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Anisotropic Fractional Gagliardo-Nirenberg, Weighted Caffarelli-Kohn-Nirenberg and Lyapunov-type Inequalities, and Applications to Riesz Potentials and *p*-sub-Laplacian Systems

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Abstract

In this paper we prove the fractional Gagliardo-Nirenberg inequality on homogeneous Lie groups. Also, we establish weighted fractional Caffarelli-Kohn-Nirenberg inequality and Lyapunov-type inequality for the Riesz potential on homogeneous Lie groups. The obtained Lyapunov inequality for the Riesz potential is new already in the classical setting of $\mathbb{R}^N$. As an application, we give two-sided estimate for the first eigenvalue of the Riesz potential. Also, we obtain Lyapunov inequality for the system of the fractional p-sub-Laplacian equations and give an application to estimate its eigenvalues.

Keywords Fractional Gagliardo-Nirenberg inequality · Fractional Caffarelli-Kohn-Nirenberg inequality · Fractional Lyapunov-type inequality · Homogeneous Lie group

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1 Introduction

1.1 Fractional Gagliardo-Nirenberg Inequality

In the works of E. Gagliardo [9] and L. Nirenberg [14] (independently), they obtained the following (interpolation) inequality

$$\|u\|^{p}_{L^{p}(\mathbb{R}^N)} \leq C\|\nabla u\|^{N(p-2)/2}_{L^2(\mathbb{R}^N)}\|u\|^{(2p-N(p-2))/2}_{L^2(\mathbb{R}^N)}, \ u \in H^1(\mathbb{R}^N), \tag{1.1}$$

where

$$\begin{cases} 2 \leq p \leq \infty \text{ for } N = 2, \\ 2 \leq p \leq \frac{2N}{N-2} \text{ for } N > 2. \end{cases}$$

The Gagliardo-Nirenberg inequality on the Heisenberg group $\mathbb{H}^n$ has the following form

$$\|u\|^{p}_{L^{p}(\mathbb{H}^n)} \leq C\|\nabla_{\mathbb{H}^n} u\|^{Q(p-2)/2}_{L^2(\mathbb{H}^n)}\|u\|^{(2p-Q(p-2))/2}_{L^2(\mathbb{H}^n)}, \tag{1.2}$$

where $\nabla_{\mathbb{H}}$ is a horizontal gradient and Q is a homogeneous dimension of $\mathbb{H}^n$. In [3], the authors established the best constant for the sub-elliptic Gagliardo-Nirenberg inequality Eq. 1.2. Consequently, in [20] the best constants in Gagliardo-Nirenberg and Sobolev inequalities were also found for general hypoelliptic (Rockland operators) on general graded Lie groups.

In [15] the authors obtained a fractional version of the Gagliardo-Nirenberg inequality in the following form:

$$\|u\|_{L^{\tau}(\mathbb{R}^N)} \leq C[u]^{a}_{s,p}\|u\|^{1-a}_{L^{\alpha}(\mathbb{R}^N)}, \ \forall u \in C^1_c(\mathbb{R}^N), \tag{1.3}$$

where $[u]_{s,p}$ is Gagliardo's seminorm defined by

$$[u]^{p}_{s,p} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy,$$

for $N \geq 1, \ s \in (0,1), \ p > 1, \ \alpha \geq 1, \ \tau > 0$, and $a \in (0,1]$ is such that

$$\frac{1}{\tau} = a\left(\frac{1}{p} - \frac{s}{N}\right) + \frac{1-a}{\alpha}.$$

In this paper we formulate the fractional Gagliardo-Nirenberg inequality on the homogeneous Lie groups. To the best of our knowledge, in this direction systematic studies on the homogeneous Lie groups started by the paper [18] in which homogeneous group versions of Hardy and Rellich inequalities were proved as consequences of universal identities.

1.2 Fractional Caffarelli-Kohn-Nirenberg Inequality

In their fundamental work [2], L. Caffarelli, R. Kohn and L. Nirenberg established:

Theorem 1.1 *Let $N \geq 1$, and let $l_1, l_2, l_3, a, b, d, \delta \in \mathbb{R}$ be such that $l_1, l_2 \geq 1$, $l_3 > 0$, $0 \leq \delta \leq 1$, and*

$$\frac{1}{l_1} + \frac{a}{N}, \ \ \frac{1}{l_2} + \frac{b}{N}, \ \ \frac{1}{l_3} + \frac{\delta d + (1-\delta)b}{N} > 0. \tag{1.4}$$

Then,

$$\||x|^{\delta d+(1-\delta)b}u\|_{L^{l_3}(\mathbb{R}^N)} \leq C\||x|^{a}\nabla u\|^{\delta}_{L^{l_1}(\mathbb{R}^N)}\||x|^{b}u\|^{1-\delta}_{L^{l_2}(\mathbb{R}^N)}, \ \ u \in C^{\infty}_c(\mathbb{R}^N), \tag{1.5}$$

if and only if

$$\frac{1}{l_3}+\frac{\delta d+(1-\delta)b}{N}=\delta\left(\frac{1}{l_1}+\frac{a-1}{N}\right)+(1-\delta)\left(\frac{1}{l_2}+\frac{b}{N}\right),$$
$$a-d\geq 0,\quad \text{if } \delta>0,$$
$$a-d\leq 1,\quad \text{if } \delta>0 \text{ and } \frac{1}{l_3}+\frac{\delta d+(1-\delta)b}{N}=\frac{1}{l_1}+\frac{a-1}{N}, \tag{1.6}$$

where C is a positive constant independent of u.

In [15] the authors proved the fractional analogues of the Caffarelli-Kohn-Nirenberg inequality in weighted fractional Sobolev spaces. Also, in [1] a fractional Caffarelli-Kohn-Nirenberg inequality for an admissible weight in $\mathbb{R}^N$ was obtained.

Recently many different versions of Caffarelli-Kohn-Nirenberg inequalities have been obtained, namely, in [24] on the Heisenberg groups, in [22] and [23] on stratified groups, in [19] and [21] on (general) homogeneous Lie groups. One of the aims of this paper is to prove the fractional weighted Caffarelli-Kohn-Nirenberg inequality on the homogeneous Lie groups.

1.3 Fractional Lyapunov-type Inequality

Historically, in Lyapunov's work [13] for the following one-dimensional homogeneous Dirichlet boundary value problem (for the second order ODE)

$$\begin{cases} u''(x)+\omega(x)u(x)=0, & x\in(a,b),\\ u(a)=u(b)=0, \end{cases} \tag{1.7}$$

it was proved that if u is a non-trivial solution of Eq. 1.7 and $\omega(x)$ is a real-valued and continuous function on $[a,b]$, then necessarily

$$\int_a^b |\omega(x)|dx>\frac{4}{b-a}. \tag{1.8}$$

Nowadays, there are many extensions of Lyapunov's inequality. In [5] the author obtains Lyapunov's inequality for the one-dimensional Dirichlet p-Laplacian

$$\begin{cases} (|u'(x)|^{p-2}u'(x))'+\omega(x)u(x)=0, & x\in(a,b), \quad 1<p<\infty,\\ u(a)=u(b)=0, \end{cases} \tag{1.9}$$

where $\omega(x)\in L^1(a,b)$, so necessarily

$$\int_a^b |\omega(x)|dx>\frac{2^p}{(b-a)^{p-1}},\quad 1<p<\infty. \tag{1.10}$$

Obviously, taking $p=2$ in Eq. 1.10, we recover the classical Lyapunov inequality Eq. 1.8.

In [10] the authors obtained interesting results concerning Lyapunov inequalities for the multi-dimesional fractional p-Laplacian $(-\Delta_p)^s$, $1<p<\infty$, $s\in(0,1)$, with a homogeneous Dirichlet boundary condition, that is,

$$\begin{cases} (-\Delta_p)^s u=\omega(x)|u|^{p-2}u, & x\in\Omega,\\ u(x)=0, & x\in\mathbb{R}^N\setminus\Omega, \end{cases} \tag{1.11}$$

where $\Omega\subset\mathbb{R}^N$ is an open set, $1<p<\infty$, and $s\in(0,1)$. Let us recall the following result of [10].

Theorem 1.2 *Let $\Omega \subset \mathbb{R}^N$ be an open set, and let $\omega \in L^\theta(\Omega)$ with $1 < \frac{N}{sp} < \theta < \infty$, be a non-negative weight. Suppose that problem* Eq. 1.11 *has a non-trivial weak solution $u \in W_0^{s,p}(\Omega)$. Then*

$$\left(\int_\Omega \omega^\theta(x)\,dx\right)^{\frac{1}{\theta}} > \frac{C}{r_\Omega^{sp-\frac{N}{\theta}}}, \tag{1.12}$$

where $C > 0$ is a universal constant and r_Ω is the inner radius of Ω.

In [4], the authors considered a system of ODE for p and q-Laplacian on the interval (a, b) with the homogeneous Dirichlet condition in the following form:

$$\begin{cases} -(|u'(x)|^{p-2}u'(x))' = f(x)|u(x)|^{\alpha-2}u(x)|v(x)|^\beta, \\ -(|v'(x)|^{q-2}v'(x))' = g(x)|u(x)|^\alpha|v(x)|^{\beta-2}v(x), \end{cases} \tag{1.13}$$

on the interval (a, b), with

$$u(a) = u(b) = v(a) = v(b) = 0, \tag{1.14}$$

where $f, g \in L^1(a, b),\ f, g \geq 0,\ p, q > 1,\ \alpha,\ \beta \geq 0$ and

$$\frac{\alpha}{p} + \frac{\beta}{q} = 1.$$

Then we have Lyapunov-type inequality for system Eq. 1.13 with homogeneous Dirichlet condition Eq. 1.14:

$$2^{\alpha+\beta} \leq (b-a)^{\frac{\alpha}{p'}+\frac{\beta}{q'}} \left(\int_a^b f(x)dx\right)^{\frac{\alpha}{p}} \left(\int_a^b g(x)dx\right)^{\frac{\beta}{q}}, \tag{1.15}$$

where $p' = \frac{p}{p-1}$ and $q' = \frac{q}{q-1}$. In [11], the authors obtained the Lyapunov-type inequality for a fractional p-Laplacian system in an open bounded subset $\Omega \subset \mathbb{R}^N$ with homogeneous Dirichlet conditions. One of our goals in this paper is to extend the Lyapunov-type inequality for the Riesz potential and for the fractional p-sub-Laplacian system on the homogeneous Lie groups. These results are given in Theorem 5.1 and 5.7. Also, we give applications of the Lyapunov-type inequality for the Riesz potential and for fractional p-sub-Laplacian system on the homogeneous Lie groups. To demonstrate our techniques we consider the Riesz potential in the Abelian case $(\mathbb{R}^N, +)$ and give two side estimates of the first eigenvalue of the Riesz potential in the Abelian case $(\mathbb{R}^N, +)$.

Summarising our main results of the present paper, we prove the following facts:

- An analogue of the fractional Gagliardo-Nirenberg inequality on the homogeneous group $\mathbb{G}$;
- An analogue of the fractional weighted Caffarelli-Kohn-Nirenberg inequality on $\mathbb{G}$;
- An analogue of the Lyapunov-type inequality for the Riesz potential on $\mathbb{G}$;
- An analogue of the Lyapunov-type inequality for the fractional p-sub-Laplacian system on $\mathbb{G}$.

