}{\Gamma(\rho,\theta)n!2^n}G_{1,2}^{2,0}\left[\theta\,\bigg|\,\begin{matrix}1-n\\0,\quad\rho-n\end{matrix}\right], & k = 2n \end{cases} \qquad (4.5)$$

for $k,n \in \mathbb{N}$, while its autocovariance reads

$$cov(B_{\alpha,\rho}^\theta(t), B_{\alpha,\rho}^\theta(s)) = \frac{e^{-\theta}\theta^{\rho-1}}{2\Gamma(\rho,\theta)}\left[t^\alpha + s^\alpha - |t-s|^\alpha\right]. \qquad (4.6)$$

Proof We first evaluate the k-th order moment of the r.v. Y_θ^ρ, for $k \in \mathbb{N}$, as follows

$$\mathbb{E}\left[(Y_\theta^\rho)^k\right] = \frac{1}{\Gamma(\rho,\theta)\Gamma(1-\rho)} \int_1^{+\infty} y^{k-1}(y-1)^{-\rho}e^{-\theta y}dy \tag{4.7}$$
$$= [\text{by } (2.9.36) \text{ in } [19]]$$
$$= \frac{1}{\Gamma(\rho,\theta)} G_{1,2}^{2,0}\left[\theta \,\middle|\, \begin{matrix} 1-k \\ 0, \quad \rho-k \end{matrix}\right].$$

By considering (4.2), together with the expression of the k-moment of the fractional Brownian motion, formula (4.5) easily follows from the independence between $B^{\alpha/2}$ and Y_θ^ρ. The autocovariance can be obtained as follows

$$cov(B_{\alpha,\rho}^\theta(t), B_{\alpha,\rho}^\theta(s)) = \mathbb{E}\left(Y_\theta^\rho\right)\mathbb{E}\left(B^{\alpha/2}(t) \cdot B^{\alpha/2}(s)\right)$$
$$= \frac{1}{2\Gamma(\rho,\theta)} G_{1,2}^{2,0}\left[\theta \,\middle|\, \begin{matrix} 0 \\ 0, \ \rho-1 \end{matrix}\right]\left[t^\alpha + s^\alpha - |t-s|^\alpha\right]$$
$$= \frac{1}{2\Gamma(\rho,\theta)} \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma(h)\Gamma(\rho-1+h)\theta^{-h}}{\Gamma(h)}dh \left[t^\alpha + s^\alpha - |t-s|^\alpha\right]$$
$$= \frac{1}{2\Gamma(\rho,\theta)} H_{0,1}^{1,0}\left[\theta \,\middle|\, \begin{matrix} - \\ (\rho-1,1) \end{matrix}\right]\left[t^\alpha + s^\alpha - |t-s|^\alpha\right],$$

which coincides with (4.6), by taking into account (1.125) in [24]. $\square$

Remark 4.3 For $\rho = 1$ and any θ, formula (4.6) reduces to the covariance of the fractional Brownian motion.

Finally, from (4.1) it is clear that the process $B_{\alpha,\rho}^\theta$ has stationary increments with characteristic function

$$\mathbb{E}\exp\left\{i\xi[B_{\alpha,\rho}^\theta(t) - B_{\alpha,\rho}^\theta(s)]\right\} = \frac{\Gamma\left(\rho,\theta+\frac{\xi^2}{2}|t-s|^\alpha\right)}{\Gamma(\rho,\theta)}, \quad \xi \in \mathbb{R},\ t,s \geq 0. \tag{4.8}$$

5 Time-Change Representation of the Gamma-Grey Brownian Motion

In this section we present a characterization of $B_{\alpha,\rho}^\theta$ as a time-changed Brownian motion, that holds in the sense of the one-dimensional distribution and in the special case where $\theta = 0$.

5.1 The Random-Time Process

We start by introducing the following process that will represent the random-time argument.

