



Asymptotics, Trace, and Density Results for Weighted Dirichlet Spaces Defined on the Half-line

Claudia Capone¹ · Agnieszka Kałamajska²

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Abstract

We give an analytic description for the completion of $C_0^\infty(\mathbb{R}_+)$, where $\mathbb{R}_+ = (0, \infty)$, in Dirichlet space $D^{1,p}(\mathbb{R}_+, \omega) := \{u : \mathbb{R}_+ \rightarrow \mathbb{R} : u \text{ is locally absolutely continuous on } \mathbb{R}_+ \text{ and } \|u\|_{L^p(\mathbb{R}_+, \omega)} < \infty\}$, for given continuous positive weight ω defined on \mathbb{R}_+ , where $1 < p < \infty$. The conditions are described in terms of the modified variants of the B_p conditions due to Kufner and Opic from 1984, which in our approach are focusing on the integrability of $\omega^{-p/(p-1)}$ near zero or near infinity. Moreover, we propose applications of our results to: obtaining new variants of Hardy inequality, interpretation of boundary value problems in ODE's defined on the half-line with solutions in $D^{1,p}(\mathbb{R}_+, \omega)$, new results from complex interpolation theory dealing with interpolation spaces between weighted Dirichlet spaces, and for deriving new Morrey type embedding theorems for our Dirichlet space.

Keywords Densities · Dirichlet space · Sobolev space · Asymptotics · Hardy inequality · Morrey inequality

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1 Introduction

In this paper we are interested in weighted Dirichlet spaces

$$D^{1,p}(\mathbb{R}_+, \omega) = \{u : \mathbb{R}_+ \rightarrow \mathbb{R} : u \text{ is locally absolutely continuous on } \mathbb{R}_+ \text{ and } \|u'\|_{L^p(\mathbb{R}_+, \omega)} < \infty\}.$$

Claudia Capone and Agnieszka Kałamajska contributed equally to this work.

✉ Agnieszka Kałamajska
A.Kalamajska@mimuw.edu.pl

Claudia Capone
c.capone@na.iac.cnr.it

¹ Istituto per le Applicazioni del Calcolo “Mauro Picone”, Consiglio Nazionale delle Ricerche, via Pietro Castellino 11, Napoli 80131, Italy

² Faculty of Mathematics, Informatics, and Mechanics, University of Warsaw, ul. Banacha 2, Warsaw 02–097, Poland

In most situations we assume that the weight $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, is continuous and $1 < p < \infty$.

In some cases we also assume that ω satisfies the localized at the endpoint variant of the general B_p -condition due to Kufner and Opic from [13]:

$$B_p(0) : \int_{(0,1)} \omega(t)^{-1/(p-1)} dt < \infty \text{ or } B_p(\infty) : \int_{(1,\infty)} \omega(t)^{-1/(p-1)} dt < \infty.$$

We address and analyze several problems related to such spaces.

Asymptotic behaviour. One of the topics of our interest is the asymptotic behaviour at the endpoints of the elements of such spaces. Assume for example that $\omega \in B_p(0)$. Among our results in this direction, we show in Theorem 3.1 that when $\omega \in B_p(0)$, $u \in D^p(\mathbb{R}_+, \omega)$ and $c \in \mathbb{R}$, then the conditions (a),(b),(c) are equivalent, where

$$(a) \exists_{t_n \searrow 0} : \lim_{t_n \rightarrow 0} (u(t_n) - c) = 0, \quad (b) \lim_{t \rightarrow 0} u(t) = c, \tag{1}$$

$$(c) \lim_{t \rightarrow 0} \frac{u(t) - c}{\Omega_\omega^0(t)} = 0, \text{ where } \Omega_\omega^0(t) := \left(\int_{(0,t)} \omega(\tau)^{\frac{-1}{p-1}} d\tau \right)^{1-\frac{1}{p}}.$$

As $\Omega_\omega^0(t) \rightarrow 0$ when $t \rightarrow 0$, we clearly have $(c) \Rightarrow (b) \Rightarrow (a)$. The nontrivial part is to prove that the converse implications hold.

Similar analysis is also provided about behaviour near infinity.

Trace operator. There are several ways to define the trace of Sobolev function, see e.g. [11], Section 6.10.5 for the classical approach. We ask about the limit (b) in (1) and we define

$$Tr^0 u := \lim_{t \rightarrow 0} u(t). \tag{2}$$

Clearly, one has to ask if such limit is well prescribed in our Dirichlet space setting. It is always so, when we assume that $\omega \in B_p(0)$, see Theorem 3.1, part iii) and it is never so, when $\omega \notin B_p(0)$, see Theorem 4.4.

The norm on Dirichlet space. Let us note that the quantity $\|u'\|_{L^p(\mathbb{R}_+, \omega)}$ annihilates all constants, therefore $\|u'\|_{L^p(\mathbb{R}_+, \omega)}$ cannot define the norm on Dirichlet space $D^{1,p}(\mathbb{R}_+, \omega)$. However, for any $a \in \mathbb{R}_+$, the quantity

$$\|u\|_{D^{1,p}(\mathbb{R}_+, \omega)}^{(a)} := |u(a)| + \|u'\|_{L^p(\mathbb{R}_+, \omega)}, \tag{3}$$

defines the norm on $D^{1,p}(\mathbb{R}_+, \omega)$ and makes it a Banach space. Moreover, all such norms $\|\cdot\|_{D^{1,p}(\mathbb{R}_+, \omega)}^{(a)} : a \in \mathbb{R}_+$, are equivalent. See Fact 2.2.

In the case of $\omega \in B_p(0)$, we can extend the definition of the norm (3) also to $a = 0$, by putting $u(0) := Tr^0(u)$ in place $u(a)$. Such modification gives also the equivalent norm. In such case the trace operator is continuous as functional on our Dirichlet space equipped with any of the proposed norms (3), including $a = 0$, see Theorem 3.1, part iv). In case of $\omega \in B_p(\infty)$, similar property holds with $a = \infty$, see Theorem 3.2.

Representation of functions. Let us focus on the case of $\omega \in B_p(0)$. Because in that case the limit $\lim_{t \rightarrow 0} u(t)$ does exist for any u in Dirichlet space and u' is integrable near zero, every element $u \in D^{1,p}(\mathbb{R}_+, \omega)$ can be represented as

$$u(t) - Tr^0 u = \int_0^t u'(\tau) d\tau = Hu'(t),$$

where on the right hand side above we deal with Hardy transform of u' , remembering that u' belongs to $L^p(\mathbb{R}_+, \omega)$. This allows to deduce several further properties, for example

applications to Hardy inequality, see Section 5.1. Similar representations hold in case of $\omega \in B_p(\infty)$. In this case we use the conjugate Hardy transform as in (35), see Theorem 3.2.

Questions about densities. Let us denote by $D_0^{1,p}(\mathbb{R}_+, \omega)$ the completion of $C_0^\infty(\mathbb{R}_+)$ in $D^{1,p}(\mathbb{R}_+, \omega)$ in any norm like (3). It is the natural question to ask about characterization of weights ω , for which $D_0^{1,p}(\mathbb{R}_+, \omega) = D^{1,p}(\mathbb{R}_+, \omega)$. If that spaces are not the same, we can ask if it is possible to characterize the space $D_0^{1,p}(\mathbb{R}_+, \omega)$ by some analytic conditions expressed in terms of the weight ω . Let us focus again on the case of $\omega \in B_p(0)$. In Theorem 4.1 we have proved that for such ω

$$D_0^{1,p}(\mathbb{R}_+, \omega) = \mathcal{B}_{p,\omega}^0(0) := \{u \in D^{1,p}(\mathbb{R}_+, \omega) : \lim_{t \rightarrow 0} u(t) = 0\} \iff \omega \notin B_p(\infty).$$

This gives the analytic description of weights for which $D_0^{1,p}(\mathbb{R}_+, \omega)$ is the kernel of trace operator as in (2), in the case of $\omega \in B_p(0)$. Let us emphasize that the trace operator $u \mapsto Tr^0 u$ in such case is well defined and continuous.

As the consequence of Theorem 3.1 and Theorem 4.4, in the case of $\omega \notin B_p(0)$, the trace operator $Tr^0(\cdot)$ at zero is not well defined. Therefore we can ask if the space $D_0^{1,p}(\mathbb{R}_+, \omega)$ could still be characterized by some analytic conditions, without assuming that $\omega \in B_p(0)$. Such characterization is provided in Theorem 4.3, which gives the precise analytic characterization of $D_0^{1,p}(\mathbb{R}_+, \omega)$, expressed in terms of the conditions $B_p(0)$ and $B_p(\infty)$. For example, as follows from Theorems: 4.3 and 4.5, among the other statements, we show that

$$D_0^{1,p}(\mathbb{R}_+, \omega) = \mathcal{B}_{p,\omega}^0(0) \iff \omega \in B_p(0) \setminus B_p(\infty).$$

Applications to: Hardy inequality, Boundary Value Problems (B.V.P.) in ODE's, generalized Morrey Theorem and to complex interpolation theory. Having more precise information about representation of function from our Dirichlet space, or about the asymptotic behaviour of the functions from given Dirichlet space near zero or infinity, one can deduce more precise variants of Hardy and conjugate Hardy inequality (see Section 5.1), or establish if the given boundary value problem, presented in term of vanishing of function near zero or infinity in the analyzed ODE, is well posed or not. The discussion is provided in Section 5.2. Moreover, in Section 5.3 we focus on certain generalization of Morrey Theorem, which deals with B_p -conditions. In Section 5.4 we have also presented some new applications of our results to complex interpolation theory, dealing with weighted Dirichlet type spaces, inspired by questions posed recently in [2].

Except the already mentioned applications, let us further explain why *the analysis of weighted Dirichlet spaces is important*.

Weighted Dirichlet spaces on general domains

$$D^{1,p}(\Omega, \rho) := \left\{ u \in D'(\Omega) : \frac{\partial u}{\partial x_i} \in L_\rho^p(\Omega), i = 1, \dots, n \right\}, \tag{4}$$

where $\Omega \subseteq \mathbb{R}^n$ is an open domain, $D'(\Omega)$ denotes the space of distributions on Ω , ρ is a given weight function defined on Ω , are one of the crucial tools in the study of degenerate PDE's, see e.g. [10, 15].

Dirichlet spaces for functions defined on the half-line are important auxiliary tools.

First of all, when asking about properties of the Dirichlet space like (4), one could consider at first the simple case of $\Omega = \mathbb{R}^n \setminus \{0\}$ (or $\Omega = \mathbb{R}^n$) and radial functions $u(x) = v(|x|)$, getting the condition $v \in D^{1,p}(\mathbb{R}_+, \omega)$ for suitable weight ω .

Dirichlet spaces related to the half-line naturally appear as a consequence of the pointwise inequalities obtained in [18, p. 193-194], inspired by the capacity estimates from [16]:

$$[-(u^*)'(t)]^q t^{r(q,n)} \lesssim \frac{d}{dt} \Psi_{q,u}(t), \quad \text{where } 1 \leq q \leq n,$$

$r(q, n) := \frac{n-q}{n} + (q - 1)$, $\Psi_{q,u}(t) := \int_{\{|u|>u^*(t)\}} |\nabla u(x)|^q dx$, u^* is the symmetric rearrangement of $u \in W_0^{1,q}(\Omega)$, Ω is a domain in \mathbb{R}^n (see page 129 in [18]). They imply the following estimates, holding for any p such that $q \leq p < \infty$, with any weight function ω defined on the half-line:

$$\int_{\mathbb{R}_+} |(u^*)'(t)|^p \omega(t) dt \lesssim \int_{\mathbb{R}_+} |\Psi'_{q,u}(t)|^{\frac{p}{q}} \frac{\omega(t)}{t^{s(q,p,n)}} dt,$$

where $s(q, p, n) := \frac{r(q,n)p}{q}$.

Dirichlet spaces related to the half-line can also serve as auxiliary tools in the a-priori estimates for solutions of PDE’s and are related to their symmetric rearrangements. This comes from the analysis, which originated in [14] and [23].

For example, the following result has been presented in [18] in Section 10 (we refer there for the missing definitions). Let us consider the entropy solution of the nonlinear elliptic equation of the form

$$\begin{cases} -\operatorname{div}(a(x, u, \nabla u)) = f v \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{5}$$

where Ω is a sufficiently regular domain in \mathbb{R}^n , $n \geq 2$, $\mu = v(x)dx$ is a probability measure on \mathbb{R}^n , $a(x, \eta, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a Carathéodory function such that for some fixed $p > 1$

$$a(x, \eta, \xi) \cdot \xi \geq v(x)|\xi|^p, \quad \text{for a.e. } x \in \Omega, \text{ and every } (\eta, \xi) \in \mathbb{R} \times \mathbb{R}^n.$$

It has been shown in Theorem 14 in [18] that if $u \in W_0^{1,1}(\Omega, v)$ is a entropy solution of (5), $\mu = v(x)dx$, $I(\cdot) = I(\mathbb{R}^n; \mu)(\cdot)$ is the isoperimetric profile of $(\mathbb{R}^n; \mu)$ (see page 130), g_μ^* means the decreasing rearrangement of g related to μ (see page 129), then

$$(-u_\mu^*)'(t) I^2(t) \leq \int_{(0,t)} f_\mu^*(s) ds.$$

From there we easily deduce that

$$\int_{\mathbb{R}_+} |(u_\mu^*)'(t)|^p \omega(t) dt \leq \int_{\mathbb{R}_+} \left(\int_{(0,t)} f_\mu^*(s) ds \right)^p \frac{\omega(t)}{I^{2p}(t)} dt,$$

In particular u_μ^* belongs to $D^{1,p}(\mathbb{R}_+, \omega)$, under suitable integrability conditions for f_μ^* .

Another example which shows the importance in considering weighted Dirichlet spaces related to the half-line is their application to the real interpolation theory via Hardy inequality in the weighted L^p setting. We refer to the pioneering work [5] and to the later contribution [6].

Novelty and link with literature. To our best knowledge, our results concerning the asymptotic behaviour near the endpoints of the interval, of functions in the non trivially weighted Dirichlet spaces, as summarized in Theorems 3.1 and 3.2, are new. In the non-weighted setting they are motivated by Morrey Theorem, see Section 5.3. However, similar type conclusions can be found also in the case of power weight in [9], on page 9.

Density results in the general Dirichlet space setting, are rather missing in the literature. In the case of $\omega \equiv 1$, they were obtained first by Sobolev in 1963 ([22]), and now they are

well understood. See also e.g. [7, Theorem 4] and references therein, where density results are obtained with respect to the semi norm $\|u'\|_{L^p(\mathbb{R}_+)}$ as in Fact 2.2, instead of the norm $\|u\|^{(a)}$ from (7). Our density analysis is based on the localized at endpoints B_p conditions as in Definition 2.1, which, in our opinion, were not considered before. However, some preliminary ideas for such conditions can be found in [13], see Remark 6.1.