The paper is organised as follows. First we give some basic discussions on fractional Sobolev spaces and related facts on homogeneous Lie groups, then in Section 3 we present the fractional Gagliardo-Nirenberg inequality on $\mathbb{G}$. The fractional weighted Caffarelli-Kohn-Nirenberg inequality on $\mathbb{G}$ is proved in Section 4. In Section 5 we discuss analogues of the Lyapunov-type inequalities for the Riesz potential and fractional p-sub-Laplacian system on $\mathbb{G}$.

2 Preliminaries

We recall that a Lie group (on $\mathbb{R}^n$) $\mathbb{G}$ with the dilation

$$D_\lambda(x) := (\lambda^{\nu_1}x_1, \ldots, \lambda^{\nu_n}x_n),\ \nu_1, \ldots, \nu_n > 0,\ D_\lambda : \mathbb{R}^n \to \mathbb{R}^n,$$

which is an automorphism of the group $\mathbb{G}$ for each $\lambda > 0$, is called a *homogeneous (Lie) group*. In this paper, for simplicity, we use the notation λx instead of the dilation $D_\lambda(x)$. The homogeneous dimension of the homogeneous group $\mathbb{G}$ is denoted by

$$Q := \nu_1 + \ldots + \nu_n.$$

A homogeneous quasi-norm on $\mathbb{G}$ is a continuous non-negative function

$$\mathbb{G} \ni x \mapsto q(x) \in [0, \infty), \tag{2.1}$$

with the properties

i) $q(x) = q(x^{-1})$ for all $x \in \mathbb{G}$,
ii) $q(\lambda x) = \lambda q(x)$ for all $x \in \mathbb{G}$ and $\lambda > 0$,
iii) $q(x) = 0$ iff $x = 0$.

Moreover, the following polarisation formula on homogeneous Lie groups will be used in our proofs: there is a (unique) positive Borel measure σ on the unit quasi-sphere $\omega_Q := \{x \in \mathbb{G} : q(x) = 1\}$, so that for every $f \in L^1(\mathbb{G})$ we have

$$\int_{\mathbb{G}} f(x)dx = \int_0^\infty \int_{\omega_Q} f(ry)r^{Q-1}d\sigma(y)dr. \tag{2.2}$$

We refer to [7] for the original appearance of such groups, and to [6] for a recent comprehensive treatment. Let $p > 1$, $s \in (0, 1)$, and let $\mathbb{G}$ be a homogeneous Lie group of homogeneous dimension Q. For a measurable function $u : \mathbb{G} \to \mathbb{R}$ we define the Gagliardo quasi-seminorm by

$$[u]_{s,p,q} = \left(\int_{\mathbb{G}}\int_{\mathbb{G}} \frac{|u(x) - u(y)|^p}{q^{Q+sp}(y^{-1} \circ x)}dxdy\right)^{1/p}. \tag{2.3}$$

Now we recall the definition of the fractional Sobolev spaces on homogeneous Lie groups denoted by $W^{s,p,q}(\mathbb{G})$. For $p \geq 1$ and $s \in (0, 1)$, the functional space

$$W^{s,p,q}(\mathbb{G}) = \{u \in L^p(\mathbb{G}) : u \text{ is measurable, } [u]_{s,p,q} < +\infty\}, \tag{2.4}$$

is called the fractional Sobolev space on $\mathbb{G}$.

Similarly, if $\Omega \subset \mathbb{G}$ is a Haar measurable set, we define the Sobolev space

$$W^{s,p,q}(\Omega) = \{u \in L^p(\Omega) : u \text{ is measurable,}$$
$$[u]_{s,p,q,\Omega} = \left(\int_\Omega\int_\Omega \frac{|u(x) - u(y)|^p}{q^{Q+sp}(y^{-1} \circ x)}dxdy\right)^{\frac{1}{p}} < +\infty\}. \tag{2.5}$$

Now we recall the definition of the weighted fractional Sobolev space on the homogeneous Lie groups denoted by

$$W^{s,p,\beta,q}(\mathbb{G}) = \{u \in L^p(\mathbb{G}) : u \text{ is measurable,}$$
$$[u]_{s,p,\beta,q} = \left(\int_{\mathbb{G}}\int_{\mathbb{G}} \frac{q^{\beta_1 p}(x)q^{\beta_2 p}(y)|u(x) - u(y)|^p}{q^{Q+sp}(y^{-1} \circ x)}dxdy\right)^{\frac{1}{p}} < +\infty\}, \tag{2.6}$$

where $\beta_1,\ \beta_2 \in \mathbb{R}$ with $\beta = \beta_1 + \beta_2$ and it depends on β_1 and β_2.

As above, for a Haar measurable set $\Omega \subset \mathbb{G}$, $p \geq 1$, $s \in (0,1)$ and $\beta_1, \ \beta_2 \in \mathbb{R}$ with $\beta = \beta_1 + \beta_2$, we define the weighted fractional Sobolev space

$$W^{s,p,\beta,q}(\Omega) = \{u \in L^p(\Omega) : u \text{ is measurable},$$
$$[u]_{s,p,\beta,q,\Omega} = \left(\int_\Omega \int_\Omega \frac{q^{\beta_1 p}(x) q^{\beta_2 p}(y)|u(x)-u(y)|^p}{q^{Q+sp}(y^{-1} \circ x)} dxdy\right)^{\frac{1}{p}} < +\infty\}. \quad (2.7)$$

Obviously, taking $\beta = \beta_1 = \beta_2 = 0$ in Eq. 2.7, we recover Eq. 2.5.

The mean of a function u is defined by

$$u_\Omega = \fint_\Omega u dx = \frac{1}{|\Omega|} \int_\Omega u dx, \quad u \in L^1(\Omega), \quad (2.8)$$

where $|\Omega|$ is the Haar measure of $\Omega \subset \mathbb{G}$.

We will also use the decomposition of $\mathbb{G}$ into quasi-annuli $A_{k,q}$ defined by

$$A_{k,q} := \{x \in \mathbb{G} : \ 2^k \leq q(x) < 2^{k+1}\}, \quad (2.9)$$

where $q(x)$ is a quasi-norm on $\mathbb{G}$.

3 Fractional Gagliargo-Nirenberg Inequality on $\mathbb{G}$

In this section we prove an analogue of the fractional Gagliardo-Nirenberg inequality on the homogeneous Lie groups. To prove Gagliardo-Nirenberg's inequality we need some preliminary results from [12], a version of a fractional Sobolev inequality on the homogeneous Lie groups.

From now on, unless specified otherwise, $\mathbb{G}$ will be a homogeneous group of homogeneous dimension Q.

Theorem 3.1 ([12], Fractional Sobolev inequality) *Let $p > 1$, $s \in (0,1)$, $Q > sp$, and let $q(\cdot)$ be a quasi-norm on $\mathbb{G}$. For any measurable and compactly supported function $u : \mathbb{G} \to \mathbb{R}$ there exists a positive constant $C = C(Q,p,s,q) > 0$ such that*

$$||u||^p_{L^{p^*}(\mathbb{G})} \leq C[u]^p_{s,p,q}, \quad (3.1)$$

where $p^* = p^*(Q,s) = \frac{Qp}{Q-sp}$.

Theorem 3.2 *Assume that $Q \geq 2$, $s \in (0,1)$, $p > 1$, $\alpha \geq 1$, $\tau > 0$, $a \in (0,1]$, $Q > sp$ and*

$$\frac{1}{\tau} = a\left(\frac{1}{p} - \frac{s}{Q}\right) + \frac{1-a}{\alpha}.$$

Then,

$$\|u\|_{L^\tau(\mathbb{G})} \leq C[u]^a_{s,p,q}\|u\|^{1-a}_{L^\alpha(\mathbb{G})}, \ \forall \ u \in C^1_c(\mathbb{G}), \quad (3.2)$$

where $C = C(s,p,Q,a,\alpha) > 0$.

Proof of Theorem 3.2 By using the Hölder inequality, for every $\frac{1}{\tau} = a\left(\frac{1}{p} - \frac{s}{Q}\right) + \frac{1-a}{\alpha}$ we get

$$\|u\|^{\tau}_{L^{\tau}(\mathbb{G})} = \int_{\mathbb{G}} |u|^{\tau} dx = \int_{\mathbb{G}} |u|^{a\tau} |u|^{(1-a)\tau} dx \leq \|u\|^{a\tau}_{L^{p^*}(\mathbb{G})} \|u\|^{(1-a)\tau}_{L^{\alpha}(\mathbb{G})}, \tag{3.3}$$

where $p^* = \frac{Qp}{Q-sp}$. From Eq. 3.3, by using the fractional Sobolev inequality (Theorem 3.1), we obtain

$$\|u\|^{\tau}_{L^{\tau}(\mathbb{G})} \leq \|u\|^{a\tau}_{L^{p^*}(\mathbb{G})} \|u\|^{(1-a)\tau}_{L^{\alpha}(\mathbb{G})} \leq C[u]^{a\tau}_{s,p,q} \|u\|^{(1-a)\tau}_{L^{\alpha}(\mathbb{G})},$$

that is,

$$\|u\|_{L^{\tau}(\mathbb{G})} \leq C[u]^{a}_{s,p,q} \|u\|^{1-a}_{L^{\alpha}(\mathbb{G})}, \tag{3.4}$$

where C is a positive constant independent of u. Theorem 3.2 is proved. □

Remark 3.3 In the Abelian case $(\mathbb{R}^N, +)$ with the standard Euclidean distance instead of the quasi-norm, from Theorem 3.2 we get the fractional Gagliardo-Nirenberg inequality which was proved in [15].

4 Weighted Fractional Caffarelli-Kohn-Nirenberg Inequality on $\mathbb{G}$

In this section we prove the weighted fractional Caffarelli-Kohn-Nirenberg inequality on the homogeneous Lie groups.

Theorem 4.1 *Assume that* $Q \geq 2$, $s \in (0, 1)$, $p > 1$, $\alpha \geq 1$, $\tau > 0$, $a \in (0, 1]$, $\beta_1, \beta_2, \beta, \mu, \gamma \in \mathbb{R}$, $\beta_1 + \beta_2 = \beta$ *and*

$$\frac{1}{\tau} + \frac{\gamma}{Q} = a\left(\frac{1}{p} + \frac{\beta - s}{Q}\right) + (1-a)\left(\frac{1}{\alpha} + \frac{\mu}{Q}\right). \tag{4.1}$$

Assume in addition that, $0 \leq \beta - \sigma$ *with* $\gamma = a\sigma + (1-a)\mu$, *and*

$$\beta - \sigma \leq s \ \text{ only if } \ \frac{1}{\tau} + \frac{\gamma}{Q} = \frac{1}{p} + \frac{\beta - s}{Q}. \tag{4.2}$$

Then for $u \in C^1_c(\mathbb{G})$ *we have*

$$\|q^{\gamma}(x)u\|_{L^{\tau}(\mathbb{G})} \leq C[u]^{a}_{s,p,\beta,q} \|q^{\mu}(x)u\|^{1-a}_{L^{\alpha}(\mathbb{G})}, \tag{4.3}$$

when $\frac{1}{\tau} + \frac{\gamma}{Q} > 0$, *and for* $u \in C^1_c(\mathbb{G} \setminus \{e\})$ *we have*

$$\|q^{\gamma}(x)u\|_{L^{\tau}(\mathbb{G})} \leq C[u]^{a}_{s,p,\beta,q} \|q^{\mu}(x)u\|^{1-a}_{L^{\alpha}(\mathbb{G})}, \tag{4.4}$$

when $\frac{1}{\tau} + \frac{\gamma}{Q} < 0$. *Here e is the identity element of* $\mathbb{G}$.