Definition 5.1 *Let $Y_\rho(t)$, $t \geq 0$, be the stochastic process defined by means of the following Laplace transform of its n-times density*

$$\mathbb{E}e^{-\sum_{k=1}^n \eta_k Y_\rho(t_k)} = \frac{\Gamma\left(\rho, \sum_{k=1}^n \eta_k t_k\right)}{\Gamma(\rho)}, \quad \eta_1, \ldots, \eta_n > 0,\ \rho \in (0,1). \tag{5.1}$$

The previous definition is well-posed, since the function (5.1) can be checked to be completely monotone (w.r.t. $\eta_1, ..., \eta_n$ and for any choice of $t_1, ..., t_n \geq 0$), by adapting the result of Lemma 3.1 in [4] to the case $\alpha = 1$. The process is, by definition, self-similar with scaling

parameter equal to one, since, by (5.1), we get that $\{aY^\rho(t), t \geq 0\} \overset{f.d.d.}{=} \{Y_\rho(at), t \geq 0\}$, for any $a > 0$. Moreover, it has stationary increments, as can be seen by taking into account that (5.1) is well-defined even for $\eta_j < 0$, for any j (by analytic continuation), so that we have that

$$\mathbb{E}e^{-\eta[Y_\rho(t_2)-Y_\rho(t_1)]} = \frac{\Gamma(\rho, \eta(t_2 - t_1))}{\Gamma(\rho)}, \qquad t_2 > t_1 \geq 0. \tag{5.2}$$

We denote by $l_\rho(y, t)$ the transition density of $Y_\rho(t)$, (for $y, t \geq 0$, $\rho \in (0, 1)$); therefore, as a consequence of the self-similarity, we have that $l_\rho(y, t) = t^{-1} l_\rho(yt^{-1})$ (where $l_\rho(\cdot)$ is given in (4.3), with $\theta = 0$) and

$$l_\rho(y, t) = \frac{1}{\Gamma(\rho)\Gamma(1-\rho)} \frac{1_{y>t}}{y(y-1)^\rho}, \qquad \rho \in (0, 1). \tag{5.3}$$

Its space-Laplace transform coincides with (5.1), for $n = 1$, i.e. $\widetilde{l}_\rho(\eta, t) = \Gamma(\rho, \eta t)/\Gamma(\rho)$.

Formula (5.2) proves also that Y_ρ has increasing trajectories, since, by (5.3), we have that $Y_\rho(t_2) - Y_\rho(t_1) \geq t_2 - t_1$ almost surely.

We recall that, in the ggBm case, the random-time argument is represented by the inverse of the β-stable subordinator. Therefore, we are interested in checking if, also in the Γ-GBM case, it is possible to define the random-time argument as the inverse of another stochastic process and to characterize the latter.

By resorting to the Doob's theorem, we can refer to the separable version of Y_ρ so that its hitting time is well-defined as follows

$$T_\rho(x) := \inf\{t \geq 0 : Y_\rho(t) > x\}, \qquad x \geq 0. \tag{5.4}$$

We now derive its transition density.

Theorem 5.1 *The space-Laplace transform of the density of the process T_ρ defined in (5.4) is given by*

$$\mathbb{E}e^{-\xi T_\rho(x)} = E_{1,1}^\rho(-\xi x), \qquad \xi > 0, \ \rho \in (0, 1), \ x \geq 0, \tag{5.5}$$

and its transition density $h_\rho(t, x) := P\{T_\rho(x) \in dt\}/dt, t, x \geq 0$, reads

$$h_\rho(t, x) = \frac{t^{\rho-1}(x-t)^{-\rho} 1_{t<x}}{\Gamma(\rho)\Gamma(1-\rho)}. \tag{5.6}$$

Proof By considering (5.4) we can write that $P\{Y_\rho(t) > x\} = P\{T_\rho(x) < t\}$, so that, taking the Laplace transform w.r.t. x and denoting by $\gamma(\rho, x) = \int_0^x e^{-w} w^{\rho-1} dw$ the lower incomplete gamma function, we have that

$$\int_0^t \widetilde{h}_\rho(z, \eta) dz = \int_0^{+\infty} e^{-\eta x} \int_x^{+\infty} l_\rho(z, t) dz dx = \frac{1}{\eta} \int_0^{+\infty} (1 - e^{-\eta z}) l_\rho(z, t) dz$$

$$= \frac{\Gamma(\rho) - \Gamma(\rho, \eta t)}{\eta \Gamma(\rho)} = \frac{\gamma(\rho, \eta t)}{\eta \Gamma(\rho)}$$

$$= \frac{1}{\eta \Gamma(\rho)} \int_0^{\eta t} e^{-w} w^{\rho-1} dw.$$

By taking also the Laplace transform w.r.t. t, we get

$$\widetilde{\widetilde{h}}_\rho(\xi, \eta) = \frac{\eta^{\rho-1}}{(\xi + \eta)^\rho}, \tag{5.7}$$

whose inverse transform (w.r.t. η) coincides with (5.5). It is easy to check that the the inverse Laplace transform (w.r.t. ξ) of the latter reads

$$h_\rho(t, x) = \frac{1_{t<x}}{t\Gamma(\rho)} G_{1,1}^{1,0}\left[\frac{t}{x} \middle| \begin{matrix} 1 \\ \rho \end{matrix}\right].$$