Most of the classical density results deal with Sobolev spaces, not Dirichlet spaces. In case of Sobolev spaces, the additional restriction on function u is provided, that is its integrability with some power. We would like to emphasise that our density results mostly deal with the norm (3), they are restricted to Dirichlet (not Sobolev) space, and characterize completely the admitted weights.

As about *the tools*, for the analysis of asymptotic behaviour we use simple computations based on Taylor's formula in 1-d. To study density, we propose the technique, which in our opinion is new in such setting. We call it the energy - caloric approximation, as it is based on the variational technique. More precisely, we first find the function which minimizes the energy functional

$$\int_{(a,b)} |u'(\tau)|^p \omega(\tau) d\tau,$$

with the given boundary data at the endpoints $\{a, b\}$. In further step we extend such local minimizers to compactly supported functions in the same Dirichlet-Sobolev class. See the considerations in Sections 4.2 and 4.3.

Organization of the paper: After the preliminary results presented in Section 2, we analyze questions about the asymptotic behaviour and trace in Section 3, while density results are presented in Section 4. Main applications: to the deriving of Hardy inequality, to the well posedness of B.V.P.'s, for deriving of Morrey-type theorems, as well as to complex interpolation theory in Dirichlet space setting, are discussed in Section 5. Some additional remarks are presented in Section 6, while in Section 7 we enclose some auxiliary computations and complementary results, for reader's convenience.

2 Notation and Preliminaries

2.1 Basic Notation

In most situations we deal with positive continuous functions $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, referred as positive weights, where, by positive expression, we mean that it is strictly larger than zero. However for our purposes we consider continuous weights only, we will sometimes formulate our statements in the more general setting.

We use standard notation: $C_0^\infty(\mathbb{R}_+)$, $L^p(\mathbb{R}_+, \omega)$, $L_{loc}^p(\mathbb{R}_+, \omega)$, $Lip(\mathbb{R}_+)$, $W^{1,p}(\mathbb{R}_+)$, $W_{loc}^{1,p}(\mathbb{R}_+)$, for smooth compactly supported functions, weighted L^p -spaces and their local variants, Lipschitz functions, the classical Sobolev spaces and their local variants. We will also use the more specific notation for the local variants of L^p and Sobolev - type spaces. For $1 \leq p < \infty$, by $L_{loc}^p([0, \infty))$ we denote all functions $f \in L_{loc}^p(\mathbb{R}_+)$ which are p -integrable near zero (shortly $\int_0 |f|^p d\tau < \infty$), while by $L_{loc}^p((0, \infty])$ we denote all functions $f \in L_{loc}^p(\mathbb{R}_+)$, which are p -integrable near infinity (shortly $\int^\infty |f|^p d\tau < \infty$). In most situations we will refer to the Lebesgue integral. However, we will also sometimes refer to the Newtonian interpretation of the integral: by writing $\int_a^b f d\tau$ we mean $F(b) - F(a)$,

where $F' = f$ a.e.. By measurable sets, we mean sets that are measurable with respect to the Lebesgue measure.

Let X be some subset of Lebesgue measurable functions defined on \mathbb{R}_+ . By X_c we will denote its subset consisting of functions with compact support in \mathbb{R}_+ . When $X \subseteq Z$, where $(Z, \|\cdot\|_Z)$ is some Banach space, then by $\overline{X}^{\|\cdot\|_Z}$ will denote the completion of X in the norm $\|\cdot\|_Z$. The symbol Z_0 will be reserved for $(C_0^\infty(\mathbb{R}_+) \cap Z)^{\|\cdot\|_Z}$.

In our estimates, we will sometimes write $f \sim 1$ if the function f defined on its respective domain can be estimated from both sides by positive constants, while the notation $F \lesssim G$, with some expressions F, G , will mean that the inequality $F \leq CG$ holds with some universal constant C .

2.2 General and Local B_p -conditions for Weights

We will deal with the following variants of the B_p -condition introduced by Kufner and Opic in [13].

Definition 2.1 (B_p -conditions) Let $\omega : \mathbb{R}_+ \rightarrow [0, \infty)$ be a measurable function which is positive almost everywhere, $1 < p < \infty$. We say that

- a) ω is a B_p -weight (shortly $\omega \in B_p$) if $\omega^{-1/(p-1)} \in L^1_{loc}((0, \infty))$, see [13];
- b) ω is a B_p -weight near zero (shortly $\omega \in B_p(0)$) if $\int_0^\infty \omega^{-1/(p-1)} d\tau < \infty$;
- c) ω is a B_p -weight near infinity (shortly $\omega \in B_p(\infty)$) if $\int_0^\infty \omega^{-1/(p-1)} d\tau < \infty$.

Note that both conditions $B_p(0)$ and $B_p(\infty)$ imply that $\omega \in B_p$. Moreover, by Hölder inequality, for any measurable set $K \subseteq [0, \infty)$

$$\int_K |f(\tau)| d\tau = \int_K |f(\tau)| \omega(\tau)^{\frac{1}{p}} \omega(\tau)^{-\frac{1}{p}} \leq \left(\int_K |f(\tau)|^p \omega(\tau) d\tau \right)^{\frac{1}{p}} \left(\int_K \omega(\tau)^{\frac{-1}{p-1}} d\tau \right)^{1-\frac{1}{p}}, \tag{6}$$

This implies the following observation.

Fact 2.1 Let ω, p be as in Definition 2.1. The following statements hold:

- i) when $\omega \in B_p$ then $L^p(\mathbb{R}_+, \omega) \subseteq L^1_{loc}(\mathbb{R}_+)$;
- ii) when $\omega \in B_p(0)$ then $L^p(\mathbb{R}_+, \omega) \subseteq L^1_{loc}([0, \infty))$;
- iii) when $\omega \in B_p(\infty)$ then $L^p(\mathbb{R}_+, \omega) \subseteq L^1_{loc}((0, \infty])$.

In our specific situation, we assume that the weight ω is continuous and positive, which guarantees that $\omega \in B_p$. The conditions $B_p(0)$ and $B_p(\infty)$, which to our best knowledge were not introduced earlier, are motivated by the general B_p condition from [13]. More precise information about B_p -conditions is provided in Remark 6.1.

2.3 Weighted Dirichlet Spaces

We start with the definition of weighted Dirichlet space.

Definition 2.2 (weighted Dirichlet space) Let $\omega : \mathbb{R}_+ \rightarrow [0, \infty)$ be positive weight, that is $\omega > 0$ a.e., $1 < p < \infty$. By $D^{1,p}(\mathbb{R}_+, \omega)$ we will denote the Dirichlet space consisting with all functions $u \in W_{loc}^{1,1}(\mathbb{R}_+)$ such that

$$\|u\|_{D^{1,p}(\mathbb{R}_+, \omega)}^* := \left(\int_{\mathbb{R}_+} |u'(t)|^p \omega(t) dt \right)^{\frac{1}{p}} < \infty.$$

Clearly, the expression $\|u\|_{D^{1,p}(\mathbb{R}_+, \omega)}^*$ annihilates constant functions, so it defines the semi norm on $D^{1,p}(\mathbb{R}_+, \omega)$ but not the norm.

We are interested in Dirichlet spaces in the case when ω is continuous and positive, and so $\omega \in B_p$. In this case, we show that the homogeneous Dirichlet space $\tilde{D}^{1,p}(\mathbb{R}_+, \omega)$ defined below is complete. The proof is enclosed in the Appendix for reader’s convenience.

Fact 2.2 (Homogeneous Dirichlet space) Let ω, p be as in Definition 2.2 and consider the relation: $u \sim v$ when $u, v \in D^{1,p}(\mathbb{R}_+, \omega)$ and $u - v \equiv c$ for some constant $c \in \mathbb{R}$. Then

$$\tilde{D}^{1,p}(\mathbb{R}_+, \omega) := D^{1,p}(\mathbb{R}_+, \omega) / \sim \text{ equipped with the norm}$$

$$\| \{u + c\}_{c \in \mathbb{R}} \|_{\tilde{D}^{1,p}(\mathbb{R}_+, \omega)}^* := \|u'\|_{L^p(\mathbb{R}_+, \omega)}$$

is a Banach space.

In the following fact we analyze the norms in Dirichlet spaces.

Fact 2.3 (The norms on $D^{1,p}(\mathbb{R}_+, \omega)$ and the topology of convergence) Let ω, p be as in Definition 2.2. Then for any $a \in (0, \infty)$ the expression

$$\|u\|_{D^{1,p}(\mathbb{R}_+, \omega)}^{(a)} := \|u'\|_{L^p(\mathbb{R}_+, \omega)} + |u(a)| \tag{7}$$

is the norm on $D^{1,p}(\mathbb{R}_+, \omega)$, which makes $D^{1,p}(\mathbb{R}_+, \omega)$ a Banach space.

Moreover, for all $a \in \mathbb{R}_+$ the norms $\| \cdot \|_{D^{1,p}(\mathbb{R}_+, \omega)}^{(a)}$ are equivalent on $D^{1,p}(\mathbb{R}_+, \omega)$ and

$$\|u_n - u\|_{D^{1,p}(\mathbb{R}_+, \omega)}^{(a)} \xrightarrow{n \rightarrow \infty} 0 \iff \tag{8}$$

$$(u'_n \rightarrow u' \text{ in } L^p(\mathbb{R}_+, \omega) \text{ and } u_n \rightarrow u \text{ uniformly on compact sets in } \mathbb{R}_+).$$

Proof. We observe that $D^{1,p}(\mathbb{R}_+, \omega) \subseteq W_{loc}^{1,1}(\mathbb{R}_+) \subseteq C(\mathbb{R}_+)$ and so the value $u(a)$ is well prescribed. In particular $\| \cdot \|_{D^{1,p}(\mathbb{R}_+, \omega)}^{(a)}$ is the norm on $D^{1,p}(\mathbb{R}_+, \omega)$.

Moreover, $(D^{1,p}(\mathbb{R}_+, \omega), \| \cdot \|_{D^{1,p}(\mathbb{R}_+, \omega)}^{(a)})$ is a Banach space, because when $\{u_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $D^{1,p}(\mathbb{R}_+, \omega)$, then, due to Fact 2.2, there exists $v \in D^{1,p}(\mathbb{R}_+, \omega)$ such that u'_n converge to v' in $L^p(\mathbb{R}_+, \omega)$.

Then for $u(t) := \int_a^t v'(\tau) d\tau + \lim_{n \rightarrow \infty} u_n(a)$ we have $\|u_n - u\|_{D^{1,p}(\mathbb{R}_+, \omega)}^{(a)} \rightarrow 0$ as $n \rightarrow \infty$.

The equivalence of norms is a consequence of the following estimate holding for any closed interval I such that $a, b \in I \subset \mathbb{R}_+, 0 < a < b < \infty$:

$$\begin{aligned} |u(b) - u(a)| &\leq \int_{(a,b)} |u'(\tau)| \omega(\tau)^{\frac{1}{p}} \omega(\tau)^{-\frac{1}{p}} d\tau \\ &\leq \left(\int_I |u'(\tau)|^p \omega d\tau \right)^{\frac{1}{p}} \left(\int_I \omega(\tau)^{\frac{-1}{p-1}} d\tau \right)^{1-\frac{1}{p}} = C_I \left(\int_{\mathbb{R}_+} |u'(\tau)|^p \omega d\tau \right)^{\frac{1}{p}}, \end{aligned} \tag{9}$$

where $C_I := \left(\int_I \omega(\tau)^{\frac{-1}{p-1}} d\tau \right)^{1-\frac{1}{p}}$. As a consequence of (9) we get (8) with any $b \in I$ and $u_n - u$ in place of u . □

More precise analysis, dealing with the conditions $B_p(0)$ and $B_p(\infty)$, will be provided in our next section.

3 Asymptotics and Trace

3.1 Analysis in the Case of $\omega \in B_p(0)$

We start with the analysis within the case of $\omega \in B_p(0)$. We obtain the following statement, which deals with trace operator defined at zero $Tr^0(\cdot)$, as in (12) below, and precisely describes the elements of weighted Dirichlet space.

Theorem 3.1 (Asymptotic behaviour and trace at zero). *Let $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $\omega \in C(\mathbb{R}_+) \cap B_p(0)$, $1 < p < \infty$ and consider the following subsets in $D^{1,p}(\mathbb{R}_+, \omega)$, defined for $c \in \mathbb{R}$:*

$$\begin{aligned} \mathcal{R}_{p,\omega}^0(c) &:= \left\{ u \in D^{1,p}(\mathbb{R}_+, \omega), u(t) = \int_0^t u'(\tau) d\tau + c : u' \in L^p(\mathbb{R}_+, \omega) \right\}; \tag{10} \\ \mathcal{A}_{p,\omega}^0(c) &:= \left\{ u \in D^{1,p}(\mathbb{R}_+, \omega) : \exists t_n \searrow 0 : \lim_{n \rightarrow \infty} u(t_n) = c \right\}; \\ \mathcal{B}_{p,\omega}^0(c) &:= \left\{ u \in D^{1,p}(\mathbb{R}_+, \omega) : \lim_{t \rightarrow 0} u(t) = c \right\}; \\ \mathcal{C}_{p,\omega}^0(c) &:= \left\{ u \in D^{1,p}(\mathbb{R}_+, \omega) : \lim_{t \rightarrow 0} \frac{u(t) - c}{\Omega_\omega^0(t)} = 0, \sup_{t>0} \frac{u(t) - c}{\Omega_\omega^0(t)} < \infty, \right\}, \end{aligned}$$

where $\Omega_\omega^0(t) = \left(\int_{(0,t)} \omega(\tau)^{\frac{-1}{p-1}} d\tau \right)^{1-\frac{1}{p}}$.

The following statements hold.

- i) For any $c \in \mathbb{R}$, the set $\mathcal{R}_{p,\omega}^0(c)$ is a closed subset in $D^{1,p}(\mathbb{R}_+, \omega)$, equipped with any norm $\|\cdot\|^{(a)}$ as in (7), where $a \in \mathbb{R}_+$.
- ii) For any $c \in \mathbb{R}$

$$\mathcal{A}_{p,\omega}^0(c) = \mathcal{B}_{p,\omega}^0(c) = \mathcal{C}_{p,\omega}^0(c) = \mathcal{R}_{p,\omega}^0(c). \tag{11}$$

- iii) For every $u \in D^{1,p}(\mathbb{R}_+, \omega)$ there is $c \in \mathbb{R}$ such that $u \in \mathcal{R}_{p,\omega}^0(c)$. In particular, the trace operator

$$Tr^0(u) := \lim_{t \rightarrow 0} u(t) =: u(0), \tag{12}$$

is well defined for every $u \in D^{1,p}(\mathbb{R}_+, \omega)$ and $D^{1,p}(\mathbb{R}_+, \omega) = \cup_{c \in \mathbb{R}} \mathcal{R}_{p,\omega}^0(c)$.
Moreover,

$$D^{1,p}(\mathbb{R}_+, \omega) = \left\{ u(t) = \int_0^t v(\tau) d\tau + c : c \in \mathbb{R}, v \in L^p(\mathbb{R}_+, \omega) \right\}.$$

- iv) The quantity

$$\|u\|_{D^{1,p}(\mathbb{R}_+, \omega)}^{(0)} := \|u'\|_{L^p(\mathbb{R}_+, \omega)} + |Tr^0(u)| \tag{13}$$

is the norm on $D^{1,p}(\mathbb{R}_+, \omega)$, which is equivalent to any norm $\|u\|_{D^{1,p}(\mathbb{R}_+, \omega)}^{(a)}$, where $a \in \mathbb{R}_+$.

