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Non-local Markovian Symmetric Forms on Infinite Dimensional Spaces

Part 2. Applications: Non Local Stochastic Quantization of Space Cut-Off Quantum Fields and Infinite Particle Systems

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Abstract

The general framework on the non-local Markovian symmetric forms on weighted l^p ($p \in [1, \infty]$) spaces constructed by Albeverio et al. (Commn. Math. Phys. **388**, 659–706, 2021) by restricting the situation where $p = 2$, is applied to probability measure spaces describing the space cut-off $P(\phi)_2$ Euclidean quantum field, the 2-dimensional Euclidean quantum fields with exponential and trigonometric potentials, and the measure associated with the field describing a system of an infinite number of classical particles. For each measure space, the Markov process corresponding to the *non-local* type stochastic quantization is constructed.

Keywords Non local Dirichlet forms on infinite dimensional spaces · Space cut-off $P(\phi)_2$ · $\exp\phi$ · $\sin\phi$-quantum field models · Euclidean quantum field · Infinite particle systems · Non-local stochastic quantization

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1 Introduction and Preliminaries

In this paper we apply a general framework on the non-local Markovian symmetric forms on weighted l^p ($p \in [1, \infty]$) spaces constructed by [16], by restricting the situation where $p = 2$, i.e., the weighted l^2-space, to the stochastic quantizations of the space cut-off $P(\phi)_2$ Euclidean quantum field, the 2-dimensional Euclidean quantum fields with exponential and trigonometric potentials, and a field of classical (infinite) particle systems. Then, for each random field, the Markov process corresponding to the *non-local* type stochastic quantization is constructed. As far as we know, there exists no considerations on the *non-local* type stochastic quantizations for such random fields through the arguments by the Dirichlet forms, that are non-local (cf., [1–5, 11, 12, 18, 21–25, 32, 34, 41, 43, 44, 57], and also for a historical aperçu on the stochastic quantizations of several random fields cf. [16] and references therein).

Thus, the main concern of the present paper is to show the explicit results on (*non-local* type) stochastic quantizations of the random fields, i.e., the probability spaces, on infinite dimensional topological vector spaces. In order to explain in simple terms the mathematical importance of this problem, before giving a description on the contents of this paper, we recall the corresponding problem defined in the framework of Markov chains with finite state spaces (also, cf. Remark 2.2 in Section 2). Let $n \in \mathbb{N}$, suppose that we are given an $n \times n$ Markovian matrix $M \equiv (p_{ij})$, $i, j = 1, \dots, n$. Thus the state space S of the Markov process in consideration is $S = \{1, \dots, n\}$. Then a probability distribution $\mu = (p_1, \dots, p_n)$, $0 \leq p_i \leq 1$, $\sum_{i=1}^n p_i = 1$, that satisfies $\mu M = \mu$ is an invariant measure of the Markov process defined through M, i.e., μ is an *eigenvector* of M with the *eigenvalue* 1. Conversely, suppose that we are given a probability distribution $\mu = (p_1, \dots, p_n)$, $0 \leq p_i \leq 1$, $\sum_{i=1}^n p_i = 1$, and suppose that we are asked to find an $n \times n$ Markovian matrix M such that $\mu M = \mu$. The latter problem is the *stochastic quantization* of the probability measure μ. It is the interpretation of the *stochastic quantization* in the framework of the Markov process with finite state space. Obvously, in general there exist many M that satisfy $\mu M = \mu$ for a given μ. *Ergodicity* of the associated Markov process is related to uniqueness of M.

Now, let us give the contents of the single sections of the present paper. In the rest of this section we recall the general framework for *non-local* Dirichlet forms on weighted l^p spaces developed in [16], which will be applied to several examples in the subsequent section. Then, in the same section in order to understand a standard procedure to construct a concrete Markov process through the general theorems in [16], we give Example 0 on the *non-local* stochastic quantization of the Euclidean free field, which has been discussed in section 5 of [16]. Here, we given the corresponding results as Theorem 5 and Theorem 6.

In Section 2, by following the procedure introduced in Example 0, we consider the *non-local* stochastic quantizations of the space cut-off exponential model of the Euclidean quantum field theory in Example 1, concluded by Theorems 7 and 8. The space cut-off $P(\phi)_2$ model and trigonometric model of the Euclidean quantum field theory constitute Example 2, concluded by Theorems 9 and 10. Example 3 is a consideration on the stochastic quantization of infinite particle systems, the corresponding resuts are given as Theorem 11.

At the end of Section 2, we give Remarks 2.1 and 2.2. Remark 2.1 includes a discussion on the *non-local* stochastic quantization of the infinite particle systems, Remark 2.2 discusses the advantages of the *non-local* stochastic quantization with respect to the *local* ones considered on *infinite dimensional* topolgical vector spaces. In Remark 2.2, we discuss how the condition of the space cut-off of the potential terms put in Example 2 have been effectively used (with a comparison of the corresponding results on Φ_3^4 *without cut-off* given in [16]), moreover an observation on the explicit representation of the *non-local* Markov process derived here by means of the related *stochastic partial differential equations* is proposed (see also Remark 11 (Fukushima decomposition) and Remark 12 (Subordination correspondences) in section 5 of [16]).

The final section is Appendix A, is where the proofs of lemmas which are given in Section 2 to discuss the stochastic quantization of the infinite particle systems.

As has been announced above we first recall the abstract results on the *non-local* Dirichlet forms defined on the Fréchet spaces provided in [16], and its application to the stochastic quantization of the Euclidean free quantum field which also has been considered in [16]. By these preparations, in the next section we proceed to construct the solutions of stochastic quantizations corresponding to the space cut-off $P(\phi)_2$ Euclidean quantum field, the 2-dimensional Euclidean fields with exponential and trigonometric potentials, and a field of classical (infinite) particle systems.

Here, we limit ourselves to recalling the results in [16] that will be applied to the stochastic quantizations mentioned above. Precisely, for the applications, we restrict our selves to the formulations on the weighted l^2 spaces and the non-local Dirichlet forms with the index $0 < \alpha \le 1$, the index characterizing the order of the non-locality which has a corresponce to the index of the α stable processes.

The *abstract* state spaces S, on which we define the Markovian symmetric forms, are the weighted l^2 spaces, denoted by $l^2_{(\beta_i)}$, with a given weight $(\beta_i)_i \in \mathbb{N}$, $\beta_i \ge 0$, $i \in \mathbb{N}$, such that

$$S = l^2_{(\beta_i)} \equiv \left\{ \mathbf{x} = (x_1, x_2, \dots) \in \mathbb{R}^{\mathbb{N}} \ : \ \|\mathbf{x}\|_{l^2_{(\beta_i)}} \equiv \left(\sum_{i=1}^{\infty} \beta_i |x_i|^2 \right)^{\frac{1}{2}} < \infty \right\}. \tag{1.1}$$

We denote by $\mathcal{B}(S)$ the Borel σ-field of S. Suppose that we are given a Borel probability measure μ on $(S, \mathcal{B}(S))$. For each $i \in \mathbb{N}$, let σ_{i^c} be the sub σ-field of $\mathcal{B}(S)$ that is generated by the Borel sets

$$B = \left\{ \mathbf{x} \in S \ \middle|\ x_{j_1} \in B_1, \dots x_{j_n} \in B_n \right\}, \quad j_k \ne i, \ \ B_k \in \mathcal{B}^1, \ \ k = 1, \dots, n, \ \ n \in \mathbb{N}, \tag{1.2}$$

where $\mathcal{B}^1$ denotes the Borel σ-field of $\mathbb{R}^1$. Thus, σ_{i^c} is the smallest σ-field that includes every B given by Eq. 1.2. *Namely, σ_{i^c} is the sub σ-field of $\mathcal{B}(S)$ generated by the variables $\mathbf{x} \setminus x_i$, i.e., the variables except of the i-th variable x_i*. For each $i \in \mathbb{N}$, let $\mu(\cdot \,|\, \sigma_{i^c})$ be the conditional probability, a one-dimensional probability distribution (i.e., a probability distribution for the i-th component x_i) valued σ_{i^c} measurable function, that is characterized by (cf. (2.4) of [24])

$$\mu\big(\{\mathbf{x} \ : \ x_i \in A\} \cap B\big) = \int_B \mu(A \,|\, \sigma_{i^c})\, \mu(d\mathbf{x}), \quad \forall A \in \mathcal{B}^1, \ \forall B \in \sigma_{i^c}. \tag{1.3}$$

Define

$$L^2(S;\mu) \equiv \left\{ f \ \middle|\ f : S \to \mathbb{R}, \text{ measurable and } \|f\|_{L^2} = \left(\int_S |f(\mathbf{x})|^2 \mu(d\mathbf{x}) \right)^{\frac{1}{2}} < \infty \right\}, \tag{1.4}$$

and

$$\mathcal{F}C_0^{\infty} = \left\{ f(x_1, \dots, x_n) \cdot \prod_{i \ge 1} I_{\mathbb{R}}(x_i) \ \middle|\ \exists f \in C_0^{\infty}(\mathbb{R}^n \to \mathbb{R}), \ n \in \mathbb{N} \right\} \subset L^2(S;\mu), \tag{1.5}$$

where $C_0^{\infty}(\mathbb{R}^n \to \mathbb{R})$ denotes the space of *real valued* infinitely differentiable functions on $\mathbb{R}^n$ with compact supports, and $I_{\mathbb{R}}(\cdot)$ denotes the indicator function.

On $L^2(S;\mu)$, for any $0 < \alpha \le 1$, let us define the Markovian symmetric form $\mathcal{E}_{(\alpha)}$ called *individually adapted Markovian symmetric form of index α to the measure μ*, the definition of which is a natural analogue of the one for α-stable type (*non local*) Dirichlet form on $\mathbb{R}^d$, $d < \infty$ (cf. (5.3), (1.4) of [39], also cf. Remark 1.1 below for the corresponding *local* Dirichlet forms on $L^2(\mathbb{R}^n; \mu)$ with *finite* $n \in \mathbb{N}$).