Indeed, by taking into account (A.3) together with formula (2.19) in [24] (since $\rho > 0$) we get

$$\int_0^{+\infty} e^{-\xi t} h_\rho(t, x) dt = \frac{1}{\Gamma(\rho)} \int_0^{+\infty} \frac{e^{-\xi t}}{t} H_{1,1}^{1,0}\left[\frac{t}{x} \middle| \begin{matrix} (1, 1) \\ (\rho, 1) \end{matrix}\right] dt$$

$$= \frac{1}{\Gamma(\rho)} H_{2,1}^{1,1}\left[\frac{1}{x\xi} \middle| \begin{matrix} (1, 1)(1, 1) \\ (\rho, 1) \end{matrix}\right]$$

$$= [\text{by property P1 in Appendix A}]$$

$$= \frac{1}{\Gamma(\rho)} H_{1,2}^{1,1}\left[x\xi \middle| \begin{matrix} (1 - \rho, 1) \\ (0, 1)(0, 1) \end{matrix}\right],$$

which coincides with (5.5) by (1.137) in [24]. Moreover, by resorting to formulae (8)-(9) in [23] and by property P2 in Appendix A, we can simplify the previous expression into (5.6). □

It is immediate to see from (5.5), that, as happens for Y_ρ, also T_ρ is self-similar, with scaling parameter equal to one, since $\{aT_\rho(t), t \geq 0\} \stackrel{f.d.d.}{=} \{T_\rho(at), t \geq 0\}$, for any $a > 0$. Moreover, the following relationship holds between the densities $h_\rho(t, x)$ and $l_\rho(x, t)$, of T_ρ and Y_ρ, respectively:

$$h_\rho(t, x) = \frac{x}{t} l_\rho(x, t), \qquad x, t \geq 0. \tag{5.8}$$

It is easy to derive the partial differential equations (p.d.e.'s) satisfied by the densities of T_ρ, given in (5.6), and of its inverse Y_ρ, given in (5.3); for this reason, we omit the proof of the following result.

Corollary 5.1 *The density of the process T_ρ satisfies the following p.d.e.*

$$\frac{\partial}{\partial t} h_\rho(t, y) = -\frac{\partial}{\partial y} h_\rho(t, y) + \frac{\rho - 1}{t} h_\rho(t, y), \quad t, y \geq 0, \tag{5.9}$$

with initial condition $h_\rho(t, 0) = \delta(t)$, while the density of Y_ρ satisfies the following p.d.e.

$$\frac{\partial}{\partial t} l_\rho(y, t) = -\frac{\partial}{\partial y} l_\rho(y, t) + \left[\frac{\rho}{t} - \frac{1}{y}\right] l_\rho(y, t), \quad t, y \geq 0, \tag{5.10}$$

with initial condition $l_\rho(y, 0) = 0$.

Remark 5.1 For $\rho = 1$, Eq. (5.9) reduces to the partial differential equation satisfied by the density of the elementary subordinator $T_1(y) = y$, which is equal to $h_1(t, y) = \delta(t - y)$, as can be easily checked by taking the Laplace transform w.r.t. y. Analogously, we have that $l_1(y, t) = t\delta(t - y)/y$, which satisfies Eq. (5.10), with $\rho = 1$. Another interesting special case is for $\rho = 1/2$. In this case the densities of the processes $T_{1/2}$ and $Y_{1/2}$ are respectively equal to

$$h_{1/2}(t, y) = \frac{1}{\pi\sqrt{t(y - t)}} 1_{t<y}, \tag{5.11}$$

which coincides with the arcsine law, and

$$l_{1/2}(y, t) = \frac{\sqrt{t}}{\pi y \sqrt{y - t}} 1_{y>t}. \tag{5.12}$$

Remark 5.2 We report in the following table, for the reader's convenience, the Laplace pairs (w.r.t. time and space) of the densities $h_\rho(t, y)$ and $l_\rho(y, t)$.