Proof. We observe that the substitution of $u - c$ in place of u , reduces the proofs of i) and ii) to the case of $c = 0$. Therefore, for those statements, we only show the case of $c = 0$.

i): Let us consider any sequence $\{u_n\}_{n \in \mathbb{N}} \subseteq \mathcal{R}_{p,\omega}^0(0)$, such that

$u_n(t) \xrightarrow{n \rightarrow \infty} u(t)$ in $D^{1,p}(\mathbb{R}_+, \omega)$. By Fact 2.3

$$u_n \rightarrow u \text{ uniformly on compact sets in } \mathbb{R}_+ \text{ and } u'_n \rightarrow u' \text{ in } L^p(\mathbb{R}_+, \omega).$$

As $\omega \in B_p(0)$, therefore the above convergence yields $u'_n \rightarrow u'$ in $L^1(0, K)$, for every $K > 0$. Since $u_n \in \mathcal{R}_{p,\omega}^0(0)$, for every $t > 0$, we have

$$u_n(t) = \int_0^t u'_n(\tau) d\tau \xrightarrow{n \rightarrow \infty} u(t) = \int_0^t u'(\tau) d\tau.$$

In particular $u \in \mathcal{R}_{p,\omega}^0(0)$ and so $\mathcal{R}_{p,\omega}^0(0)$ is closed.

ii): We start by proving the identity $\mathcal{R}_{p,\omega}^0(0) = \mathcal{B}_{p,\omega}^0(0)$.

Let $u \in \mathcal{R}_{p,\omega}^0(0)$. Then

$$u(t) = \int_0^t u'(\tau) d\tau. \tag{14}$$

Therefore $u(t) \rightarrow 0$ as $t \rightarrow 0$, hence $u \in \mathcal{B}_{p,\omega}^0(0)$. This gives $\mathcal{R}_{p,\omega}^0(0) \subseteq \mathcal{B}_{p,\omega}^0(0)$.

On the other hand, if $u \in \mathcal{B}_{p,\omega}^0(0)$, then for every $0 < \bar{t} < t < K$, we have

$$u(t) - u(\bar{t}) = \int_{\bar{t}}^t u'(\tau) d\tau.$$

Since $u' \in L^1((0, K))$ for any K , by taking the limit as $\bar{t} \rightarrow 0$, we get (14). Hence $u \in \mathcal{R}_{p,\omega}^0(0)$.

We will complete the proof of ii) by proving the identity:

$$\mathcal{A}_{p,\omega}^0(0) = \mathcal{B}_{p,\omega}^0(0) = \mathcal{C}_{p,\omega}^0(0).$$

We first show the equality

$$\mathcal{A}_{p,\omega}^0(0) = \mathcal{B}_{p,\omega}^0(0). \tag{15}$$

Clearly, $\mathcal{A}_{p,\omega}^0(0) \supseteq \mathcal{B}_{p,\omega}^0(0)$, so we have to prove the converse inclusion. To this aim, let us take $u \in \mathcal{A}_{p,\omega}^0(0)$, and let $t_n \rightarrow 0$ be such that $u(t_n) \rightarrow 0$ as $n \rightarrow \infty$. Then, for any t such that $0 < t_n < t$

$$\begin{aligned} |u(t) - u(t_n)| &\leq \int_{(t_n,t)} |u'(\tau)| d\tau = \int_{(t_n,t)} |u'(\tau)| \omega^{\frac{1}{p}}(\tau) \omega^{-\frac{1}{p}}(\tau) d\tau \\ &\leq \left(\int_{(t_n,t)} |u'(\tau)|^p \omega(\tau) d\tau \right)^{\frac{1}{p}} \left(\int_{(t_n,t)} \omega(\tau)^{\frac{-1}{p-1}} d\tau \right)^{1-\frac{1}{p}} \\ &\leq \left(\int_{(0,t)} |u'(\tau)|^p \omega(\tau) d\tau \right)^{\frac{1}{p}} \left(\int_{(0,t)} \omega(\tau)^{\frac{-1}{p-1}} d\tau \right)^{1-\frac{1}{p}}. \end{aligned} \tag{16}$$

After letting $n \rightarrow \infty$, we get

$$|u(t)| \leq \left(\int_{(0,t)} |u'(\tau)|^p \omega(\tau) d\tau \right)^{\frac{1}{p}} \cdot \Omega_\omega^0(t) \xrightarrow{t \rightarrow 0} 0, \tag{17}$$

which proves (15).

Let us show that both sets in (15) are the same as $C_{p,\omega}^0(0)$. Indeed, let us consider $u \in \mathcal{A}_{p,\omega}^0(0)$. Then, by (17), we deduce that $u(t)/\Omega_\omega^0(t) \xrightarrow{t \rightarrow 0} 0$. Hence $u \in C_{p,\omega}^0(0)$.

On the other hand, when $u \in C_{p,\omega}^0(0)$, then $u(t) \xrightarrow{t \rightarrow 0} 0$, because $1/\Omega_\omega^0(t) \xrightarrow{t \rightarrow 0} \infty$. Hence $u \in \mathcal{B}_{p,\omega}^0(0) = \mathcal{A}_{p,\omega}^0(0)$.

iii): Consider any sequence $t_n \searrow 0$. Then, for any fixed t , by using (16), we get the boundedness of $\{u(t_n)\}_{n \in \mathbb{N}}$. By Bolzano-Weierstrass Theorem, we can extract a convergent subsequence, which we will also denote by $\{u(t_n)\}_{n \in \mathbb{N}}$. Let c be its limit.

By taking the limit as $n \rightarrow \infty$ in (16) we get

$$|u(t) - c| \leq \left(\int_{(0,t)} |u'(\tau)|^p \omega(\tau) d\tau \right)^{\frac{1}{p}} \cdot \Omega_\omega^0(t), \tag{18}$$

which implies $u(t) \rightarrow c$ as $t \rightarrow 0$. Hence, any function $u \in D^{1,p}(\mathbb{R}_+, \omega)$ has the limit as $t \rightarrow 0$, which implies the well-posedness of the trace operator $Tr^0(\cdot)$.

We have also proved that any function $u \in D^{1,p}(\mathbb{R}_+, \omega)$ belongs to $\mathcal{B}_{p,\omega}^0(c)$ for some c . This, together with (11), gives the decomposition $D^{1,p}(\mathbb{R}_+, \omega) = \bigcup_{c \in \mathbb{R}} \mathcal{R}_{p,\omega}^0(c)$.

iv): Due to the existence of the limit of u at zero, we can apply the estimate (9) with $b := a > 0$ and $a := 0$, getting

$$|u(a) - Tr^0(u)| \leq C_I \left(\int_{\mathbb{R}_+} |u'(\tau)|^p \omega d\tau \right)^{\frac{1}{p}}, \text{ where}$$

$$C_I = \left(\int_{(0,a)} \omega^{-1/(p-1)} d\tau \right)^{1-\frac{1}{p}}. \tag{19}$$

Hence

$$\begin{aligned} \|u\|_{D^{1,p}(\mathbb{R}_+, \omega)}^{(a)} &= \left(\int_{\mathbb{R}_+} |u'(\tau)|^p \omega d\tau \right)^{\frac{1}{p}} + |u(a)| \\ &\leq \left(\int_{\mathbb{R}_+} |u'(\tau)|^p \omega d\tau \right)^{\frac{1}{p}} + |u(a) - Tr^0(u)| + |Tr^0(u)| \\ &\stackrel{(19)}{\leq} \left(\int_{\mathbb{R}_+} |u'(\tau)|^p \omega d\tau \right)^{\frac{1}{p}} + C_I \left(\int_{\mathbb{R}_+} |u'(\tau)|^p \omega d\tau \right)^{\frac{1}{p}} + |Tr^0(u)| \\ &= (1 + C_I) \left(\int_{\mathbb{R}_+} |u'(\tau)|^p \omega d\tau \right)^{\frac{1}{p}} + |Tr^0(u)| \\ &\leq (1 + C_I) \|u\|_{D^{1,p}(\mathbb{R}_+, \omega)}^{(0)}. \end{aligned} \tag{20}$$

On the other hand, by switching the role of a and 0 in (20), we obtain

$$\begin{aligned} \|u\|_{D^{1,p}(\mathbb{R}_+, \omega)}^{(0)} &= \left(\int_{\mathbb{R}_+} |u'(\tau)|^p \omega d\tau \right)^{\frac{1}{p}} + |Tr^0(u)| \\ &\leq \left(\int_{\mathbb{R}_+} |u'(\tau)|^p \omega d\tau \right)^{\frac{1}{p}} + |Tr^0(u) - u(a)| + |u(a)| \\ &\leq \left(\int_{\mathbb{R}_+} |u'(\tau)|^p \omega d\tau \right)^{\frac{1}{p}} + C_I \left(\int_{\mathbb{R}_+} |u'(\tau)|^p \omega d\tau \right)^{\frac{1}{p}} + |u(a)| \\ &= (1 + C_I) \left(\int_{\mathbb{R}_+} |u'(\tau)|^p \omega d\tau \right)^{\frac{1}{p}} + |u(a)| \\ &\leq (1 + C_I) \|u\|_{D^{1,p}(\mathbb{R}_+, \omega)}^{(a)}. \end{aligned}$$

This together with (20), yields the equivalence of all norms discussed, and completes the proof of the statement. \square

As a consequence of the above statement, we have the following remarks.

Remark 3.1 (Representation of $\tilde{D}^{1,p}(\mathbb{R}_+, \omega)$ for $\omega \in B_p(0)$)

Let $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \omega \in B_p(0) \cap C(\mathbb{R}_+), 1 < p < \infty$, and let $Tr^0(\cdot)$ be as in (12). Then $\mathcal{R}_{p,\omega}^0(0)$ is a Banach subspace of $D^{1,p}(\mathbb{R}_+, \omega)$ (equipped with any of the norms $\|\cdot\|_{D^{1,p}(\mathbb{R}_+, \omega)}^{(a)}$ where $a \in [0, \infty)$). Moreover, the mapping

$$D^{1,p}(\mathbb{R}_+, \omega) \ni u \mapsto u - Tr^0(u) \in \mathcal{R}_{p,\omega}^0(0)$$

is constant precisely on abstract classes in $\tilde{D}^{1,p}(\mathbb{R}_+, \omega)$ (see Fact 2.2) and defines the isometric isomorphism between $(\tilde{D}^{1,p}(\mathbb{R}_+, \omega), \|\cdot\|_{\tilde{D}^{1,p}(\mathbb{R}_+, \omega)}^*)$ and

$(\mathcal{R}_{p,\omega}^0(0), \|\cdot\|_{D^{1,p}(\mathbb{R}_+, \omega)}^{(0)})$. In particular in every abstract class in $\tilde{D}^{1,p}(\mathbb{R}_+, \omega)$, there is the representative vanishing at zero and $\tilde{D}^{1,p}(\mathbb{R}_+, \omega)$ is represented as

$$\left\{ U = \left\{ \int_0^t v(\tau) d\tau + c \right\}_{c \in \mathbb{R}} : v \in L^p(\mathbb{R}_+, \omega), \|U\|_{\tilde{D}^{1,p}(\mathbb{R}_+, \omega)}^* = \|v\|_{L^p(\mathbb{R}_+, \omega)} \right\}.$$

Remark 3.2 (Asymptotic behaviour near zero) The statement *ii* in Theorem 3.1 and (18) yield that if $\omega \in B_p(0) \cap C(\mathbb{R}_+)$ is positive, $1 < p < \infty$, then for any $u \in D^{1,p}(\mathbb{R}_+, \omega)$

$$u(t) = Tr^0(u) + a(t) \cdot \Omega_\omega^0(t),$$

where $a(t)$ is bounded, $a(t) \xrightarrow{t \rightarrow 0} 0$, and $\Omega_\omega^0(\cdot)$ is as in (10).

3.2 Analysis in the Case of $\omega \in B_p(\infty)$

Let us assume that $\omega \in B_p(\infty) \cap C(\mathbb{R}_+)$ is positive, $1 < p < \infty$. The aim of this section is to establish an analogous results to Theorem 3.1, to represent the Dirichlet space through the trace operator, but in the case $\omega \in B_p(\infty)$. The result stated below can be obtained by using very similar arguments to those used for the proof of Theorem 3.1. Since we will deal with the $B_p(\infty)$ -condition, we have to provide the analysis when t is sufficiently large. The proof is left to the reader with some general suggestions enclosed in order to treat this different setting:

- we first modify the appropriate definitions for the sets from Theorem 3.1;
- in the proofs, we substitute the previously used limit conditions: $t \searrow 0, t_n \searrow 0$ by: $t \nearrow \infty, t_n \nearrow \infty$, respectively;
- in place of (14) we deal with the representation

$$u(t) = - \int_t^\infty u'(\tau) d\tau;$$

- in place of (18) we deal with

$$|u(t) - c| \leq \left(\int_{(t,\infty)} |u'(\tau)|^p \omega(\tau) d\tau \right)^{\frac{1}{p}} \cdot \Omega_\omega^\infty(t), \tag{21}$$

which forces $c = \lim_{t \rightarrow \infty} u(t)$.

The following statement holds.