Remark 4.2 In the Abelian case $(\mathbb{R}^N, +)$ with the standard Euclidean distance instead of quasi-norm in Theorem 4.1, we get the (Euclidean) fractional Caffarelli-Kohn-Nirenberg inequality (see, e.g. [15], Theorem 1.1).

To prove the fractional weighted Caffarelli-Kohn-Nirenberg inequality on $\mathbb{G}$ we will use Theorem 3.2 in the proof of the following lemma.

Lemma 4.3 *Assume that $Q \geq 2$, $s \in (0,1)$, $p > 1$, $\alpha \geq 1$, $\tau > 0$, $a \in (0,1]$ and*

$$\frac{1}{\tau} \geq a\left(\frac{1}{p} - \frac{s}{Q}\right) + \frac{1-a}{\alpha}.$$

Let $\lambda > 0$ and $0 < r < R$ and set

$$\Omega = \{x \in \mathbb{G} : \ \lambda r < q(x) < \lambda R\}.$$

Then, for every $u \in C^1(\overline{\Omega})$, we have

$$\left(\fint_{\Omega} |u - u_{\Omega}|^{\tau} dx\right)^{\frac{1}{\tau}} \leq C_{r,R}\lambda^{\frac{a(sp-Q)}{p}} [u]^a_{s,p,q,\Omega} \left(\fint_{\Omega} |u|^{\alpha} dx\right)^{\frac{1-a}{\alpha}}, \tag{4.5}$$

where $C_{r,R}$ is a positive constant independent of u and λ.

Proof of Lemma 4.3 Without loss of generality, we assume that $0 < s' \leq s$ and $\tau' \geq \tau$ are such that

$$\frac{1}{\tau'} = a\left(\frac{1}{p} - \frac{s'}{Q}\right) + \frac{1-a}{\alpha},$$

and $\lambda = 1$, then let Ω_1 be

$$\Omega_1 = \{x \in \mathbb{G} : \ r < q(x) < R\}.$$

By using Theorem 3.2, Jensen's inequality and $[u]_{s',p,q,\Omega} \leq C[u]_{s,p,q,\Omega}$, we get

$$\begin{aligned}
\left(\fint_{\Omega_1} |u - u_{\Omega_1}|^{\tau} dx\right)^{\frac{1}{\tau}} &= \frac{1}{|\Omega_1|^{\frac{1}{\tau}}} \|u - u_{\Omega_1}\|_{\tau} \leq C_{r,R}\|u - u_{\Omega_1}\|_{L^{\tau'}(\Omega_1)} \\
&\leq C_{r,R}[u - u_{\Omega_1}]^a_{s',p,q,\Omega_1} \|u\|^{1-a}_{L^{\alpha}(\Omega_1)} \\
\leq C_{r,R}\left(\int_{\Omega_1}\int_{\Omega_1} \frac{|u(x) - u_{\Omega_1} - u(y) + u_{\Omega_1}|^p}{q^{Q+s'p}(y^{-1}\circ x)} dxdy\right)^{\frac{a}{p}} &\|u\|^{1-a}_{L^{\alpha}(\Omega_1)} \\
\leq C_{r,R}[u]^a_{s,p,q,\Omega_1} \|u\|^{1-a}_{L^{\alpha}(\Omega_1)} &\leq C_{r,R}[u]^a_{s,p,q,\Omega_1} \left(\fint_{\Omega_1} |u|^{\alpha} dx\right)^{\frac{1-a}{\alpha}},
\end{aligned} \tag{4.6}$$

where $C_{r,R} > 0$. Let us set $u(\lambda x)$ instead of $u(x)$, then

$$\begin{aligned}
\left(\fint_{\Omega_1} \left|u(\lambda x) - \fint_{\Omega_1} u(\lambda x) dx\right|^{\tau} dx\right)^{\frac{1}{\tau}} &\leq C_{r,R}\left(\int_{\Omega_1}\int_{\Omega_1} \frac{|u(\lambda x) - u(\lambda y)|^p}{q^{Q+sp}(y^{-1}\circ x)} dxdy\right)^{\frac{a}{p}} \\
&\times \left(\frac{1}{|\Omega_1|}\int_{\Omega_1} |u(\lambda x)|^{\alpha} dx\right)^{\frac{1-a}{\alpha}}.
\end{aligned} \tag{4.7}$$

Thus, we compute

$$\left(\fint_{\Omega}\left|u(x)-\fint_{\Omega}u(x)dx\right|^{\tau}dx\right)^{\frac{1}{\tau}}=\left(\frac{1}{|\Omega|}\int_{\Omega}\left|u(x)-\frac{1}{|\Omega|}\int_{\Omega}u(x)dx\right|^{\tau}dx\right)^{\frac{1}{\tau}}$$
$$=\left(\frac{1}{|\Omega|}\int_{\Omega}\left|u(\lambda y)-\frac{1}{|\Omega|}\int_{\Omega}u(\lambda y)d(\lambda y)\right|^{\tau}d(\lambda y)\right)^{\frac{1}{\tau}}$$
$$=\left(\frac{1}{|\Omega_1|}\int_{\Omega_1}\frac{\lambda^Q}{\lambda^Q}\left|u(\lambda y)-\frac{\lambda^Q}{\lambda^Q|\Omega_1|}\int_{\Omega_1}u(\lambda y)dy\right|^{\tau}dy\right)^{\frac{1}{\tau}}$$
$$=\left(\frac{1}{|\Omega_1|}\int_{\Omega_1}\left|u(\lambda y)-\frac{1}{|\Omega_1|}\int_{\Omega_1}u(\lambda y)dy\right|^{\tau}dy\right)^{\frac{1}{\tau}}$$
$$\leq C_{r,R}\left(\int_{\Omega_1}\int_{\Omega_1}\frac{|u(\lambda x)-u(\lambda y)|^p}{q^{Q+sp}(y^{-1}\circ x)}dxdy\right)^{\frac{a}{p}}\left(\frac{1}{|\Omega_1|}\int_{\Omega_1}|u(\lambda x)|^{\alpha}dx\right)^{\frac{1-a}{\alpha}}$$
$$=C_{r,R}\left(\int_{\Omega_1}\int_{\Omega_1}\frac{\lambda^{2Q}\lambda^{Q+sp}|u(\lambda x)-u(\lambda y)|^p}{\lambda^{2Q}\lambda^{Q+sp}q^{Q+sp}(y^{-1}\circ x)}dxdy\right)^{\frac{a}{p}}\left(\frac{1}{|\Omega_1|}\int_{\Omega_1}\frac{\lambda^Q}{\lambda^Q}|u(\lambda x)|^{\alpha}dx\right)^{\frac{1-a}{\alpha}}$$
$$=C_{r,R}\left(\int_{\Omega}\int_{\Omega}\frac{\lambda^{sp-Q}|u(\lambda x)-u(\lambda y)|^p}{q^{Q+sp}((\lambda y)^{-1}\circ\lambda x)}d(\lambda x)d(\lambda y)\right)^{\frac{a}{p}}\left(\frac{1}{|\Omega|}\int_{\Omega}|u(\lambda x)|^{\alpha}d(\lambda x)\right)^{\frac{1-a}{\alpha}}$$
$$=C_{r,R}\left(\int_{\Omega}\int_{\Omega}\frac{\lambda^{sp-Q}|u(x)-u(y)|^p}{q^{Q+sp}(y^{-1}\circ x)}dxdy\right)^{\frac{a}{p}}\left(\frac{1}{|\Omega|}\int_{\Omega}|u(x)|^{\alpha}dx\right)^{\frac{1-a}{\alpha}}$$
$$=C_{r,R}\lambda^{\frac{a(sp-Q)}{p}}[u]^a_{s,p,q,\Omega}\left(\frac{1}{|\Omega|}\int_{\Omega}|u(x)|^{\alpha}dx\right)^{\frac{1-a}{\alpha}}. \quad (4.8)$$

The proof of Lemma 4.3 is complete. □

Proof of Theorem 4.1 First let us consider the case Eq. 4.2, that is, $\beta-\sigma\leq s$ and $\frac{1}{\tau}+\frac{\gamma}{Q}=\frac{1}{p}+\frac{\beta-s}{Q}$. By using Lemma 4.3 with $\lambda=2^k$, $r=1$, $R=2$ and $\Omega=A_{k,q}$, we get

$$\left(\fint_{A_{k,q}}|u-u_{A_{k,q}}|^{\tau}dx\right)^{\frac{1}{\tau}}\leq C2^{\frac{ak(sp-Q)}{p}}[u]^a_{s,p,q,A_{k,q}}\left(\fint_{A_{k,q}}|u|^{\alpha}dx\right)^{\frac{1-a}{\alpha}}, \quad (4.9)$$

where $A_{k,q}$ is defined in Eq. 2.9 and $k\in\mathbb{Z}$. Now by using Eq. 4.9 we obtain

$$\int_{A_{k,q}}|u|^{\tau}dx=\int_{A_{k,q}}|u-u_{A_{k,q}}+u_{A_{k,q}}|^{\tau}dx\leq C\left(\int_{A_{k,q}}|u_{A_{k,q}}|^{\tau}dx+\int_{A_{k,q}}|u-u_{A_{k,q}}|^{\tau}dx\right)$$

$$= C\left(\int_{A_{k,q}} |u_{A_{k,q}}|^\tau dx + \frac{|A_{k,q}|}{|A_{k,q}|}\int_{A_{k,q}} |u-u_{A_{k,q}}|^\tau dx\right)$$
$$= C\left(|A_{k,q}||u_{A_{k,q}}|^\tau + |A_{k,q}|\fint_{A_{k,q}} |u-u_{A_{k,q}}|^\tau dx\right)$$
$$\leq C\left(|A_{k,q}||u_{A_{k,q}}|^\tau + 2^{\frac{ak(sp-Q)\tau}{p}}|A_{k,q}|[u]^{a\tau}_{s,p,q,A_{k,q}}\left(\frac{1}{|A_{k,q}|}\int_{A_{k,q}}|u|^\alpha dx\right)^{\frac{(1-a)\tau}{\alpha}}\right)$$
$$\leq C\left(2^{Qk}|u_{A_{k,q}}|^\tau + 2^{\frac{ak(sp-Q)\tau}{p}}2^{kQ}2^{-\frac{Q(1-a)\tau k}{\alpha}}[u]^{a\tau}_{s,p,q,A_{k,q}}\|u\|^{(1-a)\tau}_{L^\alpha(A_{k,q})}\right). \quad (4.10)$$