Process T_ρ with density $h_\rho(t, y)$	$\widetilde{h}_\rho(\xi, y) = E_{1,1}^\rho(-\xi y)$	$\widetilde{h}_\rho(t, \eta) = \frac{1}{\Gamma(\rho)}(\eta t)^{\rho-1}e^{-\eta t}$
Process Y_ρ with density $l_\rho(y, t)$	$\widetilde{l}_\rho(\eta, t) = \frac{\Gamma(\rho, \eta t)}{\Gamma(\rho)}$	$\widetilde{l}_\rho(y, \xi) = \rho E_{1,2}^{1+\rho}(-y\xi)$

The time-Laplace transform of $l_\rho(y, t)$ can be obtained taking into account (1.6.15) and (1.9.3) in [20]. It is evident from the previous corollary that, despite the expression of the Laplace transform of $h_\rho(t, y)$ and $l_\rho(y, t)$ (w.r.t. space and time, respectively) is given in terms of Mittag-Leffler functions, the p.d.e. governing the densities of both Y_ρ and T_ρ do not involve fractional operators.

5.2 Time-Changed Representation and Governing Equation

We start by considering the time-change of a standard Brownian motion $B := \{B(t), t > 0\}$ by the time-stretched process $Y_\rho(t^\alpha)$, under the assumption that the latter is independent of B, i.e.

$$B_{\alpha,\rho}(t) := B(Y_\rho(t^\alpha)), \tag{5.13}$$

for $\rho \in (0, 1]$ and $\alpha \in (0, 1]$.

As a consequence of (5.1), we can write its (one-dimensional) characteristic function as

$$\Phi_{\alpha,\rho}(\xi, t) := \mathbb{E}e^{i\xi B_{\alpha,\rho}(t)} = \frac{\Gamma\left(\rho, \xi^2 t^\alpha/2\right)}{\Gamma(\rho)}, \tag{5.14}$$

from which it is immediate to check that the following equality of the one-dimensional distribution (hereafter denoted by $\overset{d}{=}$) holds

$$B_{\alpha,\rho}(t) \overset{d}{=} \sqrt{Y_\rho} B^{\alpha/2}(t), \qquad t \geq 0, \tag{5.15}$$

where $B^{\alpha/2} := \{B^{\alpha/2}(t), t \geq 0\}$ is a fractional Brownian motion and Y_ρ is a r.v., independent of $B^{\alpha/2}$, with density $l_\rho(y)$ given in (4.3), with $\theta = 0$. We note that the moments of any order of $B_{\alpha,\rho}$ are infinite, as can be easily checked by considering (4.7).

Remark 5.3 It is well-known that, in the case of the ggBm $B^{\beta,\alpha} := \{B^{\beta,\alpha}(t), t \geq 0\}$, the following equality of the one-dimensional distribution holds $B^{\beta,\alpha}(t) \overset{d}{=} B(X^\beta(t^{\alpha/\beta}))$, $t \geq 0$, where the random time argument $X^\beta := \{X^\beta(t), t \geq 0\}$, is the inverse of a stable subordinator of index $\beta \in (0, 1)$ (see [9] and [27]). As remarked in [9] for the ggBm, also in this case the representation (5.15) holds only for the one-dimensional distribution. Indeed, for example, the two-times characteristic function of $B(Y_\rho(t^\alpha))$ reads:

$$\mathbb{E}e^{i\xi_1 B(Y_\rho(t_1^\alpha))+i\xi_2 B(Y_\rho(t_2^\alpha))} = \frac{\Gamma\left(\rho, (\xi_1^2 + \xi_1\xi_2)t_1^\alpha + (\xi_2^2 + \xi_1\xi_2)t_2^\alpha\right)}{\Gamma(\rho)},$$

and therefore it does not depend on $|t_1 - t_2|^\alpha$, on the contrary of what happens for $\mathbb{E}e^{i\xi_1 B_{\alpha,\rho}(t_1)+i\xi_2 B_{\alpha,\rho}(t_2)}$ (as can be easily seen from formula (4.1), for $n = 2$ and $\theta = 0$).

Note that, in our case, the stretching effect of time is obtained by the power of α, and does not depend on ρ. This affects also the following governing equation. We prove now that the characteristic function of $B_{\alpha,\rho}$ satisfies a time-stretched integral equation, in analogy with the ggBm.

