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# Compact Sets in Petals and Their Backward Orbits Under Semigroups of Holomorphic Functions

**Maria Kourou**[1] · **Konstantinos Zarvalis**[2]

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**Abstract**

Let $(\phi_t)_{t\geq 0}$ be a semigroup of holomorphic functions in the unit disk $\mathbb{D}$ and $K$ a compact subset of $\mathbb{D}$. We investigate the conditions under which the backward orbit of $K$ under the semigroup exists. Subsequently, the geometric characteristics, as well as, potential theoretic quantities for the backward orbit of $K$ are examined. More specifically, results are obtained concerning the asymptotic behavior of its hyperbolic area and diameter, the harmonic measure and the capacity of the condenser that $K$ forms with the unit disk.

**Keywords** Semigroup of holomorphic functions · Backward orbit · Petal · Harmonic measure · Condenser capacity · Koenigs function · Green energy · Hyperbolic area

**Mathematics Subject Classification (2010)** Primary 31A15, 30D05, 30C85 · Secondary 30C20, 47D06

## 1 Introduction

One-parameter semigroups of holomorphic self-maps of the unit disk $\mathbb{D}$ have been extensively examined in recent years. Their theory was introduced by Berkson and Porta in [4] and later expanded in several works such as [1, 5, 7, 9, 10, 12, 14, 16] and in particular, in the recent monograph [8]. A one-parameter semigroup is a family $(\phi_t)_{t\geq 0}$ of holomorphic functions in $\mathbb{D}$, where

(i) $\phi_0$ is the identity map;
(ii) $\phi_{t+s}(z) = \phi_t(\phi_s(z))$, for every $t, s \geq 0$ and $z \in \mathbb{D}$;

✉ Maria Kourou
maria.kourou@mathematik.uni-wuerzburg.de

Konstantinos Zarvalis
zarkonath@math.auth.gr

1 Department of Mathematics, Julius-Maximilians University of Wuerzburg, 97074, Wuerzburg, Germany

2 Department of Mathematics, Aristotle University of Thessaloniki, 54124, Thessaloniki, Greece

(iii) $\phi_t(z) \xrightarrow{t\to 0^+} z$, uniformly on compacta in $\mathbb{D}$.

One of the most important properties of one-parameter semigroups is the direct aspect of the continuous Denjoy-Wolff theorem. Except for semigroups of elliptic automorphisms, there exists a unique fixed point $\tau \in \overline{\mathbb{D}}$ such that

$$\lim_{t\to+\infty} \phi_t(z) = \tau, \tag{1.1}$$

for every point $z \in \mathbb{D}$. This point $\tau$ is called the Denjoy-Wolff point of the semigroup; see [1, Theorem 1.4.17]. If $\tau \in \mathbb{D}$ and $\phi_t$ is not an elliptic automorphism of $\mathbb{D}$ for any $t \geq 0$, then $(\phi_t)$ is called an *elliptic* semigroup. The *spectral value* of $\tau$ is a number $\mu \in \mathbb{C}$ with $\operatorname{Re}\mu > 0$ such that $\phi_t'(\tau) = e^{-\mu t}$, for every $t \geq 0$.

Let $\mathbb{T}$ denote the unit circle. In the case where $\tau \in \mathbb{T}$, we observe the angular derivative $\phi_t'(\tau)$. If $\phi_t'(\tau) < 1$ for one $t > 0$ (and then for all $t \geq 0$), the semigroup $(\phi_t)$ is called *hyperbolic*, whereas, if $\phi_t'(\tau) = 1$ for one $t > 0$ (and then for all $t \geq 0$), the semigroup $(\phi_t)$ is called *parabolic*. If $\phi_{t_0}$ is a hyperbolic (respectively parabolic) automorphism of $\mathbb{D}$ for some $t_0 > 0$, then $(\phi_t)$ is called a *hyperbolic (respectively parabolic) group*.

The curve $\gamma_z : [0, +\infty) \to \mathbb{D}$ with $\gamma_z(t) = \phi_t(z)$ is called the *trajectory* of a point $z \in \mathbb{D}$ and according to Eq. 1.1, $\gamma_z(t) \xrightarrow{t\to+\infty} \tau$, for every $z \in \mathbb{D}$. So, for every point in $\mathbb{D}$, its trajectory approaches a fixed point in $\overline{\mathbb{D}}$.

In [6] and [17], the trajectory of a compact subset of $\mathbb{D}$ under a non-elliptic semigroup was examined and results on its asymptotic behavior were extracted by means of several potential theoretic and geometric quantities. The main goal of the present work is to generalize those results through the investigation of the backward trajectory of a compact subset of $\mathbb{D}$ under a semigroup, which is not an elliptic group.

Following the notation of [8, Chapter 13], the *backward orbit* of a semigroup $(\phi_t)$ at a point $z \in \mathbb{D}$ is a continuous curve $\gamma_z : [0, +\infty) \to \mathbb{D}$ that satisfies $\phi_s(\gamma_z(t)) = \gamma_z(t - s)$, for every $t \in [0, +\infty)$ and every $s \in [0, t]$. A backward orbit is said to be *regular* if

$$\limsup_{t\to+\infty} d_{\mathbb{D}}(\gamma_z(t), \gamma_z(t+1)) < +\infty,$$

where $d_{\mathbb{D}}$ denotes the hyperbolic distance in $\mathbb{D}$. We state at this point some definitions related to backward orbits in order to meet the conditions under which the backward orbit of a compact set is defined.

A continuous curve $\gamma : (a, +\infty) \to \mathbb{D}$, where $a \in [-\infty, 0)$, is called a *maximal invariant curve* for $(\phi_t)$ if

$$\phi_s(\gamma(t)) = \gamma(t + s), \quad \forall s \geq 0, t \in (a, +\infty),$$

$\gamma(t) \xrightarrow{t\to+\infty} \tau$ and there exists $p \in \mathbb{T}$ such that $\gamma(t) \xrightarrow{t\to a^+} p$. The point $p$ is called the *starting point* of $\gamma$. From [8, Prop. 13.3.5], for every $z \in \mathbb{D}$, there exists a unique maximal invariant curve $\gamma_z : (a_z, +\infty) \to \mathbb{D}$, such that $\gamma_z(0) = z$. The *backward invariant set* $\mathcal{W}$ of $(\phi_t)$ is the set of all points in $\mathbb{D}$ for which $a_z = -\infty$ and it is defined as

$$\mathcal{W} := \bigcap_{t\geq 0} \phi_t(\mathbb{D}).$$

A *petal* $\Delta$ of $(\phi_t)$ is a non-empty simply connected component of the interior of $\mathcal{W}$ and satisfies the following properties:

(i) $\phi_t(\Delta) = \Delta$ for all $t \geq 0$ and $(\phi_{t|\Delta})$ is a group of automorphisms of $\Delta$
(ii) $\tau \in \partial\Delta$

(iii) there exists a boundary point $\sigma \in \partial\mathbb{D} \cap \partial\Delta$ (possibly $\sigma = \tau$) such that for every $z \in \Delta$ the curve

$$[0, +\infty) \ni t \mapsto \phi_t^{-1}(z)$$

is a backward regular orbit for $(\phi_t)$ that converges to $\sigma$. In addition, $\sigma$ is a boundary regular fixed point and in the case where $\sigma = \tau$, the semigroup is parabolic.

Moreover, as we can see in (iii), for any point $z \in \Delta$, we can denote, for the sake of simplicity, $\phi_{-t}(z) := \phi_t^{-1}(z)$ and as a result, $\phi_t(z)$ is defined for all $t \in \mathbb{R}$. More information concerning backward orbits and the characterization of petals follows in Section 2.1.

Therefore, if we suppose that $K$ is a compact subset of a petal $\Delta$ of $(\phi_t)$, then its backward orbit is

$$\gamma_K(t) := \bigcup_{z \in K} \gamma_z(t) = \phi_t^{-1}(K) = \phi_{-t}(K), \quad t \geq 0.$$

The backward orbit of every $z \in K$ is regular and so, $\gamma_K(t)$ is also regular.

With a trivial re-parametrization, we can denote the backward orbit of $K$ by $\phi_t(K)$, $t \leq 0$. As $t$ decreases and tends to $-\infty$, the compact set $\phi_t(K)$ approaches the unit circle and shrinks to a boundary fixed point. Our purpose is to determine how the geometric and potential theoretic characteristics of $\phi_t(K)$ behave during this approach.

We initiate our observations with the harmonic measure in the unit disk. Henceforward, we suppose that $(\phi_t)$ is a semigroup with Denjoy-Wolff point $\tau \in \overline{\mathbb{D}}$, which is not an elliptic group. Further suppose that $\Delta$ is a petal of $(\phi_t)$ and $K$ is a non-polar compact subset of $\Delta$; the reader may refer to Section 2.3 for polar sets. The harmonic measure $\omega(\phi_t(z), \partial\phi_t(K), \mathbb{D} \setminus \phi_t(K))$ is the Perron solution to the Dirichlet problem on $\mathbb{D} \setminus \phi_t(K)$ with given boundary values 1 on the boundary of $\phi_t(K)$ and 0 on the unit circle $\mathbb{T}$. Considering the harmonic measure $\omega(\phi_t(z), \partial\phi_t(K), \mathbb{D} \setminus \phi_t(K))$ as a function of $t \leq 0$, we obtain the following monotonicity result.

**Theorem 1.1** *Let $(\phi_t)$ be a semigroup of holomorphic functions in $\mathbb{D}$, which is not an elliptic group. Suppose $\Delta$ is a petal of $(\phi_t)$. Let $K$ be a compact non-polar subset of $\Delta$.*

*The harmonic measure $\omega(\phi_t(z), \partial\phi_t(K), \mathbb{D} \setminus \phi_t(K))$ is an increasing function of $t \in (-\infty, 0]$, for every $z \in \Delta \setminus K$.*

Therefore, the limit of the harmonic measure $\omega(\phi_t(z), \partial\phi_t(K), \mathbb{D} \setminus \phi_t(K))$, as $t \to -\infty$, exists. This way, we can get information on the asymptotic behavior of $\phi_t(K)$.

**Theorem 1.2** *Let $(\phi_t)$ be a semigroup of holomorphic functions in $\mathbb{D}$, which is not an elliptic group. Suppose $\Delta$ is a petal of $(\phi_t)$. Let $K$ be a compact non-polar subset of $\Delta$. Then*

$$\lim_{t \to -\infty} \omega(\phi_t(z), \partial\phi_t(K), \mathbb{D} \setminus \phi_t(K)) = \omega(z, \partial K, \Delta \setminus K), \quad z \in \Delta \setminus K.$$

Furthermore, we examine the change in the size of $\phi_t(K)$, as $t$ decreases and approaches $-\infty$. A natural way to do so is by observing the hyperbolic geometric characteristics of $\phi_t(K)$. Before we proceed to these characteristics, we need to examine the asymptotic behavior of the hyperbolic metric, as $t \to -\infty$.

**Theorem 1.3** *Let $(\phi_t)$ be a semigroup in $\mathbb{D}$, not an elliptic group. Suppose $\Delta$ is a petal of $(\phi_t)$. Then,*

$$\lim_{t \to -\infty} \lambda_{\mathbb{D}}(\phi_t(z))|\phi_t'(z)| = \lambda_\Delta(z), \quad z \in \Delta$$

*uniformly on compacta. Moreover,*

$$\lim_{t\to-\infty} d_{\mathbb{D}}(\phi_t(z), \phi_t(w)) = d_\Delta(z, w),$$

*for all* $z, w \in \Delta$.

At this point, we state a monotonicity property of the hyperbolic $n$-th diameter; see Section 2.3.

**Theorem 1.4** *Suppose* $(\phi_t)$ *is a semigroup of holomorphic functions in* $\mathbb{D}$*, not an elliptic group,* $\Delta$ *is a petal of* $(\phi_t)$ *and* $K \subset \Delta$ *compact. The hyperbolic* $n$*-th diameter* $d^{\mathbb{D}}_{n,h}(\phi_t(K))$ *is a decreasing function of* $t \leq 0$.

We obtain the following results concerning the asymptotic behavior of the hyperbolic area and the hyperbolic $n$-th diameter, as $t \to -\infty$.