For each $0 < \alpha \leq 1$ and $i \in \mathbb{N}$, and for the variables

$$y_i,\ y_i' \in \mathbb{R}^1,\ \ y_i \neq y_i',\ \ \mathbf{x} = (x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots) \in S \text{ and } \mathbf{x} \backslash x_i \equiv (x_1, \dots, x_{i-1}, x_{i+1}, \dots),$$

let

$$\begin{aligned}
&\Phi_\alpha(u, v; y_i, y_i', \mathbf{x} \setminus x_i) \\
&\equiv \frac{1}{|y_i - y_i'|^{\alpha+1}} \times \big\{u(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots) - u(x_1, \dots, x_{i-1}, y_i', x_{i+1}, \dots)\big\} \\
&\qquad \times \big\{v(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots) - v(x_1, \dots, x_{i-1}, y_i', x_{i+1}, \dots)\big\},
\end{aligned} \tag{1.6}$$

then define

$$\mathcal{E}^{(i)}_{(\alpha)}(u, v) \equiv \int_S \left\{ \int_{\mathbb{R}} I_{\{y\,:\,y \neq x_i\}}(y_i)\, \Phi_\alpha(u, v; y_i, x_i, \mathbf{x} \setminus x_i)\, \mu\big(dy_i \,\big|\, \sigma_{i^c}\big) \right\} \mu(d\mathbf{x}), \tag{1.7}$$

for any u, v such that the right hand side of Eq. 1.7 is finite, where for a set A and a variable y, $I_A(y)$ denotes the indicator function, and in the sequel, to simplify the notations, we denote $I_{\{y\,:\,y \neq x_i\}}(y_i)$ by, e.g., $I_{\{y_i \neq x_i\}}(y_i)$ or $I_{\{y_i \neq x_i\}}$.

By $\mathcal{D}_i$, we denote the subset of the space of real valued $\mathcal{B}(S)$-measurable functions such that the right hand side of Eq. 1.7 is finite for any $u,\ v \in \mathcal{D}_i$. Let us call $(\mathcal{E}^{(i)}{}_{(\alpha)}, \mathcal{D}_i)$ this form, $\mathcal{D}_i$ being its domain. Then define

$$\mathcal{E}_{(\alpha)}(u, v) \equiv \sum_{i \in \mathbb{N}} \mathcal{E}^{(i)}_{(\alpha)}(u, v), \qquad \forall u,\ v \in \bigcap_{i \in \mathbb{N}} \mathcal{D}_i. \tag{1.8}$$

It is easy to see that for *the Lipschiz continuous* functions $\tilde{u} \in C_0^\infty(\mathbb{R}^n \to \mathbb{R}) \subset \mathcal{F}C_0^\infty$ and $\tilde{v} \in C_0^\infty(\mathbb{R}^m \to \mathbb{R}) \subset \mathcal{F}C_0^\infty$, $n, m \in \mathbb{N}$, which are representatives of $u \in \mathcal{F}C_0^\infty$ and $v \in \mathcal{F}C_0^\infty$ respectively, $n, m \in \mathbb{N}$, $\mathcal{E}^{(i)}_{(\alpha)}(\tilde{u}, \tilde{v})$ and $\mathcal{E}_{(\alpha)}(\tilde{u}, \tilde{v})$ are finite. Actually, in [16] it is proved that Eqs. 1.7 and 1.8 are well defined for $\mathcal{F}C_0^\infty$, and hence $\mathcal{F}C_0^\infty \subset \cap_{i \in \mathbb{N}} \mathcal{D}_i$, i.e., it is shown that for any real valued $\mathcal{B}(S)$-measurable function u on S, such that $u = 0$, μ-a.e, it holds that $\mathcal{E}_{(\alpha)}(u, u) = 0$; and for any $u, v \in \mathcal{F}C_0^\infty$, there corresponds only one value $\mathcal{E}_{(\alpha)}(u, v) \in \mathbb{R}$. Moreover, in [16] it is shown that $\mathcal{E}_{(\alpha)}$ is a closable Markovian symmetric form. Precisely, the following Theorem 1 holds, which is a restatement of a result given by [16] and shall be applied to the subsequent discussions in the present paper (in [16], not only for $0 < \alpha \leq 1$ but also for $0 < \alpha < 2$, and for the state spaces S as weighted l^p spaces, $1 \leq p \leq \infty$, those theorems including the statements corresponding to Theorems 1, 2 and 3 introduced in this paper are provided):

Theorem 1 (The closability) *For the symmetric non-local forms* $\mathcal{E}_{(\alpha)}$, $0 < \alpha \leq 1$ *given by* Eq. 1.8 *the following hold:*

i) $\mathcal{E}_{(\alpha)}$ *is well-defined on* $\mathcal{F}C_0^\infty$*;*
ii) $(\mathcal{E}_{(\alpha)}, \mathcal{F}C_0^\infty)$ *is closable in* $L^2(S; \mu)$*;*
iii) $(\mathcal{E}_{(\alpha)}, \mathcal{F}C_0^\infty)$ *is Markovian.*

Thus, for each $0 < \alpha \leq 1$*, the closed extension of* $(\mathcal{E}_{(\alpha)}, \mathcal{F}C_0^\infty)$ *denoted by* $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ *with the domain* $\mathcal{D}(\mathcal{E}_{(\alpha)})$*, is a non-local Dirichlet form on* $L^2(S; \mu)$*.*

Remark 1.1 In [16], the symmetric form $\mathcal{E}_{(\alpha)}^{(i)}$ is considered for $0 < \alpha < 2$. Then, the *non-local* symmetric form $\mathcal{E}_{(\alpha)}^{(i)}$ defined by Eqs. 1.6 and 1.7, by extending the definition for $0 < \alpha < 2$, can be interpreted as *non-local* and *local* symmetric forms on the finite dimensional linear space $C_0^\infty(\mathbb{R}^d \to \mathbb{R})$ (cf., e.g., Example 4 in section 1.2 of [38], and section II-2 of [54]), the space of real valued smooth functions with compact supports on $\mathbb{R}^d$ with some finite $d \in \mathbb{N}$: For simplicity, let $d = 1$ and for the Borel probability measure μ, suppose that there exists a *smooth bounded* probability density function $\rho \in \mathcal{S}(\mathbb{R} \to \mathbb{R})$, an element of Schwartz space of rapidly decreasing functions, such that $0 < \rho(x) < \infty$, $\forall x \in \mathbb{R}$. Then $\mathcal{E}_{(\alpha)}^{(i)}$ is interpreted as follows:

$$\int_{\mathbb{R}^2} \backslash \text{“diagonal set”} \frac{(f(y)- f(x))(g(y)- g(x))}{|y-x|^{1+\alpha}} \rho(y)\,\rho(x)\,dy\,dx, \qquad f,\ g \in C_0^\infty(\mathbb{R} \to \mathbb{R}).$$

Next, for each $0 < \alpha < 2$, let

$$M(\alpha) \equiv (1 - \frac{1}{2}\alpha)^{\frac{1}{2-\alpha}},$$

then, for $f,\ g \in C_0^\infty(\mathbb{R} \to \mathbb{R})$, it holds that

$$\lim_{\alpha\uparrow 2} \int_{\mathbb{R}} \{\int_{[x-M(\alpha),\, x+M(\alpha)]} I_{\{y:y\neq x\}}(y') \frac{(f(y') - f(x))(g(y') - g(x))}{|y'-x|^{1+\alpha}} \rho(y')\,dy'\}\,\rho(x)\,dx,$$
$$= \int_{\mathbb{R}} f'(x)\,g'(x)\,(\rho(x))^2\,dx.$$

Also, for each $0 < \alpha < 2$ and each $x \in \mathbb{R}$, if we let

$$M(\alpha; x) \equiv \big(\rho(x)^{-1}(1 - \frac{1}{2}\alpha)\big)^{\frac{1}{2-\alpha}},$$

then, for $f,\ g \in C_0^\infty(\mathbb{R} \to \mathbb{R})$, it holds that

$$\lim_{\alpha\uparrow 2} \int_{\mathbb{R}} \{\int_{[x-M(\alpha;x),\, x+M(\alpha;x)]} I_{\{y:y\neq x\}}(y') \frac{(f(y') - f(x))(g(y') - g(x))}{|y'-x|^{1+\alpha}} \rho(y')\,dy'\}\,\rho(x)dx,$$
$$= \int_{\mathbb{R}} f'(x)\,g'(x)\,\rho(x)\,dx.$$

Considerations for the **infinite** dimensional situation corresponding to the above **finite** dimensional observation will be carried out in forthcoming work.

The following theorem is also a part of the main results provided by [16] on the sufficient conditions (cf. Theorems 2 and 3 below) under which the Dirichlet forms (i.e. the closed Markovian symmetric forms) defined above are *strictly quasi-regular* (cf., [22–24] and section IV-3 of [54], as well as [1] for the meaning of *“strict quasi regular”*).

Denote $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ the Dirichlet form on $L^2(S;\mu)$, with the domain $\mathcal{D}(\mathcal{E}_{(\alpha)}))$ defined through Theorem 1, obtained as the closed extension of the closable Markovian symmetric form $\mathcal{E}_{(\alpha)}$, understood as first defined on $\mathcal{F}C_0^\infty$. We shall use the same notation $\mathcal{E}_{(\alpha)}$ for the closable form and the closed form.

For each $i \in \mathbb{N}$, we denote by X_i the random variable (i.e., measurable function) on $(S, \mathcal{B}(S), \mu)$, that represents the coordinate x_i of $\mathbf{x} = (x_1, x_2, \dots)$, precisely,

$$X_i\ : S \ni \mathbf{x} \longmapsto x_i \in \mathbb{R}. \tag{1.9}$$

By making use of the random variable X_i, we have the following probabilistic expression:

$$\int_S I_B(x_i)\,\mu(d\mathbf{x}) = \mu(X_i \in B), \qquad \text{for } B \in \mathcal{B}(S). \tag{1.10}$$

Theorem 2 (The Strict Quasi-Regularity) *Let $S = l^2_{(\beta_i)}$, for $0 < \alpha \le 1$, let $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ be the closed Markovian symmetric form on $L^2(S;\mu)$ given by Theorem 1. If there exists a positive sequence $\{\gamma_i\}_i \in \mathbb{N}$ such that $\sum_{i=1}^{\infty}\gamma_i^{-1} < \infty$ (i.e., $\{\gamma_i^{-\frac{1}{2}}\}_i \in \mathbb{N}$ is a positive l^2 sequence), and an $0 < M_0 < \infty$, and both*

$$\sum_{i=1}^{\infty}(\beta_i\gamma_i)^{\frac{1+\alpha}{2}} \cdot \mu\Big(\beta_i^{\frac{1}{2}}|X_i| > M_0 \cdot \gamma_i^{-\frac{1}{2}}\Big) < \infty, \tag{1.11}$$

$$\mu\Big(\bigcup_{M\in\mathbb{N}} \in \mathbb{N}\big\{|X_i| \le M \cdot \beta_i^{-\frac{1}{2}}\,\gamma_i^{-\frac{1}{2}},\ \forall i \in \mathbb{N}\big\}\Big) = 1, \tag{1.12}$$

hold, then $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ is a strictly quasi-regular Dirichlet form.