Theorem 5.2 *Let $\rho \in (0, 1]$ and $\alpha \in (0, 1]$. Let $e_\rho^z := z^{\rho-1} E_{\rho,\rho}(z^\rho)$ be the so-called ρ-exponential function (see [20], p.50), then the characteristic function (5.14) satisfies the following integral equation*

$$\Phi_{\alpha,\rho}(\xi, t) = 1 - \frac{\alpha\xi^2}{2} \int_0^t e^{-\frac{\xi^2}{2}(t^\alpha - s^\alpha)} e_\rho^{\frac{\xi^2}{2}(t^\alpha - s^\alpha)} s^{\alpha-1} \Phi_{\alpha,\rho}(\xi, s) ds, \qquad t \geq 0, \xi \in \mathbb{R}.$$

(5.16)

Proof Let, for simplicity, $A := \xi^2/2$, then we rewrite the integral in the r.h.s. of (5.16) as

$$\frac{1}{\Gamma(\rho)} \int_0^t e^{-A(t^\alpha - s^\alpha)} \left(A(t^\alpha - s^\alpha)\right)^{\rho-1} E_{\rho,\rho}(A^\rho(t^\alpha - s^\alpha)^\rho) s^{\alpha-1} \Gamma\left(\rho, As^\alpha\right) ds$$

$$= [s = tw^{1/\alpha}]$$

$$= \frac{A^{\rho-1}t^{\alpha\rho}}{\alpha\Gamma(\rho)} \int_0^1 e^{-At^\alpha(1-w)}(1-w)^{\rho-1} E_{\rho,\rho}(A^\rho t^{\alpha\rho}(1-w)^\rho) \Gamma\left(\rho, At^\alpha w\right) dw.$$

Thus the r.h.s. of (5.16) reads

$$1 - \frac{A^\rho t^{\alpha\rho}}{\Gamma(\rho)} \int_0^1 e^{-At^\alpha(1-w)}(1-w)^{\rho-1} E_{\rho,\rho}(A^\rho t^{\alpha\rho}(1-w)^\rho) \Gamma\left(\rho, At^\alpha w\right) dw$$

$$= [\text{by (A.4)}]$$

$$= 1 - A^\rho t^{\alpha\rho} \int_0^1 e^{-At^\alpha y} y^{\rho-1} \sum_{j=0}^\infty \frac{(A^\rho t^{\alpha\rho} y^\rho)^j}{\Gamma(\rho j + \rho)} dy +$$

$$+ A^\rho t^{\alpha\rho} \int_0^1 e^{-At^\alpha(1-w)}(1-w)^{\rho-1} A^\rho t^{\alpha\rho} w^\rho e^{-At^\alpha w} \sum_{j=0}^\infty \frac{(A^\rho t^{\alpha\rho}(1-w)^\rho)^j}{\Gamma(\rho j + \rho)} \sum_{l=0}^\infty \frac{(At^\alpha w)^l}{\Gamma(\rho + l + 1)} dy$$

$$= 1 - A^\rho t^{\alpha\rho} \sum_{j=0}^\infty \frac{(A^\rho t^{\alpha\rho})^j}{\Gamma(\rho j + \rho)} \sum_{l=0}^\infty \frac{(-1)^l (At^\alpha)^l}{l!} \int_0^1 y^{l+\rho j+\rho-1} dy +$$

$$+ A^{2\rho} t^{2\alpha\rho} e^{-At^\alpha} \sum_{j=0}^\infty \frac{(A^\rho t^{\alpha\rho})^j}{\Gamma(\rho j + \rho)} \sum_{l=0}^\infty \frac{(At^\alpha)^l}{\Gamma(\rho + l + 1)} \frac{\Gamma(\rho + l + 1)\Gamma(\rho j + \rho)}{\Gamma(\rho j + 2\rho + l + 1)}$$

$$= 1 - \sum_{j=0}^\infty \frac{1}{\Gamma(\rho j + \rho)} \sum_{l=0}^\infty \frac{(-1)^l (At^\alpha)^{l+\rho j+\rho}}{l!(l + \rho j + \rho)} + \sum_{j=0}^\infty e^{-At^\alpha} \sum_{l=0}^\infty \frac{(At^\alpha)^{\rho j+2\rho+l}}{\Gamma(\rho j + 2\rho + l + 1)}$$

$$= 1 - \sum_{j=0}^\infty \frac{\Gamma(\rho j + \rho) - \Gamma(\rho j + \rho, At^\alpha)}{\Gamma(\rho j + \rho)} + \sum_{j=0}^\infty \frac{\Gamma(\rho j + 2\rho) - \Gamma(\rho j + 2\rho, At^\alpha)}{\Gamma(\rho j + 2\rho)}$$

where in the last step, we have applied (A.5) for the second term and (A.4) for the last one. After a change of index in the second sum, we easily obtain (5.14). □