Then, from Eq. 4.10 we get

$$\int_{A_{k,q}} q^{\gamma\tau}(x)|u|^\tau dx \leq 2^{(k+1)\gamma\tau}\int_{A_{k,q}}|u|^\tau dx \leq C2^{(Q+\gamma\tau)k}|u_{A_{k,q}}|^\tau$$
$$+ C2^{\gamma\tau k}2^{kQ}2^{\frac{ak(sp-Q)\tau}{p}}2^{-\frac{Q(1-a)\tau k}{\alpha}}[u]^{a\tau}_{s,p,q,A_{k,q}}\|u\|^{(1-a)\tau}_{L^\alpha(A_{k,q})} = C2^{(Q+\gamma\tau)k}|u_{A_{k,q}}|^\tau$$
$$+ C2^{\left(\gamma\tau+Q+\frac{a(sp-Q)\tau}{p}-\frac{Q(1-a)\tau}{\alpha}\right)k}\left(\int_{A_{k,q}}\int_{A_{k,q}}\frac{2^{kp\beta_1}2^{kp\beta_2}|u(x)-u(y)|^p}{2^{kp\beta}q^{Q+sp}(y^{-1}\circ x)}dxdy\right)^{\frac{a\tau}{p}}$$
$$\times\left(\int_{A_{k,q}}\frac{2^{k\alpha\mu}}{2^{k\alpha\mu}}|u(x)|^\alpha dx\right)^{\frac{(1-a)\tau}{\alpha}} \leq C2^{(Q+\gamma\tau)k}|u_{A_{k,q}}|^\tau$$
$$+ C2^{\left(\gamma\tau+Q+\frac{a(sp-Q)\tau}{p}-\frac{Q(1-a)\tau}{\alpha}-a\beta\tau-\mu\tau(1-a)\right)k}\left(\int_{A_{k,q}}\int_{A_{k,q}}\frac{q^{p\beta_1}(x)q^{p\beta_2}(y)|u(x)-u(y)|^p}{q^{Q+sp}(y^{-1}\circ x)}dxdy\right)^{\frac{a\tau}{p}}$$
$$\times\left(\int_{A_{k,q}}q^{\alpha\mu}(x)|u(x)|^\alpha dx\right)^{\frac{(1-a)\tau}{\alpha}} \leq C2^{(Q+\gamma\tau)k}|u_{A_{k,q}}|^\tau$$
$$+ C2^{\left(\gamma\tau+Q+\frac{a(sp-Q)\tau}{p}-\frac{Q(1-a)\tau}{\alpha}-a\beta\tau-\mu\tau(1-a)\right)k}[u]^{a\tau}_{s,p,\beta,q,A_{k,q}}\|q^\mu(x)u\|^{(1-a)\tau}_{L^\alpha(A_{k,q})}. \quad (4.11)$$

Here by Eq. 4.1, we have

$$\gamma\tau + Q + \frac{a(sp-Q)\tau}{p} - \frac{Q(1-a)\tau}{\alpha} - a\beta\tau - \mu\tau(1-a)$$
$$= Q\tau\left(\frac{\gamma}{Q}+\frac{1}{\tau}+\frac{a(sp-Q)}{Qp}-\frac{(1-a)}{\alpha}-\frac{a\beta}{Q}-\frac{\mu(1-a)}{Q}\right)$$
$$= Q\tau\left(a\left(\frac{1}{p}+\frac{\beta-s}{Q}\right)+(1-a)\left(\frac{1}{\alpha}+\frac{\mu}{Q}\right)+\frac{a(sp-Q)}{Qp}-\frac{(1-a)}{\alpha}-\frac{a\beta}{Q}-\frac{\mu(1-a)}{Q}\right)$$
$$= 0. \quad (4.12)$$

Thus, we obtain

$$\int_{A_{k,q}} q^{\gamma\tau}(x)|u|^\tau dx \leq C2^{(\gamma\tau+Q)k}|u_{A_{k,q}}|^\tau + C[u]^{a\tau}_{s,p,\beta,q,A_{k,q}}\|q^\mu(x)u\|^{(1-a)\tau}_{L^\alpha(A_{k,q})}, \quad (4.13)$$

and by summing over k from m to n, we get

$$\int_{\cup_{k=m}^{n} A_{k,q}} q^{\gamma\tau}(x)|u|^{\tau} dx = \int_{\{2^m < q(x) < 2^{n+1}\}} q^{\gamma\tau}(x)|u|^{\tau} dx \le C \sum_{k=m}^{n} 2^{(\gamma\tau+Q)k} |u_{A_{k,q}}|^{\tau} + C \sum_{k=m}^{n} [u]_{s,p,\beta,q,A_{k,q}}^{a\tau} \|q^{\mu}(x)u\|_{L^{\alpha}(A_{k,q})}^{(1-a)\tau}, \quad (4.14)$$

where $k, m, n \in \mathbb{Z}$ and $m \le n-2$.

To prove Eq. 4.3 let us choose n such that

$$\operatorname{supp} u \subset B_{2^n}, \quad (4.15)$$

where B_{2^n} is a quasi-ball of $\mathbb{G}$ with the radius 2^n.

The following known inequality will be used in the proof.

Lemma 4.4 (Lemma 2.2, [16]) *Let $\xi > 1$ and $\eta > 1$. Then exists a positive constant C depending ξ and η such that $1 < \zeta < \xi$,*

$$(|a| + |b|)^{\eta} \le \zeta |a|^{\eta} + \frac{C}{(\zeta - 1)^{\eta-1}} |b|^{\eta}, \quad \forall\, a, b \in \mathbb{R}. \quad (4.16)$$

Let us consider the following integral

$$\fint_{A_{k+1,q}\cup A_{k,q}} \left| u - \fint_{A_{k+1,q}\cup A_{k,q}} u \right|^{\tau} dx = \frac{1}{|A_{k+1,q}| + |A_{k,q}|} \int_{A_{k+1,q}\cup A_{k,q}} \left| u - \fint_{A_{k+1,q}\cup A_{k,q}} u \right|^{\tau} dx$$
$$= \frac{1}{|A_{k+1,q}| + |A_{k,q}|} \left(\int_{A_{k+1,q}} \left| u - \fint_{A_{k+1,q}\cup A_{k,q}} u \right|^{\tau} dx + \int_{A_{k,q}} \left| u - \fint_{A_{k+1,q}\cup A_{k,q}} u \right|^{\tau} dx \right).$$

On the other hand, a direct calculation gives

$$\fint_{A_{k+1,q}\cup A_{k,q}} \left| u - \fint_{A_{k+1,q}\cup A_{k,q}} u \right|^{\tau} dx = \frac{1}{|A_{k+1,q}| + |A_{k,q}|} \left(\int_{A_{k+1,q}} \left| u - \fint_{A_{k+1,q}\cup A_{k,q}} u \right|^{\tau} dx + \int_{A_{k,q}} \left| u - \fint_{A_{k+1,q}\cup A_{k,q}} u \right|^{\tau} dx \right)$$
$$\ge \frac{1}{|A_{k+1,q}| + |A_{k,q}|} \int_{A_{k,q}} \left| u - \fint_{A_{k+1,q}\cup A_{k,q}} u \right|^{\tau} dx$$
$$\ge \frac{1}{|A_{k+1,q}| + |A_{k,q}|} \left| \int_{A_{k,q}} \left(u - \fint_{A_{k+1,q}\cup A_{k,q}} u \right) dx \right|^{\tau}$$

$$= \frac{1}{|A_{k+1,q}|+|A_{k,q}|}\left|\int_{A_{k,q}} udx - \frac{|A_{k,q}|}{|A_{k+1,q}|+|A_{k,q}|}\int_{A_{k,q}} udx - \frac{|A_{k,q}|}{|A_{k+1,q}|+|A_{k,q}|}\int_{A_{k+1,q}} udx\right|^{\tau}$$

$$= \frac{1}{|A_{k+1,q}|+|A_{k,q}|}\left|\frac{|A_{k+1,q}|}{|A_{k+1,q}|+|A_{k,q}|}\int_{A_{k,q}} udx - \frac{|A_{k,q}|}{|A_{k+1,q}|+|A_{k,q}|}\int_{A_{k+1,q}} udx\right|^{\tau}$$

$$= \frac{1}{(|A_{k+1,q}|+|A_{k,q}|)^2}\left||A_{k+1,q}|\int_{A_{k,q}} udx - |A_{k,q}|\int_{A_{k+1,q}} udx\right|^{\tau}$$

$$= \frac{|A_{k+1,q}||A_{k,q}|}{(|A_{k+1,q}|+|A_{k,q}|)^2}\left|\frac{1}{|A_{k,q}|}\int_{A_{k,q}} udx - \frac{1}{|A_{k+1,q}|}\int_{A_{k+1,q}} udx\right|^{\tau}$$

$$= \frac{|A_{k+1,q}||A_{k,q}|}{(|A_{k+1,q}|+|A_{k,q}|)^2}|u_{A_{k+1,q}} - u_{A_{k,q}}|^{\tau} \geq C\frac{2^{Qk}2^{Q(k-1)}}{(2^{Qk}-2^{Q(k-1)})^2}|u_{A_{k+1,q}} - u_{A_{k,q}}|^{\tau}$$

$$\geq C\frac{2^{2Qk}2^{-Q}}{2^{2kQ}(1+2^{-Q})^2}|u_{A_{k+1,q}} - u_{A_{k,q}}|^{\tau} \geq C|u_{A_{k+1,q}} - u_{A_{k,q}}|^{\tau}. \quad (4.17)$$

From Eq. 4.17 and Lemma 4.3, we obtain

$$|u_{A_{k+1,q}} - u_{A_{k,q}}|^{\tau} \leq C\fint_{A_{k+1,q}\cup A_{k,q}}\left|u - \fint_{A_{k+1,q}\cup A_{k,q}} u\right|^{\tau} dx$$

$$\leq C2^{\frac{ak(sp-Q)}{p}}[u]^{\tau a}_{s,p,q,A_{k+1,q}\cup A_{k,q}}\left(\fint_{A_{k+1,q}\cup A_{k,q}}|u|^{\alpha}dx\right)^{\frac{(1-a)\tau}{\alpha}}. \quad (4.18)$$

By using this fact, taking $\tau = 1$ we have

$$|u_{A_{k,q}}| \leq |u_{A_{k+1,q}} - u_{A_{k,q}}| + |u_{A_{k+1,q}}|$$

$$\leq |u_{A_{k+1,q}}| + C2^{\frac{ak(sp-Q)}{p}}[u]^{a}_{s,p,q,A_{k+1,q}\cup A_{k,q}}\left(\fint_{A_{k+1,q}\cup A_{k,q}}|u|^{\alpha}dx\right)^{\frac{(1-a)}{\alpha}}, \quad (4.19)$$

and by using Lemma 4.4 with $\eta = \tau$, $\zeta = 2^{\gamma\tau+Q}c$, where $c = \frac{2}{1+2^{\gamma\tau+Q}} < 1$, since $\gamma\tau + Q > 0$, we have

$$2^{(\gamma\tau+Q)k}|u_{A_{k,q}}|^{\tau} \leq c2^{(k+1)(\gamma\tau+Q)}|u_{A_{k+1,q}}|^{\tau} + C[u]^{\tau a}_{s,p,\beta,q,A_{k+1,q}\cup A_{k,q}}\|q^{\mu}(x)u\|^{(1-a)\tau}_{L^{\alpha}(A_{k+1,q}\cup A_{k,q})}.$$

By summing over k from m to n and by using Eq. 4.15 we have

$$\sum_{k=m}^{n} 2^{(\gamma\tau+Q)k}|u_{A_{k,q}}|^{\tau} \leq \sum_{k=m}^{n} c2^{(k+1)(\gamma\tau+Q)}|u_{A_{k+1,q}}|^{\tau}$$

$$+ C\sum_{k=m}^{n}[u]^{\tau a}_{s,p,\beta,q,A_{k+1,q}\cup A_{k,q}}\|q^{\mu}(x)u\|^{(1-a)\tau}_{L^{\alpha}(A_{k+1,q}\cup A_{k,q})}. \quad (4.20)$$