Theorem 3.2 (Asymptotic behaviour and trace at infinity) *Let $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \omega \in C(\mathbb{R}_+) \cap B_p(\infty), 1 < p < \infty$. For any $c \in \mathbb{R}$, let us consider the following subsets in $D^{1,p}(\mathbb{R}_+, \omega)$:*

$$\mathcal{R}_{p,\omega}^\infty(c) := \left\{ u \in D^{1,p}(\mathbb{R}_+, \omega), u(t) = \int_t^\infty v(\tau) d\tau + c : v \in L^p(\mathbb{R}_+, \omega) \right\}; \tag{22}$$

$$\mathcal{A}_{p,\omega}^\infty(c) := \left\{ u \in D^{1,p}(\mathbb{R}_+, \omega) : \exists t_n \nearrow \infty : \lim_{n \rightarrow \infty} u(t_n) = c \right\};$$

$$\mathcal{B}_{p,\omega}^\infty(c) := \left\{ u \in D^{1,p}(\mathbb{R}_+, \omega) : \lim_{t \rightarrow \infty} u(t) = c \right\};$$

$$\mathcal{C}_{p,\omega}^\infty(c) := \left\{ u \in D^{1,p}(\mathbb{R}_+, \omega) : \lim_{t \rightarrow \infty} \frac{u(t) - c}{\Omega_\omega^\infty(t)} = 0, \sup_{t>0} \frac{u(t) - c}{\Omega_\omega^\infty(t)} < \infty, \right\},$$

where $\Omega_\omega^\infty(t) = \left(\int_{(t,\infty)} \omega(\tau)^{\frac{-1}{p-1}} d\tau \right)^{1-\frac{1}{p}}$.

The following statements hold.

- i) For any $c \in \mathbb{R}$, the set $\mathcal{R}_{p,\omega}^\infty(c)$ is a closed subset in $D^{1,p}(\mathbb{R}_+, \omega)$, equipped with any norm $\| \cdot \|_{D^{1,p}(\mathbb{R}_+, \omega)}^{(a)}$ as in (7), where $a \in \mathbb{R}_+$.
- ii) For any $c \in \mathbb{R}$

$$\mathcal{A}_{p,\omega}^\infty(c) = \mathcal{B}_{p,\omega}^\infty(c) = \mathcal{C}_{p,\omega}^\infty(c) = \mathcal{R}_{p,\omega}^\infty(c).$$

- iii) For every $u \in D^{1,p}(\mathbb{R}_+, \omega)$ there is $c \in \mathbb{R}$ such that $u \in \mathcal{R}_{p,\omega}^\infty(c)$. In particular, the operator

$$Tr^\infty(u) := \lim_{t \rightarrow \infty} u(t) =: u(\infty), \tag{23}$$

is well defined for every $u \in D^{1,p}(\mathbb{R}_+, \omega)$ and $D^{1,p}(\mathbb{R}_+, \omega) = \cup_{c \in \mathbb{R}} \mathcal{R}_{p,\omega}^\infty(c)$.

Moreover,

$$D^{1,p}(\mathbb{R}_+, \omega) = \left\{ u(t) = \int_t^\infty v(\tau) d\tau + c : c \in \mathbb{R}, v \in L^p(\mathbb{R}_+, \omega) \right\}.$$

- iv) The quantity

$$\|u\|_{D^{1,p}(\mathbb{R}_+, \omega)}^{(\infty)} = \|u'\|_{L^p(\mathbb{R}_+, \omega)} + |Tr^\infty(u)|$$

is the norm on $D^{1,p}(\mathbb{R}_+, \omega)$, which is equivalent to any norm $\|u\|^{(a)}$, where $a \in \mathbb{R}_+$.

Remark 3.3 (Representation of $\tilde{D}^{1,p}(\mathbb{R}_+, \omega)$ for $\omega \in B_p(\infty)$) Let $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \omega \in B_p(\infty) \cap C(\mathbb{R}_+)$ and $1 < p < \infty$, and let $Tr^\infty(\cdot)$ be as in (23). Then $\mathcal{R}_{p,\omega}^\infty(0)$ is a Banach subspace of $D^{1,p}(\mathbb{R}_+, \omega)$ (equipped with any of the norms $\|\cdot\|_{D^{1,p}(\mathbb{R}_+, \omega)}^{(a)}$ where $a \in (0, \infty]$). Moreover, the mapping

$$D^{1,p}(\mathbb{R}_+, \omega) \ni u \mapsto u - Tr^\infty(u) \in \mathcal{R}_{p,\omega}^\infty(0)$$

is constant precisely on abstract classes in $\tilde{D}^{1,p}(\mathbb{R}_+, \omega)$ (see Fact 2.2) and defines the isometric isomorphism between $\tilde{D}^{1,p}(\mathbb{R}_+, \omega)$ and $(\mathcal{R}_{p,\omega}^\infty(0), \|\cdot\|_{D^{1,p}(\mathbb{R}_+, \omega)}^{(\infty)})$. In particular in every abstract class in $\tilde{D}^{1,p}(\mathbb{R}_+, \omega)$ there is the representative vanishing at infinity and $\tilde{D}^{1,p}(\mathbb{R}_+, \omega)$ represents as

$$\left\{ U = \left\{ \int_t^\infty v(\tau) d\tau + c \right\}_{c \in \mathbb{R}} : v \in L^p(\mathbb{R}_+, \omega), \|U\|_{\tilde{D}^{1,p}(\mathbb{R}_+, \omega)}^* = \|v\|_{L^p(\mathbb{R}_+, \omega)} \right\}.$$

Remark 3.4 (Asymptotic behaviour near infinity) The statement ii) in Theorem 3.2 and (21) yield that if $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \omega \in B_p(\infty) \cap C(\mathbb{R}_+)$ and $1 < p < \infty$, then for every $u \in D^{1,p}(\mathbb{R}_+, \omega)$

$$u(t) = Tr^\infty(u) + a(t) \cdot \Omega_\omega^\infty(t),$$

where $a(t)$ is bounded, $a(t) \xrightarrow{t \rightarrow \infty} 0$, and $\Omega_\omega^\infty(\cdot)$ is as in (22).

4 Characterization of $D_0^{1,p}(\mathbb{R}_+, \omega)$ and Density Results

4.1 The Space $D_0^{1,p}(\mathbb{R}_+, \omega)$ and First Density Results

We start with the following definition.

Definition 4.1 [The space $D_0^{1,p}(\mathbb{R}_+, \omega)$, the case of $\omega \in B_p$]

When $\omega \in B_p, 1 < p < \infty$, by $D_0^{1,p}(\mathbb{R}_+, \omega)$ we will denote the subset of all functions $u \in D^{1,p}(\mathbb{R}_+, \omega)$, for which there exists a sequence $\{\phi_n\}_{n \in \mathbb{N}} \subseteq C_0^\infty(\mathbb{R}_+)$, which satisfies:

$$\phi_n' \rightarrow u' \text{ in } L^p(\mathbb{R}_+, \omega) \text{ and } \phi_n \rightarrow u \text{ uniformly on compact sets in } \mathbb{R}_+, \text{ as } n \rightarrow \infty.$$

By Fact 2.3, $D_0^{1,p}(\mathbb{R}_+, \omega)$ is the same as the completion of $C_0^\infty(\mathbb{R}_+)$ in the space $D^{1,p}(\mathbb{R}_+, \omega)$ equipped with any of the norms $\|\cdot\|_{D^{1,p}(\mathbb{R}_+, \omega)}^{(a)}$, where $a \in \mathbb{R}_+$ can be taken arbitrary. In particular, it is the Banach subspace of $(D^{1,p}(\mathbb{R}_+, \omega), \|\cdot\|_{D^{1,p}(\mathbb{R}_+, \omega)}^{(a)})$, with an arbitrary $a \in \mathbb{R}_+$.

The following fact is rather obvious to the specialists, but for reader’s convenience we submit its proof.

Lemma 4.1 Let $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \omega \in C(\mathbb{R}_+)$ and $1 < p < \infty$. Then for any $a \in \mathbb{R}_+$

$$D_0^{1,p}(\mathbb{R}_+, \omega) = \overline{Lip_c}^{\|\cdot\|_{D^{1,p}(\mathbb{R}_+, \omega)}^{(a)}} = \overline{(D^{1,p}(\mathbb{R}_+, \omega))}_c^{\|\cdot\|_{D^{1,p}(\mathbb{R}_+, \omega)}^{(a)}}.$$

Proof. Clearly,

$$C_0^\infty(\mathbb{R}_+) \subseteq Lip_c(\mathbb{R}_+) \subseteq (D^{1,p}(\mathbb{R}_+, \omega))_c.$$

Hence, it suffices to show that $\overline{(D^{1,p}(\mathbb{R}_+, \omega))_c}^{\|\cdot\|_{D^{1,p}(\mathbb{R}_+, \omega)}^{(a)}} \subseteq D_0^{1,p}(\mathbb{R}_+, \omega)$. For that, take $u \in (D^{1,p}(\mathbb{R}_+, \omega))_c$ with the support $[a, b] \subseteq \mathbb{R}_+$. As on compactly supported sets $\omega \sim 1$, therefore $u \in D^{1,p}(\mathbb{R}_+)$ and u is compactly supported. By standard convolution arguments, the convolutions $u_\epsilon(x) := \phi_\epsilon * u$, with the classical mollifier functions $\phi_\epsilon(x) = \epsilon^{-1}\phi(x/\epsilon)$, where $\phi \in C_0^\infty(\mathbb{R})$, $0 \leq \phi \leq 1$, $\text{supp } \phi \subseteq [-1, 1]$ and $\int_{\mathbb{R}} \phi dx = 1$, converge to u in the topology of $D^{1,p}(\mathbb{R}_+, \omega)$. Moreover, their supports are subsets of $J := [a/2, 3/2b]$ for the sufficiently small ϵ 's. Again, as $\omega \sim 1$ on J , therefore u_ϵ 's converge to u also in $D^{1,p}(\mathbb{R}_+, \omega)$. This shows that $u \in D_0^{1,p}(\mathbb{R}_+, \omega)$. \square

In the preceding sections we will analyze independently the cases: $\omega \in B_p(0)$ and $\omega \in B_p(\infty)$.

4.2 The Case of $\omega \in B_p(0)$

Let $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $\omega \in B_p(0) \cap C(\mathbb{R}_+)$ and $1 < p < \infty$, and let us consider the space $D_0^{1,p}(\mathbb{R}_+, \omega)$ as in Definition 4.1. According to Theorem 3.1, part iv), it is the completion of $C_0^\infty(\mathbb{R}_+)$ in $D^{1,p}(\mathbb{R}_+, \omega)$ equipped with the norm $\|\cdot\|^{(0)}$. As $C_0^\infty(\mathbb{R}_+) \subseteq \mathcal{R}_{p,\omega}^0(0)$ and by Theorem 3.1 $\mathcal{R}_{p,\omega}^0(0)$ is a closed subspace in $D^{1,p}(\mathbb{R}_+, \omega)$, we deduce that

$$D_0^{1,p}(\mathbb{R}_+, \omega) \subseteq \mathcal{R}_{p,\omega}^0(0), \text{ when } \omega \in B_p(0) \cap C(\mathbb{R}_+) \text{ is positive.} \tag{24}$$

We address the question about density:

$$\text{When } D_0^{1,p}(\mathbb{R}_+, \omega) = \mathcal{R}_{p,\omega}^0(0) ?$$

The statement given below answers to this question.

Theorem 4.1 (*Characterization of weights for $D_0^{1,p}(\mathbb{R}_+, \omega) = \mathcal{R}_{p,\omega}^0(0)$*)

Let $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $\omega \in B_p(0) \cap C(\mathbb{R}_+)$, $1 < p < \infty$. Then

$$D_0^{1,p}(\mathbb{R}_+, \omega) = \mathcal{R}_{p,\omega}^0(0) \iff \omega \notin B_p(\infty).$$

The proof will be based on the following lemma, whose proof is submitted in the Appendix for reader's convenience.

Lemma 4.2 (*Energy minimizer with the nontrivial constraint at the left end*).

Let $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $\omega \in C(\mathbb{R}_+)$, $0 < k < K < \infty$, $0 \neq a \in \mathbb{R}$, $1 < p < \infty$, and consider energy functional

$$E_\omega(\phi) := \int_{(k,K)} |\phi'(t)|^p \omega(t) dt, \quad \phi \in W^{1,p}((k, K)), \quad \phi(k) = a, \quad \phi(K) = 0. \tag{25}$$

Then the minimum of $E_\omega(\cdot)$ is achieved at

$$\phi_{(k,K,a)}(t) := a \left(\int_{(k,K)} \omega^{\frac{-1}{p-1}}(\tau) d\tau \right)^{-1} \int_{(t,K)} \omega^{\frac{-1}{p-1}}(\tau) d\tau.$$

Now we are in position to prove Theorem 4.1.

Proof of Theorem 4.1: " \Leftarrow " (E_ω -caloric approximation):

We have called this part of the proof " ω -caloric approximation", because the construction

of the approximation sequence involves the energy minimizers of (25).

Assume that $\omega \in (C(\mathbb{R}_+) \cap B_p(\infty)) \setminus B_p(0)$. Thanks to (24) we only have to prove that $\mathcal{R}_{p,\omega}^0(0) \subseteq D_0^{1,p}(\mathbb{R}_+, \omega)$.

The proof follows by two steps.

STEP 1. REDUCTION ARGUMENT. We show that it suffices to prove that any $u \in D^{1,p}(\mathbb{R}_+, \omega)$, such that

$$u \equiv 0 \text{ near zero and } u \equiv 1 \text{ on } (k, \infty), \text{ for some } k > 0, \tag{26}$$

belongs to $D_0^{1,p}(\mathbb{R}_+, \omega)$.

Indeed, let us take $u \in \mathcal{R}_{p,\omega}^0(0)$. We notice that functions in the form

$$\tilde{u}_n(t) := \int_0^t u'(\tau) \chi_{(\frac{1}{n}, n)}(\tau) d\tau \tag{27}$$

converge to u in $(D^{1,p}(\mathbb{R}_+, \omega), \|\cdot\|^{(0)})$ (see (13)), they are zero near zero and constant near infinity. Clearly, if that constant equals zero, then according to Lemma 4.1, we have $\tilde{u}_n \in D_0^{1,p}(\mathbb{R}_+, \omega)$. In the other case, we are left with the proof that $\tilde{u}_n \in D_0^{1,p}(\mathbb{R}_+, \omega)$. Obviously, it suffices to consider $C\tilde{u}_n$ instead of \tilde{u}_n , with constant C such that $C\tilde{u}_n \equiv 1$ near infinity.

STEP 2. PROOF IN THE SPECIAL CASE. We prove that any $u \in D^{1,p}(\mathbb{R}_+, \omega)$ as in (26) belongs to $D_0^{1,p}(\mathbb{R}_+, \omega)$.