Equation (5.16) reduces, for $\rho = 1$, to the equation satisfied by the characteristic function of the fractional Brownian motion, i.e.

$$\frac{\partial}{\partial t} u(\xi, t) = -\frac{\alpha}{2} t^{\alpha-1} \xi^2 u(\xi, t).$$

(5.17)

On the other hand, for $\rho < 1$, it can be compared with that satisfied by the characteristic function of the ggBm (see Proposition 4.1 in [26]); in this case the presence of the variable ξ also in the integral's kernel does not allow to obtain, by the Fourier inversion, a master equation for the density of the process, as happens for the ggBm.

We then provide an alternative result, which leads to the governing equation of the marginal density of $B_{\alpha,\rho}$. In this case, we will resort to the equality in distribution (5.15).

Theorem 5.3 *Let $\rho \in (0, 1]$ and $\alpha \in (0, 1]$. The density*

$$f_{B_{\alpha,\rho}}(x, t) = \frac{1}{\sqrt{4\pi t^\alpha}} \int_0^{+\infty} \tau^{-1/2} \exp\left\{-\frac{x^2}{4t\tau^\alpha}\right\} l_\rho(\tau)d\tau, \qquad x \in \mathbb{R}, \ t \geq 0, \qquad (5.18)$$

where $l_\rho(\cdot)$ is given in (4.3), satisfies the following integro-differential equation

$$\frac{\partial}{\partial t} f(x, t) = \frac{\alpha\rho}{t} [f(x, t) - f(x, 0)] + \frac{\alpha}{2t} \frac{\partial^2}{\partial x^2} \int_0^t z^\alpha \frac{\partial}{\partial z} f(x, z)dz, \qquad (5.19)$$

with initial conditions $f(x, 0) = \delta(x)$ and $f(0, t) = 0$.

Proof Let $\widehat{f}(\xi) := \int_{-\infty}^{+\infty} e^{i\xi x} f(x)dx$ denote the Fourier transform, then by transforming (5.19), w.r.t. x, and considering (5.14), we can check that $\Phi_{\alpha,\rho}(\xi, t)$ satisfies the following equation

$$\frac{\partial}{\partial t} \widehat{f}(\xi, t) = \frac{\alpha\rho}{t} \left[\widehat{f}(\xi, t) - 1\right] - \frac{\alpha}{2} t^{\alpha-1} \xi^2 \widehat{f}(\xi, t) + \frac{\alpha^2 \xi^2}{2t} \int_0^t z^{\alpha-1} \widehat{f}(\xi, z)dz, \qquad (5.20)$$

where we have taken into account the initial condition, $\widehat{f}(\xi, 0) = 1$. We then rewrite the r.h.s. of (5.20) as follows:

$$\frac{\alpha\rho}{t} \left[\frac{\Gamma\left(\rho, \xi^2 t^\alpha/2\right)}{\Gamma(\rho)} - 1\right] - \frac{1}{2}\alpha t^{\alpha-1}\xi^2 \frac{\Gamma\left(\rho, \xi^2 t^\alpha/2\right)}{\Gamma(\rho)} + \frac{\alpha^2 \xi^2}{2t} \int_0^t z^{\alpha-1} \frac{\Gamma\left(\rho, \xi^2 z^\alpha/2\right)}{\Gamma(\rho)} dz \ (5.21)$$

$$= -\frac{\alpha\rho}{t\Gamma(\rho)} \gamma\left(\rho, \xi^2 t^\alpha/2\right) - \frac{1}{2}\alpha t^{\alpha-1}\xi^2 \frac{\Gamma\left(\rho, \xi^2 t^\alpha/2\right)}{\Gamma(\rho)} + \frac{\alpha\xi^2}{2t\Gamma(\rho)} \int_0^{t^\alpha} \int_{\xi^2 w/2}^{+\infty} e^{-y} y^{\rho-1} dy dw$$