Next, from [16], we quote a theorem corresponding to the Markov processes associated to the non-local Dirichlet forms defined above.

Let $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$, $0 < \alpha \le 1$, be the family of strictly quasi-regular Dirichlet forms on $L^2(S;\mu)$ with the state space S defined by Theorems 1 and 2.

For the strictly quasi-regular Dirichlet form $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ there exists a properly associated S-valued Hunt process (seem Definitions IV-1.5, 1.8 and 1.13, Theorem V-2.13 and Proposition V-2.15 of [54])

$$\mathbb{M} \equiv \Big(\Omega, \mathcal{F}, (X_t)_{t\ge 0}, (P_{\mathbf{x}})_{\mathbf{x}\in S_\triangle}\Big). \tag{1.13}$$

$\triangle$ is a point adjoined to S as an isolated point of $S_\triangle \equiv S \cup \{\triangle\}$. Let $(T_t)_{t\ge 0}$ be the strongly continuous contraction semigroup associated with $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$, and $(p_t)_{t\ge 0}$ be the corresponding transition semigroup of kernels of the Hunt process $(X_t)_{t\ge 0}$, then for any $u \in \mathcal{F}C_0^\infty \subset \mathcal{D}(\mathcal{E}_{(\alpha)})$ the following holds:

$$\frac{d}{dt}\int_S \big(p_t u\big)(\mathbf{x})\,\mu(d\mathbf{x}) = \frac{d}{dt}\big(T_t u, 1\big)_{L^2(S;\mu)} = \mathcal{E}_{(\alpha)}(T_t u, 1) = 0. \tag{1.14}$$

By this, we see that

$$\int_S \big(p_t u\big)(\mathbf{x})\,\mu(d\mathbf{x}) = \int_S u(\mathbf{x})\,\mu(d\mathbf{x}), \quad \forall t \ge 0, \quad \forall u \in \mathcal{F}C_0^\infty, \tag{1.15}$$

and hence, by the density of $\mathcal{F}C_0^\infty$ in $L^2(S;\mu)$

$$\int_S P_{\mathbf{x}}(X_t \in B)\,\mu(d\mathbf{x}) = \mu(B), \qquad \forall B \in \mathcal{B}(S), \quad \forall t \ge 0. \tag{1.16}$$

Thus, the following Theorem 3 holds.

Theorem 3 (Associated Markov process) *Let $0 < \alpha \le 1$, and let $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ be a strictly quasi-regular Dirichlet form on $L^2(S;\mu)$ that is defined through Theorem 2. Then to $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$, there exists a properly associated S-valued Hunt process $\mathbb{M}$ defined by* Eq. 1.13*, the invariant measure of which is μ (cf.* Eq. 1.16*).*

As the final preparations for the main discussions given in the next section, we recall the formulation corresponding to the stochastic quantization of the Euclidean free quantum field, discussed in the section 5 of [16].

Let us recall the Bochner-Minlos theorem stated in a general framework. Let E be a nuclear space (cf., e.g., Chapters 47-51 of [68], and [8, 46]). Suppose in particular that E is a countably Hilbert space,

characterized by a sequence of *real* Hilbert norms $\| \ \|_n$, $n \in \mathbb{N} \cup \{0\}$ such that $\| \ \|_0 < \| \ \|_1 < \cdots < \| \ \|_n < \cdots$. Let E_n be the completion of E with respect to the norm $\| \ \|_n$, then by definition $E = \bigcap_{n \geq 0} E_n$ and $E_0 \supset E_1 \supset \cdots \supset E_n \supset \cdots$. Define

$$E_n^* \equiv \text{the dual space of } E_n, \text{ and assume the identification } E_0^* = E_0\,.$$

Then we have

$$E \subset \cdots \subset E_{n+1} \subset E_n \subset \cdots \subset E_0 = E_0^* \subset \cdots \subset E_n^* \subset E_{n+1}^* \subset \cdots \subset E^*.$$

Since by assumption E is a nuclear space, for any $m \in \mathbb{N} \cup \{0\}$ there exists an $n \in \mathbb{N} \cup \{0\}$, $n > m$, such that the (canonical) injection $T_m^n : E_n \to E_m$ is a trace class (nuclear class) positive operator. The Bochner-Minlos theorem (cf. [45]) is given as follows:

Theorem 4 (Bochner-Minlos Theorem) *Let $C(\varphi)$, $\varphi \in E$, be a complex valued function on E such that*

i) *$C(\varphi)$ is continuous with respect to the norm $\| \cdot \|_m$ for some $m \in \mathbb{N} \cup \{0\}$;*

ii) ***(positive definiteness)** for any $k \in \mathbb{N}$,*

$$\sum_{i,j=1}^{k} \bar{\alpha}_i \alpha_j C(\varphi_i - \varphi_j) \geq 0, \qquad \forall \alpha_i \in \mathbb{C},\ \forall \varphi_i \in E,\ i = 1, \ldots, k;$$

(where $\bar{\alpha}$ means complex conjugate of α).

iii) ***(normalization)** $C(0) = 1$.*

Then, there exists a unique Borel probability measure ν on E^* such that

$$C(\varphi) = \int_{E^*} e^{i<\phi,\varphi>} \nu(d\phi), \qquad \varphi \in E.$$

Moreover, if the (canonical) injection $T_m^n : E_n \to E_m$, for all $n > m$, is a Hilbert-Schmidt operator, then the support of ν is in E_n^*, where $<\phi, \varphi>= {}_{E^*}<\phi, \varphi>_E$ is the dualization between $\phi \in E^*$ and $\varphi \in E$.

Remark 1.2 The assumption on the continuity of $C(\varphi)$ on E given in i) of the above Theorem 4 can be replaced by the continuity of $C(\varphi)$ at the *origin* in E, which is equivalent to i) under the assumption that $C(\varphi)$ satisfies ii) and iii) in Theorem 4 (cf. e.g., [50]). Namely, under the assumption of ii) and iii), the following is equivalent to i): For any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|C(\varphi) - 1| < \epsilon, \qquad \forall \varphi \in E \quad \text{with} \quad \|\varphi\|_m < \delta.$$

This can be seen as follows: Assume that ii) and iii) hold. For ii), let $k = 3$, $\alpha_1 = \alpha$, $\alpha_2 = -\alpha$, $\alpha_3 = \beta$, $\varphi_1 = 0$, $\varphi_2 = \varphi$ and $\varphi_3 = \psi$, then by the assumption ii), the positive definiteness of C, we have

$$\begin{aligned}&\alpha\overline{\alpha} \cdot (2C(0) - C(\varphi) - C(-\varphi))\\ &\quad + \alpha\overline{\beta} \cdot (C(-\psi-\varphi) - C(-\psi)) + \overline{\alpha}\beta \cdot (C(\psi+\varphi) - C(\psi)) + \beta\overline{\beta} \cdot C(0) \geq 0.\end{aligned}$$

By making use of the fact that $C(-\varphi) = \overline{C(\varphi)}$, which follows from ii), and the assumption iii), from the above inequality we have

$$0 \le \det \begin{pmatrix} 2 - C(\varphi) - \overline{C(\varphi)} & \overline{C(\psi+\varphi) - C(\psi)} \\ C(\psi+\varphi) - C(\psi) & 1 \end{pmatrix}.$$

From this it follows that

$$|C(\psi+\varphi) - C(\psi)|^2 \le 2\,|C(\varphi) - 1|.$$

By making use of the support property of ν by means of the Hilbert-Schmidt operators given by Theorem 4, we can present a framework by which Theorems 1, 2, 3 and 4 can be applied to the *stochastic quantization* of Euclidean quantum fields.

Now, we define an adequate countably Hilbert nuclear space $\mathcal{H}_0 \supset \mathcal{S}(\mathbb{R}^d \to \mathbb{R}) \equiv \mathcal{S}(\mathbb{R}^d)$, for a given $d \in \mathbb{N}$. Let

$$\mathcal{H}_0 \equiv \left\{ f \;:\; \|f\|_{\mathcal{H}_0} = \left((f, f)_{\mathcal{H}_0}\right)^{\frac{1}{2}} < \infty,\;\; f : \mathbb{R}^d \to \mathbb{R},\;\; \text{measurable} \right\} \supset \mathcal{S}(\mathbb{R}^d), \quad (1.17)$$

where

$$(f, g)_{\mathcal{H}_0} \equiv (f, g)_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} f(x) g(x)\, dx. \quad (1.18)$$

Let us consider the following *pseudo differential operators* on $\mathcal{S}(\mathbb{R}^d \to \mathbb{R}) \equiv \mathcal{S}(\mathbb{R}^d)$

$$H \equiv (|x|^2 + 1)^{\frac{d+1}{2}} (-\Delta + 1)^{\frac{d+1}{2}} (|x|^2 + 1)^{\frac{d+1}{2}}, \quad (1.19)$$

$$H^{-1} \equiv (|x|^2 + 1)^{-\frac{d+1}{2}} (-\Delta + 1)^{-\frac{d+1}{2}} (|x|^2 + 1)^{-\frac{d+1}{2}}, \quad (1.20)$$

with Δ the d-dimensional Laplace operator. For each $n \in \mathbb{N}$, define

$$\mathcal{H}_n \equiv \text{the completion of } \mathcal{S}(\mathbb{R}^d) \text{ with respect to the norm } \|f\|_n, \quad f \in \mathcal{S}(\mathbb{R}^d), \quad (1.21)$$

where $\|f\|_n^2 = (f, f)_n$ (in the case where $n = 1$, to denote the $\mathcal{H}_1$ norm we use the exact notation $\|\;\|_{\mathcal{H}_1}$, in order to avoid a confusion between the notation of some L^1 or l^1 norms) with the corresponding scalar product

$$(f, g)_n = (H^n f, H^n g)_{\mathcal{H}_0}, \qquad f, g \in \mathcal{S}(\mathbb{R}^d). \quad (1.22)$$

Moreover we define, for $n \in \mathbb{N}$:

$$\mathcal{H}_{-n} \equiv \text{the completion of } \mathcal{S}(\mathbb{R}^d) \text{ with respect to the norm} \|f\|_{-n}, \; f \in \mathcal{S}(\mathbb{R}^d), \quad (1.23)$$

where $\|f\|_{-n}^2 = (f, f)_{-n}$, with

$$(f, g)_{-n} = ((H^{-1})^n f, (H^{-1})^n g)_{\mathcal{H}_0}, \qquad f, g \in \mathcal{S}(\mathbb{R}^d). \quad (1.24)$$

Then obviously, for $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\|f\|_n \le \|f\|_{n+1}, \qquad \|f\|_{-n-1} \le \|f\|_{-n}, \quad (1.25)$$

and by taking the inductive limit and setting $\mathcal{H} = \bigcap_{n\in\mathbb{N}} \mathcal{H}_n$, we have the following inclusions:

$$\mathcal{H} \subset \cdots \subset \mathcal{H}_{n+1} \subset \mathcal{H}_n \subset \cdots \subset \mathcal{H}_0 \subset \cdots \subset \mathcal{H}_{-n} \subset \mathcal{H}_{-n-1} \subset \cdots \subset \mathcal{H}^*. \quad (1.26)$$

The (topological) dual space of $\mathcal{H}_n$ is $\mathcal{H}_{-n}$, $n \in \mathbb{N}$.