Let $u \in D^{1,p}(\mathbb{R}_+, \omega)$ be as in (26). For any $n \in \mathbb{N}$ and $k < t_n$, let

$$u_n(t) := \begin{cases} u(t) & \text{if } t < k \\ \phi_{(k,t_n,1)}(t) & \text{if } t \in [k, t_n] \\ 0 & \text{if } t > t_n \end{cases}$$

where $\phi_{(k,t_n,1)}$ is as in Lemma 4.2 and $t_n \nearrow \infty$. Clearly, the u_n 's are compactly supported. We will show that

$$u_n \xrightarrow{n \rightarrow \infty} u \text{ in } (D^{1,p}(\mathbb{R}_+, \omega), \|\cdot\|_{D^{1,p}(\mathbb{R}_+, \omega)}^{(0)}), \tag{28}$$

which, together with Lemma 4.1, will close the assertion for this part of the statement. We have

$$\begin{aligned} \int_{\mathbb{R}_+} |(u_n - u)'(t)|^p \omega(t) dt &= \int_{(k,t_n)} |(1 - \phi_{(k,t_n,1)}(t))'|^p \omega(t) dt \\ &= \int_{(k,t_n)} |\phi_{(k,t_n,1)}'(t)|^p \omega(t) dt = \int_{(k,t_n)} \frac{\omega^{\frac{-p}{p-1}}(t)}{\left(\int_{(k,t_n)} \omega^{\frac{-1}{p-1}}(\tau) d\tau\right)^p} \cdot \omega(t) dt \\ &= \left(\int_{(k,t_n)} |\omega^{\frac{-1}{p-1}}(\tau)|^p d\tau\right)^{-p} \int_{(k,t_n)} \omega^{\frac{-1}{p-1}}(\tau) d\tau = \left(\int_{(k,t_n)} \omega^{\frac{-1}{p-1}}(\tau) d\tau\right)^{1-p}. \end{aligned} \tag{29}$$

As $\omega \notin B_p(\infty)$ and $p > 1$,

$$\int_{(k,t_n)} \omega^{\frac{-1}{p-1}}(\tau) d\tau \xrightarrow{n \rightarrow \infty} \infty, \text{ consequently } \left(\int_{(k,t_n)} \omega^{\frac{-1}{p-1}}(\tau) d\tau\right)^{1-p} \xrightarrow{n \rightarrow \infty} 0.$$

This implies (28).

“ \implies ”:

Suppose that $D_0^{1,p}(\mathbb{R}_+, \omega) = \mathcal{R}_{p,\omega}^0(0)$. We will show that $\omega \notin B_p(\infty)$.

Clearly the function

$$u(t) := \begin{cases} 0 & \text{if } t \leq 1 \\ t - 1 & \text{if } t \in [1, 2] \\ 1 & \text{if } t > 2 \end{cases}$$

belongs to $\mathcal{R}_{p,\omega}^0(0) (= D_0^{1,p}(\mathbb{R}_+, \omega))$, and so there is the sequence $\{u_n\}_{n \in \mathbb{N}} \subseteq C_0^\infty(\mathbb{R}_+)$ such that $u_n \rightarrow u$ in $(D^{1,p}(\mathbb{R}_+, \omega), \|\cdot\|_{D^{1,p}(\mathbb{R}_+, \omega)}^{(2)})$, and in all the equivalent norms. In particular

$$\xi_n := u_n(2) \xrightarrow{n \rightarrow \infty} 1 \text{ and } E_\omega(u_n) := \int_{(2,t_n)} |u'_n(\tau)|^p \omega(\tau) d\tau \xrightarrow{n \rightarrow \infty} 0,$$

for any $t_n > 2$. Let $\{t_n\}_{n \in \mathbb{N}}$ be any sequence such that $\text{supp } u_n \subseteq (0, t_n)$ and $t_n \nearrow \infty$ as $n \rightarrow \infty$. According to Lemma 4.2, the energy $E_\omega(\phi_{(2,t_n,\xi_n)})$ cannot be larger than $E_\omega(u_n)$. Therefore we also have

$$I_n := E_\omega(\phi_{(2,t_n,\xi_n)}) = \int_{(2,t_n)} |(\phi_{(2,t_n,\xi_n)})'(\tau)|^p \omega(\tau) d\tau \xrightarrow{n \rightarrow \infty} 0.$$

Since

$$I_n = |\xi_n|^p \int_{(2,t_n)} \left(\frac{\omega^{-\frac{1}{p-1}}}{\int_{(2,t_n)} \omega^{-\frac{1}{p-1}}(\tau) d\tau} \right)^p \omega(t) dt = |\xi_n|^p \left(\int_{(2,t_n)} \omega^{-\frac{1}{p-1}}(\tau) d\tau \right)^{1-p},$$

it follows that the above converges to zero if and only if

$$\lim_{t \rightarrow \infty} \int_{(2,t)} \omega^{-\frac{1}{p-1}}(\tau) d\tau = \infty,$$

equivalently $\omega \notin B_p(\infty)$. This completes the proof of the statement. □

4.3 Analysis in the Case of $\omega \in B_p(\infty)$

Let us assume that $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \omega \in B_p(\infty) \cap C(\mathbb{R}_+)$ and $1 < p < \infty$. Our aim is to analyze the properties of the space $D_0^{1,p}(\mathbb{R}_+, \omega)$, the completion of $C_0^\infty(\mathbb{R}_+)$ in $D^{1,p}(\mathbb{R}_+, \omega)$ equipped with the norm $\|\cdot\|_{D^{1,p}(\mathbb{R}_+, \omega)}^{(\infty)}$, defined in Theorem 3.2. As $C_0^\infty(\mathbb{R}_+) \subseteq \mathcal{R}_{p,\omega}^\infty(0)$, where the latter space, due to Theorem 3.2, is a closed subspace in $D^{1,p}(\mathbb{R}_+, \omega)$, we deduce that

$$D_0^{1,p}(\mathbb{R}_+, \omega) \subseteq \mathcal{R}_{p,\omega}^\infty(0). \tag{30}$$

The goal of this section is the following characterization theorem.

Theorem 4.2 (*Characterization of weights for $D_0^{1,p}(\mathbb{R}_+, \omega) = \mathcal{R}_{p,\omega}^\infty(0)$*)

Let $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \omega \in B_p(\infty) \cap C(\mathbb{R}_+)$ and $1 < p < \infty$. Then

$$D_0^{1,p}(\mathbb{R}_+, \omega) = \mathcal{R}_{p,\omega}^\infty(0) \iff \omega \notin B_p(0).$$

The proof is an easy modification of the proof of Theorem 4.1, so we only sketch it. We start by stating the following result similar to that of Lemma 4.2. Its proof can be easily obtained either by the internal symmetry argument, or by suitable modification of the proof of Lemma 4.2. We leave it to the reader.

Lemma 4.3 (Energy minimizer for the nontrivial constraint at the right end).

Let $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \omega \in B_p \cap C(\mathbb{R}_+)$ and $1 < p < \infty, 0 < k < K < \infty, 0 \neq a \in \mathbb{R}$, and consider energy functional

$$\tilde{E}_\omega(\phi) := \int_{(k,K)} |\phi'(t)|^p \omega(t) dt, \quad \phi \in W^{1,p}((k, K)), \phi(k) = 0, \phi(K) = a.$$

Then the minimum of $\tilde{E}_\omega(\cdot)$ is achieved at

$$\tilde{\phi}_{(k,K,a)}(t) := a \left(\int_{(k,K)} \omega^{\frac{-1}{p-1}}(\tau) d\tau \right)^{-1} \int_{(k,t)} \omega^{\frac{-1}{p-1}}(\tau) d\tau.$$

We are in position to sketch the proof of Theorem 4.2.

Proof of Theorem 4.2. “ \Leftarrow ” (\tilde{E}_ω - caloric approximation):

Assume that $\omega \in B_p(\infty) \setminus B_p(0)$. As (30) holds, we only have to prove that $\mathcal{R}_{p,\omega}^\infty(0) \subseteq D_0^{1,p}(\mathbb{R}_+, \omega)$.

Let $u \in \mathcal{R}_{p,\omega}^\infty(0)$. We will show that $u \in D_0^{1,p}(\mathbb{R}_+, \omega)$.

The proof consists of two steps.

STEP 1. REDUCTION ARGUMENT. We show that it suffices to prove that any $u \in D^{1,p}(\mathbb{R}_+, \omega)$, such that

$$u \equiv 1 \text{ on some } (0, c) \text{ where } c > 0, \quad u \equiv 0 \text{ near } \infty, \tag{31}$$

belongs to $D_0^{1,p}(\mathbb{R}_+, \omega)$.

To this aim, we note that functions

$$\tilde{u}_n(t) := - \int_t^\infty u'(\tau) \chi_{(\frac{1}{n}, n)}(\tau) d\tau$$

are proportional to functions as in (31), and they converge to u in $(D^{1,p}(\mathbb{R}_+, \omega), \|\cdot\|^{(\infty)})$.

STEP 2. PROOF IN THE SPECIAL CASE.

We prove that any $u \in D^{1,p}(\mathbb{R}_+, \omega)$ as in (31) belongs to $D_0^{1,p}(\mathbb{R}_+, \omega)$. To this purpose, let us consider the following sequence dealing with $0 < s_n < c, s_n \searrow 0$:

$$u_n(t) := \begin{cases} u(t) & \text{if } t > c \\ \tilde{\phi}_{(s_n,c,1)}(t) & \text{if } t \in [s_n, c] \\ 0 & \text{if } t < s_n, \end{cases}$$

where $\tilde{\phi}_{(s_n,c,1)}$ is as in Lemma 4.3. Obviously, the u_n 's are compactly supported and so they belong to $D_0^{1,p}(\mathbb{R}_+, \omega)$, by Lemma 4.1. By similar computations as in (29), we get

$$\int_{\mathbb{R}_+} |(u_n - u)'(t)|^p \omega(t) dt = \left(\int_{(s_n,c)} \omega^{\frac{-1}{p-1}}(\tau) d\tau \right)^{1-p} \xrightarrow{n \rightarrow \infty} 0,$$

because $\omega \notin B_p(0)$. Therefore $u \in D_0^{1,p}(\mathbb{R}_+, \omega)$.

“ \implies ”:

We prove that the condition $D_0^{1,p}(\mathbb{R}_+, \omega) = \mathcal{R}^{1,p}(\infty)$ forces the condition $\omega \notin B_p(0)$. To this aim, assume that $D_0^{1,p}(\mathbb{R}_+, \omega) = \mathcal{R}^{1,p}(\infty)$ and let

$$u(t) := \begin{cases} 1 & \text{if } t \leq 1 \\ 2 - t & \text{if } t \in [1, 2] \\ 0 & \text{if } t > 2 \end{cases}$$

Then $u \in \mathcal{R}_{p,\omega}^\infty(0) \subseteq D_0^{1,p}(\mathbb{R}_+, \omega)$. Hence, there exists a sequence $\{u_n\}_{n \in \mathbb{N}} \subseteq C_0^\infty(\mathbb{R}_+)$ such that $u_n \rightarrow u$ in $(D^{1,p}(\mathbb{R}_+, \omega), \|\cdot\|_{D^{1,p}(\mathbb{R}_+, \omega)}^{(1)})$. In particular

$$\xi_n := u_n(1) \xrightarrow{n \rightarrow \infty} 1 \text{ and } \tilde{E}_\omega(u_n) := \int_{(s_n, 1)} |u_n'(\tau)|^p \omega(\tau) d\tau \xrightarrow{n \rightarrow \infty} 0,$$

where $\text{supp } u_n \subseteq (s_n, \infty)$, for some $s_n \searrow 0$. By Lemma 4.3, the energy $I_n := \tilde{E}_\omega(\tilde{\phi}_{(s_n, 1, \xi_n)})$ cannot be larger than $\tilde{E}_\omega(u_n)$, therefore

$$I_n := |\xi_n|^p \left(\int_{(s_n, 1)} \omega^{\frac{-1}{p-1}}(\tau) d\tau \right)^{1-p} \xrightarrow{n \rightarrow \infty} 0.$$

Consequently $\omega \notin B_p(0)$, which completes the proof. □

4.4 Analytic Description of $D_0^{1,p}(\mathbb{R}_+, \omega)$ in General Case

Our main statement in this section reads as follows.

Theorem 4.3 (Description of $D_0^{1,p}(\mathbb{R}_+, \omega)$ for all admitted weights). *Let $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $\omega \in C(\mathbb{R}_+)$ and $1 < p < \infty$. Then we have.*

i) If $\omega \in B_p(0) \setminus B_p(\infty)$, then

$$D_0^{1,p}(\mathbb{R}_+, \omega) = \mathcal{R}_{p,\omega}^0(0).$$

ii) If $\omega \in B_p(\infty) \setminus B_p(0)$, then

$$D_0^{1,p}(\mathbb{R}_+, \omega) = \mathcal{R}_{p,\omega}^\infty(0).$$

iii) If $\omega \notin B_p(0) \cup B_p(\infty)$, then

$$D_0^{1,p}(\mathbb{R}_+, \omega) = D^{1,p}(\mathbb{R}_+, \omega).$$

iv) If $\omega \in B_p(0) \cap B_p(\infty)$, then

$$D_0^{1,p}(\mathbb{R}_+, \omega) = \mathcal{R}_{p,\omega}^0(0) \cap \mathcal{R}_{p,\omega}^\infty(0).$$

Proof. i) and ii): Statements i) and ii) have been already obtained in Theorems 4.1 and 4.2. We are left with the proofs of parts iii) and iv).

iii): Assume that $\omega \notin B_p(0) \cup B_p(\infty)$. Let $u \in D^{1,p}(\mathbb{R}_+, \omega)$ and let us consider the Lipschitz resolution of the unity on \mathbb{R}_+ : ϕ_0, ϕ_1 , defined by

$$\phi_1(t) = 1\chi_{(0,1)} + (-t + 2)\chi_{[1,2]}, \quad \phi_0(t) := 1 - \phi_1(t).$$

We have to prove that $u \in D_0^{1,p}(\mathbb{R}_+, \omega)$. As $u = \phi_0 u + \phi_1 u$, it suffices to consider the following cases: a) $u \equiv 0$ near 0 and b) $u \equiv 0$ near ∞ .

In case a), suppose that $u \equiv 0$ on $(0, a]$ for some $a > 0$. Then functions as in (27) converge to u in $(D^{1,p}(\mathbb{R}_+, \omega), \|\cdot\|_{D^{1,p}(\mathbb{R}_+, \omega)}^{(a)})$, they are zero on $(0, a]$ and constant when $t > n$. Therefore the proof reduces to the case of $u \equiv 0$ near zero and $u \equiv Const$ near ∞ . Then we repeat all the arguments from the proof of Theorem 4.1, Step 2, in the proof of the implication “ \Leftarrow ”.

In case b), the argument at the beginning of the proof of Theorem 4.2 reduces that case to the situation when $u \equiv Const$ near 0 and $u \equiv 0$ near to infinity. Then we use precisely the same arguments as in the proof of Theorem 4.2, Step 2 in part “ \Leftarrow ”.

iv): Let $\omega \in B_p(0) \cap B_p(\infty)$.