$$= -\frac{\alpha\rho}{t\Gamma(\rho)} \gamma\left(\rho, \xi^2 t^\alpha/2\right) - \frac{1}{2}\alpha t^{\alpha-1}\xi^2 \frac{\Gamma\left(\rho, \xi^2 t^\alpha/2\right)}{\Gamma(\rho)} + \frac{\alpha}{t\Gamma(\rho)} \int_0^{\xi^2 t^\alpha/2} e^{-y} y^\rho dy +$$

$$+ \frac{\alpha\xi^2 t^{\alpha-1}}{2\Gamma(\rho)} \int_{\xi^2 t^\alpha/2}^{+\infty} e^{-y} y^{\rho-1} dy$$

$$= -\frac{\alpha}{t\Gamma(\rho)} \left[\gamma\left(\rho+1, \xi^2 t^\alpha/2\right)\right] - \frac{\alpha}{t\Gamma(\rho)} \left(\frac{\xi^2 t^\alpha}{2}\right)^\rho e^{-\xi^2 t^\alpha/2} + \frac{\alpha}{t\Gamma(\rho)} \gamma\left(\rho+1, \xi^2 t^\alpha/2\right),$$

where we have applied the following well-known relationship between upper incomplete and lower incomplete gamma functions $\Gamma(\rho) = \Gamma(\rho, x) + \gamma(\rho, x)$ and the recurrence formula

$$\gamma(\rho+1, x) = \rho\gamma(\rho, x) - x^\rho e^{-x}$$

(see [13], p.951). It is now easy to check that (5.21) coincides with the first derivative of $\Phi_{\alpha,\rho}(\xi, t)$ (w.r.t. t) and then equation (5.20) holds. □

The knowledge of the p.d.e. governing the density (5.18) can be then used in order to simulate the trajectories of the tempered Γ-GBM. Note that, for $\rho = 1$, equation (5.20) reduces to (5.17), since, in this case $\widehat{f}(\xi, t) = e^{-\xi^2 t^\alpha/2}$ and thus $\int_0^t z^{\alpha-1} \widehat{f}(\xi, z)dz = t^\alpha \widehat{f}(\xi, t)/\alpha$.

Appendix A. Special Functions

We present here some definitions and results concerning special functions that are needed in our analysis.

Let us recall the definition of the *H-function* (see, for example, [24], p.21):

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right.\right] := \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\prod_{j=1}^{m} \Gamma(b_j + B_j s) \prod_{j=1}^{n} \Gamma(1 - a_j - A_j s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j - B_j s) \prod_{j=n+1}^{p} \Gamma(a_j + A_j s)} z^{-s} ds,$$

(A.1)

with $z \neq 0$, $m, n, p, q \in \mathbb{N}_0$, for $0 \leq m \leq q$, $0 \leq n \leq p$, $a_j, b_j \in \mathbb{R}$, for $i = 1, ..., p$, $j = 1, ..., q$ and $\mathcal{L}$ is a contour such that the following condition is satisfied

$$(b_j + \alpha) \neq (a_l - k - 1), \qquad j = 1, ..., m, \ l = 1, ..., n, \ \alpha, k = 0, 1, ... \quad \text{(A.2)}$$

We need the following well-known properties of the H-function.

P1 For any $z \neq 0$, we have that

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right.\right] = H_{q,p}^{n,m}\left[\frac{1}{z}\left|\begin{matrix}(1 - b_q, B_q)\\(1 - a_p, A_q)\end{matrix}\right.\right]$$

(see equation (1.58) in [24]).

P2 For any $\sigma \in \mathbb{C}$, we have that

$$z^{\sigma} H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right.\right] = H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_p + \sigma A_p, A_p)\\(b_q + \sigma B_q, B_q)\end{matrix}\right.\right]$$

(see equation (1.60) in [24]).

We recall that the *Meijer G-function* is a special case of the H-function (see [18]), i.e.

$$G_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1, ..., a_p)\\(b_1, ..., b_q)\end{matrix}\right.\right] = H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1, 1) \ ... \ (a_p, 1)\\(b_1, 1) \ ... \ (b_q, 1)\end{matrix}\right.\right], \quad \text{(A.3)}$$

and that the function $G_{p,p}^{p,0}\left[x\left|\begin{matrix}...\\...\end{matrix}\right.\right]$ vanishes, for any $|x| > 1$, $p \in \mathbb{N}$ (see [18], Property 3).