By the operator H^{-1} given by Eq. 1.20 on $\mathcal{S}(\mathbb{R}^d)$ we can define, on each $\mathcal{H}_n$, $n \in \mathbb{N}$, the bounded symmetric (hence self-adjoint) operators

$$(H^{-1})^k, \quad k \in \mathbb{N} \cup \{0\} \quad (1.27)$$

(we use the same notations for the operators on $\mathcal{S}(\mathbb{R}^d)$ and on $\mathcal{H}_n$). Hence, for the canonical injection

$$T_n^{n+k} : \mathcal{H}_{n+k} \longrightarrow \mathcal{H}_n, \qquad k, n \in \mathbb{N} \cup \{0\}, \tag{1.28}$$

it holds that

$$\|T_n^{n+k} f\|_n = \|(H^{-1})^k f\|_{\mathcal{H}_0}, \qquad \forall f \in \mathcal{H}_{n+k},$$

where by a simple calculation by means of the Fourier transform, and by Young's inequality, we see that for each $n \in \mathbb{N} \cup \{0\}$, H^{-1} on $\mathcal{H}_n$ is a Hilbert-Schmidt operator and hence $(H^{-1})^2$ on $\mathcal{H}_n$ is a trace class operator.

Now, by applying to the strictly positive self-adjoint Hilbert-Schmidt (hence compact) operator H^{-1}, on $\mathcal{H}_0 = L^2(\mathbb{R}^d \to \mathbb{R})$ the *Hilbert-Schmidt theorem* (cf., e.g., Theorem VI 16, Theorem VI 22 of [63], and also [64]) we have that there exists an orthonormal base (O.N.B.) $\{\varphi_i\}_{i\in\mathbb{N}}$ of $\mathcal{H}_0$ such that

$$H^{-1}\varphi_i = \lambda_i\, \varphi_i, \qquad i \in \mathbb{N}, \tag{1.29}$$

where $\{\lambda_i\}_{i\in\mathbb{N}}$ are the corresponding eigenvalues such that

$$0 < \cdots < \lambda_2 < \lambda_1 \le 1, \quad \text{which satisfy} \quad \sum_{i\in\mathbb{N}} (\lambda_i)^2 < \infty, \quad \text{i.e.,} \quad \{\lambda_i\}_{i\in\mathbb{N}} \in l^2, \tag{1.30}$$

and $\{\varphi_i\}_{i\in\mathbb{N}}$ is indexed adequately corresponding to the finite multiplicity of each λ_i, $i \in \mathbb{N}$. By the definition (1.21), (1.22), (1.23) and (1.24) (cf. also Eq. 1.27), for each $n \in \mathbb{N} \cup \{0\}$,

$$\{(\lambda_i)^n \varphi_i\}_{i\in\mathbb{N}} \quad \text{is an O.N.B. of} \quad \mathcal{H}_n \tag{1.31}$$

and

$$\{(\lambda_i)^{-n} \varphi_i\}_{i\in\mathbb{N}} \quad \text{is an O.N.B. of} \quad \mathcal{H}_{-n} \tag{1.32}$$

Thus, by denoting $\mathbb{Z}$ the set of integers, by the Fourier series expansion of functions in $\mathcal{H}_m$, $m \in \mathbb{Z}$ (cf. Eqs. 1.21–1.24), such that for $f \in \mathcal{H}_m$, we have

$$f = \sum_{i\in\mathbb{N}} a_i (\lambda_i^m \varphi_i), \quad \text{with} \quad a_i \equiv \big(f, (\lambda_i^m \varphi_i)\big)_m, \; i \in \mathbb{N}. \tag{1.33}$$

In particular for $f \in \mathcal{S}(\mathbb{R}^d) \subset \mathcal{H}_m$, it holds that $a_i = {\lambda_i}^{-m}(f, \varphi_i)_{\mathcal{H}_0}$. Moreover we have

$$\sum_{i\in\mathbb{N}} a_i^2 = \|f\|_m^2,$$

that yields an *isometric isomorphism* τ_m for each $m \in \mathbb{Z}$ such that

$$\tau_m \;:\; \mathcal{H}_m \ni f \longmapsto (\lambda_1^m a_1, \lambda_2^m a_2, \ldots) \in l^2_{(\lambda_i^{-2m})}, \tag{1.34}$$

where $l^2_{(\lambda_i^{-2m})}$ is the weighted l^2 space defined by Eq. 1.1 with $p = 2$, and $\beta_i = \lambda_i^{-2m}$. Precisely, for $f = \sum_{i\in\mathbb{N}} a_i (\lambda_i^m \varphi_i) \in \mathcal{H}_m$ and $g = \sum_{i\in\mathbb{N}} b_i (\lambda_i^m \varphi_i) \in \mathcal{H}_m$, with $a_i \equiv \big(f, (\lambda_i^m \varphi_i)\big)_m$, $b_i \equiv \big(g, (\lambda_i^m \varphi_i)\big)_m$, $i \in \mathbb{N}$, by τ_m the following holds (cf. Eqs. 1.31 and 1.32):

$$(f, g)_m = \sum_{i\in\mathbb{N}} a_i \cdot b_i = \sum_{i\in\mathbb{N}} \lambda_i^{-m} (\lambda_i^m a_i) \cdot \lambda_i^{-m} (\lambda_i^m b_i) = \big(\tau_m f, \tau_m g\big)_{l^2_{(\lambda_i^{-2m})}}.$$

By the map τ_m we can identify, in particular, the two systems of Hilbert spaces given by Eqs. 1.35 and 1.36 through the following diagram:

$$\ldots \quad \mathcal{H}_2 \subset \quad \mathcal{H}_1 \subset \mathcal{H}_0 = L^2(\mathbb{R}^d) \subset \mathcal{H}_{-1} \subset \mathcal{H}_{-2} \quad \ldots \tag{1.35}$$

$$\begin{array}{ccccccccccc} & \| & & \| & & \| & & \| & & \| & \\ \cdots & l^2_{(\lambda_i^{-4})} & \subset & l^2_{(\lambda_i^{-2})} & \subset & l^2 & \subset & l^2_{(\lambda_i^{2})} & \subset & l^2_{(\lambda_i^{4})} & \cdots . \end{array} \tag{1.36}$$

Example 0. (The Euclidean Free Quantum Field) This fundamental example introduced in [59], which the following has been considered in [16], shows how the abstract Theorems 1, 2 and 3, from which we can construct weighted l^2-space valued *non-local* symmetric Markov processes through the *non-local* Dirichlet forms, can be applied to construct *separable Hilbert space* (cf. Eqs. 1.35 and 1.36) valued Markov processes, which is a *stochastic quantization* of a given *physical* random field.

Let ν_0 be the Euclidean free field probability measure on $\mathcal{S}' \equiv \mathcal{S}'(\mathbb{R}^d)$. It is characterized by the (generalized) characteristic function $C(\varphi)$ in Theorem 4 of ν_0 given by

$$C(\varphi) = \exp(-\frac{1}{2}(\varphi, (-\Delta + m_0^2)^{-1}\varphi)_{L^2(\mathbb{R}^d)}). \qquad \text{for } \varphi \in \mathcal{S}(\mathbb{R}^d \to \mathbb{R}), \tag{1.37}$$

Equivalently, ν_0 is the centered Gaussian probability measure on $\mathcal{S}'$, the covariance of which is given by

$$\int_{\mathcal{S}}' < \phi, \varphi_1 > \cdot < \phi, \varphi_1 > \nu_0(d\phi) = \big(\varphi_1, (-\Delta + m_0^2)^{-1}\varphi_2\big)_{L^2(\mathbb{R}^d)}, \qquad \varphi_1,\ \varphi_2 \in \mathcal{S}(\mathbb{R}^d \to \mathbb{R}), \tag{1.38}$$

where Δ is the d-dimensional Laplace operator and $m_0 > 0$ (for $d \geq 3$, we can also allow for $m_0 = 0$) is a given mass for this scalar field. $\phi(f) =< \phi, f >$, $f \in \mathcal{S}(\mathbb{R}^d \to \mathbb{R})$ is the coordinate process ϕ to ν_0 (for the Euclidean free field cf. [58, 59, 62] and, e.g., [6, 26, 40, 66]). By Eq. 1.37, the functional $C(\varphi)$ is continuous with respect to the norm of the space $\mathcal{H}_0 = L^2(\mathbb{R}^d)$, and the kernel of $(-\Delta + m_0^2)^{-1}$, which is the Fourier inverse transform of $(|\xi|^2 + m_0^2)^{-1}$, $\xi \in \mathbb{R}^d$, is explicitly given by Bessel functions (cf., e.g., section 2–5 of [56]). Then, by Theorem 4 and Eq. 1.28 the support of ν_0 can be taken to be in the separable Hilbert spaces $\mathcal{H}_{-n}$, $n \geq 1$ (cf. Eqs. 1.35 and 1.36).

Let us apply Theorems 1, 2 and 3 to this random field. We start the consideration from the case where $\alpha = 1$, a simplest situation. Then, we shall state the corresponding results for the cases where $0 < \alpha < 1$.