By (24) and (30) we have $D_0^{1,p}(\mathbb{R}_+, \omega) \subseteq \mathcal{R}_{p,\omega}^0(0) \cap \mathcal{R}_{p,\omega}^\infty(0)$. Therefore it suffices to show that the converse inclusion holds. For that, assume that $u \in \mathcal{R}_{p,\omega}^0(0) \cap \mathcal{R}_{p,\omega}^\infty(0)$. Note that in particular u is bounded and, by Fact 2.1, $u' \in L_{loc}^1([0, \infty)) \cap L_{loc}^1((0, \infty]) = L^1((0, \infty))$. Moreover,

$$u(t) = \int_0^t u'(\tau) d\tau = - \int_t^\infty u'(\tau) d\tau.$$

Thus we have

$$\int_0^\infty u'(\tau) d\tau = 0 \text{ and } u' \in L^1(\mathbb{R}_+).$$

For each $n \in \mathbf{N}$, consider

$$u_n(t) := \int_0^t \chi_{(\frac{1}{n}, n)}(u'(\tau) - c_n) d\tau, \text{ where } c_n = \frac{1}{n - \frac{1}{n}} \int_{\frac{1}{n}}^n u'(\tau) d\tau.$$

Then

$$u'_n(\tau) = \chi_{(\frac{1}{n}, n)}(u'(\tau) - c_n), \int_0^\infty u'_n(\tau) d\tau = 0, u'_n \in L^1(\mathbb{R}_+) \cap L^p(\mathbb{R}_+, \omega).$$

Consequently

$$u_n(t) = \int_0^t u'_n(\tau) d\tau = - \int_t^\infty u'_n(\tau) d\tau, u'_n \in L^p(\mathbb{R}_+, \omega).$$

Moreover, we have $u_n \xrightarrow{n \rightarrow \infty} u$ in $D^{1,p}(\mathbb{R}_+, \omega)$ and u_n is supported in $[\frac{1}{n}, n]$. This together with Lemma 4.1 implies that $u \in D_0^{1,p}(\mathbb{R}_+, \omega)$ and ends the proof of the statement. \square

4.5 Sharpness in Theorems: 3.1, 3.2 and 4.3

This section is devoted to the proof of sharpness in statements *i*) in Theorems 3.1 and 3.2, and the converse implications in Theorem 4.3.

We have the following result.

Theorem 4.4 (Sharpness in Theorems 3.1, 3.2, parts *i*)).

Let $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \omega \in C(\mathbb{R}_+)$ and $1 < p < \infty$. Then the following statements hold.

- i) $\mathcal{R}_{p,\omega}^0(0)$ is a closed subset in $D^{1,p}(\mathbb{R}_+, \omega) \iff \omega \in B_p(0)$.
- ii) $\mathcal{R}_{p,\omega}^\infty(0)$ is a closed subset in $D^{1,p}(\mathbb{R}_+, \omega) \iff \omega \in B_p(\infty)$.

Proof. The implications “ \Leftarrow ” were already proven in Theorem 3.1, 3.2, parts i). We are left with the proofs of the converse implications.

“ \Rightarrow ”:

i): We argue by contradiction. Assume that the implication does not hold, that is $\mathcal{R}_{p,\omega}^0(0)$ is a closed subset in $D^{1,p}(\mathbb{R}_+, \omega)$, but $\omega \notin B_p(0)$. We have either a) $\omega \in B_p(\infty)$ or b) $\omega \notin B_p(\infty)$.

If a) holds, then by part ii) of Theorem 4.3, $D_0^{1,p}(\mathbb{R}_+, \omega) = \mathcal{R}_{p,\omega}^\infty(0)$.

The function

$$u(x) := \begin{cases} 1 & \text{if } x \in (0, 1) \\ -x + 2 & \text{if } x \in [1, 2) \\ 0 & \text{if } x \geq 2 \end{cases} \tag{32}$$

belongs to $\mathcal{R}_{p,\omega}^\infty(0)$, which is $D_0^{1,p}(\mathbb{R}_+, \omega)$. Hence, it can be approximated in $D^{1,p}(\mathbb{R}_+, \omega)$ by functions with compact support in \mathbb{R}_+ . Such functions take the value zero near the origin, so they belong to $\mathcal{R}_{p,\omega}^0(0)$. But their limit, the function $u(\cdot)$, does not belong to $\mathcal{R}_{p,\omega}^0(0)$, as $u \equiv 1$ near the origin. Therefore $\mathcal{R}_{p,\omega}^0(0)$ cannot be closed. The condition a) is not possible.

Let us suppose that b) holds. By Theorem 4.3, part iii), we have $D_0^{1,p}(\mathbb{R}_+, \omega) = D^{1,p}(\mathbb{R}_+, \omega)$. Thus, the function u defined in (32) belongs to $D_0^{1,p}(\mathbb{R}_+, \omega)$. We can argue as in the previous case, obtaining the contradiction that $\mathcal{R}_{p,\omega}^0(0)$ is closed. Therefore necessarily $\omega \in B_p(0)$.

ii): Let us assume that the implication “ \Rightarrow ” does not hold, that is $\mathcal{R}_{p,\omega}^\infty(0)$ is a closed subset in $D^{1,p}(\mathbb{R}_+, \omega)$ but $\omega \notin B_p(\infty)$.

We have either a) $\omega \in B_p(0)$ or b) $\omega \notin B_p(0)$. Both conditions, by Theorem 4.3, imply that $D_0^{1,p}(\mathbb{R}_+, \omega) = \mathcal{R}_{p,\omega}^0(0)$ or $D_0^{1,p}(\mathbb{R}_+, \omega) = D^{1,p}(\mathbb{R}_+, \omega)$, respectively.

Let us consider the function

$$u(x) := \begin{cases} 0 & \text{if } x \in (0, 1) \\ x - 1 & \text{if } x \in [1, 2) \\ 1 & \text{if } x \geq 2 \end{cases}$$

It belongs to $D_0^{1,p}(\mathbb{R}_+, \omega)$. By arguments as in the proof of part i), we get a contradiction in both cases: a) and b), which proves ii).

The proof of the statement is complete. □

Let us proceed by proving the converse implications in Theorem 4.3.

We state the following

Theorem 4.5 (Sharpness of conditions on weights in Theorem 4.3).

Let $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \omega \in C(\mathbb{R}_+)$ and $1 < p < \infty$. Then we have.

- i) $D_0^{1,p}(\mathbb{R}_+, \omega) = \mathcal{R}_{p,\omega}^0(0) \iff \omega \in B_p(0) \setminus B_p(\infty)$.
- ii) $D_0^{1,p}(\mathbb{R}_+, \omega) = \mathcal{R}_{p,\omega}^\infty(0) \iff \omega \in B_p(\infty) \setminus B_p(0)$.
- iii) $D_0^{1,p}(\mathbb{R}_+, \omega) = D^{1,p}(\mathbb{R}_+, \omega) \iff \omega \notin B_p(0) \cup B_p(\infty)$.
- iv) $D_0^{1,p}(\mathbb{R}_+, \omega) = \mathcal{R}_{p,\omega}^0(0) \cap \mathcal{R}_{p,\omega}^\infty(0) \iff \omega \in B_p(0) \cap B_p(\infty)$.

Proof. For each of the statements we only have to prove the implication “ \Rightarrow ”.

“ \implies :”

i): As $D_0^{1,p}(\mathbb{R}_+, \omega) = \mathcal{R}_{p,\omega}^0(0)$, therefore $\mathcal{R}_{p,\omega}^0(0)$ is a closed subset in $D^{1,p}(\mathbb{R}_+, \omega)$. Hence, by statement i) in Theorem 4.4, we get $\omega \in B_p(0)$. We have only two possibilities: a) $\omega \in B_p(0) \setminus B_p(\infty)$ or b) $\omega \in B_p(0) \cap B_p(\infty)$. We will show that condition b) cannot hold. We argue by contradiction. If the condition b) was true then, by Theorem 4.3, it would imply $D_0^{1,p}(\mathbb{R}_+, \omega) = \mathcal{R}_{p,\omega}^0(0) \cap \mathcal{R}_{p,\omega}^\infty(0)$, and consequently $\mathcal{R}_{p,\omega}^0(0) = \mathcal{R}_{p,\omega}^0(0) \cap \mathcal{R}_{p,\omega}^\infty(0)$. It would follow that $\mathcal{R}_{p,\omega}^0(0) \setminus \mathcal{R}_{p,\omega}^\infty(0) = \emptyset$, while the function

$$u(x) := \begin{cases} 0 & \text{if } x \in (0, 1) \\ x - 1 & \text{if } x \in [1, 2) \\ 1 & \text{if } x \geq 2 \end{cases} \tag{33}$$

belongs to $\mathcal{R}_{p,\omega}^0(0) \setminus \mathcal{R}_{p,\omega}^\infty(0)$. We arrive at a contradiction, therefore only the condition a) can be true. This proves the statement i).

ii): We argue similarly as before. As we have $D_0^{1,p}(\mathbb{R}_+, \omega) = \mathcal{R}_{p,\omega}^\infty(0)$, therefore $\mathcal{R}_{p,\omega}^\infty(0)$ is closed. Hence, by statement ii) in Theorem 4.4, we get $\omega \in B_p(\infty)$.

We have only two possibilities: either a) $\omega \in B_p(\infty) \setminus B_p(0)$ or b) $\omega \in B_p(0) \cap B_p(\infty)$. We will show that condition b) cannot hold.

Indeed, if b) would hold, then, from Theorem 4.3, statement iv), we would deduce that $\mathcal{R}_{p,\omega}^\infty(0) = \mathcal{R}_{p,\omega}^0(0) \cap \mathcal{R}_{p,\omega}^\infty(0)$. This would imply $\mathcal{R}_{p,\omega}^\infty(0) \setminus \mathcal{R}_{p,\omega}^0(0) = \emptyset$, while the function

$$u(x) := \begin{cases} 1 & \text{if } x \in (0, 1) \\ -x + 2 & \text{if } x \in [1, 2) \\ 0 & \text{if } x \geq 2 \end{cases} \tag{34}$$

belongs to $\mathcal{R}_{p,\omega}^\infty(0) \setminus \mathcal{R}_{p,\omega}^0(0)$. The contradiction shows that the condition a) holds and this completes the proof of the statement ii).

iii): By contradiction, let us assume that $D_0^{1,p}(\mathbb{R}_+, \omega) = D^{1,p}(\mathbb{R}_+, \omega)$ and $\omega \in B_p(0) \cup B_p(\infty)$. Then we have either: a) or b) or c), where a) $\omega \in B_p(0) \setminus B_p(\infty)$, b) $\omega \in B_p(\infty) \setminus B_p(0)$, c) $\omega \in B_p(0) \cap B_p(\infty)$.

According to Theorem 4.3, we get either $D_0^{1,p}(\mathbb{R}_+, \omega) = \mathcal{R}_{p,\omega}^0(0)$, or $D_0^{1,p}(\mathbb{R}_+, \omega) = \mathcal{R}_{p,\omega}^\infty(0)$, or $D_0^{1,p}(\mathbb{R}_+, \omega) = \mathcal{R}_{p,\omega}^0(0) \cap \mathcal{R}_{p,\omega}^\infty(0)$, respectively. Consequently, we would have either $D^{1,p}(\mathbb{R}_+, \omega) = \mathcal{R}_{p,\omega}^0(0)$, or $D^{1,p}(\mathbb{R}_+, \omega) = \mathcal{R}_{p,\omega}^\infty(0)$, or $D^{1,p}(\mathbb{R}_+, \omega) = \mathcal{R}_{p,\omega}^0(0) \cap \mathcal{R}_{p,\omega}^\infty(0)$, respectively. However, those identities cannot be true. For example, the function $u \equiv 1$ belongs to $D^{1,p}(\mathbb{R}_+, \omega)$, while it does not belong to any of the sets: $\mathcal{R}_{p,\omega}^0(0)$, $\mathcal{R}_{p,\omega}^\infty(0)$, $\mathcal{R}_{p,\omega}^0(0) \cap \mathcal{R}_{p,\omega}^\infty(0)$. The contradiction proves iii).

iv): Let us suppose that the implication does not hold, that is $D_0^{1,p}(\mathbb{R}_+, \omega) = \mathcal{R}_{p,\omega}^0(0) \cap \mathcal{R}_{p,\omega}^\infty(0)$, but $\omega \notin B_p(0) \cap B_p(\infty)$. Then either a) or b) or c) holds, where a) $\omega \in B_p(0) \setminus B_p(\infty)$, b) $\omega \in B_p(\infty) \setminus B_p(0)$, c) $\omega \notin B_p(0) \cup B_p(\infty)$.

By Theorem 4.3, those conditions imply: $\mathcal{R}_{p,\omega}^0(0) \cap \mathcal{R}_{p,\omega}^\infty(0) = \mathcal{R}_{p,\omega}^0(0)$, or $\mathcal{R}_{p,\omega}^0(0) \cap \mathcal{R}_{p,\omega}^\infty(0) = \mathcal{R}_{p,\omega}^\infty(0)$, or $\mathcal{R}_{p,\omega}^0(0) \cap \mathcal{R}_{p,\omega}^\infty(0) = D^{1,p}(\mathbb{R}_+, \omega)$, respectively.

In first two situations we would have $\mathcal{R}_{p,\omega}^0(0) \setminus \mathcal{R}_{p,\omega}^\infty(0) = \emptyset$ or $\mathcal{R}_{p,\omega}^\infty(0) \setminus \mathcal{R}_{p,\omega}^0(0) = \emptyset$, which are false, thanks to either (33) or (34), respectively. The third situation cannot be true also, because the function $u \equiv 1$ belongs to $D^{1,p}(\mathbb{R}_+, \omega)$, but it does not belong to $\mathcal{R}_{p,\omega}^0(0) \cap \mathcal{R}_{p,\omega}^\infty(0)$. In any case we get the contradiction, which proves the validity of iv) and finishes the proof of the statement. \square

5 Applications

Let us present several example applications of our results.

5.1 Application to Hardy Inequality

Let us consider classical Hardy and the conjugate Hardy operators, respectively:

$$Hv(t) := \int_{(0,t)} v(\tau) d\tau, \quad v \in L^1_{loc}([0, \infty)), \quad H^*v(t) := \int_{(t,\infty)} v(\tau) d\tau, \quad v \in L^1_{loc}((0, \infty]). \tag{35}$$

In Theorems 3.1 and 3.2 we have shown that Hardy type operators are isometric embeddings between $L^p(\mathbb{R}_+, \omega)$ and $\mathcal{R}^0_{p,\omega}(0)$ or $\mathcal{R}^\infty_{p,\omega}(0)$, respectively. Precisely, it follows from Theorem 3.1 and 3.2 that

$$H : L^p(\mathbb{R}_+, \omega) \xrightarrow{\text{isometry}} (\mathcal{R}^0_{p,\omega}(0), \|\cdot\|_{D^{1,p}(\mathbb{R}_+, \omega)}^{(0)}), \quad \text{when } \omega \in B_p(0),$$

$$H^* : L^p(\mathbb{R}_+, \omega) \xrightarrow{\text{isometry}} (\mathcal{R}^\infty_{p,\omega}(0), \|\cdot\|_{D^{1,p}(\mathbb{R}_+, \omega)}^{(\infty)}), \quad \text{when } \omega \in B_p(\infty).$$

In the first case the inverse is $u \mapsto u'$, while in the second case it is $-u'$.