By using Eq. 4.20, we compute

$$(1-c)\sum_{k=m}^{n}2^{(\gamma\tau+Q)k}|u_{A_{k,q}}|^{\tau}\le 2^{(\gamma\tau+Q)m}|u_{A_{m,q}}|^{\tau}+(1-c)\sum_{k=m+1}^{n}2^{(\gamma\tau+Q)k}|u_{A_{k,q}}|^{\tau}$$

$$\le C\sum_{k=m}^{n}[u]^{\tau a}_{s,p,\beta,q,A_{k+1,q}\cup A_{k,q}}\|q^{\mu}(x)u\|^{(1-a)\tau}_{L^{\alpha}(A_{k+1,q}\cup A_{k,q})}. \quad (4.21)$$

This yields

$$\sum_{k=m}^{n}2^{(\gamma\tau+Q)k}|u_{A_{k,q}}|^{\tau}\le C\sum_{k=m}^{n}[u]^{\tau a}_{s,p,\beta,q,A_{k+1,q}\cup A_{k,q}}\|q^{\mu}(x)u\|^{(1-a)\tau}_{L^{\alpha}(A_{k+1,q}\cup A_{k,q})}. \quad (4.22)$$

From Eqs. 4.14 and 4.22, we have

$$\int_{\{2^m<q(x)<2^{n+1}\}}q^{\gamma\tau}(x)|u|^{\tau}dx\le C\sum_{k=m}^{n}[u]^{\tau a}_{s,p,\beta,q,A_{k+1,q}\cup A_{k,q}}\|q^{\mu}(x)u\|^{(1-a)\tau}_{L^{\alpha}(A_{k+1,q}\cup A_{k,q})}. \quad (4.23)$$

Let $s,t\ge 0$ be such that $s+t\ge 1$. Then for any $x_k, y_k\ge 0$, we have

$$\sum_{k=m}^{n}x_k^s y_k^t\le\left(\sum_{k=m}^{n}x_k\right)^s\left(\sum_{k=m}^{n}y_k\right)^t. \quad (4.24)$$

By using this inequality in Eq. 4.23 with $s=\frac{\tau a}{p}$, $t=\frac{(1-a)\tau}{\alpha}$, $\frac{a}{p}+\frac{1-a}{\alpha}\ge\frac{1}{\tau}$ and $s\ge\beta-\sigma$, we obtain

$$\int_{\{q(x)>2^m\}}q^{\gamma\tau}(x)|u|^{\tau}dx\le C[u]^{a\tau}_{s,p,\beta,q,\cup_{k=m}^{\infty}A_{k,q}}\|q^{\mu}(x)u\|^{(1-a)\tau}_{L^{\alpha}(\cup_{k=m}^{\infty}A_{k,q})}. \quad (4.25)$$

Inequality Eq. 4.3 is proved.

Let us prove Eq. 4.4. The strategy of the proof is similar to the previous case. Choose m such that

$$\operatorname{supp} u\cap B_{2^m}=\emptyset. \quad (4.26)$$

From Lemma 4.3 we have

$$|u_{A_{k+1,q}}-u_{A_{k,q}}|^{\tau}\le C2^{\frac{a\tau k(sp-Q)}{p}}[u]^{\tau a}_{s,p,q,A_{k+1,q}\cup A_{k,q}}\left(\fint_{A_{k+1,q}\cup A_{k,q}}|u|^{\alpha}dx\right)^{\frac{(1-a)\tau}{\alpha}}.$$

By Lemma 4.4 and choosing $c=\frac{1+2^{\gamma\tau+Q}}{2}<1$, since $\gamma\tau+Q<0$, we have

$$2^{(\gamma\tau+Q)(k+1)}|u_{A_{k+1,q}}|^{\tau}\le c2^{k(\gamma\tau+Q)}|u_{A_{k,q}}|^{\tau}+C[u]^{\tau a}_{s,p,\beta,q,A_{k+1,q}\cup A_{k,q}}\|q^{\mu}(x)u\|^{(1-a)\tau}_{L^{\alpha}(A_{k+1,q}\cup A_{k,q})},$$

and by summing over k from m to n and by using Eq. 4.26 we obtain

$$\sum_{k=m}^{n}2^{(\gamma\tau+Q)k}|u_{A_{k,q}}|^{\tau}\le C\sum_{k=m-1}^{n-1}[u]^{\tau a}_{s,p,\beta,q,A_{k+1,q}\cup A_{k,q}}\|q^{\mu}(x)u\|^{(1-a)\tau}_{L^{\alpha}(A_{k+1,q}\cup A_{k,q})}. \quad (4.27)$$

From Eqs. 4.14 and 4.27, we establish that

$$\int_{\{2^m<q(x)<2^{n+1}\}}q^{\gamma\tau}(x)|u|^{\tau}dx\le C\sum_{k=m-1}^{n-1}[u]^{\tau a}_{s,p,\beta,q,A_{k+1,q}\cup A_{k,q}}\|q^{\mu}(x)u\|^{(1-a)\tau}_{L^{\alpha}(A_{k+1,q}\cup A_{k,q})}. \quad (4.28)$$

Now by using Eq. 4.24 we get

$$\int_{\{q(x)<2^{n+1}\}} q^{\gamma\tau}(x)|u|^{\tau}dx \leq C[u]^{\tau a}_{s,p,\beta,q,\cup_{k=-\infty}^{n}A_{k,q}}\|q^{\mu}(x)u\|^{(1-a)\tau}_{L^{\alpha}(\cup_{k=-\infty}^{n}A_{k,q})}. \tag{4.29}$$

The proof of the case $s \geq \beta-\sigma$ is complete.

Let us prove the case of $\beta-\sigma > s$. Without loss of generality, we assume that

$$[u]_{s,p,\beta,q} = \|u\|_{L^{\alpha}(\mathbb{G})} = 1, \tag{4.30}$$

where

$$\frac{1}{p}+\frac{\beta-s}{Q} \neq \frac{1}{\alpha}+\frac{\mu}{Q}.$$

We also assume that $a_1 > 0,\ 1 > a_2$ and $\tau_1,\ \tau_2 > 0$ with

$$\frac{1}{\tau_2} = \frac{a_2}{p}+\frac{1-a_2}{\alpha}, \tag{4.31}$$

and

$$\text{if} \quad \frac{a}{p}+\frac{1-a}{\alpha}-\frac{as}{Q} > 0, \quad \text{then} \ \frac{1}{\tau_1} = \frac{a_1}{p}+\frac{1-a_1}{\alpha}-\frac{a_1 s}{Q},$$

$$\text{if} \quad \frac{a}{p}+\frac{1-a}{\alpha}-\frac{as}{Q} \leq 0, \quad \text{then} \ \frac{1}{\tau} > \frac{1}{\tau_1} \geq \frac{a_1}{p}+\frac{1-a_1}{\alpha}-\frac{a_1 s}{Q}. \tag{4.32}$$

Taking $\gamma_1 = a_1\beta+(1-a_1)\mu$ and $\gamma_2 = a_2(\beta-s)+(1-a_2)\mu$, we obtain

$$\frac{1}{\tau_1}+\frac{\gamma_1}{Q} \geq a_1\left(\frac{1}{p}+\frac{\beta-s}{Q}\right)+(1-a_1)\left(\frac{1}{\alpha}+\frac{\mu}{Q}\right) \tag{4.33}$$

and

$$\frac{1}{\tau_2}+\frac{\gamma_2}{Q} = a_2\left(\frac{1}{p}+\frac{\beta-s}{Q}\right)+(1-a_2)\left(\frac{1}{\alpha}+\frac{\mu}{Q}\right). \tag{4.34}$$

Let a_1 and a_2 be such that

$$|a-a_1| \ \text{and} \ |a-a_2| \ \text{are small enough,} \tag{4.35}$$

$$a_2 < a < a_1, \ \text{if} \ \frac{1}{p}+\frac{\beta-s}{Q} > \frac{1}{\alpha}+\frac{\mu}{Q}, \tag{4.36}$$

$$a_1 < a < a_2, \ \text{if} \ \frac{1}{p}+\frac{\beta-s}{Q} < \frac{1}{\alpha}+\frac{\mu}{Q}. \tag{4.37}$$

By using Eqs. 4.35–4.37 in Eqs. 4.33, 4.34 and 4.1, we establish

$$\frac{1}{\tau_1}+\frac{\gamma_1}{Q} > \frac{1}{\tau}+\frac{\gamma}{Q} > \frac{1}{\tau_2}+\frac{\gamma_2}{Q} > 0. \tag{4.38}$$

From Eq. 4.32 in the case $\frac{a}{p}+\frac{1-a}{\alpha}-\frac{as}{Q} > 0$ with $a > 0$, $\beta-\sigma > s$ and Eq. 4.35, we get

$$\frac{1}{\tau}-\frac{1}{\tau_1} = (a-a_1)\left(\frac{1}{p}-\frac{s}{Q}-\frac{1}{\alpha}\right)+\frac{a}{Q}(\beta-\sigma) > 0, \tag{4.39}$$

and

$$\frac{1}{\tau}-\frac{1}{\tau_2} = (a-a_2)\left(\frac{1}{p}-\frac{1}{\alpha}\right)+\frac{a}{Q}(\beta-\sigma-s) > 0. \tag{4.40}$$

From Eqs. 4.32, 4.39 and 4.40, we have

$$\tau_1 > \tau, \quad \tau_2 > \tau.$$

Thus, using this, Eq. 4.35 and Hölder's inequality, we obtain

$$\|q^{\gamma}(x)u\|_{L^{\tau}(\mathbb{G}\setminus B_1)} \leq C\|q^{\gamma_1}(x)u\|_{L^{\tau_1}(\mathbb{G})}, \tag{4.41}$$

and

$$\|q^{\gamma}(x)u\|_{L^{\tau}(B_1)} \leq C\|q^{\gamma_2}(x)u\|_{L^{\tau_2}(\mathbb{G})}, \tag{4.42}$$

where B_1 is the unit quasi-ball. By using the previous case, we establish

$$\|q^{\gamma_1}(x)u\|_{L^{\tau_1}(\mathbb{G})} \leq C[u]^{a_1}_{s,p,\beta,q}\|q^{\mu}(x)u\|^{1-a_1}_{L^{\alpha}(\mathbb{G})} \leq C, \tag{4.43}$$

and

$$\|q^{\gamma_2}(x)u\|_{L^{\tau_2}(\mathbb{G})} \leq C[u]^{a_2}_{s,p,\beta,q}\|q^{\mu}(x)u\|^{1-a_2}_{L^{\alpha}(\mathbb{G})} \leq C. \tag{4.44}$$

The proof of Theorem 4.1 is complete. □

Remark 4.5 By taking in Eq. 4.4 $a = 1$, $\tau = p$, $\beta_1 = \beta_2 = 0$, and $\gamma = -s$, we get an analogue of the fractional Hardy inequality on homogeneous Lie groups (Theorem 2.9, [12]).

Remark 4.6 In the Abelian case $(\mathbb{R}^N, +)$ with the standard Eucledian distance instead of the quasi-norm and by taking in Eq. 4.4 $a = 1$, $\tau = p$, $\beta_1 = \beta_2 = 0$, and $\gamma = -s$, we get the fractional Hardy inequality (Theorem 1.1, [8]).