**Theorem 1.5** *Suppose* $(\phi_t)$ *is a semigroup of holomorphic functions in* $\mathbb{D}$*, not an elliptic group,* $\Delta$ *is a petal of* $(\phi_t)$ *and* $K \subset \Delta$ *compact. Then*

$$\lim_{t\to-\infty} \mathrm{A}^{\mathbb{D}}_h(\phi_t(K)) = \mathrm{A}^{\Delta}_h K$$

*and*

$$\lim_{t\to-\infty} d^{\mathbb{D}}_{n,h}(\phi_t(K)) = d^{\Delta}_{n,h} K.$$

Last but not least, we pursue to extend the results for condenser capacity of [6] in the case of backward orbits. The ordered pair $(\mathbb{D}, \phi_t(K))$ forms a condenser, as $\mathbb{D}$ is a subdomain of $\widehat{\mathbb{C}}$ and $\phi_t(K)$ is a compact subset of $\mathbb{D}$. In the case where $K$ is non-polar, then so is $\phi_t(K)$; [21, Corollary 3.6.6]. This allows us to measure the size of the condenser by means of its capacity; more information on condensers follows in Sections 2.3 and 2.4. We obtain the following result concerning the convergence of the capacity of $(\mathbb{D}, \phi_t(K))$, as $t \to -\infty$.

**Theorem 1.6** *Suppose* $(\phi_t)$ *is a semigroup of holomorphic functions in* $\mathbb{D}$*, not an elliptic group,* $\Delta$ *is a petal of* $(\phi_t)$ *and* $K \subset \Delta$ *compact and non-polar. Then*

$$\lim_{t\to-\infty} \mathrm{cap}(\mathbb{D}, \phi_t(K)) = \mathrm{cap}(\Delta, K).$$

The above results are restricted in the case where the backward orbit of $K$ is regular. The question that arises is how the characteristics of $\phi_t(K)$ change, provided the backward orbit of $K$ is non-regular. In this case, we can talk about "degenerate petals", which are basically non-regular maximal invariant curves of $(\phi_t)$; we provide information on the variety of the forms petals can take in Section 2.1. We obtain the following outcome:

**Theorem 1.7** *Let* $(\phi_t)$ *be a semigroup in* $\mathbb{D}$*, which is not a group. Suppose* $\Delta$ *is a degenerate petal of* $(\phi_t)$ *and* $K$ *is a compact non-polar subset of* $\Delta$.

(i) $\lim_{t\to-\infty} \omega(\phi_t(z), \phi_t(K), \mathbb{D}) = 0$*, for all* $z \in \Delta \setminus K$,

(ii) $\lim_{t\to-\infty} \lambda_{\mathbb{D}}(\phi_t(z))|\phi_t'(z)| = +\infty$*, for all* $z \in \Delta$,

(iii) $\lim_{t\to-\infty} d_{\mathbb{D}}(\phi_t(z), \phi_t(w)) = +\infty$*, for all* $z, w \in \Delta$ *with* $z \neq w$,

(iv) $\lim_{t\to-\infty} g_{\mathbb{D}}(\phi_t(z), \phi_t(w)) = 0$*, for all* $z, w \in \Delta$ *with* $z \neq w$,

(v) $\lim_{t\to-\infty} \mathrm{A}_h^{\mathbb{D}}(\phi_t(K)) = 0$,

(vi) $\lim_{t\to-\infty} d_{n,h}^{\mathbb{D}}(\phi_t(K)) = 1$,

(vii) $\lim_{t\to-\infty} \operatorname{cap}(\mathbb{D}, \phi_t(K)) = +\infty$.

The article is structured as follows. In Section 2, we state some basic tools of one-parameter semigroups, potential theory and hyperbolic geometry, which will be used in proving the above theorems. In Section 3, Theorems 1.1 and 1.2 are proved concerning the monotonicity and the asymptotic behavior of harmonic measure. Afterwards, the asymptotic behavior of hyperbolic metric, hyperbolic area and hyperbolic $n$-th diameter is examined in Section 4, whereas in Section 5, similar results are obtained regarding the condenser capacity. Meanwhile in Section 6, the case of a non-regular backward orbit of a compact set is investigated.

## 2 Preparation for the Proofs

### 2.1 Koenigs Function-Petals-Backward Orbits

For every one-parameter non-elliptic semigroup, there exists a Riemann mapping $h$ fixing the origin such that $\Omega := h(\mathbb{D})$ is a simply connected domain and also, *convex in the positive direction*. This means that $\{w + s : s \geq 0\} \subset \Omega$, for every $w \in \Omega$. The function $h$ is unique up to a real-valued constant and it is called the *Koenigs function* of the semigroup. A major property of the Koenigs function is that it linearizes the trajectories of the points in $\mathbb{D}$ under the semigroup; basically

$$h(\phi_t(z)) = h(z) + t, \tag{2.1}$$

for all $z \in \mathbb{D}$, and $t \geq 0$.

The Koenigs function can also be defined for one-parameter elliptic semigroups, which are not groups. In this case, $h(\tau) = 0$, where $\tau \in \mathbb{D}$ is the Denjoy-Wolff-point of $(\phi_t)$ and

$$h(\phi_t(z)) = e^{-\mu t} h(z), \quad \forall z \in \mathbb{D} \tag{2.2}$$

where $\mu$ is the spectral value of $\tau$. The Koenigs function associated to an elliptic semigroup is unique up to multiplication by a complex-valued constant and maps the trajectories onto spirals

$$\operatorname{spir}_\mu[c] := \{e^{-\mu t} c : t \geq 0\}. \tag{2.3}$$

The domain $\Omega$ is $\mu$*-spirallike* with respect to 0, since $0 \in \Omega$ and $e^{-\mu t}\Omega \subseteq \Omega$, for all $t \geq 0$. In addition, every point $w \in \mathbb{C} \setminus \{0\}$ can be written as $w = e^{-\mu t + i\theta}$, for some $t \in \mathbb{R}$ and $\theta \in [-\pi, \pi)$. The $\mu$*-spirallike argument* of $w$ is defined as $\operatorname{Arg}_\mu(w) := \theta$.

The properties of the Koenigs function can be further generalized to the backward orbits, supposing they exist. Before we move on to petals and the convergence of backward orbits, we need the following definition concerning the fixed points of a one-parameter semigroup.

**Definition 2.1** [8, Chapters 12 & 14] Let $(\phi_t)$ be a semigroup of holomorphic self-maps of $\mathbb{D}$, which is not an elliptic group. A boundary fixed point $\sigma$ of $(\phi_t)$ is called *regular*, if the angular derivative of $\phi_t$ at $\sigma$ is finite, for all $t \geq 0$.

Suppose $\sigma \in \mathbb{T}\setminus\{\tau\}$ is a fixed point of $(\phi_t)$. If $\phi_t'(\sigma) = e^{-\lambda t} < +\infty$, for some $\lambda \in (-\infty, 0)$, then $\sigma$ is a *repelling fixed point* of $(\phi_t)$ and $\lambda$ is called the *spectral value* of $\sigma$. In the case where $\sigma$ is a non-regular point, it is called *super-repelling fixed point* of $(\phi_t)$. A super-repelling fixed point is *of the first type* if it is the starting point of a maximal invariant curve of $(\phi_t)$.

As stated in the Introduction in the case of regular backward orbits, for every point $z \in \Delta$, where $\Delta$ is a petal of $(\phi_t)$, the backward orbit exists and converges to a boundary regular fixed point. Hence it converges either to the Denjoy-Wolff point, assuming that $(\phi_t)$ is non-elliptic, or to a repelling boundary fixed point of $(\phi_t)$.

In the case where $(\phi_t)$ is a non-elliptic group, the backward orbit $\gamma_z$ is regular. If the group is hyperbolic and $\sigma \in \mathbb{T}\setminus\{\tau\}$ is the repelling fixed point, then $\gamma_z(t)$ converges non-tangentially to $\sigma$, as $t \to -\infty$. On the other hand, if the group is parabolic, the backward orbit $\gamma_z(t)$ converges tangentially to the Denjoy-Wolff point $\tau$, as $t \to -\infty$.

If $(\phi_t)$ is a non-elliptic semigroup, not a group, for every backward orbit $\gamma_z$, there exists $\sigma \in \mathbb{T}$ (possible even $\sigma = \tau$) that is a fixed point of $(\phi_t)$ with $\gamma_z(t) \xrightarrow{t\to-\infty} \sigma$. In the case where $\sigma \in \mathbb{T}\setminus\{\tau\}$ is a repelling fixed point, then $\gamma_z(t)$ is regular and $\gamma_z(t) \xrightarrow{t\to-\infty} \sigma$ non-tangentially.

In the case of an elliptic semigroup, not a group, a backward orbit is either identical to $\tau$ or a curve that converges to a boundary fixed point $\sigma$ of $(\phi_t)$. If $\sigma$ is a repelling fixed point of $(\phi_t)$, then the backward orbit is regular and it converges non-tangentially to $\sigma$.

Furthermore, there exists the following characterization of petals of a one-parameter semigroup, which is not an elliptic group. Let $\Delta$ be a petal of $(\phi_t)$. The petal $\Delta$ is called *hyperbolic* if $\partial\Delta$ contains a repelling fixed point of $(\phi_t)$. If $\partial\Delta \setminus \{\tau\}$ contains no boundary fixed points, then $\Delta$ is a *parabolic petal*. More specifically, only parabolic semigroups can have parabolic petals. The boundary of $\Delta$ can contain at most two fixed points of $(\phi_t)$; see [8, Prop.13.4.10]. In Fig. 1, we observe all the possible cases on hyperbolic and parabolic petals.

Let $h$ be the associated Koenigs function of $(\phi_t)$. The image of a petal under $h$ is a maximal domain in $\Omega$. More specifically, for a non-elliptic semigroup, $h(\Delta)$ is a maximal horizontal strip or a maximal horizontal half-plane in $\Omega$, supposing that $\Delta$ is hyperbolic or parabolic, respectively; see Fig. 2.

Suppose $(\phi_t)$ is an elliptic semigroup with Denjoy-Wolff point $\tau \in \mathbb{D}$ and spectral value $\mu$. Let $\Delta$ be a petal associated to the repelling point $\sigma$, with spectral value $\lambda \in (-\infty, 0)$. The image $h(\Delta)$ is a maximal $\mu$-spirallike sector in $\Omega$ of center $e^{i\theta_0}$, for some $\theta_0 \in [-\pi, \pi)$, and amplitude $2a := -\frac{|\mu|^2\pi}{\lambda \operatorname{Re}\mu}$; i.e.

$$h(\Delta) = \operatorname{Spir}[\mu, 2a, \theta_0] := \bigcup_{\theta\in(\theta_0-a,\theta_0+a)} \operatorname{spir}_\mu[e^{i\theta}] \cap (\mathbb{C}\setminus\{0\}).$$

Furthermore, petals have a direct connection with regular backward orbits. When it comes to non-regular backward orbits, it makes no sense to talk about petals in the way they were defined in the Introduction. A non-regular backward orbit for one-parameter semigroups, which are not groups, can fall into one of the following three cases (see Fig. 4):

(i) either it is part of the boundary of a hyperbolic petal, in which case it converges tangentially to a repelling fixed point of the semigroup,
(ii) either it is part of the boundary of a parabolic petal in which case it converges tangentially to the Denjoy-Wolff point of the semigroup, a situation that can arise solely in non-elliptic semigroups,

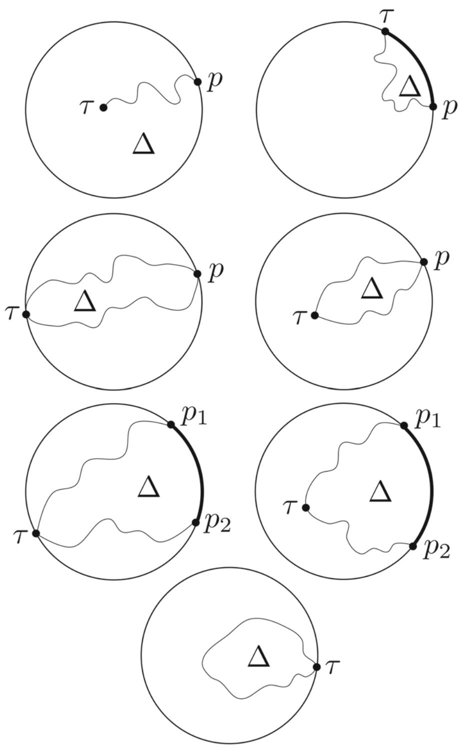



**Fig. 1** Hyperbolic & parabolic petals

(iii) or it converges to a super-repelling fixed point of the semigroup, in which case this super-repelling fixed point is of the first type and the convergence can be either tangential or non-tangential.

Hence, in any of the above cases there exists a "degenerate" petal. Suppose $(\phi_t)$ is a semigroup of holomorphic self-maps of $\mathbb{D}$, which is not an elliptic group, and $\gamma : [0, +\infty) \to \mathbb{D}$ is a non-regular backward orbit for $(\phi_t)$. Through the Koenigs function $h$, we move to the associated planar domain $\Omega$. Then, as with the regular backward orbits, the image $h(\gamma[0, +\infty))$ is either a half-line that converges to $\infty$ through the negative direction or a half-spiral that converges to $\infty$. Remembering that the images of both hyperbolic and parabolic petals through $h$ are maximal domains in $\Omega$, we can see that the image under $h$ of this degenerate petal is just the line or the spiral containing the set $h(\gamma[0, +\infty))$, respectively. As a result, the image of such a petal under the Koenigs function is either $\{h(\gamma(0)) + t : t \in \mathbb{R}\}$ or $\{e^{-\mu t}h(\gamma(0)) : t \in \mathbb{R}\}$, depending on the type of $(\phi_t)$.