Such identification can be further used to obtain the extended variants of Hardy type inequality, where the class of admissible functions is defined in terms of limits of u at 0 or at ∞ .

The necessary and sufficient conditions for boundedness of Hardy operator H and conjugate Hardy operator H^* as acting from $L^p(\mathbb{R}_+, \omega)$ to $L^q(\mathbb{R}_+, h)$, where $1 < p, q < \infty$ are known, see e.g. [12, 15, 19, 21]. For reader’s convenience we enclose them in the Appendix in Theorems 7.1 and 7.2. Let us call them (C) - in the case of conditions for H , and (C*) - in the case of conditions for H^* , respectively.

We have the following example statement, which deepens our understanding of the Hardy inequality. As the $B_p(0)$ condition seems not known before, in our opinion, the result is new.

Theorem 5.1 (Analysis of Hardy inequality). *Suppose that the pair of weight functions (h, ω) , with positive $\omega \in C(\mathbb{R}_+)$, satisfies the condition (C) as in Theorem 7.1, $1 < q, p < \infty$.*

Then the following statements hold.

i) *We have $h \in L^1_{loc}((0, \infty])$ and $\omega \in B_p(0)$. In particular:*

- *the operator $Tr^0(u) := \lim_{t \rightarrow 0} u(t)$ is well defined for every $u \in D^{1,p}(\mathbb{R}_+, \omega)$;*
- *the set $\mathcal{R}^0_{p,\omega}(0) = \{u \in D^{1,p}(\mathbb{R}_+, \omega) : Tr^0(u) = 0\}$ is closed subspace in $D^{1,p}(\mathbb{R}_+, \omega)$;*
- *the inequality*

$$\left(\int_{\mathbb{R}_+} |u(t)|^q h(t) dt \right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}_+} |u'(t)|^p \omega(t) dt \right)^{\frac{1}{p}}, \tag{36}$$

holds for every $u \in \mathcal{R}^0_{p,\omega}(0)$, with constant $C > 0$ independent on u .

ii) *When $h \notin L^1(\mathbb{R}_+)$, then inequality (36) with right hand side finite holds precisely on the set $\mathcal{R}^0_{p,\omega}(0)$. In particular, the continuous embedding of Banach spaces $X \subseteq L^q(\mathbb{R}_+, h)$, which is proven to hold for $X = \mathcal{R}^0_{p,\omega}(0)$, cannot be extended to any subspace $X \subseteq D^{1,p}(\mathbb{R}_+, \omega)$ which is an essential superset of $\mathcal{R}^0_{p,\omega}(0)$.*

iii) When $h \in L^1(\mathbb{R}_+)$, then inequality (36) extends to inequality

$$\|u\|_{L^q(\mathbb{R}_+,h)} \leq C \left(\|u'\|_{L^p(\mathbb{R}_+, \omega)} + |Tr^0(u)| \right), \quad u \in D^{1,p}(\mathbb{R}_+, \omega), \tag{37}$$

with constant C independent on u . Moreover, (36) holds on every subspace $V \subseteq D^{1,p}(\mathbb{R}_+, \omega)$ which does not contain the nonzero constant functions.

Proof. i): The fact that $h \in L^1_{loc}((0, \infty])$ and $\omega \in B_p(0)$ follows from Conditions (C), see Theorem 7.1. The fact that $\mathcal{R}^0_{p,\omega}(0)$ is closed subspace in $D^{1,p}(\mathbb{R}_+, \omega)$ and that the operator $Tr^0(\cdot)$ is well defined on $D^{1,p}(\mathbb{R}_+, \omega)$ follows from Theorem 3.1. By the same theorem, $\mathcal{R}^0_{p,\omega}(0)$ is precisely the range of Hardy transforms under the action of Hardy operator H applied to $L^p(\mathbb{R}_+, \omega)$. Therefore, under Conditions (C), (36) holds on $\mathcal{R}^0_{p,\omega}(0)$. Now we prove the remaining statements.

ii): We already know that (36) holds on $\mathcal{R}^0_{p,\omega}(0)$. On the other hand, when $u \notin \mathcal{R}^0_{p,\omega}(0)$ then $Tr^0(u) = c \neq 0$, and in such case the left hand side in (36) cannot be finite as h is not integrable near zero.

iii): By triangle inequality $\|u\|_{L^q(\mathbb{R}_+,h)} \leq \|u - Tr^0(u)\|_{L^q(\mathbb{R}_+,h)} + |Tr^0(u)|_{L^q(\mathbb{R}_+,h)}$ and by the application of (36) to $u - Tr^0(u) \in \mathcal{R}^0_{p,\omega}(0)$, we easily get (37). Let V be any subspace of $D^{1,p}(\mathbb{R}_+, \omega)$ which does not contain nonzero constant functions. Then the seminorm $\|u\|^*_{D^{1,p}(\mathbb{R}_+, \omega)}$ is the norm on V and so $X_1 := (V, \|\cdot\|^*_{D^{1,p}(\mathbb{R}_+, \omega)})$, as well as $X_2 := (V, \|\cdot\|^{(0)}_{D^{1,p}(\mathbb{R}_+, \omega)})$ (see Theorem 3.1), are Banach spaces. Moreover, the identity operator $id : X_2 \rightarrow X_1$ is a continuous linear bijection. Let us apply Banach Inverse Mapping Theorem ([20]):

Theorem 5.2 (Banach’s Inverse Mapping Theorem). *Let X, Y be Banach spaces and let $T : X \mapsto Y$ be a linear bounded operator. If T is bijective, then $T^{-1} : Y \mapsto X$ is bounded.*

It guarantees that the inverse, $id : X_1 \rightarrow X_2$ is bounded. This implies that the norms $\|\cdot\|^*_{D^{1,p}(\mathbb{R}_+, \omega)}$ and $\|\cdot\|^{(0)}_{D^{1,p}(\mathbb{R}_+, \omega)}$ are comparable on V . Thus, we can substitute the norm $\|\cdot\|^{(0)}_{D^{1,p}(\mathbb{R}_+, \omega)}$ by the norm $\|\cdot\|^*_{D^{1,p}(\mathbb{R}_+, \omega)}$ in (37), when dealing with $u \in V$, with the eventual change of constant in the estimate. Therefore the second statement in iii) follows.

Note that if we consider for example $V := \{u \in D^{1,p}(\mathbb{R}_+, \omega) : u(1) = 0\}$, which is proper closed subspace of $D^{1,p}(\mathbb{R}_+, \omega)$ which does not contain the nonzero constant functions, then inequality (36) holds on V . On the other hand, the space V cannot be compared with $\mathcal{R}^0_{p,\omega}(0)$ in the sense that $\mathcal{R}^0_{p,\omega}(0) \setminus V \neq \emptyset$ and $V \setminus \mathcal{R}^0_{p,\omega}(0) \neq \emptyset$. \square

Remark 5.1 (Possible analysis of conjugate Hardy inequality). Similar considerations based on the analysis of Hardy conjugate transform H^* lead to the validity of (36), equipped with the condition $\lim_{t \rightarrow \infty} u(t) = 0$, under Conditions (C*) for the admitted weights (see Theorem 7.2).

Remark 5.2 (Links with the solvability of second order elliptic ODEs). As shown in [4], when $\Omega \subseteq \mathbb{R}^n$ is an open domain, $1 < p < \infty$, $\rho \in B_p(\Omega)$, that is $\rho^{-1/(p-1)}$ is locally integrable in Ω , then the validity of the Poincaré inequality

$$\int_{\Omega} |u|^p \rho dx \leq C \int_{\Omega} |\nabla u|^p \rho dx, \quad u \in W_0^{1,p}(\Omega, \rho),$$

is equivalent to the solvability of degenerate elliptic PDE’s

$$\operatorname{div}(\rho(x)|\nabla u|^{p-2}\nabla u) = f \in (W_0^{1,p}(\Omega, \rho))^*,$$

where $W_0^{1,p}(\Omega, \rho)$ is the completion of smooth compactly supported functions in $W^{1,p}(\Omega, \rho)$, X^* is the dual space to X . One could consider the case of $\Omega = \mathbb{R}_+$ and ask about the validity of inequalities

$$\int_{\mathbb{R}_+} |u(x)|^p \rho dx \leq C \int_{\mathbb{R}_+} |u'(x)|^p \rho dx, \quad u \in D_0^{1,p}(\mathbb{R}_+, \rho),$$

to deduce the solvability of ODE's like

$$\left(\rho(x)|u'(x)|^{p-2}u'(x) \right)' = f \in \left(W_0^{1,p}(\mathbb{R}_+, \rho) \right)^*.$$

5.2 Application to Formulation of Dirichlet Boundary Conditions for Solutions of ODE's

The following remark contributes to the interpretation and well-posedness of boundary conditions of Dirichlet type, in various problems dealing with ODE's.

Remark 5.3 Our analysis allows to interpret precisely Dirichlet type boundary conditions for $u \in D^{1,p}(\mathbb{R}_+, \omega)$ with $1 < p < \infty$:

$$\lim_{t \rightarrow 0} u(t) = c \text{ when } \omega \in B_p(0), \quad \text{or} \quad \lim_{t \rightarrow \infty} u(t) = c \text{ when } \omega \in B_p(\infty). \quad (38)$$

We already know (see Theorems: 3.1 and 3.2) that, for $u \in D^{1,p}(\mathbb{R}_+, \omega)$, in both cases the above conditions can be equivalently stated as

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{u(t) - c}{\left(\int_{(0,t)} \omega(\tau)^{-\frac{1}{p-1}} d\tau \right)^{1-\frac{1}{p}}} &= 0, \quad \text{when } \omega \in B_p(0), \\ \lim_{t \rightarrow \infty} \frac{u(t) - c}{\left(\int_{(t,\infty)} \omega(\tau)^{-\frac{1}{p-1}} d\tau \right)^{1-\frac{1}{p}}} &= 0, \quad \text{when } \omega \in B_p(\infty). \end{aligned} \quad (39)$$

As in later conditions the denominators converge to zero, (39) is stronger than (38) if we do not assume that $u \in D^{1,p}(\mathbb{R}_+, \omega)$. We can now confirm that the boundary conditions defined by (39) are well posed and equivalent for functions in the respective Dirichlet space $D^{1,p}(\mathbb{R}_+, \omega)$.

5.3 Generalization of Morrey's Inequality

Morrey's inequality in 1-dimension says that when $1 < p < \infty$ then for any $u \in D^{1,p}(\mathbb{R}_+, \omega \equiv 1)$

$$\|u\|_{C^{0,1-\frac{1}{p}}(\mathbb{R}_+)} = \sup_{x,y \in \mathbb{R}_+} \frac{|u(x) - u(y)|}{|x - y|^{1-\frac{1}{p}}} \leq \|u'\|_{L^p(\mathbb{R}_+)}, \quad (40)$$

see e.g [1], Lemma 4.28 on page 99.

Remark 5.4 Using a simple modification of inequalities (15), we deduce that when $\omega \in C(\mathbb{R}_+)$ $\omega > 0$, $1 < p < \infty$, then for any $u \in D^{1,p}(\mathbb{R}_+, \omega)$

$$\|u\|_{C_\omega^{0,1-\frac{1}{p}}(\mathbb{R}_+)} := \sup_{x,y \in \mathbb{R}_+} \frac{|u(x) - u(y)|}{d_{w,p}(x,y)^{1-\frac{1}{p}}} \leq \|u'\|_{L^p(\mathbb{R}_+, \omega)},$$

where $d_{w,p}(x, y) := \int_x^y \omega^{-\frac{1}{p-1}}(\tau) d\tau$ replaces $\text{dist}(x, y)^{1-\frac{1}{p}} = |x - y|$. Observe that the function $d_{w,p}(x, y)$ obeys the properties of distance function on \mathbb{R}_+ and replaces $|x - y|$ in (40).

5.4 Application to Complex Interpolation Theory for Weighted Dirichlet Spaces

In the paper [2], on page 2434, in the third question, the authors have asked about certain complex interpolation results for the weighted homogeneous Sobolev spaces, which in our setting we call Dirichlet spaces:

$$D^{1,p}(U, \omega) := \left\{ u \in L^1_{loc}(U) : \frac{\partial u}{\partial x_i} \in L^p(U, \omega), \text{ for } i = 1, \dots, n \right\},$$

where $U \subseteq \mathbb{R}^n$ is an open set and $\omega : U \rightarrow \mathbb{R}_+$ is a given weight. The authors have focused on the case of $p \in [1, \infty)$, $n = 1$, and $U = \mathbb{R}$ for the special class of weights, which satisfy the compact boundedness condition as in Definition 1.3 on page 2383. That condition is satisfied by every positive continuous function defined on \mathbb{R} . In that case the mappings

$$\psi \mapsto \int_0^x \psi(t)dt \text{ and its inverse } \phi \mapsto \phi' \tag{41}$$

give the isomorphic identification between the two Banach couples

$$((D^{1,p_0}(\mathbb{R}, \omega_0), (D^{1,p_1}(\mathbb{R}, \omega_1))) \text{ and } ((L^{p_0}(\mathbb{R}, \omega_0), (L^{p_1}(\mathbb{R}, \omega_1))).$$

Let $(X, Y)_\theta$ denote the complex interpolation pair between Banach spaces X, Y . It is deduced from the Calderón type generalization of Stein-Weiss Theorem, as in [2], in Remark 3.2 on page 2397, that one has:

$$((L^{p_0}(\mathbb{R}, \omega_0), L^{p_1}(\mathbb{R}, \omega_1))_\theta = L^{p_\theta}(\mathbb{R}, \omega_\theta),$$

where $\omega_\theta^{\frac{1}{p_\theta}} = \omega_0^{\frac{1-\theta}{p_0}} \omega_1^{\frac{\theta}{p_1}}$, $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.
From there it follows that

$$((D^{1,p_0}(\mathbb{R}, \omega_0), D^{1,p_1}(\mathbb{R}, \omega_1))_\theta = D^{1,p_\theta}(\mathbb{R}, \omega_\theta).$$

The precise arguments are submitted in Section A.4 on pages 2439 and 2440 in [2].