Now we consider the critical case $\frac{1}{\tau} + \frac{\gamma}{Q} = 0$.

Theorem 4.7 *Assume that $Q \geq 2$, $s \in (0, 1)$, $p > 1$, $\alpha \geq 1$, $\tau > 1$, $a \in (0, 1]$, $\beta_1, \beta_2, \beta, \mu, \gamma \in \mathbb{R}$, $\beta_1 + \beta_2 = \beta$,*

$$\frac{1}{\tau} + \frac{\gamma}{Q} = a\left(\frac{1}{p} + \frac{\beta - s}{Q}\right) + (1-a)\left(\frac{1}{\alpha} + \frac{\mu}{Q}\right). \tag{4.45}$$

Assume in addition that, $0 \leq \beta - \sigma \leq s$ with $\gamma = a\sigma + (1-a)\mu$.
If $\frac{1}{\tau} + \frac{\gamma}{Q} = 0$ and $\operatorname{supp} u \subset B_R$, then, we have

$$\left\| \frac{q^{\gamma}(x)}{\ln \frac{2R}{q(x)}} u \right\|_{L^{\tau}(\mathbb{G})} \leq C[u]^{a}_{s,p,\beta,q}\|q^{\mu}(x)u\|^{1-a}_{L^{\alpha}(\mathbb{G})}, \quad u \in C^1_c(\mathbb{G}), \tag{4.46}$$

where $B_R = \{x \in \mathbb{G} : q(x) < R\}$ is the quasi-ball and $0 < r < R$.

Proof of Theorem 4.7 The proof is similar to the proof of Theorem 4.1. In Eq. 4.13, summing over k from m to n and fixing $\varepsilon > 0$, we have

$$\int_{\{q(x)>2^m\}} \frac{q^{\gamma\tau}(x)}{\ln^{1+\varepsilon}\left(\frac{2R}{q(x)}\right)}|u|^{\tau}dx \leq C\sum_{k=m}^{n}\frac{1}{(n+1-k)^{1+\varepsilon}}|u_{A_{k,q}}|^{\tau}$$
$$+ C\sum_{k=m}^{n}[u]^{a\tau}_{s,p,\beta,q,A_{k,q}}\|q^{\mu}(x)u\|^{(1-a)\tau}_{L^{\alpha}(A_{k,q})}. \tag{4.47}$$

From Lemma 4.3, we have

$$|u_{A_{k+1,q}} - u_{A_{k,q}}| \leq C2^{\frac{ak(sp-Q)}{p}}[u]^{a}_{s,p,q,A_{k+1,q}\cup A_{k,q}}\left(\fint_{A_{k+1,q}\cup A_{k,q}}|u|^{\alpha}dx\right)^{\frac{1-a}{\alpha}}.$$

By using Lemma 4.4 with $\zeta=\frac{(n+1-k)^{\varepsilon}}{(n+\frac{1}{2}-k)^{\varepsilon}}$ we get

$$\frac{|u_{A_{k,q}}|^{\tau}}{(n+1-k)^{\varepsilon}} \leq \frac{|u_{A_{k+1,q}}|^{\tau}}{(n+\frac{1}{2}-k)^{\varepsilon}} + C(n+1-k)^{\tau-1-\varepsilon}[u]^{a\tau}_{s,p,\beta,q,A_{k+1,q}\cup A_{k,q}}\|q^{\mu}(x)u\|^{(1-a)\tau}_{L^{\alpha}(A_{k+1,q}\cup A_{k,q})}. \quad (4.48)$$

For $\varepsilon>0$ and $n\geq k$, we have

$$\frac{1}{(n-k+1)^{\varepsilon}}-\frac{1}{(n-k+\frac{3}{2})^{\varepsilon}} \sim \frac{1}{(n-k+1)^{1+\varepsilon}}. \quad (4.49)$$

By using this fact, Eqs. 4.48, 4.49 and $\varepsilon=\tau-1$, we obtain

$$\sum_{k=m}^{n}\frac{|u_{A_{k,q}}|^{\tau}}{(n+1-k)^{\tau}} \leq C\sum_{k=m}^{n}[u]^{a\tau}_{s,p,\beta,q,A_{k+1,q}\cup A_{k,q}}\|q^{\mu}(x)u\|^{(1-a)\tau}_{L^{\alpha}(A_{k+1,q}\cup A_{k,q})}. \quad (4.50)$$

From Eqs. 4.47 and 4.50, we establish

$$\int_{\{q(x)>2^m\}}\frac{q^{\gamma\tau}(x)}{\ln^{\tau}\frac{2R}{q(x)}}|u|^{\tau}dx \leq C\sum_{k=m}^{n}[u]^{a\tau}_{s,p,\beta,q,A_{k+1,q}\cup A_{k,q}}\|q^{\mu}(x)u\|^{(1-a)\tau}_{L^{\alpha}(A_{k+1,q}\cup A_{k,q})}. \quad (4.51)$$

By using Eqs. 4.24 with 4.45 and $0\leq\beta-\sigma\leq s$, where $s=\frac{\tau a}{p}$, $t=\frac{(1-a)\tau}{\alpha}$, we have $s+t\geq 1$ and we arrive at

$$\int_{\{q(x)>2^m\}}\frac{q^{\gamma\tau}(x)}{\ln^{\tau}\frac{2R}{q(x)}}|u|^{\tau}dx \leq C\sum_{k=m}^{n}[u]^{a\tau}_{s,p,\beta,q,\cup_{k=m}^{\infty}A_{k,q}}\|q^{\mu}(x)u\|^{(1-a)\tau}_{L^{\alpha}(\cup_{k=m}^{\infty}A_{k,q})}. \quad (4.52)$$

Theorem 4.7 is proved. □

5 Lyapunov-type Inequalities for the Fractional Operators on $\mathbb{G}$

In this section we prove the Lyapunov-type inequality for the Riesz potential and for the fractional p-sub-Laplacian system on homogeneous Lie groups. Note that the Lyapunov-type inequality for the Riesz operator is new even in the Abelian case $(\mathbb{R}^N,+)$. Also, we give applications of the Lyapunov-type inequality, more precisely, we give two side estimates for the first eigenvalue of the Riesz potential of the fractional p-sub-Laplacian system.

Let us consider the Riesz potential on a Haar measurable set $\Omega\subset\mathbb{G}$ that can be defined by the formula

$$\mathfrak{R}u(x)=\int_{\Omega}\frac{u(y)}{q^{Q-2s}(y^{-1}\circ x)}dy,\quad 0<2s<Q. \quad (5.1)$$

The (weighted) Riesz potential can be also defined by

$$\mathfrak{R}(\omega u)(x)=\int_{\Omega}\frac{\omega(y)u(y)}{q^{Q-2s}(y^{-1}\circ x)}dy,\quad 0<2s<Q. \quad (5.2)$$

Theorem 5.1 *Let $\Omega\subset\mathbb{G}$ be a Haar measurable set and let $Q\geq 2>2s>0$ and let $1<p<2$. Assume that $\omega\in L^{\frac{p}{2-p}}(\Omega)$, $\frac{1}{q^{Q-2s}(y^{-1}\circ x)}\in L^{\frac{p}{p-1}}(\Omega\times\Omega)$ and $C_0=$*

$\left\| \frac{1}{q^{Q-2s}(y^{-1}\circ x)} \right\|_{L^{\frac{p}{p-1}}(\Omega\times\Omega)}$. *Let* $u \in L^{\frac{p}{p-1}}(\Omega)$, $u \neq 0$, *satisfy*

$$\mathfrak{R}(\omega u)(x) = \int_\Omega \frac{\omega(y)u(y)}{q^{Q-2s}(y^{-1}\circ x)} dy = u(x), \text{ for a.e. } x \in \Omega. \tag{5.3}$$

Then

$$\|\omega\|_{L^{\frac{p}{2-p}}(\Omega)} \geq \frac{1}{C_0}. \tag{5.4}$$

Proof of Theorem 5.1 In Eq. 5.3, by using Hölder's inequality for $p, \theta > 1$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{\theta} + \frac{1}{\theta'} = 1$, we have

$$\begin{aligned} |u(x)| &= \left| \int_\Omega \frac{\omega(y)u(y)}{q^{Q-2s}(y^{-1}\circ x)} dy \right| \leq \left(\int_\Omega |\omega(y)u(y)|^p dy \right)^{\frac{1}{p}} \left(\int_\Omega \left| \frac{1}{q^{Q-2s}(y^{-1}\circ x)} \right|^{p'} dy \right)^{\frac{1}{p'}} \\ &\leq \left(\int_\Omega |\omega(y)|^{p\theta} dy \right)^{\frac{1}{p\theta}} \left(\int_\Omega |u(y)|^{\theta' p} dy \right)^{\frac{1}{\theta' p}} \left(\int_\Omega \left| \frac{1}{q^{Q-2s}(y^{-1}\circ x)} \right|^{p'} dy \right)^{\frac{1}{p'}} \\ &= \|\omega\|_{L^{p\theta}(\Omega)} \|u\|_{L^{p\theta'}(\Omega)} \left(\int_\Omega \left| \frac{1}{q^{Q-2s}(y^{-1}\circ x)} \right|^{p'} dy \right)^{\frac{1}{p'}}. \end{aligned} \tag{5.5}$$

Let p' be such that $p' = p\theta'$ and then $\theta = \frac{1}{2-p}$. Thus, we get

$$|u(x)| \leq \|\omega\|_{L^{\frac{p}{2-p}}(\Omega)} \|u\|_{L^{\frac{p}{p-1}}(\Omega)} \left(\int_\Omega \left| \frac{1}{q^{Q-2s}(y^{-1}\circ x)} \right|^{\frac{p}{p-1}} dy \right)^{\frac{p-1}{p}}. \tag{5.6}$$

From Eq. 5.6 we calculate

$$\begin{aligned} \|u\|_{L^{\frac{p}{p-1}}(\Omega)} &\leq \|\omega\|_{L^{\frac{p}{2-p}}(\Omega)} \|u\|_{L^{\frac{p}{p-1}}(\Omega)} \left(\int_\Omega \int_\Omega \left| \frac{1}{q^{Q-2s}(y^{-1}\circ x)} \right|^{\frac{p}{p-1}} dxdy \right)^{\frac{p-1}{p}} \\ &= C_0 \|\omega\|_{L^{\frac{p}{2-p}}(\Omega)} \|u\|_{L^{\frac{p}{p-1}}(\Omega)}. \end{aligned} \tag{5.7}$$

Finally, since $u \neq 0$, this implies

$$\|\omega\|_{L^{\frac{p}{2-p}}(\Omega)} \geq \frac{1}{C_0}. \tag{5.8}$$

Theorem 5.1 is proved. □

Let us consider the following spectral problem for the Riesz potential:

$$\mathfrak{R}u(x) = \int_\Omega \frac{u(y)}{q^{Q-2s}(y^{-1}\circ x)} dy = \lambda u(x), \quad x \in \Omega, \ \ 0 < 2s < Q. \tag{5.9}$$

We recall the Rayleigh quotient for the Riesz potential:

$$\lambda_1(\Omega) = \sup_{u\neq 0} \frac{\int_\Omega \int_\Omega \frac{u(x)u(y)}{q^{Q-2s}(y^{-1}\circ x)} dxdy}{\|u\|^2_{L^2(\Omega)}}, \tag{5.10}$$

where $\lambda_1(\Omega)$ is the first eigenvalue of the Riesz potential.