Now, we take ν_0 as a Borel probability measure on $\mathcal{H}_{-2}$. By Eqs. 1.34, 1.35 and 1.36, by taking $m = -2$, τ_{-2} defines an isometric isomorphism such that

$$\tau_{-2} : \mathcal{H}_{-2} \ni f \longmapsto (a_1, a_2, \ldots) \in l^2_{(\lambda_i^4)}, \quad \text{with} \quad a_i \equiv (f, \lambda_i^{-2}\varphi_i)_{-2}, \ \ i \in \mathbb{N}. \tag{1.39}$$

Define a probability measure μ on $l^2_{(\lambda_i^4)}$ such that

$$\mu(B) \equiv \nu_0 \circ \tau_{-2}^{-1}(B) \quad \text{for} \quad B \in \mathcal{B}(l^2_{(\lambda_i^4)}).$$

We set $S = l^2_{(\lambda_i^4)}$ in Theorems 1, 2 and 3, then it follows that the weight β_i satisfies $\beta_i = \lambda_i^4$. We can take $\gamma_i^{-\frac{1}{2}} = \lambda_i$ in Theorem 2, then, from Eq. 1.30 we have

$$\sum_{i=1}^{\infty} \beta_i \gamma_i \cdot \mu\Big(\beta_i^{\frac{1}{2}} |X_i| > M \cdot \gamma_i^{-\frac{1}{2}}\Big) \leq \sum_{i=1}^{\infty} \beta_i \gamma_i = \sum_{i=1}^{\infty} (\lambda_i)^2 < \infty \tag{1.40}$$

Equation 1.40 shows that the condition (1.11) holds.

Also, as has been mentioned above, since $\nu_0(\mathcal{H}_{-n}) = 1$, for any $n \geq 1$, we have

$$1 = \nu_0(\mathcal{H}_{-1}) = \mu(l^2_{(\lambda_i^2)}) = \mu\Big(\bigcup_{M \in \mathbb{N}} \{|X_i| \leq M\beta_i^{-\frac{1}{2}}\gamma_i^{-\frac{1}{2}},\ \forall i \in \mathbb{N}\}\Big), \quad \text{for } \beta_i = \lambda_i^4,\ \ \gamma_i^{-\frac{1}{2}} = \lambda_i.$$

This shows that the condition (1.12) is satisfied.

Thus, by Theorems 2 and 3, for $\alpha = 1$, there exists an $l^2_{(\lambda_i^4)}$-valued Hunt process

$$\mathbb{M} \equiv \big(\Omega, \mathcal{F}, (X_t)_{t\geq 0}, (P_{\mathbf{x}})_{\mathbf{x}\in S_\triangle}\big), \tag{1.41}$$

associated to the non-local Dirichlet form $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$.

We can now define an $\mathcal{H}_{-2}$-valued process $(Y_t)_{t\geq 0}$ such that

$$(Y_t)_{t\geq 0} \equiv \big(\tau_{-2}^{-1}(X_t)\big)_{t\geq 0}. \tag{1.42}$$

Equivalently, by Eq. 1.39 for $X_t = (X_1(t), X_2(t), \dots) \in l^2_{(\lambda_i^4)}$, $P_{\mathbf{x}} - a.e.$, by setting $A_i(t)$ such that $A_i(t) \equiv \lambda_i X_i(t)$ (cf. Eqs. 1.33 and 1.34), we see that Y_t is also given by

$$Y_t = \sum_{i\in\mathbb{N}} A_i(t)(\lambda_i^{-2}\varphi_i) = \sum_{i\in\mathbb{N}} X_i(t)\varphi_i \in \mathcal{H}_{-2}, \qquad \forall t \geq 0,\ \ P_{\mathbf{x}} - a.e., \ \text{ for any} x \in S_\triangle. \tag{1.43}$$

Thus, by using Eqs. 1.16 and 1.39 we have proven the following:

Theorem 5 (The free field case for $\alpha = 1$) *Let us adopt the definitions given by* Eqs. 1.17–1.24*,* 1.29 *and* 1.30*, for a given $d \in \mathbb{N}$. Let ν_0 be the Euclidean free field probability measure on $\mathcal{S}'(\mathbb{R}^d \to \mathbb{R})$ defined through* Eq. 1.37*. For $\alpha = 1$, let $(Y_t)_{t\geq 0}$ be defined by* Eq. 1.42*, more explicitly by* Eq. 1.43*, then Y_t is an $\mathcal{H}_{-2}$-valued Hunt process that is a stochastic quantization (according to the definition we gave to this term) with respect to the non-local Dirichlet form $(\tilde{\mathcal{E}}_{(\alpha)}, \mathcal{D}(\tilde{\mathcal{E}}_{(\alpha)}))$ on $L^2(\mathcal{H}_{-2}, \nu_0)$, that is defined through $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$, by making use of τ_{-2}. In particular $(Y_t)_{t\geq 0}$ is a non-local Markov process with invariant measure ν_0.*

For the cases where $0 < \alpha < 1$, we can also apply Theorems 1, 2 and 3, and then have the corresponding result to Eq. 1.43. For this purpose we have only to notice that for $\alpha \in (0, 1)$ if we take ν_0 as a Borel probability measure on $\mathcal{H}_{-3}$, and set $S \equiv l^2_{(\lambda_i^6)}$, $\beta_i \equiv \lambda_i^6$, $\gamma_i \equiv \lambda_i$, and define

$$\tau_{-3} : \mathcal{H}_{-3} \ni f \longmapsto (\lambda_1^{-3}a_1, \lambda_1^{-3}a_2, \dots) \in l^2_{(\lambda_i^6)}, \tag{1.44}$$

$$\text{with} \quad a_i \equiv (f, \lambda_i^{-3}\varphi_i)_{-3}, \ \ i \in \mathbb{N},$$

(cf. Eqs. 1.34, 1.35, 1.36 and 1.39), then

$$\sum_{i=1}^{\infty}(\beta_i\gamma_i)^{\frac{\alpha+1}{2}} = \sum_{i=1}^{\infty}(\lambda_i)^{2(\alpha+1)} < \infty.$$

As a consequence, for $\alpha \in (0, 1)$, we then see that by the this setting (1.11) and (1.12) also hold (cf. Eq. 1.40 together with the formula given below (1.40)).

Define a probability measure μ on $l^2_{(\lambda_i^6)}$ such that

$$\mu(B) \equiv \nu_0 \circ \tau_{-3}^{-1}(B) \quad \text{for} \quad B \in \mathcal{B}(l^2_{(\lambda_i^6)}). \tag{1.45}$$

Then we have an analogue of Eq. 1.43 as follows: By Theorems 2 and 3, for each $0 < \alpha < 1$, there exists an $l^2_{(\lambda_i^6)}$-valued Hunt process

$$\mathbb{M} \equiv \big(\Omega, \mathcal{F}, (X_t))_{t\geq 0}, (P_{\mathbf{x}})_{\mathbf{x}\in S_\triangle}\big), \tag{1.46}$$

associated to the non-local Dirichlet form $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$. By making use of $\mathbb{M}$ we can define an $\mathcal{H}_{-3}$-valued process $(Y_t)_{t\geq 0}$ such that $(Y_t)_{t\geq 0} \equiv \big(\tau_{-2}^{-1}(X_t)\big)_{t\geq 0}$, explicitly, by

(1.44) for $X_t = (X_1(t), X_2(t), \dots) \in l^2_{(\lambda_i^6)}$, $P_{\mathbf{x}} - a.e.\, x \in S_\triangle$, by setting $A_i(t)$ such that $X_i(t) = \lambda_i^{-3} A_i(t)$ (cf. Eqs. 1.33 and 1.34), then Y_t is given by

$$Y_t = \sum_{i \in \mathbb{N}} A_i(t)(\lambda_i^{-3}\varphi_i) = \sum_{i \in \mathbb{N}} X_i(t)\varphi_i \in \mathcal{H}_{-3}, \qquad \forall t \geq 0,\ P_{\mathbf{x}} - a.e.,\ x \in S_\triangle. \quad (1.47)$$

Thus, by using Eqs. 1.16 and 1.44 we have proven the following:

Theorem 6 (The Free Field Case for $0 < \alpha < 1$) *Let us adopt the definitions given by* Eqs. 1.17–1.24*,* 1.29 *and* 1.30*, for a given $d \in \mathbb{N}$. Let ν_0 be the Euclidean free field probability measure on $\mathcal{S}'(\mathbb{R}^d \to \mathbb{R})$ defined through* Eq. 1.37*. For each $0 < \alpha < 1$, let $(Y_t)_{t \geq 0}$ be defined by* Eq. 1.47*, then Y_t is an $\mathcal{H}_{-3}$-valued Hunt process that is a stochastic quantization with respect to the non-local Dirichlet form $\big(\tilde{\mathcal{E}}_{(\alpha)}, \mathcal{D}\big(\tilde{\mathcal{E}}(\alpha)\big)\big)$ on $L^2(\mathcal{H}_{-3}, \nu_0)$, that is defined through $\big(\mathcal{E}_{(\alpha)}, \mathcal{D}\big(\mathcal{E}_{(\alpha)}\big)\big)$ (by making use of τ_{-3} via* Eq. 1.45*). In particular $(Y_t)_{t \geq 0}$ is a non-local Markov process with invariant measure ν_0.*

The diffusion case $\alpha = 2$ was already discussed in [22, 24] (and references therein).

2 Other Applications; Quantum Field Models with Interactions, Infinite Particle Systems

Following analogous arguments to the one used for Example 0 (see the previous section) in the present section we shall treat the following problems, related and complementary to those of [16]:

1. *Non-local* type stochastic quantization of the (truncated) Høegh-Krohn exponential model with $d = 2$ (for the considerations on this random field, cf., e.g., [3, 4, 7–9, 15, 20, 26, 29, 37, 47–49, 52, 66].
2. *Non-local* type stochastic quantization of the (space cut-off) $P(\phi)_2$ and the Albeverio Høegh-Krohn trigonometric model with $d = 2$ (for the considerations on this random field, cf., e.g., [9, 13, 14, 26, 28–31, 36]).
3. *Non-local* type stochastic quantization of the random fields of classical infinite particle systems (for the considerations on this random field, and its *local* type stochastic quantizations by means of local Dirichlet form arguments, cf., e.g., [17, 60, 65, 67, 69]).