The theory presented above is based on [8, Chapter 13], where the reader may find a detailed overview on the backward orbits in conjunction with the classification of one-parameter semigroups and the geometry of petals.

## 2.2 Hyperbolic and Quasi-Hyperbolic Metric

The *hyperbolic metric* in $\mathbb{D}$ is $\lambda_{\mathbb{D}}(z)|dz| = (1 - |z|^2)^{-1}|dz|$, where $\lambda_{\mathbb{D}}$ denotes its density. Suppose $f : \mathbb{D} \to U$ is a conformal mapping, where $U$ is a simply connected domain of $\mathbb{C}$.

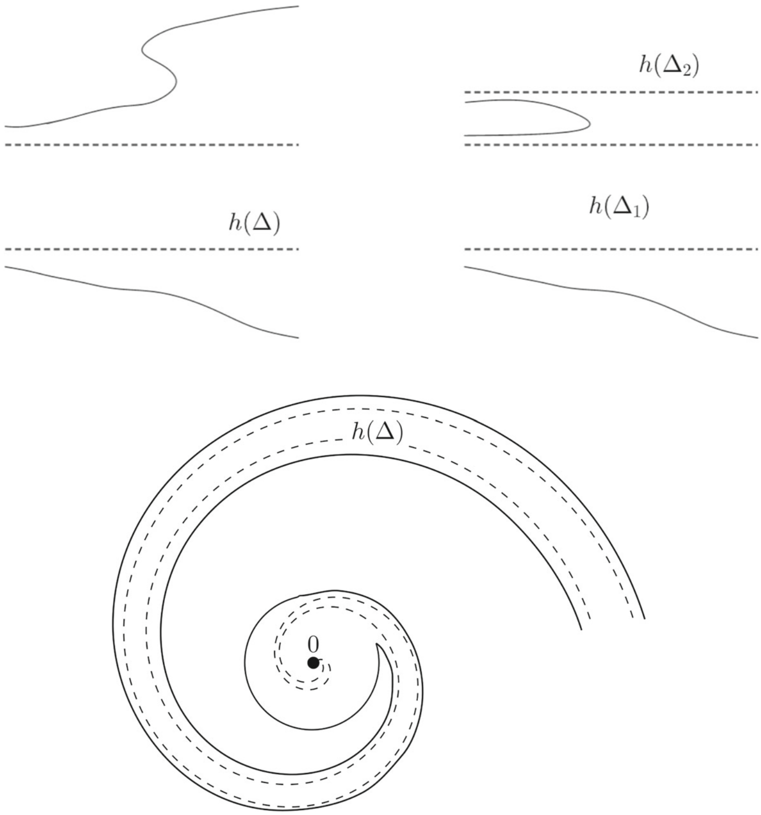



**Fig. 2** Images of Petals under $h$

The hyperbolic density in $U$ is

$$\lambda_U(f(z))\left|f'(z)\right| = \lambda_{\mathbb{D}}(z), \quad z \in \mathbb{D}. \tag{2.4}$$

In the case where $U$ is a simply connected subdomain of the unit disk, then for $z \in U$, $\lambda_{\mathbb{D}}(z) \leq \lambda_U(z)$. The *hyperbolic distance* between two points $a, b \in U$ is

$$d_U(a, b) = \inf_{\gamma \subset U} \int_\gamma \lambda_U(z)|dz|,$$

where $\gamma$ is any rectifiable curve that lies in $U$ and joins $a$, $b$. The infimum is attained for the hyperbolic geodesic arc that joins $a$, $b$. For instance, the hyperbolic distance in the unit disk, for $z, w \in \mathbb{D}$ is equal to

$$d_{\mathbb{D}}(z, w) = \operatorname{arctanh}\left|\frac{z-w}{1-\bar{z}w}\right|.$$

Hyperbolic distance is invariant under conformal mappings. Hyperbolic distance is invariant under conformal mappings; for every choice of $z, w \in \mathbb{D}$ it is true that $d_{\mathbb{D}}(z, w) = d_U(f(z), f(w))$. Then $d_{\mathbb{D}}(z, w) = d_U(f(z), f(w))$, for every choice of $z, w \in \mathbb{D}$.

In a simply connected domain $U$, [8, Theorem 5.3.1] provides a lower bound of the hyperbolic distance. Set $\delta_U(z) := \operatorname{dist}(z, \partial U)$ the Euclidean distance of $z$ from the boundary. Then

$$d_U(z,w) \geq \frac{1}{4}\log\left(1+\frac{|z-w|}{\min\{\delta_U(z),\delta_U(w)\}}\right), \tag{2.5}$$

for $z, w \in U$. Moreover, for $z \in U$, the quasi-hyperbolic metric is defined as

$$\lambda^\star(z)|dz| = \frac{|dz|}{\operatorname{dist}(z,\partial U)};$$

see e.g. [19, p.92]. The following inequality connects hyperbolic and quasi-hyperbolic densities:

$$\frac{1}{4}\lambda_U^\star(z) \leq \lambda_U(z) \leq \lambda_U^\star(z), \tag{2.6}$$

for all $z \in U$.

In addition, if $K$ is a compact subset of $\mathbb{D}$, its hyperbolic area is given by the formula

$$\mathrm{A}_h^{\mathbb{D}}(K) = \int_K \lambda_{\mathbb{D}}(z)^2 dA(z),$$

where $A$ is the Lebesgue area measure. Let us note that the hyperbolic area is also conformally invariant. The reader may refer to [3, 19] for further properties of the hyperbolic metric.

### 2.3 Euclidean and Hyperbolic *n*-th Diameter—Capacities and Condensers

Let $K$ be a compact subset of $\mathbb{C}$. The *Euclidean n-th diameter* of $K$ is

$$d_n(K) = \sup_{w_1,\dots,w_n\in K} \prod_{1\leq\mu<\nu\leq n} |w_\mu - w_\nu|^{\frac{2}{n(n-1)}} \tag{2.7}$$

and the supremum is attained, since $K$ is compact, for an $n$-tuple of points, which is called a *Fekete n-tuple* for $K$; see [21, Definition 5.5.1]. We should point out that a Fekete $n$-tuple is not unique for the compact set $K$. Its *logarithmic capacity* cap $K$ is equal to the limit of $d_n(K)$, as $n \to +\infty$. Sets of zero logarithmic capacity are called *polar sets* and they are negligible from the point of view of potential theory.

Furthermore, for a compact set $K \subset \mathbb{D}$, its *hyperbolic n-th diameter* is defined as

$$d_{n,h}^{\mathbb{D}}(K) = \sup_{w_1,\dots,w_n\in K} \prod_{1\leq\mu<\nu\leq n} \left|\frac{w_\mu - w_\nu}{1-\overline{w}_\mu w_\nu}\right|^{\frac{2}{n(n-1)}},$$

where the supremum is attained for an $n$-tuple of points. The *hyperbolic capacity* of $K$ is defined to be

$$\operatorname{caph} K = \lim_{n\to+\infty} d_{n,h}^{\mathbb{D}}(K)$$

and it is a conformally invariant quantity, due to the conformal invariance of the hyperbolic distance.

Another potential theoretic and conformally invariant quantity is the capacity of a condenser. A *condenser* is an ordered pair $(D, K)$, where $D$ is a proper domain of $\widehat{\mathbb{C}}$ and $K$ is a compact subset of $D$. Suppose both $\partial D$ and $K$ are non-polar. In the special case where $D$ is a simply connected domain, the *capacity* of $(D, K)$ can be defined as

$$\operatorname{cap}(D,K) := -\frac{2\pi}{\log \operatorname{caph}_D K}, \tag{2.8}$$

where $\operatorname{caph}_D K$ denotes the hyperbolic capacity of $K$ in $D$; see [11, Theorem 1.22].

As stated above, the condenser capacity is conformally invariant. Suppose $D$ and $G$ are simply connected domains in $\mathbb{C}$. If $f : D \to G$ is holomorphic and $K \subset D$, then

$$\operatorname{cap}(D, K) \geq \operatorname{cap}(G, f(K)). \tag{2.9}$$

Equality holds if and only if $f$ is conformal. For more information on condensers, the reader may refer to [11].

## 2.4 Green Potential—Green Capacity

Let $D$ be a domain of the extended complex plane $\widehat{\mathbb{C}}$. The *Green function* of $D$ with *pole* at $w \in D$ is denoted by $g_D(\cdot, w)$ and satisfies the following conditions:

(i) $g_D(\cdot, w)$ is positive harmonic on $D \setminus \{w\}$ and bounded outside every neighborhood of $w$
(ii) $g_D(\cdot, w) + \log|\cdot - w|$ is harmonic on $D$
(iii) $g_D(w, w) = \infty$, and for $z \to w$

$$g_D(z, w) = \begin{cases} \log|z| + \mathcal{O}(1), & w = \infty \\ -\log|z - w| + \mathcal{O}(1), & w \neq \infty \end{cases}$$

(vi) $g_D(\cdot, w) = 0$ on the boundary $\partial D$.

For every simply connected domain $D$, there exists the following relation between its Green function and the hyperbolic distance:

$$g_D(z, w) = -\log\tanh d_D(z, w), \tag{2.10}$$

for $z, w \in \mathbb{D}$; see [21, p.109]. If the boundary of a domain $D$ is non-polar, then the Green function $g_D$ exists and it is unique. In this case, the domain $D$ is called *Greenian*. The Green function is symmetric, for every $z, w \in D$. Moreover, the Green function is conformally invariant; e.g. [21, Theorem 4.4.4].

Let $D$ be a Greenian domain of $\widehat{\mathbb{C}}$ with Green function $g_D(x, y)$, for $x, y \in D$. For a measure $\mu$ with compact support in $D$, its *Green energy* is defined as the integral

$$I_D[\mu] := \iint g_D(x, y)d\mu(x)d\mu(y).$$

Suppose $K$ is a compact subset of $D$. The *Green energy* of $K$ with respect to $D$ is defined as

$$V(K, D) = \inf_{\mu} I_D[\mu],$$

where the infimum is taken over all the Borel measures $\mu$ with compact support $K$ such that $\mu(K) = 1$. If $V(K, D) < +\infty$, the infimum is attained for a Borel measure $\mu$, which is called *Green equilibrium measure* of $K$. The Green energy of a compact set is directly associated with condenser capacity.

*Remark 2.1* If $K$ and $\partial D$ have positive logarithmic capacity, the capacity of the condenser $(D, K)$ is proportional to the Green energy of the compact set $K$ and it is true that

$$\operatorname{cap}(D, K) = \frac{2\pi}{V(K, D)}. \tag{2.11}$$

More information on Green energy and the aspects of Green function can be found in [2, 15].

### 2.5 Harmonic Measure

Let $D$ be a proper subdomain of $\widehat{\mathbb{C}}$ with non-polar boundary and $\mathcal{B}(\partial D)$ be the $\sigma$-algebra of the Borel sets of $\partial D$. Suppose $E \in \mathcal{B}(\partial D)$. The *harmonic measure* of $E$ at a point $z \in D$ is the solution of the generalized Dirichlet problem in $D$ with boundary values 1 on $E$ and 0 on $\partial D \setminus E$. For a fixed $E \in \mathcal{B}(\partial D)$, $\omega(\cdot, E, D)$ is a harmonic and bounded function on $D$. Moreover, for a fixed point $z \in D$, the map

$$\omega(z, \cdot, D) : \mathcal{B}(\partial D) \to [0, 1] \quad \text{with} \quad E \mapsto \omega(z, E, D)$$

is a Borel probability measure on $\partial D$. In addition, if $\zeta$ is a regular point of $\partial D$ which lies outside the relative boundary of $E$ in $\partial D$, then $\lim_{z \to \zeta} \omega(z, E, D) = \mathcal{X}_E(\zeta)$, where $\mathcal{X}_E(\cdot)$ denotes the characteristic function of $E$.

A major property of the harmonic measure is its conformal invariance; see [21, §4.3]. For the sake of simplicity, if $E$ is a compact subset of $D$ with positive logarithmic capacity, from now on we will use the notation $\omega(z, E, D) := \omega(z, \partial E, D \setminus E)$. Let us state the following property for the harmonic measure.