So, a direct consequence of Theorem 5.1 is

Theorem 5.2 *Let $\Omega \subset \mathbb{G}$ be a Haar measurable set and $Q \geq 2 > 2s > 0$ and let $1 < p < 2$. Assume that $\frac{1}{q^{Q-2s}(y^{-1}\circ x)} \in L^{\frac{p}{p-1}}(\Omega \times \Omega)$. Then for the spectral problem* Eq. 5.9, *we have*

$$\lambda_1(\Omega) \leq C_0|\Omega|^{\frac{2-p}{p}}, \tag{5.11}$$

where $C_0 = \left\| \frac{1}{q^{Q-2s}(y^{-1}\circ x)} \right\|_{L^{\frac{p}{p-1}}(\Omega\times\Omega)}$.

Proof of Theorem 5.2 By using Eq. 5.10, Theorem 5.1 and $\omega = \frac{1}{\lambda_1(\Omega)}$, we obtain

$$\lambda_1(\Omega) \leq C_0|\Omega|^{\frac{2-p}{p}}. \tag{5.12}$$

Theorem 5.2 is proved. □

In the Abelian group $(\mathbb{R}^N, +)$ we have the following consequences. To the best of our knowledge, these results seem new (even in this Euclidean case).

Let us consider the Riesz potential on $\Omega \subset \mathbb{R}^N$:

$$\mathfrak{R}u(x) = \int_\Omega \frac{u(y)}{|x-y|^{N-2s}}dy, \quad 0 < 2s < N, \tag{5.13}$$

and the weighted Riesz potential

$$\mathfrak{R}(\omega u)(x) = \int_\Omega \frac{\omega(y)u(y)}{|x-y|^{N-2s}}dy, \quad 0 < 2s < N. \tag{5.14}$$

Then we have following theorem:

Theorem 5.3 *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a measurable set with $|\Omega| < \infty$, $1 < p < 2$ and let $N \geq 2 > 2s > 0$. Assume that $\omega \in L^{\frac{p}{2-p}}(\Omega)$, $\frac{1}{|x-y|^{N-2s}} \in L^{\frac{p}{p-1}}(\Omega \times \Omega)$ and let $S = \left\| \frac{1}{|x-y|^{N-2s}} \right\|_{L^{\frac{p}{p-1}}(\Omega\times\Omega)}$. Assume that $u \in L^{\frac{p}{p-1}}(\Omega)$, $u \neq 0$, satisfies*

$$\mathfrak{R}(\omega u)(x) = u(x), \quad x \in \Omega.$$

Then

$$\|\omega\|_{L^{\frac{p}{2-p}}(\Omega)} \geq \frac{1}{S}. \tag{5.15}$$

Proof of Theorem 5.3 In Theorem 5.1 we set $\mathbb{G} = (\mathbb{R}^N, +)$ and take the standard Euclidean distance instead of the quasi-norm. □

Let us consider the spectral problem for Eq. 5.13:

$$\mathfrak{R}u(x) = \int_\Omega \frac{u(y)}{|x-y|^{N-2s}}dy = \lambda u(x), \quad 0 < 2s < N, \tag{5.16}$$

Theorem 5.4 *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a set with $|\Omega| < \infty$, $1 < p < 2$ and $N \geq 2 > 2s > 0$ and $1 < p < 2$. Assume that $\omega \in L^{\frac{p}{2-p}}(\Omega)$, $\frac{1}{|x-y|^{N-2s}} \in L^{\frac{p}{p-1}}(\Omega \times \Omega)$ and $S = \left\| \frac{1}{|x-y|^{N-2s}} \right\|_{L^{\frac{p}{p-1}}(\Omega\times\Omega)}$. Then for the spectral problem* Eq. 5.16 *we have,*

$$\lambda_1(\Omega) \leq \lambda_1(B) \leq S|B|^{\frac{2-p}{p}}, \tag{5.17}$$

where $B \subset \mathbb{R}^N$ is an open ball, $\lambda_1(\Omega)$ is the first eigenvalue of the spectral problem Eq. 5.16 *with $|\Omega| = |B|$.*

Proof of Theorem 5.4 The proof of $\lambda_1(B) \leq S|B|^{\frac{2-p}{p}}$ is the same as the proof of Theorem 5.2. From [17] we have

$$\lambda_1(B) \geq \lambda_1(\Omega).$$

The proof of Theorem 5.4 is complete. □

In [12] the authors proved a Lyapunov-type inequality for the fractional p-sub-Laplacian with the homogeneous Dirichlet condition. Here we establish Lyapunov-type inequality for the fractional p-sub-Laplacian system for the homogeneous Dirichlet problem. Namely, let us consider the fractional p-sub-Laplacian system:

$$\begin{cases} (-\Delta_{p_1,q})^{s_1} u_1(x) = \omega_1(x)|u_1(x)|^{\alpha_1-2}u_1(x)|u_2(x)|^{\alpha_2}\ldots|u_n(x)|^{\alpha_n}, \ x \in \Omega, \\ (-\Delta_{p_2,q})^{s_2} u_2(x) = \omega_2(x)|u_1(x)|^{\alpha_1}|u_2(x)|^{\alpha_2-2}u_2(x)\ldots|u_n(x)|^{\alpha_n}, \ x \in \Omega, \\ \qquad\qquad\ldots \\ (-\Delta_{p_n,q})^{s_n} u_n(x) = \omega_n(x)|u_1(x)|^{\alpha_1}|u_2(x)|^{\alpha_2}\ldots|u_n(x)|^{\alpha_n-2}u_n(x), \ x \in \Omega, \end{cases} \tag{5.18}$$

with homogeneous Dirichlet conditions

$$u_i(x) = 0, \quad x \in \mathbb{G} \setminus \Omega, \quad i = 1, \ldots, n, \tag{5.19}$$

where $\Omega \subset \mathbb{G}$ is a Haar measurable set, $\omega_i \in L^1(\Omega)$, $\omega_i \geq 0$, $s_i \in (0, 1)$, $p_i \in (1, \infty)$ and $(-\Delta_{p,q})^s$ is the fractional p-sub-Laplacian on $\mathbb{G}$ defined by

$$(-\Delta_{p_i,q})^{s_i} u_i(x) = 2 \lim_{\delta \searrow 0} \int_{\mathbb{G}\setminus B_q(x,\delta)} \frac{|u_i(x) - u_i(y)|^{p_i-2}(u_i(x) - u_i(y))}{q^{Q+s_i p_i}(y^{-1} \circ x)} dy, \quad x \in \mathbb{G},$$
$$i = 1, \ldots, n. \tag{5.20}$$

Here $B_q(x, \delta)$ is a quasi-ball with respect to q, with radius δ, centred at $x \in \mathbb{G}$, and α_i are positive parameters such that

$$\sum_{i=1}^{n} \frac{\alpha_i}{p_i} = 1. \tag{5.21}$$

To prove a Lyapunov-type inequality for the system we need some preliminary results from [12], the so-called fractional Hardy inequality on the homogeneous Lie groups.

Theorem 5.5 ([12], Fractional Hardy inequality) *For all $u \in C_c^\infty(\mathbb{G})$ we have*

$$C \int_{\mathbb{G}} \frac{|u(x)|^p}{q^{ps}(x)} dx \leq [u]^p_{s,p,q}, \tag{5.22}$$

where $p \in (1, \infty)$, $s \in (0, 1)$, and C is a positive constant.

We denote by $r_{\Omega,q}$ the inner quasi-radius of Ω, that is,

$$r_{\Omega,q} = \max\{q(x) : \ x \in \Omega\}. \tag{5.23}$$

Definition 5.6 We say that $(u_1, \dots, u_n) \in \prod_{i=1}^n W_0^{s_i,p_i}(\Omega)$ is a weak solution of Eqs. 5.18–5.19 if for all $(v_1, \dots, v_n) \in \prod_{i=1}^n W_0^{s_i,p_i}(\Omega)$, we have

$$\int_{\mathbb{G}}\int_{\mathbb{G}} \frac{|u_i(x)-u_i(y)|^{p_i-2}(u_i(x)-u_i(y))(v_i(x)-v_i(y))}{q^{Q+s_ip_i}(y^{-1}\circ x)}dxdy$$
$$= \int_\Omega \omega_i(x)\left(\prod_{j=1}^{i-1}|u_j(x)|^{\alpha_j}\right)\left(\prod_{j=i+1}^{n}|u_j(x)|^{\alpha_j}\right)|u_i(x)|^{\alpha_i-2}u_i(x)v_i(x)dx, \quad (5.24)$$

for every $i = 1, \dots, n$.

Now we present the following analogue of the Lyapunov-type inequality for the fractional p-sub-Laplacian system on $\mathbb{G}$.

Theorem 5.7 *Let $s_i \in (0,1)$ and $p_i \in (1,\infty)$ be such that $Q > s_ip_i$ for all $i = 1, \dots, n$. Let $\omega_i \in L^\theta(\Omega)$ be a non-negative weight and assume that*

$$1 < \max_{i=1,\dots,n}\left\{\frac{Q}{s_ip_i}\right\} < \theta < \infty.$$

If Eqs. 5.18–5.19 *admits a nontrivial weak solution, then*

$$\prod_{i=1}^n \|\omega_i\|_{L^\theta(\Omega)}^{\frac{\theta\alpha_i}{p_i}} \geq C r_{\Omega,q}^{Q-\theta\sum_{j=1}^n s_j\alpha_j}, \quad (5.25)$$

where $C > 0$ is a positive constant.

Remark 5.8 In Theorem 5.7, by taking $n = 1$ and $\alpha_1 = p$, we establish the Lyapunov-type inequality for the fractional p-sub-Laplacian on $\mathbb{G}$ (see, e.g. [12, Theorem 3.1]).