Example 1. (The Space cut-off Høegh-Krohn Exponential Model With $d = 2$.) Let ν_0 be the Euclidean free field measure on $\mathcal{S}' \equiv \mathcal{S}'(\mathbb{R}^2)$, discussed in Example 0 with $m_0 > 0$ (precisely, see Eqs. 1.37 and 1.38 with $d = 2$). Here, for simplicity we set the mass term $m_0 = 1$. Let a_0 be a given real number and g a given *positive* valued function on $\mathbb{R}^2$ such that

$$|a_0| < \sqrt{4\pi}, \qquad g \in L^2(\mathbb{R}^2 \to \mathbb{R}) \cap L^1(\mathbb{R}^2 \to \mathbb{R}). \quad (2.1)$$

a_0 is called "charge" parameter, g (Euclidean) *space cut-off*. Define a measurable function $V(\cdot)$ on the measure space $(S', \mathcal{B}(\mathcal{S}'), \nu_0)$, as follows:

$$V(\phi) \equiv V_{a_0 \cdot g}(\phi) \equiv \sum_{n=0}^{\infty} \frac{(a_0)^n}{n!} < g, : \phi^n :>, \quad (2.2)$$

(often written as $: \exp(a_0 < g, \phi >) :$), where $< \cdot, \cdot >$ denotes the dualization between a test function and a distribution, and $: \phi^n :$ denotes the n-th Wick monomial of ϕ, the

$S'(\mathbb{R}^2 \to \mathbb{R})$-valued random variable of which probability distribution is the free field measure ν_0 (cf. e.g., [66]). Then, it is shown (cf. e.g., [10], THEOREM V.24 and PROPOSITION VIII.43 of [26, 66], the first work in this direction being for the "time zero" version [10]) that

$$V(\phi) \in L^2(\mathcal{S}'; \nu_0), \quad V(\phi) \geq 0, \quad \nu_0 - a.e., \qquad e^{-V(\phi)} \in L^p(\mathcal{S}'; \nu_0), \quad \forall p \geq 1, \tag{2.3}$$

also

$$V(\phi) \in L^p(\mathcal{S}'; \nu_0) \qquad \text{if} \quad p < \frac{4\pi}{a_0^2}.$$

By Eq. 2.3, the positivity of $V(\phi)$, it holds that

$$0 \leq e^{-V(\phi)} \leq 1, \qquad \nu_0 - a.e.. \tag{2.4}$$

Thus, by Eq. 2.4, it is possible to define a probability measure $\nu_{\exp}$ on $\mathcal{S}'$ such that

$$\nu_{\exp}(d\phi) \equiv \frac{1}{Z} e^{-V(\phi)}\, \nu_0(d\phi), \tag{2.5}$$

where Z is the normalizing constant such that

$$Z \equiv Z_{a_0,g} \equiv \int_{\mathcal{S}'} e^{-V(\phi)}\, \nu_0(d\phi). \tag{2.6}$$

Now, we shall look at the support property of the measure $\nu_{\exp}$ through the Bochner-Minlos theorem (see Theorem 4 in the previous section), and then apply Theorems 1, 2 and 3 in the previous section quoted from [16], to the random field $(\mathcal{S}', \mathcal{B}(\mathcal{S}'), \nu_{\exp})$. To this end, we consider the characteristic function $C(\varphi)$ of $\nu_{\exp}$:

$$C(\varphi) \equiv \int_{\mathcal{S}'} e^{i<\phi,\varphi>}\, \nu_{\exp}(d\phi), \qquad \varphi \in S(\mathbb{R}^2 \to \mathbb{R}). \tag{2.7}$$

The existence (i.e., the well definedness) of $C(\varphi)$ and its continuity property can be guaranteed and shown as follows: We first have the following evaluation:

$$\begin{aligned}
&|\int_{\mathcal{S}'} \sum_{k=0}^{\infty} \frac{(i<\phi,\varphi>)^k}{k!}\, \nu_{\exp}(d\phi) - 1| \\
&= |\frac{1}{Z}\int_{\mathcal{S}'} \sum_{k=1}^{\infty} \frac{(i<\phi,\varphi>)^k}{k!}\, e^{-V(\phi)}\, \nu_0(d\phi)| \\
&\leq |\frac{1}{Z}\sum_{k=1}^{\infty} \frac{i^k}{k!}\int_{S'} <\phi,\varphi>^k\ e^{-V(\phi)}\, \nu_0(d\phi)| \\
&\leq \frac{1}{Z}\sum_{k=1}^{\infty} \frac{|i^k|}{k!}\int_{\mathcal{S}'} |<\phi,\varphi>^k\ e^{-V(\phi)}|\, \nu_0(d\phi) \\
&\leq \frac{1}{Z}\sum_{k=1}^{\infty} \frac{1}{k!}\int_{\mathcal{S}'} |<\phi,\varphi>^k|\, \nu_0(d\phi) \\
&\leq \frac{1}{Z}\{\sum_{l=1}^{\infty} \frac{1}{(2l)!} E_{\nu_0}[<\phi,\varphi>^{2l}] \\
&\quad + \sum_{l=1}^{\infty} \frac{1}{(2l-1)!} E_{\nu_0}[|<\phi,\varphi>|^l \cdot |<\phi,\varphi>|^{l-1}]\},
\end{aligned} \tag{2.8}$$

where, Eq. 2.5 is used for the first equality, and Fubini Theorem and Eq. 2.4 are applied for the first inequality, and Eq. 2.4 is again used for the third inequality, and $E_{\nu_0}[\cdot]$ denotes the expectation with respect to the probability measure ν_0. Then, by denoting

$$|||\varphi|||^2 \equiv ((-\Delta+1)^{-1}\varphi, \varphi)_{L^2(\mathbb{R}^2)}$$

(from the definition of the Euclidean free field, cf., Eq. 1.38) for the first term of the right hand side of the last inequality of Eq. 2.8, it holds that

$$E_{\nu_0}[<\phi, \varphi>^{2l}] = (2l-1)!!|||\varphi|||^{2l},$$

and for the second term (with $l \geq 2$) of the right hand side of the last inequality of Eq. 2.8, by the Cauchy-Schwarz inequality, it holds that

$$\begin{aligned} E_{\nu_0}[|<\phi,\varphi>|^l \cdot |<\phi,\varphi>|^{l-1}] &\leq (E_{\nu_0}[<\phi,\varphi>^{2l}] \cdot E_{\nu_0}[<\phi,\varphi>^{2l-2}])^{\frac{1}{2}} \\ &= ((2l-1)!!|||\varphi|||^{2l} \cdot (2l-3)!!|||\varphi|||^{2l-2})^{\frac{1}{2}}, \end{aligned}$$

also, since,

$$\frac{(2l-1)!!}{(2l)!} = \frac{2^{-l}}{l!} \qquad \text{and} \qquad \frac{((2l-1)!! \cdot (2l-3)!!)^{\frac{1}{2}}}{(2l-1)!} \leq \frac{2^{-l+1}}{(l-1)!},$$

we then see the the right hand side of Eq. 2.8 is dominated by

$$\begin{aligned} &\frac{1}{Z}\left\{\sum_{l=1}^{\infty}\frac{2^{-l}}{l!}|||\varphi|||^{2l} + E_{\nu_0}[|<\phi,\varphi>|] + \sum_{l=2}^{\infty}\frac{2^{-l+1}}{(l-1)!}|||\varphi|||^{2l-1}\right\} \\ &\leq \frac{1}{Z}(2(e^{\frac{1}{2}|||\varphi|||^2}-1) + |||\varphi|||). \end{aligned} \tag{2.9}$$

Equations 2.8 with 2.9 shows that $C(\varphi)$ is continuous *at the origin* with respect to the norm

$$|||\varphi|||^2 \equiv ((-\Delta+1)^{-1}\varphi, \varphi)_{L^2(\mathbb{R}^2)} \leq \|\varphi\|^2_{L^2(\mathbb{R}^2)}, \qquad \varphi \in \mathcal{S}(\mathbb{R}^2 \to \mathbb{R}),$$

and hence, by Remark 1.2, $C(\varphi)$ has the same continuity as the characteristic function of the Euclidean free field measure given by Eqs. 1.37 and 1.38 (though, by the arguments of the present evaluation, in Eq. 2.9, the sign of the exponent is positive). Then, by Theorem 4 (with Remark 1.2) and Eq. 1.28 the support of $\nu_{\exp}$ can be taken to be in the Hilbert spaces $\mathcal{H}_{-n}$, $n \geq 1$ (cf. Eqs. 1.35 and 1.36).

Thus, by repeating the same arguments for the Euclidean free field (cf. Eqs. 1.39–1.43), and by applying Theorems 1, 2 and 3 in Section 1, we have the analogous results on the non-local stochastic quantization for the space cut-off Høegh-Krohn field $\nu_{\exp}$ with $d=2$ to those for the Euclidean free field with $d=2$ (see Theorems 5 and 6) as follows:

Theorem 7 (The Space cut-off Høegh-Krohn Model With $d=2$, $\alpha=1$) *Let us adopt the definitions given by* Eqs. 1.17–1.24, 1.29 *and* 1.30, *for* $d=2$. *Let* ν_{exp} *be the Borel probability measure on* $\mathcal{S}'(\mathbb{R}^2 \to \mathbb{R})$ *defined through* Eq. 2.5. *By* Eq. 1.39 *with* $d=2$, *define a Borel probabilty measure* $\mu_{\exp}$ *on* $l^2_{(\lambda_i^4)}$ *such that* $\mu_{exp}(B) = \nu_{exp} \circ \tau_{-2}^{-1}(B)$ *for* $B \in \mathcal{B}(l^2_{(\lambda_i^4)})$. *Then, by Theorems 2 and 3, the following hold:*

i) *For* $\alpha = 1$, *on* $L^2(l^2_{(\lambda_i^4)}, \mu_{exp})$ *the non-local Dirichlet form* $(E_\alpha, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ *is well defined and there exists an associated* $l^2_{(\lambda_i^4)}$*-valued Hunt process* $\mathbb{M} \equiv \big(\Omega, \mathcal{F}, (X_t)_{t\geq 0}, (P_{\mathbf{x}})_{\mathbf{x}\in S_\triangle}\big)$.