Let us consider the *upper-incomplete gamma function*, defined as $\Gamma(\rho, x) := \int_x^{+\infty} e^{-t} t^{\rho-1} dt$. We recall its following series representations

$$\Gamma(\rho, x) = \Gamma(\rho)\left(1 - x^{\rho} e^{-x} \sum_{j=0}^{\infty} \frac{x^j}{\Gamma(\rho + j + 1)}\right) \quad \text{(A.4)}$$

and

$$\Gamma(\rho, x) = \Gamma(\rho) - \sum_{j=0}^{\infty} \frac{(-1)^j x^{\rho+j}}{j!(\rho + j)}, \quad \text{(A.5)}$$

for $x > 0$ and $\rho \neq 0, -1, -2, ...$ (see [1]).

Finally, we recall the definition of the *Mittag-Leffler function with three parameters* (also called Prabhakar function), for any $x \in \mathbb{C}$,

$$E_{\alpha,\beta}^{\gamma}(x) := \sum_{j=0}^{\infty} \frac{(\gamma)_j x^j}{j!\Gamma(\alpha j + \beta)}, \qquad \alpha, \beta, \gamma \in \mathbb{C}, \ \text{Re}(\alpha) > 0,$$

where $(\gamma)_j := \Gamma(\gamma + j)/\Gamma(\gamma)$, together with the n-order differentiation formula (see [12] and [11], for details), for any $n \in \mathbb{N}$, $\lambda \in \mathbb{C}$, $x \in \mathbb{R}^+$:

$$\frac{d^n}{dx^n} x^{\beta-1} E_{\alpha,\beta}^{\gamma}\left(\lambda x^{\alpha}\right) = x^{\beta-n-1} E_{\alpha,\beta-n}^{\gamma}\left(\lambda x^{\alpha}\right). \tag{A.6}$$

Moreover, it is proved in [23] that the Prabhakar function is completely monotone on $\mathbb{R}^+$ (i.e. $f(\cdot) = E_{\alpha,\beta}^{\gamma}(\cdot)$ is infinitely differentiable and such that $f : (0, +\infty) \to \mathbb{R}$ with $(-1)^k f^{(k)}(x) \geq 0$ for any $k \in \mathbb{N}$, $x > 0$) for the parameters inside the following ranges: $0 < \alpha \leq 1$ and $0 < \alpha\gamma \leq \beta \leq 1$.

Appendix B. Holomorphic Property on Locally Convex Spaces

We recall some definitions and theorems on complex analysis in infinite dimensional convex spaces, for further details see [10]. We define here the complexification of a real Hilbert space as a direct sum $\mathcal{H}_{\mathbb{C}} = \mathcal{H} \oplus i\mathcal{H} = \{\xi_1 + i\xi_2 | \xi_1, \xi_2 \in \mathcal{H}\}$.

Definition B.1 *Given a real Hilbert space $\mathcal{H}$, the scalar product in the complexification $\mathcal{H}_{\mathbb{C}}$ can be rewritten by using the bilinear extension of the scalar product in $\mathcal{H}$:*

$$\langle h, g \rangle_{\mathcal{H}_{\mathbb{C}}} = \langle \bar{h}, g \rangle_{\mathcal{H}} \quad \text{for } h, g \in \mathcal{H}_{\mathbb{C}}$$

Definition B.2 *Let be E a vector space on $\mathbb{C}$. U is said "finitely open" if $U \cap F$ is open w.r.t. the Euclidean topology on F, for each finite dimensional subspace F of E.*

Definition B.3 *Let E be a vector space on $\mathbb{C}$, $U \subset E$ a finitely open subset and F a locally convex space. A function $f : U \subset E \to F$ is said "Gateaux" or "G-holomorphic" if $\forall \xi \in U$, $\forall \eta \in E$ and $\phi \in F'$, the function $\mathbb{C} \ni \lambda \to \phi(f(\xi + \lambda\eta)) \in \mathbb{C}$ is holomorphic on some neighborhood of 0 in $\mathbb{C}$.*

Note that we will apply this definition to functions in $\mathbb{C}$, so we have that $F' = \mathbb{C}$, so it is sufficient to check the holomorphic property on f itself. The following lemma is useful in the proof of Theorem 2.1, for further details see [22].

Lemma B.1 Let $U \subset \mathcal{S}_{\mathbb{C}}$ be open and $f : U \to \mathbb{C}$. Then f is holomorphic, if and only if it is G-holomorphic and locally bounded, i.e. each point $\xi \in U$ has a neighborhood whose image under f is bounded.

Declarations

The authors have no competing interests to declare that are relevant to the content of this article.

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