Proof of Theorem 5.7 For all $i = 1, \dots, n$, let us define

$$\xi_i = \gamma_ip_i + (1-\gamma_i)p_i^*, \quad (5.26)$$

and

$$\gamma_i = \frac{\theta - \frac{Q}{s_ip_i}}{\theta - 1}, \quad (5.27)$$

where $p_i^* = \frac{Q}{Q-s_ip_i}$ is the Sobolev conjugate exponent as in Theorem 3.1. Notice that for all $i = 1, \dots, n$ we have $\gamma_i \in (0,1)$ and $\xi_i = p_i\theta'$, where $\theta' = \frac{\theta}{\theta-1}$. Then for every $i \in \{1, \dots, n\}$ we get

$$\int_\Omega \frac{|u_i(x)|^{\xi_i}}{r_{\Omega,q}^{\gamma_is_ip_i}}dx \leq \int_\Omega \frac{|u_i(x)|^{\xi_i}}{q^{\gamma_is_ip_i}(x)}dx,$$

and by using Hölder's inequality with the following exponents $\nu_i = \frac{1}{\gamma_i}$ and $\frac{1}{\nu_i} + \frac{1}{\nu_i'} = 1$, we get

$$\int_\Omega \frac{|u_i(x)|^{\xi_i}}{q^{\gamma_is_ip_i}(x)}dx = \int_\Omega \frac{|u_i(x)|^{\gamma_ip_i}|u_i(x)|^{(1-\gamma_i)p_i^*}}{q^{\gamma_is_ip_i}(x)}dx$$
$$\leq \left(\int_\Omega \frac{|u_i(x)|^{p_i}}{q^{s_ip_i}(x)}dx\right)^{\gamma_i}\left(\int_\Omega |u_i(x)|^{p_i^*}dx\right)^{1-\gamma_i}. \quad (5.28)$$

On the other hand, from Theorem 3.1, we obtain

$$\left(\int_\Omega |u_i(x)|^{p_i^*}dx\right)^{1-\gamma_i} \le C[u_i]_{s_i,p_i,q}^{p_i^*(1-\gamma_i)},$$

and from Theorem 5.5, we have

$$\left(\int_\Omega \frac{|u_i(x)|^{p_i}}{q^{s_ip_i}(x)}dx\right)^{\gamma_i} \le C[u_i]_{s_i,p_i,q}^{p_i\gamma_i}.$$

Thus, from Eq. 5.28 and by taking $u_i(x)=v_i(x)$ in Eq. 5.24, we get

$$\int_\Omega \frac{|u_i(x)|^{\xi_i}}{q^{\gamma_is_ip_i}(x)} \le C([u_i]_{s_i,p_i,q,\Omega}^{p_i})^{\frac{\xi_i}{p_i}} \le C([u_i]_{s_i,p_i,q}^{p_i})^{\frac{\xi_i}{p_i}}$$

$$= C\left(\int_\Omega \omega_i(x)\prod_{j=1}^n |u_j|^{\alpha_j}dx\right)^{\frac{\xi_i}{p_i}} = C\left(\int_\Omega \omega_i(x)\prod_{j=1}^n |u_j|^{\alpha_j}dx\right)^{\theta'},$$

for every $i=1,\dots,n$. Therefore, by using Hölder's inequality with exponents θ and θ', we obtain

$$\int_\Omega \frac{|u_i(x)|^{\xi_i}}{q^{\gamma_is_ip_i}(x)}dx \le C\|\omega_i\|_{L^\theta(\Omega)}^{\frac{\theta}{\theta-1}} \int_\Omega \prod_{j=1}^n |u_j(x)|^{\alpha_j\theta'}dx.$$

By using Hölder's inequality and Eq. 5.21, we get

$$\int_\Omega \prod_{j=1}^n |u_j(x)|^{\alpha_j\theta'}dx \le \prod_{j=1}^n \left(\int_\Omega |u_j|^{\theta' p_j}dx\right)^{\frac{\alpha_j}{p_j}}.$$

This implies that

$$\int_\Omega \frac{|u_i(x)|^{\xi_i}}{q^{\gamma_is_ip_i}(x)}dx \le C\|\omega_i\|_{L^\theta(\Omega)}^{\frac{\theta}{\theta-1}} \prod_{j=1}^n \left(\int_\Omega |u_j|^{\theta' p_j}dx\right)^{\frac{\alpha_j}{p_j}}.$$

So we establish

$$\int_\Omega \frac{|u_i(x)|^{\xi_i}}{r_{\Omega,q}^{\gamma_is_ip_i}}dx \le \int_\Omega \frac{|u_i(x)|^{\xi_i}}{q^{\gamma_is_ip_i}(x)}dx$$

$$\le C\|\omega_i\|_{L^\theta(\Omega)}^{\frac{\theta}{\theta-1}} \prod_{j=1}^n \left(\int_\Omega |u_j|^{\theta' p_j}dx\right)^{\frac{\alpha_j}{p_j}}.$$

Thus, for every $e_i>0$ we have

$$\left(\int_\Omega \frac{|u_i(x)|^{\xi_i}}{r_{\Omega,q}^{\gamma_is_ip_i}}dx\right)^{e_i} = \frac{1}{r_{\Omega,q}^{e_i\gamma_is_ip_i}}\left(\int_\Omega |u_i(x)|^{\xi_i}dx\right)^{e_i}$$

$$\le C\|\omega_i\|_{L^\theta(\Omega)}^{\frac{e_i\theta}{\theta-1}} \prod_{j=1}^n \left(\int_\Omega |u_j|^{\theta' p_j}dx\right)^{\frac{e_i\alpha_j}{p_j}},$$

so that

$$\frac{1}{r_{\Omega,q}^{\sum_{j=1}^n \gamma_js_jp_je_j}}\prod_{i=1}^n \left(\int_\Omega |u_i(x)|^{\theta' p_i}dx\right)^{e_i}$$

$$\leq C \left(\prod_{i=1}^{n} \|\omega_i\|_{L^\theta(\Omega)}^{\frac{e_i\theta}{\theta-1}} \right) \left(\prod_{i=1}^{n} \left(\int_\Omega |u_i(x)|^{\theta' p_i} dx \right)^{\frac{\alpha_i \sum_{j=1}^n e_j}{p_i}} \right).$$

This yields

$$\frac{1}{r_{\Omega,q}^{\sum_{j=1}^n \gamma_j s_j p_j e_j}} \leq C \left(\prod_{i=1}^{n} \|\omega_i\|_{L^\theta(\Omega)}^{\frac{e_i\theta}{\theta-1}} \right) \left(\prod_{i=1}^{n} \left(|u_i(x)|^{\theta' p_i} dx \right)^{\frac{\alpha_i \sum_{j=1}^n e_j}{p_i} - e_i} \right), \tag{5.29}$$

where C is a positive constant. Then, we choose $e_i,\ i = 1, \dots, n$, such that

$$\frac{\alpha_i \sum_{j=1}^n e_j}{p_i} - e_i = 0, \quad i = 1, \dots, n.$$

Consequently, from Eq. 5.21 we have the solution of this system

$$e_i = \frac{\alpha_i}{p_i}, \quad i = 1, \dots, n. \tag{5.30}$$

From Eqs. 5.29, 5.27 and 5.30 we arrive at

$$\prod_{i=1}^{n} \|\omega_i\|_{L^\theta(\Omega)}^{\frac{\theta\alpha_i}{p_i}} \geq C r_{\Omega,q}^{Q-\theta \sum_{j=1}^n s_j \alpha_j}. \tag{5.31}$$

Theorem 5.7 is proved. □

Now, let us discuss an application of the Lyapunov-type inequality for the fractional p-sub-Laplacian system on $\mathbb{G}$. In order to do it we consider the spectral problem for the fractional p-sub-Laplacian system in the following form:

$$\begin{cases} (-\Delta_{p_1,q})^{s_1} u_1(x) = \lambda_1 \alpha_1 \varphi(x) |u_1(x)|^{\alpha_1 - 2} u_1(x) |u_2(x)|^{\alpha_2} \dots |u_n(x)|^{\alpha_n}, & x \in \Omega, \\ (-\Delta_{p_2,q})^{s_2} u_2(x) = \lambda_2 \alpha_2 \varphi(x) |u_1(x)|^{\alpha_1} |u_2(x)|^{\alpha_2 - 2} u_2(x) \dots |u_n(x)|^{\alpha_n}, & x \in \Omega, \\ \dots \\ (-\Delta_{p_n,q})^{s_n} u_n(x) = \lambda_n \alpha_n \varphi(x) |u_1(x)|^{\alpha_1} |u_2(x)|^{\alpha_2} \dots |u_n(x)|^{\alpha_n - 2} u_n(x), & x \in \Omega, \end{cases} \tag{5.32}$$

with

$$u_i(x) = 0, \quad x \in \mathbb{G} \setminus \Omega, \quad i = 1, \dots, n, \tag{5.33}$$

where $\Omega \subset \mathbb{G}$ is a Haar measurable set, $\varphi \in L^1(\Omega)$, $\varphi \geq 0$ and $s_i \in (0, 1)$, $p_i \in (1, \infty)$, $i = 1, \dots, n$.

Definition 5.9 We say that $\lambda = (\lambda_1, \dots, \lambda_n)$ is an eigenvalue if the problem Eqs. 5.32–5.33 admits at least one nontrivial weak solution $(u_1, \dots, u_n) \in \prod_{i=1}^n W_0^{s_i,p_i}(\Omega)$.

Theorem 5.10 *Let $s_i \in (0, 1)$ and $p_i \in (1, \infty)$ be such that $Q > s_i p_i$, for all $i = 1, \dots, n$, and*

$$1 < \max_{i=1,\dots,n} \left\{ \frac{Q}{s_i p_i} \right\} < \theta < \infty.$$

Let $\varphi \in L^\theta(\Omega)$ with $\|\varphi\|_{L^\theta(\Omega)} \neq 0$. Then, we have

$$\lambda_k \geq \frac{C}{\alpha_k} \left(\frac{1}{\prod_{i=1, i\neq k}^n \lambda_i^{\frac{\alpha_i}{p_i}}} \right)^{\frac{p_k}{\alpha_k}} \left(\frac{1}{r_{\Omega,q}^{\theta \sum_{i=1}^n \alpha_i s_i - Q} \prod_{i=1, i\neq k}^n \alpha_i^{\frac{\theta\alpha_i}{p_i}} \int_\Omega \varphi^\theta(x) dx} \right)^{\frac{p_k}{\theta\alpha_k}}, \tag{5.34}$$

where C is a positive constant and $k = 1, \ldots, n$.

Proof of Theorem 5.10 In Theorem 5.7 by taking $\omega_k = \lambda_k \alpha_k \varphi(x),\ \ k = 1, \ldots, n$, we have

$$\alpha_k^{\frac{\theta\alpha_k}{p_k}} \lambda_k^{\frac{\theta\alpha_k}{p_k}} \prod_{i=1, i\neq k}^{n} (\alpha_i \lambda_i)^{\frac{\theta\alpha_i}{p_i}} \prod_{i=1}^{n} \|\varphi\|_{L^\theta(\Omega)}^{\frac{\theta\alpha_i}{p_i}} \geq C r_{\Omega,q}^{Q-\theta\sum_{j=1}^{n} s_j\alpha_j}.$$

Thus, using Eq. 5.21 we obtain

$$\alpha_k^{\frac{\theta\alpha_k}{p_k}} \lambda_k^{\frac{\theta\alpha_k}{p_k}} \prod_{i=1, i\neq k}^{n} (\alpha_i \lambda_i)^{\frac{\theta\alpha_i}{p_i}} \int_\Omega \varphi^\theta(x)dx \geq C r_{\Omega,q}^{Q-\theta\sum_{j=1}^{n} s_j\alpha_j}.$$

This implies

$$\lambda_k^{\frac{\theta\alpha_k}{p_k}} \geq \frac{C}{\alpha_k^{\frac{\theta\alpha_k}{p_k}} r_{\Omega,q}^{\theta\sum_{j=1}^{n} s_j\alpha_j - Q} \prod_{i=1,i\neq k}^{n} (\alpha_i\lambda_i)^{\frac{\theta\alpha_i}{p_i}} \int_\Omega \varphi^\theta(x)dx},\ \ k = 1, \ldots, n.$$

Finally, we get that

$$\lambda_k \geq \frac{C}{\alpha_k} \left(\frac{1}{\prod_{i=1,i\neq k}^{n} \lambda_i^{\frac{\alpha_i}{p_i}}} \right)^{\frac{p_k}{\alpha_k}} \left(\frac{1}{r_{\Omega,q}^{\theta\sum_{i=1}^{n}\alpha_i s_i - Q} \prod_{i=1,i\neq k}^{n} \alpha_i^{\frac{\theta\alpha_i}{p_i}} \int_\Omega \varphi^\theta(x)dx} \right)^{\frac{p_k}{\theta\alpha_k}},$$
$$k = 1, \ldots, n. \quad (5.35)$$

Theorem 5.10 is proved. □

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