**Proposition 2.1** (Strong Markov Property for Harmonic Measure) [20, p.88] *Suppose $\Omega$ is a Greenian domain in $\mathbb{C}$ and $S$ is a subdomain of $\Omega$. Let $E \subset \partial\Omega \cap \partial S$. Set $A := \partial S \cap \Omega$. Then for $z \in \Omega$,*

$$\omega(z, E, \Omega) = \omega(z, E, S) + \int_A \omega(s, E, \Omega) \cdot \omega(z, ds, S).$$

Respectively, we state the following relation between harmonic measure and the Green function.

**Proposition 2.2** (Strong Markov Property for Green function) [20, p.111] *Suppose $\Omega$ is a Greenian domain in $\mathbb{C}$ and $S$ is a subdomain of $\Omega$. Set $A := \partial S \cap \Omega$. Then for $z, w \in S$,*

$$g_\Omega(z, w) = g_S(z, w) + \int_A g_\Omega(\alpha, z) \cdot \omega(w, d\alpha, S).$$

Another property of great significance for harmonic measure is its probabilistic interpretation. Suppose $D$ is a domain in $\mathbb{C}$ and $E$ a Borel subset of $\partial D$. Let $B_t, t > 0$, be a Brownian motion in the complex plane starting from a point $z \in D$. Let $t_0 = \inf\{t > 0 : B_t \notin D\}$ be the first exit time of $B_t$ from $D$. The harmonic measure $\omega(z, E, D)$ is the probability of $B_{t_0} \in E$. Information and detailed theory on harmonic measure can be found in [13] and [21, Chapter 4].

## 3 Harmonic Measure—Proofs of Theorems 1.1 and 1.2

In the course of the following paragraphs, we suppose that $(\phi_t)$ is a semigroup of holomorphic self-maps of $\mathbb{D}$, not an elliptic group, with associated Koenigs function $h$. We further suppose that $\Delta$ is a petal of $(\phi_t)$ and $\Omega$ is the associated Koenigs domain of the semigroup. Take $K$ to be a compact non-polar subset of $\Delta$.

*Proof of Theorem 1.1.* Fix a point $z \in \Delta \setminus K$. Suppose that $t \leq s \leq 0$. Because of the difference in the nature of the Koenigs function between elliptic and non-elliptic semigroups, we need to make the following distinction:

(i) *Non-elliptic Semigroups:* By the convexity of the associated planar domain $\Omega$, we immediately get $\Omega - t \subset \Omega - s$. Thus, by the conformal invariance and the monotonicity property of the harmonic measure and the fact that $h(\Delta) \subset \Omega - t$, we have

$$\begin{aligned}\omega(\phi_s(z), \phi_t(K), \mathbb{D}) &= \omega(h(z) + s, h(K) + s, \Omega) \\ &= \omega(h(z), h(K), \Omega - s) \\ &\geq \omega(h(z), h(K), \Omega - t) \\ &= \omega(h(z) + t, h(K) + t, \Omega) \\ &= \omega(\phi_t(z), \phi_t(K), \mathbb{D}),\end{aligned}$$

and we get the desired result.

(ii) *Elliptic Semigroups:* Suppose that $\mu \in \mathbb{C}, \operatorname{Re}\mu > 0$, is the spectral value of $(\phi_t)$. Following a similar procedure, we get

$$\begin{aligned}\omega(\phi_s(z), \phi_s(z), \mathbb{D}) &= \omega(e^{-\mu s} h(z), e^{-\mu s} h(K), \Omega) \\ &= \omega(h(z), h(K), e^{\mu s}\Omega) \\ &\geq \omega(h(z), h(K), e^{\mu t}\Omega) \\ &= \omega(\phi_t(z), \phi_t(K), \mathbb{D}),\end{aligned}$$

as $e^{\mu t} \cdot \Omega \subset e^{\mu s} \cdot \Omega$, because $\Omega$ is $\mu$-spirallike with respect to 0. Therefore, Theorem 1.1 is proved.

□

*Proof of Theorem 1.2.* Once again, we need to distinguish two separate cases, since the geometry of $\Omega$ is totally different between elliptic and non-elliptic semigroups. However, the proofs are adequately similar. For this reason, we will present the proof for non-elliptic semigroups with every detail and then, only mention the key differences for the elliptic case, in order to avoid the redundancy.

(i) *Non-elliptic Semigroups:* Fix a point $z \in \Delta \setminus K$. Using the monotonicity property of the harmonic measure and considering consecutively the fact that $\phi_t$ is an automorphism of $\Delta$, for all $t \leq 0$, we see that

$$\begin{aligned}\omega(\phi_t(z), \phi_t(K), \mathbb{D}) &\geq \omega(\phi_t(z), \phi_t(K), \Delta) \\ &= \omega(z, K, \Delta),\end{aligned}$$

for all $t \leq 0$. Therefore,

$$\lim_{t \to -\infty} \omega(\phi_t(z), \phi_t(K), \mathbb{D}) \geq \omega(z, K, \Delta), \tag{3.1}$$

since the limit exists due to the monotonicity established in Theorem 1.1. Now, we need a similar reverse inequality. Due to conformal invariance and the properties of the Koenigs function, for $t \leq 0$ and $z \in \Delta \setminus K$,

$$\omega(\phi_t(z), \phi_t(K), \mathbb{D}) = \omega(h(\phi_t(z)), h(\phi_t(K)), \Omega) = \omega(h(z) + t, h(K) + t, \Omega).$$

Since $K$ is compact, there exists a $t_0 \leq 0$ such that $h(z)+t \notin h(K)$, for all $t \leq t_0$. We will deal first with the case when the petal $\Delta$ is hyperbolic. Then $h(\Delta)$ is a maximal strip contained in $\Omega$.

In combination with the fact that $\Omega$ is convex in the positive direction, there exists $t_1 \leq 0$ such that the sets

$$\partial\Omega \cap \{\zeta : \operatorname{Im}\zeta > \operatorname{Im} h(z), \operatorname{Re}\zeta = \operatorname{Re} h(z) + t\} \quad \text{and} \quad \partial\Omega \cap \{\zeta : \operatorname{Im}\zeta < \operatorname{Im} h(z), \operatorname{Re}\zeta = \operatorname{Re} h(z) + t\}$$

are both non-empty, for all $t \leq t_1$. For $t < 2\min\{t_0, t_1\}$, we denote by $p_t^+$ the point of $\partial\Omega$ such that

$$\left|p_t^+ - \left(h(z) + \frac{t}{2}\right)\right| = \min\left\{\left|\zeta - \left(h(z) + \frac{t}{2}\right)\right| : \zeta \in \partial\Omega, \operatorname{Im}\zeta > \operatorname{Im} h(z), \operatorname{Re}\zeta = \operatorname{Re} h(z) + \frac{t}{2}\right\}.$$

Similarly, we denote by $p_t^-$ the point of $\partial\Omega$ such that

$$\left|p_t^- - \left(h(z) + \frac{t}{2}\right)\right| = \min\left\{\left|\zeta - \left(h(z) + \frac{t}{2}\right)\right| : \zeta \in \partial\Omega, \operatorname{Im}\zeta < \operatorname{Im} h(z), \operatorname{Re}\zeta = \operatorname{Re} h(z) + \frac{t}{2}\right\}.$$

Then we define the sets

$$L_t^+ := \{\zeta : \operatorname{Re}\zeta \leq \operatorname{Re} p_t^+, \operatorname{Im}\zeta = \operatorname{Im} p_t^+\} \text{ and } L_t^- := \{\zeta : \operatorname{Re}\zeta \leq \operatorname{Re} p_t^-, \operatorname{Im}\zeta = \operatorname{Im} p_t^-\}$$

and we construct the domain $\Omega_t := \mathbb{C} \setminus (L_t^+ \cup L_t^-)$ (see Fig. 3).

Since the families of points $(p_t^+)_t, (p_t^-)_t \subset \partial\Omega$, by convexity it is clear that $\Omega \subset \Omega_t$, for all sufficiently small $t$. According to the domain monotonicity of the harmonic

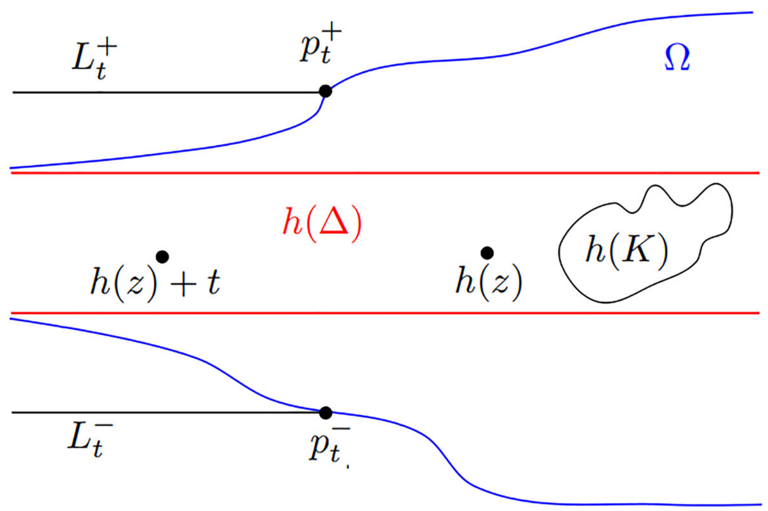



**Fig. 3** The construction of $\Omega_t$

measure, we obtain

$$\begin{aligned}\omega(h(z)+t, h(K)+t, \Omega) &\leq \omega(h(z)+t, h(K)+t, \Omega_t)\\ &= \omega(h(z), h(K), \Omega_t - t).\end{aligned}$$

It is easy to see that $h(\Delta) \subset \Omega_t - t$, for all sufficiently small $t$. Moreover,

$$\partial h(K) \subset \partial(h(\Delta) \setminus h(K)) \cap \partial((\Omega_t - t) \setminus h(K))$$

and

$$\partial(h(\Delta) \setminus h(K)) \cap ((\Omega_t - t) \setminus h(K)) = \partial h(\Delta).$$

Since $z \in \Delta \setminus K$, we have $h(z) \in h(\Delta) \setminus h(K)$ and applying Proposition 2.1, we have

$$\begin{aligned}\omega(h(z), h(K), \Omega_t - t) &= \omega(h(z), h(K), h(\Delta))\\ &\quad + \int_{\partial h(\Delta)} \omega(\zeta, h(K), \Omega_t - t) \cdot \omega(h(z), d\zeta, h(\Delta) \setminus h(K)).\end{aligned}$$

By the construction above, we can see that

$$\lim_{t \to -\infty} (\operatorname{Re} p_t^+ - t) = \lim_{t \to -\infty} (\operatorname{Re} p_t^- - t) = +\infty, \tag{3.2}$$

while for $h(\Delta) = \{w : a < \operatorname{Im} w < b\}$,

$$\lim_{t \to -\infty} \operatorname{Im} p_t^+ = b \quad \text{and} \quad \lim_{t \to -\infty} \operatorname{Im} p_t^- = a,$$

since the strip $h(\Delta)$ is maximal in $\Omega$. Hence the boundary of $\Omega_t - t$, as $t$ decreases, approaches $\partial h(\Delta)$. We will first prove that for any $\zeta \in \partial h(\Delta)$, we have $\omega(\zeta, h(K), \Omega_t - t) \xrightarrow{t \to -\infty} 0$.