ii) *For* $X_t = (X_1(t), X_2(t), \ldots) \in l^2_{(\lambda_i^4)}$, $P_{\mathbf{x}}$ *− a.e., define* $Y_t = \sum_{i\in\mathbb{N}} X_i(t)\varphi_i$ $(\forall t \geq 0$, $P_{\mathbf{x}}$ *− a.e., for any* $x \in S_\triangle$*, then* Y_t *is an* $\mathcal{H}_{-2}$*-valued Hunt process that is a stochastic quantization with respect to the non-local Dirichlet form* $(\tilde{\mathcal{E}}_{(\alpha)}, \mathcal{D}(\tilde{\mathcal{E}}_{(\alpha)}))$ *on* $L^2(\mathcal{H}_{-2}, \nu_{exp})$*, that is defined through* $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)})))$*, by making use of* τ_{-2}*. In particular* $(Y_t)_{t\geq 0}$ *is a non-local Markov process with invariant measure* ν_{exp}*.*

Theorem 8 (The Space cut-off Høegh-Krohn Model With $d = 2, 0 < \alpha < 1$) *Let us adopt the definitions given by* Eqs. 1.17–1.24, 1.29 *and* 1.30*, for* $d = 2$*. Let* $\nu_{\exp}$ *be the Borel probability measure on* $\mathcal{S}'(\mathbb{R}^2 \to \mathbb{R})$ *defined through* Eq. 2.5*. By* Eq. 1.44 *with* $d = 2$*, define a Borel probabilty measure* $\mu_{\exp}$ *on* $l^2_{(\lambda_i^6)}$ *such that* $\mu_{exp}(B) = \nu_{exp} \circ \tau_{-2}^{-1}(B)$ *for* $B \in \mathcal{B}(l^2_{(\lambda_i^6)})$*. Then, by Theorems 2 and 3, the following hold:*

i) *For each* $0 < \alpha < 1$*, on* $L^2(l^2_{(\lambda_i^6)}, \mu_{exp})$ *the non-local Dirichlet form* $(\mathcal{E}_\alpha, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ *is well defined and there exists an associated* $l^2_{(\lambda_i^6)}$*-valued Hunt process* $\mathbb{M} \equiv \big(\Omega, \mathcal{F}, (X_t)_{t\geq 0}, (P_{\mathbf{x}})_{\mathbf{x}\in S_\triangle}\big)$*.*

ii) *For* $X_t = (X_1(t), X_2(t), \ldots) \in l^2_{(\lambda_i^6)}$, $P_{\mathbf{x}}$ *− a.e., define* $Y_t = \sum_{i\in\mathbb{N}} X_i(t)\varphi_i$ $(\forall t \geq 0$, $P_{\mathbf{x}}$ *− a.e., for any* $x \in S_\triangle$*, then* Y_t *is an* $\mathcal{H}_{-3}$*-valued Hunt process that is a stochastic quantization with respect to the non-local Dirichlet form* $(\tilde{\mathcal{E}}_{(\alpha)}, \mathcal{D}(\tilde{\mathcal{E}}_{(\alpha)}))$ *on* $L^2(\mathcal{H}_{-3}, \nu_{exp})$*, that is defined through* $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)})))$*, by making use of* τ_{-3}*. In particular* $(Y_t)_{t\geq 0}$ *is a non-local Markov process with invariant measure* $\nu_{\exp}$*.*

Example 2. (The Space cut-off $P(\phi)_2$ and the Albeverio Høegh-Krohn trigonometric Model With d = 2.) Let us consider once more the Euclidean free field measure ν_0 on $\mathcal{S}' \equiv \mathcal{S}'(\mathbb{R}^2 \to \mathbb{R})$, discussed in Example 0 (precisely, see Eqs. 1.37 and 1.38 with $d = 2$). As in Example 1, for simplicity we set the mass term $m_0 = 1$.

Let $\nu_{P(\Phi)}$, $\nu_{\sin}$ and $\nu_{\cos}$ be the probability measures on $\mathcal{S}'$ that are defined by (cf. Eq. 2.5)

$$\nu_{P(\Phi)}(d\phi) \equiv \frac{1}{Z_{P(\Phi)}} e^{-V_{P(\Phi)}(\phi)} \nu_0(d\phi), \tag{2.10}$$

and

$$\nu_{\sin}(d\phi) \equiv \frac{1}{Z_{\sin}} e^{-V_{\sin}(\phi)} \nu_0(d\phi), \qquad \nu_{\cos}(d\phi) \equiv \frac{1}{Z_{\cos}} e^{-V_{\cos}(\phi)} \nu_0(d\phi), \tag{2.11}$$

with

$$\begin{gathered} V_{P(\Phi)}(\phi) \equiv \lambda \; < g, :\phi^{2n}:>, \\ \text{for some given} \quad \lambda \geq 0, \;\; n \in \mathbb{N}, \\ g \in L^2(\mathbb{R}^2 \to \mathbb{R}) \cap L^1(\mathbb{R}^2 \to \mathbb{R}) \text{ satisfying } g \geq 0, \end{gathered} \tag{2.12}$$

and

$$V_{\sin}(\phi) \equiv \lambda \sum_{k=0}^{\infty} \frac{(-1)^k (a_0)^{2k+1}}{(2k+1)!} < g, :\phi^{2k+1}:>, \quad V_{\cos}(\phi) \equiv \lambda \sum_{k=0}^{\infty} \frac{(-1)^k (a_0)^{2k}}{(2k)!} < g, :\phi^{2k}:>,$$

$$\text{for some given} \quad \lambda \geq 0, \;\; |a_0| < \sqrt{4\pi}, \quad g \;\; \text{as in (2.12)}, \tag{2.13}$$

respectively, where $Z_{P(\Phi)}$, $Z_{\sin}$ and $Z_{\cos}$ are the corresponding normalizing constants such that

$$Z_{P(\Phi)} \equiv \int_{\mathcal{S}'} e^{-V_{P(\phi)}} \nu_0(d\phi), \quad Z_{\sin} \equiv \int_{\mathcal{S}'} e^{-V_{\sin}} \nu_0(d\phi), \quad Z_{\cos} \equiv \int_{\mathcal{S}'} e^{-V_{cos}} \nu_0(d\phi), \tag{2.14}$$

respectively. Under the above settings, it is known that (for the polynomial potential case cf., e.g., [35, 58, 40, 66], and for the trigonometric potential case cf. [9, 13, 26, 36, 37, 40]), one has that $W \in \bigcap_{r\geq 1} L^r(\mathcal{S}'; \nu_0)$ for any

$$W = e^{-V_{P(\Phi)}}, \quad e^{-V_{\sin}}, \quad e^{-V_{\cos}}. \tag{2.15}$$

The bound (2.4), i.e., $0 \leq e^{-V} \leq 1$, which holds for $V_{\exp}$ in Example 1, does not hold for these $V's$ in the present example, but by Eq. 2.15, similar to the case of $V_{\exp}$, we can show that the characteristic functions (cf. Theorem 4) of these probability measures, $\nu_{P(\Phi)}$, $\nu_{\sin}$ and $\nu_{\cos}$ satisfy the *comparable continuity* as the characteristic function of the Euclidean free field measure. Thus by Theorem 4 and Eq. 1.28 the support of $\nu_{P(\Phi)}$, $\nu_{\sin}$ and $\nu_{\cos}$ can be taken to be in the Hilbert spaces $\mathcal{H}_{-n}$, $n \geq 1$ (cf. Eqs. 1.35 and 1.36). Then, for the present random fields, we can repeat the arguments for the Euclidean free field (cf. Eqs. 1.39–1.41), getting corresponding results as the the ones of Example 1.

The corresponding continuity of the characteristic functions of the probability measures $\nu_{P(\Phi)}$, $\nu_{\sin}$ and $\nu_{\cos}$ can be seen as follows.

Denote by $F(\phi)$, $\phi \in \mathcal{S}'(\mathbb{R}^2 \to \mathbb{R})$, one of the *positive* random variables$e^{-V_{P(\Phi)}}/Z_{P(\phi)}$, $e^{-V_{\sin}}/Z_{\sin}$ and $e^{-V_{\cos}}/Z_{\cos}$ on the probability space $(\mathcal{S}', \mathcal{B}(\mathcal{S}'), \nu_0)$. Then by Eq. 2.15, by applying the Hölder's inequality (twice) (with $p = \frac{4}{3}$, $q = 4$), we have (cf. Eq. 2.8), for $\varphi \in \mathcal{S} \equiv \mathcal{S}(\mathbb{R}^2 \to \mathbb{R})$,

$$\begin{aligned}
&|\int_{S'} \sum_{k=0}^{\infty} \frac{(i <\phi, \varphi>)^k}{k!} F(\phi)\, \nu_0(d\phi) - 1| \\
&\leq \sum_{k=1}^{\infty} \frac{1}{k!} \int_{S'} \big| <\phi, \varphi>^k \big| \, F(\phi)\, \nu_0(d\phi) \\
&\leq \sum_{k=1}^{\infty} \{ \Big(\int_{S'} \big| <\phi, \varphi> \big|^{\frac{4}{3}k} \nu_0(d\phi) \Big)^{\frac{3}{4}} \Big(\int_{S'} (F(\phi))^4\, \nu_0(d\phi) \Big)^{\frac{1}{4}} \} \\
&\leq \sum_{k=1}^{\infty} \{ \Big(\int_{S'} \big| <\phi, \varphi> \big|^{4k} \nu_0(d\phi) \Big)^{\frac{1}{4}} \Big(\int_{S'} (F(\phi))^4\, \nu_0(d\phi) \Big)^{\frac{1}{4}} \}.
\end{aligned} \tag{2.16}$$

Also, by denoting $r \equiv |||\varphi||| \equiv ((-\Delta + 1)^{-1}\varphi, \varphi)$, note that the following evaluations hold:

$$\begin{aligned}
\frac{1}{k!} \Big(\int_{S'} (<\phi, \varphi>)^{4k} \nu_0(d\phi) \Big)^{\frac{1}{4}} &= \frac{1}{k!} ((4k-1)!!\, r^{2k})^{\frac{1}{4}} \leq \frac{1}{k!} r^{\frac{k}{2}} ((4k)!!)^{\frac{1}{4}} \\
&\leq \frac{1}{k!} r^{\frac{k}{2}} \big(((2k)!!)^2 \big)^{\frac{1}{4}} = \frac{1}{(k!)^{\frac{1}{2}}} (4r)^{\frac{k}{2}},
\end{aligned} \tag{2.17}$$

and

$$(k!)^{-\frac{1}{2}} \leq 2^{-\frac{k}{2}} \frac{1}{(l-1)!}, \qquad \text{for} \quad k = 2l, \; l \in \mathbb{N}, \tag{2.18}$$

$$(k!)^{-\frac{1}{2}} \leq 2^{-\frac{k}{2}} \frac{1}{(l-2)!}, \qquad \text{for} \quad k = 2l-1, \; l \in \mathbb{N}, \; l \geq 2. \tag{2.19}$$