Consider now the case of $U = \mathbb{R}_+$, $p_0, p_1 \in (1, \infty)$ and weights ω_0, ω_1 such that either a) or b) holds when a) $\omega_0, \omega_1 \in B_p(0)$ or b) $\omega_0, \omega_1 \in B_p(\infty)$ are positive and continuous, $1 < p < \infty$. In case a), the mappings (41), while in case b), the mapping

$$\psi \mapsto - \int_x^\infty \psi(t)dt \text{ and its inverse } \phi \mapsto \phi'.$$

give the isomorphic identification between the two Banach couples

$$((D^{1,p_0}(\mathbb{R}_+, \omega_0), (D^{1,p_1}(\mathbb{R}_+, \omega_1))) \text{ and } ((L^{p_0}(\mathbb{R}_+, \omega_0), (L^{p_1}(\mathbb{R}_+, \omega_1))).$$

From there, by the same arguments as in [2], we deduce that

$$((D^{1,p_0}(\mathbb{R}_+, \omega_0), D^{1,p_1}(\mathbb{R}_+, \omega_1))_\theta = D^{1,p_\theta}(\mathbb{R}_+, \omega_\theta),$$

where $\omega_\theta^{\frac{1}{p_\theta}} = \omega_0^{\frac{1-\theta}{p_0}} \omega_1^{\frac{\theta}{p_1}}$, $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. As local B_p conditions: $B_p(0)$ and $B_p(\infty)$ have not been analyzed earlier (see Remark 6.1), in our opinion the result is new.

6 Perspectives for Further Development, Remarks, and Open Questions

Let us collect some remarks, focusing on the link with literature and further possible extensions.

Remark 6.1 (Local B_p conditions in literature). In general the localized B_p conditions are missing in the literature. However, in [13], having positive almost everywhere weight ω defined on open set $\Omega \subseteq \mathbb{R}^n$, the authors consider the so-called “exceptional set” $M_p(\Omega) :=$

$$\left\{ x \in \Omega : \int_{\Omega \cap V(x)} \omega^{-1/(p-1)}(y) dy = \infty, \text{ for every neighborhood } V(x) \text{ of } x \right\}.$$

In our case $\Omega = \mathbb{R}_+$ and $\omega \in B_p(\mathbb{R}_+)$, so $M_p(\Omega) = \emptyset$, but the extension of the above definition also to $x \in \bar{\Omega}$ in place of $x \in \Omega$, would lead in our situation to the validation of conditions $B_p(0)$ and $B_p(\infty)$.

Our results can be extended further in several directions. Let us propose some of them.

Remark 6.2 (Possible extensions).

(a) *The choice of another domain.* All the results that we will stated deal with functions defined on the half line. However, without major changes in the proofs, one can consider instead any interval (a, b) in place of \mathbb{R}_+ . We have focused on functions defined on \mathbb{R}_+ to make our presentation simpler.

(b) *Possible discontinuities inside the interval.* We have assumed in all our statements, that weight function ω is positive and continuous inside the interval \mathbb{R}_+ . It would be interesting to know how much this assumption can be weakened.

(c) *Higher order Dirichlet spaces.* Instead of $D^{1,p}(\mathbb{R}_+, \omega)$, one could consider higher order Dirichlet spaces

$$D^{k,p}(\mathbb{R}_+, \omega) = \{u : \mathbb{R}_+ \rightarrow \mathbb{R} : u \text{ is locally absolutely continuous on } \mathbb{R}_+ \text{ and } \|u^{(k)}\|_{L^p(\mathbb{R}_+, \omega)} < \infty\}.$$

where $k \in \mathbb{N}$ and $u^{(k)}$ is the distributional derivative of u , and ask similar questions.

(d) *Fractional order Dirichlet spaces.* Instead of $D^{1,p}(\mathbb{R}_+, \omega)$, one could consider fractional order Dirichlet spaces, where the derivative u' is replaced by the fractional one, $u^{(\alpha)}$, where $0 < \alpha < 1$. One can use the Caputo, Riemann-Liouville, or Grünwald-Leitnikov derivatives, as discussed for example in the book [17].

Remark 6.3 (About Muckenhoupt weights). In many papers the authors deal with Muckenhoupt weights. Let $\omega : \mathbb{R} \rightarrow \mathbb{R}_+$ be the Muckenhoupt weight, $1 < p < \infty$. Then, by definition, ω satisfies the A_p Muckenhoupt condition (see [19])

$$\sup_I \left(\int_I \frac{1}{|I|} \int_I \omega(t) dt \right) \left(\int_I \frac{1}{|I|} \int_I \omega(t)^{-1/(p-1)} dt \right)^{p-1} < \infty,$$

where I are intervals in \mathbb{R} . However our weights are continuous inside \mathbb{R}_+ , they do not need to satisfy the A_p condition. For example, such a one is $\omega(t) = t^{p-1}$, because $\omega^{-1/(p-1)}$ is not integrable near zero.

According to the discussion made in Section 5.4, we address an open question, which naturally arises from our discussion.

Open Question 6.1 Suppose that ω_0, ω_1 are continuous positive weights defined on \mathbb{R}_+ , $1 < p < \infty$, such that $\omega_0 \in B_p(0) \setminus B_p(\infty)$ and $\omega_1 \in B_p(\infty) \setminus B_p(0)$. What is the complex interpolation space

$$((D^{1,p_0}(\mathbb{R}_+, \omega_0), D^{1,p_1}(\mathbb{R}_+, \omega_1)))_\theta,$$

where $\theta \in (0, 1)$?

Remark 6.4 (Similar questions in the Sobolev space setting). Our results can be linked with therecent result by Kaczmarek and second author [8], where the authors deal with power weighted Sobolev spaces (with $\omega(x) = x^\alpha, \alpha \in \mathbb{R}$)

$$W^{1,p}(\mathbb{R}_+, x^\alpha) := \{u \in W_{loc}^{1,1}(\mathbb{R}_+) : \|u\|_{W^{1,p}(\mathbb{R}_+, x^\alpha)} := \|u\|_{L^p(\mathbb{R}_+, x^\alpha)} + \|u'\|_{L^p(\mathbb{R}_+, x^\alpha)}\},$$

and derive similar results, such as e.g. the analysis of trace operator, asymptotic behaviour near endpoints: 0 and ∞ , density results, applications to complex interpolation theory. As Sobolev spaces and Dirichlet spaces are not the same, despite similarities, our approach requires different analysis and it cannot be considered as direct generalization of results by Kaczmarek and second author.

7 Appendix

7.1 The Complementary Proofs

Proof of Fact 2.2. The proof is based on modification of arguments from [13], where Sobolev spaces instead of Dirichlet spaces were considered.

Let $U_n := \{u_n + c\}_{c \in \mathbb{R}}$ be a Cauchy sequence in $\tilde{D}^{1,p}(\mathbb{R}_+, \omega)$. Then for any fixed $a \in \mathbb{R}_+$ the function

$$v_n := \int_a^t u'_n(\tau) d\tau$$

is the representative of each U_n in its class in $\tilde{D}^{1,p}(\mathbb{R}_+, \omega)$. As $\{u'_n\}_{n \in \mathbb{N}}$ is the Cauchy sequence in $L^p(\mathbb{R}_+, \omega)$, which is complete, so $u'_n \xrightarrow{n \rightarrow \infty} g$ in $L^p(\mathbb{R}_+, \omega)$ for some $g \in L^p(\mathbb{R}_+, \omega)$. The arguments as in (6) allow to conclude that, in the case of $\omega \in B_p$, we have $u'_n \xrightarrow{n \rightarrow \infty} g$ in $L^1_{loc}(\mathbb{R}_+)$, which gives $v_n \xrightarrow{n \rightarrow \infty} v := \int_a^t g(\tau) d\tau \in D^{1,p}(\mathbb{R}_+, \omega)$ uniformly on compact sets. This gives

$$U_n \xrightarrow{n \rightarrow \infty} U := \{v + c\}_{c \in \mathbb{R}} \text{ in } \tilde{D}^{1,p}(\mathbb{R}_+, \omega),$$

because $\|U_n - U\|_{\tilde{D}^{1,p}(\mathbb{R}_+, \omega)}^* =$

$$= \|\{v_n - v + c\}_{c \in \mathbb{R}}\|_{\tilde{D}^{1,p}(\mathbb{R}_+, \omega)}^* = \|(v_n - v)'\|_{L^p(\mathbb{R}_+, \omega)} = \|u'_n - g\|_{L^p(\mathbb{R}_+, \omega)} \xrightarrow{n \rightarrow \infty} 0.$$

We have shown that the space $\tilde{D}^{1,p}(\mathbb{R}_+, \omega)$ is complete. □

Proof of Lemma 4.2. An easy verification shows that $\phi_{(k, K, a)}$ belongs to the admissible class for the functional, that is the non-weighted Sobolev space $W^{1,p}((k, K))$. We will show that $\phi_{(k, K, a)}$ is the unique minimizer of (25). We give the proof for $a = 1$, because $\phi_{(k, K, a)}(t) = a\phi_{k, K, 1}(t)$. As $E(\cdot)$ is convex functional and the admissible subset of $W^{1,p}((k, K))$ is convex

closed set, Direct Methods in the Calculus of Variations (see e.g. [3]), give existence of unique minimizer of $E(\cdot)$. Let us call such a minimizer ϕ_0 . Let $T(t) := \frac{t-K}{k-K}$ and

$$\tilde{E}(v) := \int_{(k,K)} |(T(t) + v(t))'|^p \omega(t) dt = \int_{(k,K)} \left| \frac{1}{k-K} + v'(t) \right|^p \omega(t) dt,$$

where $v \in W^{1,p}((k, K), \omega)$, $v(k) = v(K) = 0$.

We have

$$\phi_0 \text{ is minimizer of } E(\cdot) \iff v_0 := \phi_0 - T \text{ is minimizer of } \tilde{E}(\cdot).$$

By Direct Methods in Calculus of Variations, because the functional is nontrivial, coercive and convex on $W^{1,p}((k, K))$ and $\omega \sim 1$ on $[k, K]$, we deduce that there exists a unique minimizer of \tilde{E} . To find the minimizer, we compute Euler-Lagrange equation corresponding to the minimizer.

For any $v \in C_0^\infty((k, K))$ we have

$$\begin{aligned} 0 &= \frac{d}{ds} \tilde{E}(v_0 + sv)|_{s=0} \\ &= p \int_{(k,K)} \left\{ \left| \frac{1}{k-K} + v_0'(t) \right|^{p-1} \operatorname{sgn} \left(\frac{1}{k-K} + v_0'(t) \right) \omega(t) \right\} v'(t) dt. \end{aligned}$$

As ω is continuous, the function inside brackets $\{\cdot\}$ is integrable over (k, K) and its weak derivative is zero. Thus this function is constant and hence

$$\left| \frac{1}{k-K} + v_0'(t) \right|^{p-1} \left\{ \operatorname{sgn} \left(\frac{1}{k-K} + v_0'(t) \right) \right\} = \frac{Const}{\omega(t)}. \tag{42}$$

Denote $\Phi_r(a) := |a|^{r-1}a = |a|^r \operatorname{sign} a$, where $r > 0, a \in \mathbb{R}$. Then Φ_r is invertible and $\Phi_r^{-1}(a) = \Phi_{\frac{1}{r}}(a)$. Applying $\Phi_{\frac{1}{p-1}}$ to both sides in (42), we get

$$\frac{1}{k-K} + v_0'(t) = \Phi_{\frac{1}{p-1}} \left(\frac{Const}{\omega(t)} \right) = \frac{c_1}{\omega(t)^{\frac{1}{p-1}}}, \text{ for some } c_1 \in \mathbb{R}.$$

This implies

$$\begin{aligned} v_0'(t) &= -\frac{1}{k-K} + c_1 \omega(t)^{-1/(p-1)}, \text{ consequently} \\ v_0(t) &= \int_k^t v_0'(\tau) d\tau = -\frac{1}{k-K}(t-k) + c_1 \int_k^t \omega(\tau)^{-1/(p-1)} d\tau, \end{aligned}$$

for some constant c_1 . Recalling that $v_0(K) = 0$, we deduce that

$$c_1 = - \left(\int_{(k,K)} \omega(\tau)^{-1/(p-1)} d\tau \right)^{-1},$$

which allows to conclude the statement. □

7.2 Some Results about Hardy and Hardy Conjugate Operators

Recall that we deal with Hardy and conjugate Hardy operators as in (35).

Theorem 7.1 (Conditions (C), [21]). For positive weights $\omega, h : \mathbb{R}_+ \rightarrow [0, \infty) \cup \{\infty\}$ and $1 < p, q < \infty$, the Hardy operator

$$H : L^p(\mathbb{R}_+, \omega) \rightarrow L^q(\mathbb{R}_+, h)$$

is bounded if and only if i) or ii) holds, where

i) $1 < p \leq q < \infty$ and

$$E_1 := \sup_{t \in (0, \infty)} \left(\int_{(t, \infty)} h(\tau) d\tau \right)^{\frac{1}{q}} \left(\int_{(0, t)} \omega(\tau)^{-\frac{1}{p-1}} d\tau \right)^{\frac{1}{p}} < \infty,$$

ii) $1 < q < p < \infty$ and

$$E_2 := \int_{\mathbb{R}_+} \left(\int_{(t, \infty)} h(\tau) d\tau \right)^{\frac{p}{p-q}} \left(\int_{(0, t)} \omega(\tau)^{-\frac{1}{p-1}} d\tau \right)^{\frac{p(q-1)}{p-q}} \omega(t)^{-\frac{1}{p-1}} dt < \infty,$$

$$E_3 := \int_{\mathbb{R}_+} \left(\int_{(t, \infty)} h(\tau) d\tau \right)^{\frac{q}{p-q}} h(t) \left(\int_{(0, t)} \omega(\tau)^{-\frac{1}{p-1}} d\tau \right)^{\frac{q(p-1)}{p}} dt < \infty.$$

Theorem 7.2 (Conditions (C*), Theorems: 7.4 and 7.6 in [12]). For positive weights $\omega, h : \mathbb{R}_+ \rightarrow [0, \infty) \cup \{+\infty\}$ and $1 < p, q < \infty$, the conjugate Hardy operator

$$H^* : L^p(\mathbb{R}_+, \omega) \rightarrow L^q(\mathbb{R}_+, h)$$

is bounded if and only if i) or ii) holds, where

i) $1 < p \leq q < \infty$ and

$$A := \sup_{t \in \mathbb{R}_+} A(t) < \infty, \text{ and } \lim_{t \rightarrow 0} A(t) = \lim_{t \rightarrow \infty} A(t) = 0, \text{ where}$$

$$A(t) := \left(\int_{(0, t)} h(\tau) d\tau \right)^{\frac{1}{q}} \left(\int_{(t, \infty)} \omega(\tau)^{-\frac{1}{p-1}} d\tau \right)^{1-\frac{1}{p}}.$$

ii) $1 < q < p < \infty$ and

$$A := \int_{\mathbb{R}_+} \left(\int_{(0, t)} h(\tau) d\tau \right)^{\frac{p}{p-q}} \left(\int_{(t, \infty)} \omega(\tau)^{-\frac{1}{p-1}} d\tau \right)^{\frac{p(q-1)}{p-q}} \omega(t)^{-\frac{1}{p-1}} dt.$$

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