The set $h(K)$ is compact and so, there exists a horizontal strip $S$ such that $h(K) \subset S \subsetneq h(\Delta)$. We denote by $\partial S^+$ and $\partial S^-$ the upper and lower boundary components of $S$, respectively. Set $A_t^+$ and $A_t^-$ to be the horizontal lines that contain $L_t^+$ and $L_t^-$, respectively. Define $S_1^t$ to be the horizontal strip bounded by $A_t^+$ and $\partial S^+$ and $S_2^t$ the one bounded by $\partial S^-$ and $A_t^-$. Without loss of generality, we take a point $\zeta \in \partial h(\Delta) \cap S_1^t$. If $\zeta \in S_2^t$, the proof follows in the same manner. We write $S_1^t = \{w \in \mathbb{C} : a_1 < \operatorname{Im} w < \operatorname{Im} p_t^+\}$. According to [21, p. 100], the harmonic measure

$$\omega(\zeta, A_t^+, S_1^t) = \frac{\operatorname{Im} \zeta - a_1}{\operatorname{Im} p_t^+ - a_1} = \frac{b - a_1}{\operatorname{Im} p_t^+ - a_1}.$$

However, $\operatorname{Im} p_t^+ \to b$, as $t \to -\infty$, from the construction of $\Omega_t$'s. As a result, $\omega(\zeta, A_t^+, S_1^t) \xrightarrow{t \to -\infty} 1$. Furthermore, from Eq. 3.2, we obtain that the half-line $L_t^+ - t$ expands towards $\infty$, in the positive direction. Hence, when $t \to -\infty$, $L_t^+ - t$ and $A_t^+$ coincide and so,

$$\lim_{t \to -\infty} \left[\omega(\zeta, A_t^+, S_1^t) - \omega(\zeta, L_t^+ - t, S_1^t)\right] = 0$$

that implies $\omega(\zeta, L_t^+ - t, S_1^t) \xrightarrow{t \to -\infty} 1$. However $S_1^t \subset (\Omega_t - t) \setminus h(K)$ and according to the subordination principle of harmonic measure,

$$\omega(\zeta, L_t^+ - t, S_1^t) \leq \omega(\zeta, L_t^+ - t, (\Omega_t - t) \setminus h(K)) \Rightarrow \lim_{t \to -\infty} \omega(\zeta, L_t^+ - t, (\Omega_t - t) \setminus h(K)) \geq 1.$$

From the properties of harmonic measure, $\omega(\zeta, L_t^+ - t, (\Omega_t - t) \setminus h(K)) \xrightarrow{t \to -\infty} 1$ and we deduce that

$$\lim_{t \to -\infty} \omega(\zeta, h(K), \Omega_t - t) = 0.$$

The choice of $\zeta$ was arbitrary and hence, the convergence is true for all $\zeta \in \partial h(\Delta)$. Next, we examine the uniform convergence. Consider the quantity

$$\sup_{\zeta \in \partial h(\Delta)} \omega(\zeta, h(K), \Omega_t - t).$$

Since $h(K)$ is compact, we have

$$\lim_{\Omega_t - t \ni \zeta \to \infty} \omega(\zeta, h(K), \Omega_t - t) = 0.$$

Therefore, the above supremum is attained on some point of $\partial h(\Delta)$. We denote by $\zeta_t$ such a point and we distinguish two cases: either $\operatorname{Re} \zeta_t > \operatorname{Re} p_t^+ - t$, for all $t < 2\min\{t_0, t_1\}$ or $\operatorname{Re} \zeta_t \leq \operatorname{Re} p_t^+ - t$, for all $t < 2\min\{t_0, t_1\}$. Of course, these two cases might alternate as $t \to -\infty$, but, in such a scenario, we can just consider two subfamilies of $(\zeta_t)$ and proceed in the same manner. In the first case, it is clear that $\zeta_t \xrightarrow{t \to -\infty} \infty$, something which directly implies that $\omega(\zeta_t, h(K), \Omega_t - t) \xrightarrow{t \to -\infty} 0$. In the second case, as $t \to -\infty$, one of the half-lines $L_t^+ - t$, $L_t^- - t$, and thus the boundary of $\Omega_t - t$, is getting arbitrarily close to $\zeta_t$. Using the above construction of horizontal strips, we are led again to $\omega(\zeta_t, h(K), \Omega_t - t) \xrightarrow{t \to -\infty} 0$. Therefore, we get

$$\sup_{\zeta \in \partial h(\Delta)} \omega(\zeta, h(K), \Omega_t - t) \xrightarrow{t \to -\infty} 0$$

and the harmonic measure converges uniformly to 0 on $\partial h(\Delta)$. Let $\epsilon > 0$. Then, there exists $t_2 < 2\min\{t_0, t_1\}$ such that $\omega(\zeta, h(K), \Omega_t - t) < \epsilon$, for all $\zeta \in \partial h(\Delta)$ and all $t \leq t_2$. Returning to the Strong Markov Property, for $t \leq t_2$, we get

$$\begin{aligned}
\omega(h(z), h(K), \Omega_t - t) &< \omega(h(z), h(K), h(\Delta)) + \int_{\partial h(\Delta)} \epsilon \cdot \omega(h(z), d\zeta, h(\Delta) \setminus h(K)) \\
&= \omega(h(z), h(K), h(\Delta)) + \epsilon \cdot \omega(h(z), \partial h(\Delta), h(\Delta) \setminus h(K)) \\
&\leq \omega(h(z), h(K), h(\Delta)) + \epsilon,
\end{aligned}$$

since the harmonic measure attains values in $[0, 1]$. Therefore,

$$\limsup_{t \to -\infty} \omega(h(z), h(K), \Omega_t - t) \leq \omega(h(z), h(K), h(\Delta)) = \omega(z, K, \Delta).$$

With the use of conformal invariance of the harmonic measure, we obtain

$$\begin{aligned}
\lim_{t \to -\infty} \omega(\phi_t(z), \phi_t(K), \mathbb{D}) &= \lim_{t \to -\infty} \omega(h(z) + t, h(K) + t, \Omega) \\
&\leq \limsup_{t \to -\infty} \omega(h(z), h(K), \Omega_t - t) \\
&\leq \omega(z, K, \Delta).
\end{aligned}$$

All in all, $\omega(\phi_t(z), \phi_t(K), \mathbb{D}) \xrightarrow{t \to -\infty} \omega(z, K, \Delta)$ and we have the desired result.

If the petal $\Delta$ is parabolic, then $h(\Delta)$ is a horizontal half-plane. In this case, only one of the families $(p_t^+)$, $(p_t^-)$ exists. Without loss of generality, we can assume that it is $(p_t^+)$. Then, we construct the half-lines $(L_t^+)$ in the same manner as before and consider the simply connected domains $\Omega_t = \mathbb{C} \setminus L_t^+$, for all suitable $t \leq 0$. The set $h(K)$ is compact and so, there exists a horizontal half-plane $H$ such that $h(K) \subset H \subsetneq h(\Delta)$. The horizontal line $\partial H$ along with $A_t$, which is also a horizontal line that contains $L_t^+$, are the boundary components of a horizontal strip that contains $\partial h(\Delta)$. Following the same procedure as in the hyperbolic case, we obtain that the harmonic measure in $\Omega_t - t \setminus h(K)$ with respect to $\partial h(K)$ converges uniformly to 0 on $\partial h(\Delta)$. Then, we continue with the proof exactly as above and deduce the desired result.

(ii) *Elliptic Semigroups:* Fix $z \in \Delta \setminus K$. In a similar fashion as in the previous case, we can prove that

$$\lim_{t\to-\infty} \omega(\phi_t(z), \phi_t(K), \mathbb{D}) \geq \omega(z, K, \Delta).$$

For the "reverse" inequality, we need to make the analogous construction. Since $\Omega$ is $\mu$-spirallike with respect to 0, there exists a $t_1 \leq 0$ such that the sets $\partial\Omega \cap \{\zeta : \mathrm{Arg}_\mu(\zeta) > \mathrm{Arg}_\mu(h(z)), |\zeta| = |e^{-\mu t}h(z)|\}$ and $\partial\Omega \cap \{\zeta : \mathrm{Arg}_\mu(\zeta) < \mathrm{Arg}_\mu(h(z)), |\zeta| = |e^{-\mu t}h(z)|\}$ are both non-empty, for all $t \leq t_1$. This time, we denote by $p_t^+$ the point of $\partial\Omega$ such that

$$\left|p_t^+ - e^{-\mu\frac{t}{2}}h(z)\right| = \min\left\{\left|\zeta - e^{-\mu\frac{t}{2}}h(z)\right| : \zeta \in \partial\Omega, \mathrm{Arg}_\mu(\zeta) > \mathrm{Arg}_\mu(h(z)), |\zeta| = \left|e^{-\mu\frac{t}{2}}h(z)\right|\right\}.$$

Similarly, we denote by $p_t^-$ the point of $\partial\Omega$ such that

$$\left|p_t^- - e^{-\mu\frac{t}{2}}h(z)\right| = \min\left\{\left|\zeta - e^{-\mu\frac{t}{2}}h(z)\right| : \zeta \in \partial\Omega, \mathrm{Arg}_\mu(\zeta) < \mathrm{Arg}_\mu(h(z)), |\zeta| = \left|e^{-\mu\frac{t}{2}}h(z)\right|\right\}.$$

Then, we define the half-spirals $S_t^+ = \{\zeta : |\zeta| \geq |p_t^+|, \mathrm{Arg}_\mu(\zeta) = \mathrm{Arg}_\mu(p_t^+)\}$ and $S_t^- = \{\zeta : |\zeta| \geq |p_t^-|, \mathrm{Arg}_\mu(\zeta) = \mathrm{Arg}_\mu(p_t^-)\}$. Finally, we construct the simply connected domains $\Omega_t := \mathbb{C} \setminus (S_t^+ \cup S_t^-)$, for all $t \leq t_1$. It is clear that $\Omega \subset \Omega_t$, for all $t \leq t_1$. As a result,

$$\begin{aligned}\omega(\phi_t(z), \phi_t(K), \mathbb{D}) &= \omega(e^{-\mu t}h(z), e^{-\mu t}h(K), \Omega)\\ &\leq \omega(e^{-\mu t}h(z), e^{-\mu t}h(K), \Omega_t)\\ &= \omega(h(z), h(K), e^{\mu t}\Omega_t).\end{aligned}$$

Using Markov Property and following the analogous steps as in the non-elliptic case, we can prove that

$$\limsup_{t\to-\infty} \omega(h(z), h(K), e^{\mu t}\Omega_t) \leq \omega(h(z), h(K), h(\Delta)) = \omega(z, K, \Delta).$$

Combining, we find that

$$\lim_{t\to-\infty} \omega(\phi_t(z), \phi_t(K), \mathbb{D}) \leq \omega(z, K, \Delta) \leq \lim_{t\to-\infty} \omega(\phi_t(z), \phi_t(K), \mathbb{D}),$$

which yields the desired result.

□

## 4 Hyperbolic Geometric Quantities—Proofs of Theorems 1.3, 1.4 and 1.5

For the proof of Theorem 1.3, we will need to provide some information concerning a notion of domain convergence. Firstly, a domain $D \subset \mathbb{C}$ is called *hyperbolic* if its complement contains at least two points. Now, let $\{D_n\}_{n\in\mathbb{N}}$ be a sequence of such domains. The *pre-kernel* $D_0$ of $\{D_n\}_{n\in\mathbb{N}}$ is said to be the set of all points $w$ for which there exists

$r > 0$ such that $\overline{D(w, r)} \subseteq D_n$, for all sufficiently large $n$. If $z_0 \in D_0$, then the *kernel of* $\{D_n\}_{n\in\mathbb{N}}$ *with respect to* $z_0$ is the connected component $D$ of $D_0$ that contains $z_0$. In addition, we say that $\{D_n\}_{n\in\mathbb{N}}$ *converges to D with respect to* $z_0$ if $z_0 \in D_0$ and each subsequence of $\{D_n\}_{n\in\mathbb{N}}$ has the same kernel $D$ with respect to $z_0$. For more information on kernels and kernel convergence see [19, §1.4]. Making use of the above definitions, we need the following Lemma concerning the convergence of hyperbolic density of a sequence of domains.

**Lemma 4.1** [22] *Suppose that* $\{D_n\}_{n\in\mathbb{N}}$ *is a sequence of hyperbolic domains that converges to the hyperbolic domain D with respect to* $z_0$. *Suppose* $w \in D$. *Then the limit of* $\lambda_{D_n}(w)$, *as* $n \to +\infty$, *exists and*

$$\lim_{n\to+\infty} \lambda_{D_n}(w) = \lambda_D(w).$$

The following Lemma provides information on the asymptotic behavior of Euclidean distance to the boundary along a non-regular backward orbit.

**Lemma 4.2** *Let* $(\phi_t)$ *be a semigroup of holomorphic self-maps of* $\mathbb{D}$, *not a group. Suppose* $\gamma : [0, +\infty) \to \mathbb{D}$ *is a non-regular backward orbit of* $(\phi_t)$. *Then*

$$\lim_{t\to-\infty} \delta_\Omega(h(\phi_t(z)) = 0, \quad z \in \gamma([0, +\infty)).$$

*Proof* Suppose $z = \gamma(0)$ and observe that $\gamma(t) = \phi_{-t}(z)$, for all $t \geq 0$. As seen in Section 2.1, non-regular backward orbits are either boundary components of petals or backward orbits landing at a super-repelling boundary fixed point.

For a non-elliptic semigroup $(\phi_t)$, the image of a petal under $h$ is a maximal horizontal strip or a maximal horizontal half-plane. Furthermore, from [8, Corollary 13.6.7], we may observe that for a backward orbit landing at a super-repelling point $\xi \in \mathbb{T}$, $\operatorname{Im} h(z) \xrightarrow{z\to\xi} a$, for some $a \in \mathbb{R}$. Due to $\Omega$ being convex in the positive direction, we conclude that in both cases $\delta_\Omega(h(z) + t) \to 0$, as $t \to -\infty$.

Suppose now that $(\phi_t)$ is an elliptic semigroup. Let $\mu \in \mathbb{C}$ with $\operatorname{Re} \mu > 0$ be the spectral value of $\tau \in \mathbb{D}$. The image of a petal of $(\phi_t)$ under $h$ is a maximal spirallike sector in $\Omega$. In the case where the backward orbit lands at a super-repelling point $\xi \in \mathbb{T}$, $\operatorname{Arg}_\mu(h(z)) \xrightarrow{z\to\xi} \theta_0$, for some $\theta_0 \in [-\pi, \pi]$; see [8, Corollary 13.6.4]. Bearing also in mind that $\Omega$ is $\mu$-spirallike, we are led to $\delta_\Omega(e^{-\mu t} h(z)) \to 0$, as $t \to -\infty$. □

*Remark 4.1* Suppose $\gamma : (a, +\infty) \to \mathbb{D}$, $a > -\infty$, is a maximal invariant curve with starting point $p := \lim_{t\to a^+} \gamma(t)$ a non-fixed boundary point of $(\phi_t)$. Let $z := \gamma(0)$. Then

$$\angle \lim_{t\to a^+} h(\phi_t(z)) = \angle \lim_{z\to p} h(z) =: \widetilde{p} \in \partial\Omega$$

and hence, $\delta_\Omega(h(\phi_t(z))) \to 0$, as $t \to a^+$.

Note that for such maximal invariant curves on the boundary of a petal and for $t \leq a$, $h(z) + t$ or $e^{-\mu t} h(z)$, respectively, are boundary points of $\Omega$ and thus, their distance to the boundary coincides with zero.

The following Lemma handles the asymptotic behavior of the hyperbolic distance on boundary components of a petal.

**Lemma 4.3** *Suppose $(\phi_t)$ is a semigroup of holomorphic self-maps of $\mathbb{D}$, which is not an elliptic group. Let $\Delta$ be a petal of $(\phi_t)$ and $z \in \Delta$. Then $d_{\phi_t(\mathbb{D})}(\cdot, z)^{-1}$ converges uniformly to 0 on $\partial\Delta$, as $t \to +\infty$.*

*Proof* For $\zeta \in \partial\Delta \cap \mathbb{T}$, then $\zeta \in \partial\phi_t(\mathbb{D})$, for all $t \geq 0$ and so, $d_{\phi_t(\mathbb{D})}(\zeta, z) = d_{\mathbb{D}}(\zeta, z) = +\infty$. Hence we examine the boundary components of $\Delta$ that lie in $\mathbb{D}$. These are either non-regular backward orbits or maximal invariant curves with a boundary non-fixed starting point. We denote such a boundary component by $\gamma : (a, +\infty) \to \mathbb{D}$, $a \in [-\infty, 0)$.

Let $h$ be the associated Koenigs function of $(\phi_t)$ and $\Omega$ the associated planar domain. For $t \leq 0$, set

$$\Omega_t := h(\phi_{-t}(\mathbb{D})) = \begin{cases} \Omega - t, & (\phi_t) \text{ is non-elliptic} \\ e^{\mu t}\Omega, & (\phi_t) \text{ is elliptic.} \end{cases}$$

The family of domains $(\Omega_t)_{t\leq 0}$ is decreasing, as $t$ decreases, and converges to $h(\Delta)$, with respect to all points in $h(\Delta)$. Let $w = h(z) \in h(\Delta)$. From Eq. 2.5 we obtain

$$d_{\Omega_t}(h(\zeta), w) = d_\Omega(h(\phi_t(\zeta)), h(\phi_t(z))) \geq \frac{1}{4}\log\left(1 + \frac{|h(z) - h(\zeta)|}{\min\{\delta_\Omega(h(\phi_t(\zeta))), \delta_\Omega(h(\phi_t(z)))\}}\right), \tag{4.1}$$

for all $\zeta \in \partial\Delta \cap \mathbb{D}$. Due to Lemma 4.2 and Remark 4.1,

$$d_\Omega(h(\phi_t(\zeta)), h(\phi_t(z))) \xrightarrow{t\to a} +\infty.$$

Let us note that according to Remark 4.1, we can substitute $a$ with $-\infty$. Due to conformal invariance and with a trivial re-parametrization, this is equivalent to $d_{\phi_t(\mathbb{D})}(\cdot, z)$ converging pointwise to $+\infty$, as $t \to +\infty$. Its reciprocal is pointwise decreasing in $t \geq 0$, since $(\phi_t(\mathbb{D}))_{t\geq 0}$ is a decreasing family of domains, and it converges pointwise to 0 on $\partial\Delta$. Furtheremore $d_{\phi_t(\mathbb{D})}(\cdot, z)^{-1}$ is continuous and $\partial\Delta$ is a compact set. From Dini's Theorem, $d_{\phi_t(\mathbb{D})}(\cdot, z)^{-1}$ converges uniformly to 0 on $\partial\Delta$. $\square$

*Proof of Theorem 1.3.* Suppose $(\phi_t)$ is a semigroup of holomorphic functions, not an elliptic group. Let $\mathcal{W}$ be the backward invariant set of $(\phi_t)$ and $\Delta$ be a petal of $(\phi_t)$. Recall that $\Delta$ is a simply connected component of the interior of $\mathcal{W}$.

First we examine the convergence of the hyperbolic density. For every $t \geq 0$, $\Delta \subset \phi_t(\mathbb{D})$ and thus, $\lambda_{\phi_t(\mathbb{D})}(z) \leq \lambda_\Delta(z)$, for every $z \in \Delta$, due to the domain monotonicity property of the hyperbolic density. Since $\phi_t$ is a holomorphic self-map of $\mathbb{D}$, for $t, s > 0$ we have that $\phi_{t+s}(\mathbb{D}) = \phi_s(\phi_t(\mathbb{D})) \subset ... \subset \phi_t(\mathbb{D}) \subset .. \subset \phi_1(\mathbb{D})$ and hence $(\phi_t(\mathbb{D}))_{t>0}$ is a decreasing family of subdomains of $\mathbb{D}$. Therefore $\lambda_{\phi_t(\mathbb{D})}(z)$ is an increasing function of $t \geq 0$ and the limit, as $t \to +\infty$, exists.

Set $n \in \mathbb{N}$. Then $\{\phi_n(\mathbb{D})\}_n$ is a decreasing sequence of domains. It is easy to observe that $\phi_n(\mathbb{D})$ converges to $\Delta$, with respect to all points in $\Delta$. From Lemma 4.1, we are led to $\lambda_{\phi_n(\mathbb{D})}(w) \xrightarrow{n\to+\infty} \lambda_\Delta(w)$, for every $w \in \Delta$. Due to the uniqueness of the limit, we also conclude that $\lambda_{\phi_t(\mathbb{D})}(w) \xrightarrow{t\to+\infty} \lambda_\Delta(w)$, for every $w \in \Delta$, which is equivalent to $\lambda_{\mathbb{D}}(\phi_t(w))|\phi_t'(w)| \xrightarrow{t\to-\infty} \lambda_\Delta(w)$; see Eq. 2.4. Now we examine the uniform convergence. Fix $w \in \Delta$. According to [18, Theorem 1], since $\Delta \subset \phi_t(\mathbb{D})$, we obtain

$$1 \leq \frac{\lambda_\Delta(w)}{\lambda_{\phi_t(\mathbb{D})}(w)} \leq 1 + \frac{2}{e^{2R_t(w)} - 1}, \tag{4.2}$$

where

$$R_t(w) = d_{\phi_t(\mathbb{D})}(w, \phi_t(\mathbb{D}) \setminus \Delta) = \inf_{\partial\Delta\cap\mathbb{D}} d_{\phi_t(\mathbb{D})}(w, \cdot).$$

The infimum above is attained for a specific point on $\partial\Delta \cap \mathbb{D}$. Since the family $(\phi_t(\mathbb{D}))_t$ is decreasing, then $R_t(w)$ is pointwise increasing and hence its limit, as $t \to +\infty$, exists. From Lemma 4.3,

$$R_t(w) \xrightarrow{t\to+\infty} +\infty \tag{4.3}$$

and for every $\epsilon > 0$, there exists $t_0 \le 0$ such that for every $t \le t_0$,

$$\frac{1}{d_{\phi_t(\mathbb{D})}(\zeta, w)} < \epsilon \Rightarrow d_{\phi_t(\mathbb{D})}(\zeta, w) > \frac{1}{\epsilon} := M, \quad \forall \zeta \in \partial\Delta$$

and so, $R_t(w) > M$. The choice of $w$ was arbitrary and thus, Eq. 4.3 is true for all $w \in \Delta$. We are interested in the case where $w$ lies in a compact subset of $\Delta$. We can rewrite Eq. 4.2 in the following way:

$$0 \le \lambda_\Delta(w) - \lambda_{\phi_t(\mathbb{D})}(w) \le \frac{2\lambda_{\phi_t(\mathbb{D})}(w)}{e^{2R_t(w)} - 1}.$$

The hyperbolic density $\lambda_{\phi_t(\mathbb{D})}$ is bounded on compacta in $\Delta$. Hence, there exists a constant $c > 0$ such that

$$0 \le \lambda_\Delta(w) - \lambda_{\phi_t(\mathbb{D})}(w) \le \frac{2c}{e^{2M} - 1}.$$

However $M$ is arbitrarily large, so we let $M \to +\infty$ and it follows that

$$\lambda_{\phi_t(\mathbb{D})}(w) \xrightarrow{t\to+\infty} \lambda_\Delta(w), \tag{4.4}$$

locally uniformly in $\Delta$. Due to conformal invariance, we conclude that $\lambda_{\mathbb{D}}(\phi_t(\cdot))|\phi_t'(\cdot)|$ converges locally uniformly to $\lambda_\Delta(\cdot)$ in $\Delta$, as $t \to -\infty$.

Next, we move on to the convergence of the hyperbolic distance. Let $z, w \in \Delta$. Suppose $h$ is the associated Koenigs function of $(\phi_t)$ and $\Omega$ the associated planar domain. Due to the monotonicity property of the hyperbolic distance, we have

$$d_\Omega(h(\phi_t(z)), h(\phi_t(w))) \le d_{h(\Delta)}(h(\phi_t(z)), h(\phi_t(w))) = d_{h(\Delta)}(h(z), h(w)),$$

for all $t \le 0$. Therefore,

$$\limsup_{t\to-\infty} d_\Omega(h(\phi_t(z)), h(\phi_t(w))) \le d_{h(\Delta)}(h(z), h(w)). \tag{4.5}$$

As in the proof of Lemma 4.3, for $t \le 0$, we set

$$\Omega_t := h(\phi_{-t}(\mathbb{D})) = \begin{cases} \Omega - t, & (\phi_t) \text{ is non-elliptic} \\ e^{\mu t}\Omega, & (\phi_t) \text{ is elliptic.} \end{cases}$$

Since $h(\Delta) \subset \Omega_t$ for all $t \le 0$, it is also true that $h(z), h(w) \in \Omega_t$, for all $t \le 0$. So, let $\delta_t : [0, 1] \to \Omega_t$ be the hyperbolic geodesic arc of $\Omega_t$ such that $\delta_t(0) = h(z)$ and $\delta_t(1) = h(w)$. Suppose that there exists a sequence of real numbers $\{t_n\} \subset (-\infty, 0]$ with $\lim_{n\to+\infty} t_n = -\infty$, such that $\delta_{t_n}[0, 1] \cap \Omega_t \setminus h(\Delta) \neq \emptyset$. As a result, for any $n \in \mathbb{N}$, there exists at least one $x_n \in (0, 1)$ such that $\delta_{t_n}(x_n) \in \Omega_t \setminus h(\Delta)$. Due to the conformal invariance of the hyperbolic distance and the fact that $\delta_{t_n}$ is a geodesic arc, we have

$$\begin{aligned} d_\Delta(z, w) &= d_{h(\Delta)}(h(z), h(w)) \\ &\ge d_{\Omega_{t_n}}(h(z), h(w)) \\ &\ge d_{\Omega_{t_n}}(h(z), \delta_{t_n}(x_n)) \\ &\ge \frac{1}{4}\log\left(1 + \frac{|\delta_{t_n}(x_n) - h(z)|}{\min\{\delta_{\Omega_{t_n}}(h(z)), \delta_{\Omega_{t_n}}(\delta_{t_n}(x_n))\}}\right), \end{aligned} \tag{4.6}$$

where the last inequality follows from Eq. 2.5. As far as non-elliptic semigroups are concerned, we distinguish the following two cases:

(i) either $\inf \mathrm{Re}(\delta_{t_n}(x_n)) \in \mathbb{R}$ and $\sup \mathrm{Re}(\delta_{t_n}(x_n)) = +\infty$,
(ii) or both $\inf \mathrm{Re}(\delta_{t_n}(x_n)), \sup \mathrm{Re}(\delta_{t_n}(x_n)) \in \mathbb{R}$.

Any other case where $\inf \mathrm{Re}(\delta_{t_n}(x_n)) = -\infty$ can be treated in a similar manner as (i). On the other hand, for elliptic semigroups we distinguish the following two cases:

(i) either $\inf |\delta_{t_n}(x_n)| \in [0, +\infty)$ and $\sup |\delta_{t_n}(x_n)| = +\infty$,
(ii) or both $\inf |\delta_{t_n}(x_n)|, \sup |\delta_{t_n}(x_n)| \in [0, +\infty)$.

In cases (i) above, it is clear that $\delta_{t_n}(x_n) \xrightarrow{n\to+\infty} \infty$, while $\delta_{\Omega_{t_n}}(h(z))$ remains bounded from above. Therefore, due to Eq. 4.6, we are led to $d_\Delta(z, w) = +\infty$. Contradiction! In cases (ii), we can see that $\delta_{\Omega_{t_n}}(\delta_{t_n}(x_n)) \xrightarrow{n\to+\infty} 0$ (because of Lemma 4.2 and the fact that $t_n \xrightarrow{n\to+\infty} -\infty$), while $|\delta_{t_n}(x_n) - h(z)|$ remains bounded from below. Therefore, once again, we are led to $\delta_\Delta(z, w) = +\infty$. Contradiction!

Thus, there can be no such sequence $\{t_n\}$, which means that there exists a $T \le 0$ such that $\delta_t([0, 1]) \subset h(\Delta)$, for all $t \le T$. Bearing in mind the uniform convergence of the hyperbolic density on compacta, for every $\epsilon > 0$, there exists $t_0 \le T$ such that, for all $t \le t_0$,

$$\begin{aligned} d_\Delta(z, w) &= d_{h(\Delta)}(h(z), h(w)) \\ &\le \int_{\delta_t} \lambda_{h(\Delta)}(\zeta)|d\zeta| \\ &< \int_{\delta_t} \left[\lambda_{\Omega_t}(\zeta) + \epsilon\right] |d\zeta| \\ &= d_{\Omega_t}(h(z), h(w)) + \epsilon \cdot \mathrm{length}(\delta_t), \end{aligned}$$

where $\mathrm{length}(\delta_t)$ is the Euclidean length of $\delta_t$. The curve $\delta_t$ is the geodesic joining $h(z)$ and $h(w)$ in $\Omega_t$ and so, its pre-image under the Koenigs function is the geodesic joining $\phi_t(z)$ and $\phi_t(w)$ in $\mathbb{D}$. We denote by $C$ the line segment or the spiral arc joining $h(z)$ and $h(w)$ in $\Omega_t$, in the case where $(\phi_t)$ is non-elliptic or elliptic, respectively. According to the Gehring-Hayman Theorem [19, §4.1–4.6], there exists an absolute constant $K$ such that

$$\mathrm{length}(\delta_t) \le K \cdot \mathrm{length}(C) < +\infty.$$

Therefore, we obtain that

$$\liminf_{t\to-\infty} d_{\Omega_t}(h(z), h(w)) \ge d_{h(\Delta)}(h(z), h(w)),$$

which combined with Eq. 4.5 and the conformal invariance of the hyperbolic metric, gives us the desired result. □

*Proof of Theorem 1.4.* Fix $t, s \le 0$. The hyperbolic $n$-th diameter

$$d_{n,h}^{\mathbb{D}}(\phi_t(K))^{\frac{n(n-1)}{2}} = \max_{z_1,\dots,z_n\in\phi_t(K)} \prod_{1\le\mu<\nu\le n} \tanh d_{\mathbb{D}}(z_\mu, z_\nu) = \prod_{1\le\mu<\nu\le n} \tanh d_{\mathbb{D}}(y_\mu, y_\nu),$$

where $y_1, \dots, y_n$ is a Fekete $n$-tuple of $\phi_t(K)$. The hyperbolic $n$-th diameter of $\phi_{t+s}(K)$ is equal to

$$d_{n,h}^{\mathbb{D}}(\phi_{t+s}(K))^{\frac{n(n-1)}{2}} = \max_{z_1,\dots,z_n\in\phi_t(K)} \prod_{1\le\mu<\nu\le n} \tanh d_{\mathbb{D}}(\phi_s(z_\mu), \phi_s(z_\nu)).$$

Using Schwarz-Pick lemma, we get

$$\begin{aligned} d^{\mathbb{D}}_{n,h}(\phi_t(K))^{\frac{n(n-1)}{2}} &= \prod_{1\le\mu<\nu\le n} \tanh d_{\mathbb{D}}(y_\mu, y_\nu) \\ &\le \prod_{1\le\mu<\nu\le n} \tanh d_{\mathbb{D}}(\phi_s(y_\mu), \phi_s(y_\nu)) \\ &\le d^{\mathbb{D}}_{n,h}(\phi_{t+s}(K))^{\frac{n(n-1)}{2}}, \end{aligned}$$

for every choice of $t$. Therefore, $d^{\mathbb{D}}_{n,h}(\phi_t(K))$ is a decreasing function of $t \le 0$. □

*Proof of Theorem 1.5.* The hyperbolic area of $\phi_t(K)$ in $\mathbb{D}$ is

$$\mathrm{A}^{\mathbb{D}}_h(\phi_t(K)) = \int_{\phi_t(K)} \lambda_{\mathbb{D}}(z)^2 dA(z) = \int_K \lambda_{\mathbb{D}}(\phi_t(z))^2 |\phi_t'(z)|^2 dA(z),$$

where $A$ is the Lebesgue area measure. Due to the uniform convergence of the hyperbolic metric on compacta in Theorem 1.3, we obtain

$$\lim_{t\to-\infty} \mathrm{A}^{\mathbb{D}}_h(\phi_t(K)) = \lim_{t\to-\infty} \int_K \lambda_{\mathbb{D}}(\phi_t(z))^2 |\phi_t'(z)|^2 dA(z) = \int_K \lambda_\Delta(z)^2 dA(z) = \mathrm{A}^{\Delta}_h(K).$$

Concerning the hyperbolic $n$-th diameter, suppose that $\phi_t(z_1)$, ..., $\phi_t(z_n)$ is a Fekete $n$-tuple of the compact set $\phi_t(K)$ in $\mathbb{D}$. Then

$$d^{\mathbb{D}}_{n,h}(\phi_t(K))^{\frac{n(n-1)}{2}} = \prod_{1\le\mu<\nu\le n} \tanh d_{\mathbb{D}}(\phi_t(z_\mu), \phi_t(z_\nu)). \tag{4.7}$$

From Theorem 1.4, the limit of the hyperbolic $n$-th diameter, as $t \to -\infty$, exists and we have

$$\begin{aligned} \lim_{t\to-\infty} d^{\mathbb{D}}_{n,h}(\phi_t(K))^{\frac{n(n-1)}{2}} &= \prod_{1\le\mu<\nu\le n} \lim_{t\to-\infty} \tanh d_{\mathbb{D}}(\phi_t(z_\mu), \phi_t(z_\nu)) \\ &= \prod_{1\le\mu<\nu\le n} \tanh d_\Delta(z_\mu, z_\nu) \\ &\le d^{\Delta}_{n,h}(K)^{\frac{n(n-1)}{2}}, \end{aligned} \tag{4.8}$$

where the second equality holds due to Theorem 1.3. Furthermore, suppose now that $y_1$, ..., $y_n$ is a Fekete $n$-tuple of $K$ in the petal $\Delta$. Then

$$\begin{aligned} d^{\Delta}_{n,h}(K)^{\frac{n(n-1)}{2}} &= \prod_{1\le\mu<\nu\le n} \tanh d_\Delta(y_\mu, y_\nu) \\ &= \prod_{1\le\mu<\nu\le n} \tanh\left(\lim_{t\to-\infty} d_{\mathbb{D}}(\phi_t(y_\mu), \phi_t(y_\nu))\right) \\ &= \lim_{t\to-\infty} \prod_{1\le\mu<\nu\le n} \tanh d_{\mathbb{D}}(\phi_t(y_\mu), \phi_t(y_\nu)) \\ &\le \lim_{t\to-\infty} d^{\mathbb{D}}_{n,h}(\phi_t(K))^{\frac{n(n-1)}{2}}, \end{aligned} \tag{4.9}$$

where the second equality holds due to Theorem 1.3. Combining Eqs. 4.8 and 4.9, we get

$$\lim_{t\to-\infty} d^{\mathbb{D}}_{n,h}(\phi_t(K)) = d^{\Delta}_{n,h}(K).$$

□

## 5 Condenser Capacity—Proof of Theorem 1.6

Before the proof of Theorem 1.6, we state the following Lemma concerning the convergence of the Green function.

**Lemma 5.1** *Suppose $(\phi_t)$ is a semigroup of holomorphic self-maps of $\mathbb{D}$, which is not an elliptic group. Let $\Delta$ be a petal of $(\phi_t)$ and $z \in \Delta$. The Green function $g_{\phi_t(\mathbb{D})}(\cdot, z)$ converges uniformly to 0 on $\partial\Delta$, as $t \to +\infty$.*

*Proof* Recall that $(\phi_t(\mathbb{D}))_{t\geq 0}$ is a decreasing family of domains. Hence $g_{\phi_t(\mathbb{D})}(\cdot, z)$ is pointwise decreasing on $\partial\mathbb{D}$, due to the Subordination Principle of the Green function; see [21, Theorem 4.4.4]. For $\zeta \in \partial\Delta \cap \mathbb{T}$, $\zeta \in \partial\phi_t(\mathbb{D})$, for all $t \geq 0$. Hence $g_{\phi_t(\mathbb{D})}(\zeta, z) = 0$, since the Green function vanishes on the boundary. For $t \leq 0$, we denote $\Omega_t$ as in the proof of Lemma 4.3. Using the connection between the hyperbolic distance and the Green function in a simply connected domain Eq. 2.10, we obtain

$$\operatorname{arctanh} e^{-g_{\Omega_t}(h(\zeta),h(z))} = d_{\Omega_t}(h(\zeta), h(z)) \xrightarrow{t\to-\infty} +\infty \tag{5.1}$$

due to Lemma 4.3. Hence $g_{\Omega_t}(\cdot, h(z))$ converges pointwise to 0 on $\partial\Delta$, as $t \to -\infty$, or equivalently $g_{\phi_t(\mathbb{D})}(\cdot, z)$ converges pointwise to 0 on $\partial\Delta$, as $t \to +\infty$. However, the Green function is continuous and $\partial\Delta$ is compact. From Dini's Theorem, we obtain that the convergence on $\partial\Delta$ is uniform. □

*Proof of Theorem 1.6.* As described in Section 2.3, the pairs $(\mathbb{D}, \phi_t(K))$ and $(\Delta, K)$ form condensers in $\mathbb{C}$, for all $t \leq 0$. Due to inclusion, $\operatorname{cap}(\mathbb{D}, \phi_t(K)) \leq \operatorname{cap}(\Delta, K)$, for all $t \leq 0$, and thus,

$$\limsup_{t\to+\infty} \operatorname{cap}(\phi_t(\mathbb{D}), K) \leq \limsup_{t\to-\infty} \operatorname{cap}(\mathbb{D}, \phi_t(K)) \leq \operatorname{cap}(\Delta, K), \tag{5.2}$$

where the first inequality holds due to Eq. 2.9. In fact it will be proved that equality holds in Eq. 5.2. In order to do so, we work with the condenser $(\phi_t(\mathbb{D}), K)$, where $t \geq 0$. Fix $z, w \in K$. Since $\Delta \subset \phi_t(\mathbb{D})$, from the Strong Markov Property for the Green function (Proposition 2.2), we have

$$g_{\phi_t(\mathbb{D})}(z, w) = g_\Delta(z, w) + \int\limits_{\partial\Delta\cap\mathbb{D}} g_{\phi_t(\mathbb{D})}(\zeta, z) \cdot \omega(w, d\zeta, \Delta). \tag{5.3}$$

According to Lemma 5.1, for every $\epsilon > 0$, there exists a $t_0 \geq 0$, such that for every $t \geq t_0$, $g_{\phi_t(\mathbb{D})}(\zeta, z) < \epsilon$, for all $\zeta \in \partial\Delta$. Hence Eq. 5.3 may be written as

$$\begin{aligned} g_{\phi_t(\mathbb{D})}(z, w) &< g_\Delta(z, w) + \int\limits_{\partial\Delta\cap\mathbb{D}} \epsilon \cdot \omega(w, d\zeta, \Delta) \\ &\leq g_\Delta(z, w) + \epsilon \cdot \omega(w, \partial\Delta, \Delta) \\ &= g_\Delta(z, w) + \epsilon. \end{aligned} \tag{5.4}$$

The set $K$ is compact, so, it has a unique Green equilibrium measure in each domain. Suppose $\mu_t$ is the Green equilibrium measure of $K$ in $\phi_t(\mathbb{D})$ and $\mu$ the Green equilibrium measure of $K$ in $\Delta$. Integrating in Eq. 5.4 with respect to $\mu$, we obtain

$$\begin{aligned}\iint_{K^2} g_\Delta(z,w)d\mu(z)d\mu(w)+\epsilon\cdot\mu(K)^2 &> \iint_{K^2} g_{\phi_t(\mathbb{D})}(z,w)d\mu(z)d\mu(w)\\ &> \iint_{K^2} g_{\phi_t(\mathbb{D})}(z,w)d\mu_t(z)d\mu_t(w),\end{aligned}$$

according to the properties of Green equilibrium measures. Bearing also in mind that $\mu(K)=1$, it follows

$$V(K,\phi_t(\mathbb{D}))<V(K,\Delta)+\epsilon. \tag{5.5}$$

The inequality Eq. 5.5 leads to

$$\limsup_{t\to+\infty} V(K,\phi_t(\mathbb{D}))\le V(K,\Delta)\Rightarrow \limsup_{t\to+\infty}\frac{2\pi}{\operatorname{cap}(\phi_t(\mathbb{D}),K)}\le\frac{2\pi}{\operatorname{cap}(\Delta,K)}.$$

Since $K$ is non-polar, $\operatorname{cap}(\phi_t(\mathbb{D}),K)>0$ and we conclude that

$$\liminf_{t\to+\infty}\operatorname{cap}(\phi_t(\mathbb{D}),K)\ge\operatorname{cap}(\Delta,K). \tag{5.6}$$

Combining Eq. 5.2 with Eq. 5.6 results to $\operatorname{cap}(\phi_t(\mathbb{D}),K)\xrightarrow{t\to+\infty}\operatorname{cap}(\Delta,K)$, or equivalently,

$$\lim_{t\to-\infty}\operatorname{cap}(\mathbb{D},\phi_t(K))=\operatorname{cap}(\Delta,K).$$

□

*Remark 5.1* From Theorem 1.6 and Eq. 2.8, we can conclude the following about the asymptotic behavior of the hyperbolic capacity:

$$\lim_{t\to-\infty}\operatorname{caph}\phi_t(K)=\operatorname{caph}_\Delta K.$$

## 6 Non-regular Backward Orbits—Proof of Theorem 1.7

As stated in Section 2.1, a non-regular backward orbit $\gamma:[0,+\infty)\to\mathbb{D}$ determines a degenerate petal, whose image under the associated Koenigs function of $(\phi_t)$ is a horizontal line or a spiral. If we denote by $\Delta$ this degenerate petal, we get $h(\Delta)=\{h(\gamma(0))+t:t\in\mathbb{R}\}$ or $h(\Delta)=\{e^{-\mu t}h(\gamma(0)):t\in\mathbb{R}\}$, depending on the type of the petal.

Suppose $K$ is a compact subset of $\Delta$. In order to avoid polar sets, we assume that $K$ is a continuum on $\Delta$. Hence $h(K)$ is either a line segment or a spiral arc. If $K$ has a different form, the proof is analogous. In Fig. 4, we see all the possible cases when $h(K)$ lies on the image of a degenerate petal.

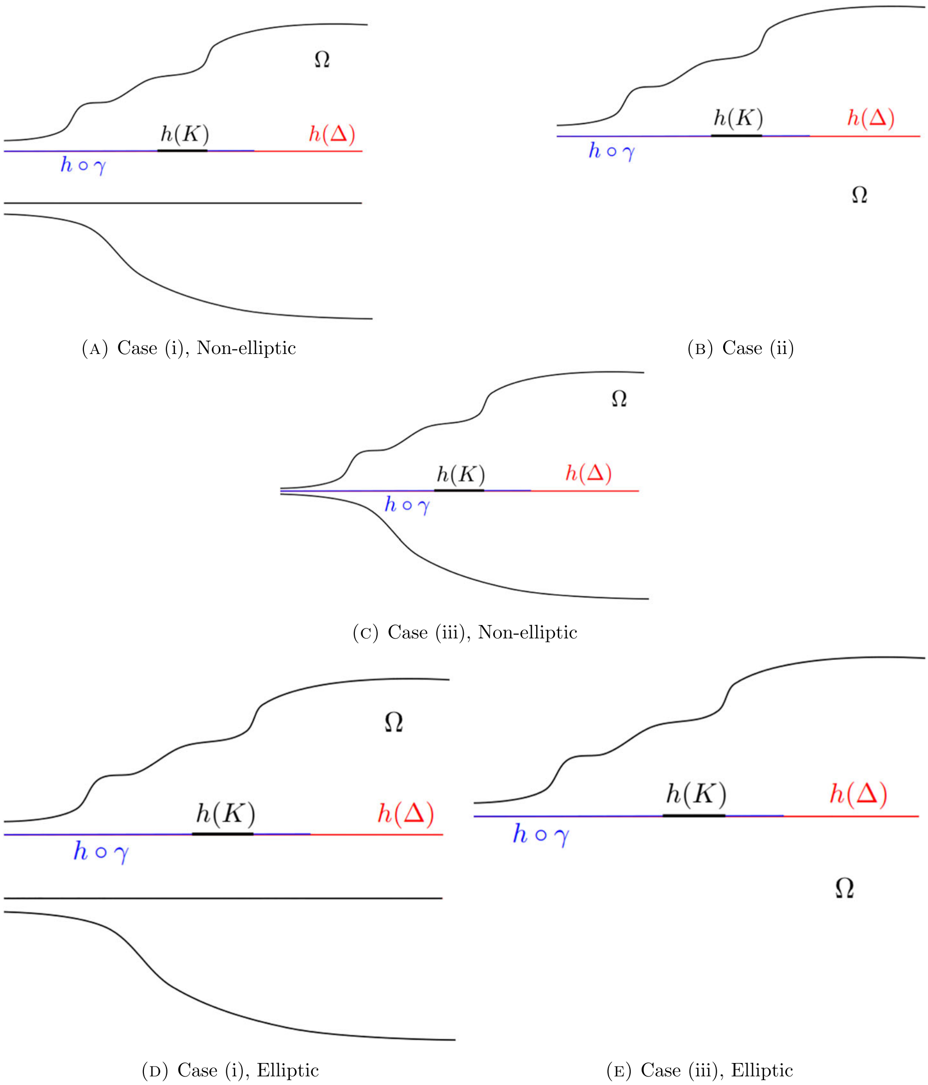
An important property that all the cases of non-regular backward orbits possess is the fact that

$$\lim_{t\to+\infty}\delta_\Omega(h(\gamma(t)))=0; \tag{6.1}$$

thanks to Lemma 4.2.

*Proof of Theorem 1.7.* As in the proof of Lemma 4.3, for $t\le 0$, we let

$$\Omega_t:=h(\phi_{-t}(\mathbb{D}))=\begin{cases}\Omega-t, & (\phi_t)\text{ is non-elliptic}\\ e^{\mu t}\Omega, & (\phi_t)\text{ is elliptic.}\end{cases}$$




(A) Case (i), Non-elliptic (B) Case (ii)

(C) Case (iii), Non-elliptic

(D) Case (i), Elliptic (E) Case (iii), Elliptic

**Fig. 4** Non-regular backward orbit of $K$

Suppose $K$ is a compact non-polar subset of $\Delta$. Then,

(i) Let $z \in \Delta \setminus K$. Then, for $t \leq 0$, by the conformal invariance of the harmonic measure, we have

$$\begin{aligned}\omega(\phi_t(z), \phi_t(K), \mathbb{D}) &= \omega(h(\phi_t(z)), h(\phi_t(K)), \Omega) \\ &= \omega(h(z), h(K), \Omega_t).\end{aligned}$$

From Lemma 4.2, $\lim\limits_{t \to -\infty} \delta_{\Omega_t}(h(z)) = 0$, while the distance between $h(z)$ and $h(K)$ remains constant, regardless of $t$. Hence, the probability that a Brownian motion starting from $h(z)$ hits $\partial h(K)$ before the boundary of $\Omega_t$, for small values of $t$, is negligible. In conclusion, $\omega(h(z), h(K), \Omega_t) \xrightarrow{t\to-\infty} 0$.

(ii) Let $z \in \Delta$. For $t \leq 0$, we find that

$$\lambda_{\Omega_t}(h(z)) \geq \frac{1}{4\delta_{\Omega_t}(h(z))} \xrightarrow{t\to-\infty} +\infty,$$

due to Lemma 4.2 and Eq. 2.6. Consequently, by conformal invariance

$$\lim_{t\to-\infty} \lambda_{\mathbb{D}}(\phi_t(z))|\phi_t'(z)| = +\infty.$$

(iii) Let $z, w \in \Delta$ with $z \neq w$. With the help of the conformal invariance of the hyperbolic distance and Eq. 2.5, we have for $t \leq 0$,

$$\begin{aligned} d_{\mathbb{D}}(\phi_t(z), \phi_t(w)) &= d_{\Omega}(h(\phi_t(z)), h(\phi_t(w))) \\ &= d_{\Omega_t}(h(z), h(w)) \\ &\geq \frac{1}{4}\log\left(1 + \frac{|h(z) - h(w)|}{\min\{\delta_{\Omega_t}(h(z)), \delta_{\Omega_t}(h(w))\}}\right). \end{aligned}$$

The last inequality clearly implies that

$$\lim_{t\to-\infty} d_{\mathbb{D}}(\phi_t(z), \phi_t(w)) = +\infty.$$

(iv) Keeping in mind that for any $z, w \in \mathbb{D}$, it is true that

$$g_{\mathbb{D}}(z, w) = -\log\tanh d_{\mathbb{D}}(z, w),$$

the desired result is a direct corollary of (iii).

(v) Once again, by conformal invariance,

$$A_h^{\mathbb{D}}(\phi_t(K)) = A_h^{\Omega}(h(\phi_t(K)) = A_h^{\Omega_t}(h(K)),$$

for all $t \leq 0$. As we mentioned before, the compact set $h(K)$ can be considered as a line segment or a spiral arc. Therefore, trivially, its hyperbolic area with respect to $\Omega_t$ is zero, for all $t \leq 0$. As a result, we directly get $\lim_{t\to-\infty} A_h^{\mathbb{D}}(\phi_t(K)) = 0$.

(vi) Let $n \in \mathbb{N}$. For $t \leq 0$, let $\phi_t(z_1), \phi_t(z_2), ..., \phi_t(z_n)$ be a Fekete n-tuple for $\phi_t(K)$. Then,

$$d_{n,h}^{\mathbb{D}}(\phi_t(K)) = \prod_{1\leq\mu<\nu\leq n} \tanh d_{\mathbb{D}}(\phi_t(z_\mu), \phi_t(z_\nu))^{\frac{2}{n(n-1)}}.$$

Therefore, using (iii), we get the desired result.

(vii) Suppose $\mu_t$ is the equilibrium measure of $h(K)$ in $\Omega_t$ and $\mu$ the equilibrium measure of $h(K)$ in $\Omega$. According to Eq. 2.9,

$$\begin{aligned} \frac{1}{\mathrm{cap}(\Omega, h(\phi_t(K))} &= \iint g_{\Omega_t}(h(z), h(w))d\mu_t(h(z))d\mu_t(h(w)) \\ &\leq \iint g_{\Omega_t}(h(z), h(w))d\mu(h(z))d\mu(h(w)). \end{aligned}$$

The family of domains $(\Omega_t)_{t\leq 0}$ lies in $\Omega$ and according to the Subordination Principle of the Green function [21, §4.4],

$$g_{\Omega_t}(h(z), h(w)) \leq g_{\Omega}(h(z), h(w)),$$

where $g_{\Omega}(h(z), h(w))$ is integrable, as

$$\iint g_{\Omega}(h(z), h(w))d\mu(h(z))d\mu(h(w)) = V(h(K), \Omega) < +\infty.$$

As a result, we can apply the reverse inequality in Fatou's Lemma and taking lim sup, as $t \to -\infty$, we obtain

$$\begin{aligned}\limsup_{t\to-\infty} \frac{1}{\operatorname{cap}(\Omega, h(\phi_t(K))} &\leq \limsup_{t\to-\infty} \iint g_{\Omega_t}(h(z), h(w))d\mu(h(z))d\mu(h(w)) \\ &\leq \iint \limsup_{t\to-\infty} g_{\Omega_t}(h(z), h(w))d\mu(h(z))d\mu(h(w)) \\ &\underset{(iv)}{=} 0.\end{aligned}$$

Since the capacity is always non-negative and conformally invariant, we obtain that

$$\limsup_{t\to-\infty} \frac{1}{\operatorname{cap}(\mathbb{D}, \phi_t(K))} = 0 \Rightarrow \liminf_{t\to-\infty} \operatorname{cap}(\mathbb{D}, \phi_t(K)) = +\infty,$$

which provides us with the desired result.

□

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## Declarations

**Conflict of Interest** The authors declare that there is no conflict of interest. All data generated or analysed during this study are included in this published article (and its supplementary information files).



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