Then, by applying Eqs. 2.17, 2.18 and 2.19 to the right hand side of Eq. 2.16 we see that

$$
\begin{aligned}
&|\int_{\mathcal{S}'} \sum_{k=0}^{\infty} \frac{(i<\phi,\varphi>)^k}{k!} F(\phi)\,\nu_0(d\phi) - 1| \\
&\leq \sum_{k=1}^{\infty} \frac{1}{(k!)^{\frac{1}{2}}} (4r)^{\frac{k}{2}} \cdot \Big(\int_{\mathcal{S}'} (F(\phi))^4\,\nu_0(d\phi)\Big)^{\frac{1}{4}} \\
&\leq \{(4r)^{\frac{1}{2}} + \sum_{l=1}^{\infty} \frac{1}{(l-1)!} 2^{-l}(4r)^l \\
&\quad + \sum_{l=2}^{\infty} \frac{1}{(l-2)!} 2^{-l+\frac{1}{2}} (4r)^{l-\frac{1}{2}}\} \cdot \Big(\int_{\mathcal{S}'} (F(\phi))^4\,\nu_0(d\phi)\Big)^{\frac{1}{4}} \\
&= 2r^{\frac{1}{2}}\{1 + e^{2r} r^{\frac{1}{2}} \big(1 + 2^{\frac{1}{2}} r^{\frac{1}{2}}\big)\} \cdot \Big(\int_{\mathcal{S}'} (F(\phi))^4\,\nu_0(d\phi)\Big)^{\frac{1}{4}}.
\end{aligned}
\tag{2.20}
$$

Equations 2.14 and 2.15, 2.16 with Eq. 2.20 show that the characteristic functions of $\nu_{P(\phi)}$, $\nu_{\sin}$ and $\nu_{\cos}$ are continuous *at the origin* with respect to the norm $r = |||\varphi||| \equiv \big((-\Delta + 1)^{-1}\varphi, \varphi\big)^{\frac{1}{2}}$ for $\varphi \in \mathcal{S}'$. Hence, by Remark 1.2, the the characteristic functions (cf. Theorem 4) of $\nu_{P(\Phi)}$, $\nu_{\sin}$ and $\nu_{\cos}$ satisfy the *comparable continuity* as the characteristic function of the Euclidean free field measure.

Thus we can apply Theorems 1, 2 and 3 to the non-local stochastic quantization of the random fields $\nu_{P(\Phi)}$, $\nu_{\sin}$ and $\nu_{\cos}$, then for these random fields we also get the analogous statements as the one for the Euclidean free field with $d = 2$ and the one for the space cut-off Høegh-Krohn model in Example 1 as follows:

Theorem 9 (The Space cut-off $P(\phi)_2$ and the Albeverio Høegh-Krohn Trigonometric Model With $\mathbf{d = 2}$, $\alpha = 1$) *Let us adopt the definitions given by* Eqs. 1.17–1.24, 1.29 *and* 1.30, *for $d = 2$. Let ν be one of the Borel probability measures $\nu_{P(\Phi)}$, $\nu_{\sin}$ or $\nu_{\cos}$ on $\mathcal{S}'(\mathbb{R}^2 \to \mathbb{R})$ defined through* Eqs. 2.10 *and* 2.11 *respectively. By* Eq. 1.39 *with $d = 2$, define a Borel probabilty measure μ on $l^2_{(\lambda_i^4)}$ such that $\mu(B) = \nu \circ \tau_{-2}^{-1}(B)$ for $B \in \mathcal{B}(l^2_{(\lambda_i^4)})$. Then, by Theorems 2 and 3, the following hold:*

i) *For $\alpha = 1$, on $L^2(l^2_{(\lambda_i^4)}, \mu)$ the non-local Dirichlet form $(\mathcal{E}_\alpha, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ is well defined and there exists an associated $l^2_{(\lambda_i^4)}$-valued Hunt process* $\mathbb{M} \equiv \big(\Omega, \mathcal{F}, (X_t)_t \geq 0, (P_{\mathbf{x}})_{\mathbf{x}\in S_\triangle}\big)$.

ii) *For $X_t = (X_1(t), X_2(t), \dots) \in l^2_{(\lambda_i^4)}$, $P_{\mathbf{x}} - a.e.$, define $Y_t = \sum_{i\in\mathbb{N}} X_i(t)\varphi_i$ ($\forall t \geq 0$, $P_{\mathbf{x}} - a.e.$, for any $x \in S_\triangle$, then Y_t is an $\mathcal{H}_{-2}$-valued Hunt process that is a stochastic quantization with respect to the non-local Dirichlet form $(\tilde{\mathcal{E}}_{(\alpha)}, \mathcal{D}(\tilde{\mathcal{E}}_{(\alpha)})))$ on $L^2(\mathcal{H}_{-2}, \nu)$, that is defined through $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$, by making use of τ_{-2}. In particular $(Y_t)_{t\geq 0}$ is a non-local Markov process with invariant measure ν.*

Theorem 10 (The Space cut-off $P(\phi)_2$ and the Albeverio Høegh-Krohn Trigonometric Model With $\mathbf{d = 2}$, $0 < \alpha < 1$) *Let us adopt the definitions given by* Eqs. 1.17–1.24, 1.29 *and* 1.30, *for $d = 2$. Let ν be one of the Borel probability measures $\nu_{P(\Phi)}$, $\nu_{\sin}$ or $\nu_{\cos}$ on $\mathcal{S}'(\mathbb{R}^2 \to \mathbb{R})$ defined through* Eqs. 2.10 *and* 2.11 *respectively. By* Eq. 1.44 *with $d = 2$,* *define a Borel probabilty measure* μ *on* $l^2_{(\lambda_i^6)}$ *such that* $\mu(B) = \nu \circ \tau_{-6}^{-1}(B)$ *for* $B \in B(l^2_{(\lambda_i^6)})$. *Then, by Theorems 2 and 3, the following hold:*

i) *For each* $0 < \alpha < 1$, *on* $L^2(l^2_{(\lambda_i^6)}, \mu)$ *the non-local Dirichlet form* $(\mathcal{E}_\alpha, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ *is well defined and there exists an associated* $l^2_{(\lambda_i^6)}$*-valued Hunt process* $\mathbb{M} \equiv \big(\Omega, \mathcal{F}, (X_t)_{t\geq 0}, (P_{\mathbf{x}})_{\mathbf{x}\in S_\triangle}\big)$.

ii) *For* $X_t = (X_1(t), X_2(t), \dots) \in l^2_{(\lambda_i^6)}$, $P_{\mathbf{x}} - a.e.$, *define* $Y_t = \sum_{i\in\mathbb{N}} X_i(t)\varphi_i$ $(\forall t \geq 0$, $P_{\mathbf{x}} - a.e.$, *for any* $x \in S_\triangle$, *then* Y_t *is an* $\mathcal{H}_{-3}$*-valued Hunt process that is a stochastic quantization with respect to the non-local Dirichlet form* $(\tilde{\mathcal{E}}_{(\alpha)}, \mathcal{D}(\tilde{\mathcal{E}}_{(\alpha)})))$ *on* $L^2(H_{-3}, \nu)$, *that is defined through* $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$, *by making use of* τ_{-3}. *In particular* $(Y_t)_{t\geq 0}$ *is a non-local Markov process with invariant measure* ν.

Example 3. (Non-Local Stochastic Quantization for Classical Infinite Particle Systems.)

In this example we apply Theorems 1, 2 and 3 to the random fields of classical statistical mechanics considered by [65].

On the local type stochastic quantizations for such random fields, the various considerations have been already made through the arguments of *local Dirichlet forms* (for the fundamental formulations cf. [17, 60, 67, 69], and for the corresponding extended considerations cf. [33, 61] and references therein, also cf. [53] where a first consideration on the stochastic quantization of such a random field through the arguments of an infinite system of stochastic differential equations is presented). But, as far as we know, there exists no considerations on the *non-local* type stochastic quantizations for such classical particle systems through the arguments by *non-local Dirichlet forms*.

We first recall the configuration space for the classical infinite particle systems (for an original formulation, cf. [65], and also cf. [69] for their interpretation as a subspace of Radon measures, which will be used in the subsequent discussions of the present example (the notations, e.g., $\mathcal{Y}$, adopted here are diffent to the ones in [65] and [69])). Define

$$\mathcal{Y} \equiv \{\mathbb{Y} \,|\, \mathbb{Y} : \mathbb{R}^d \to \mathbb{Z}_+ \text{ such that} \sum_{y\in K} \mathbb{Y}(y) < \infty \text{ for any compact} K \subset \mathbb{R}^d\},$$

$$\sigma[\mathcal{Y}] \equiv \text{the } \sigma - \text{field generated by } \{\mathbb{Y} \,|\, \sum_{y\in B} \mathbb{Y}(y) = m\},$$
$$B \text{ running over the bounded Borel set of } \mathbb{R}^d,\ m \in \mathbb{Z}_+, \tag{2.21}$$

where $d \in \mathbb{N}$ is a given dimension, and $\mathbb{Z}_+$ is the set of non-negative integers, $\mathbb{Z}_+ \equiv \mathbb{N} \cup \{0\}$.

On the measurable space $(\mathcal{Y}, \sigma[\mathcal{Y}])$, suppose that we are given a probability measure ν that satisfies (cf. Corollary 2.8, Prop. 5.2 of [65], and cf. also (2.19) of [69]):

$$\nu\Big(\bigcup_{N\in\mathbb{N}} U_N \Big) = 1, \tag{2.22}$$

and for some given $\gamma > 0$ and real δ,

$$\nu\big(U_N^c\big) \leq \sum_{l=0}^{\infty} \big\{ \exp[-(\gamma N^2 - e^\delta)] \big\}^{l+1}, \tag{2.23}$$

where, for $N \in \mathbb{N}$,

$$U_N \equiv \Big\{ \mathbb{Y} \in \mathcal{Y} \,|\, \forall l \in \mathbb{Z}_+, \sum_{r:|r|\leq l} n(\mathbb{Y}, r)^2 \leq N^2(2l+1)^d \Big\} \tag{2.24}$$

with

$$n(\mathbb{Y}, r) \equiv \sum_{y \in Q_r} \mathbb{Y}(y), \tag{2.25}$$

$$Q_r \equiv \left\{ y = (y^1, \cdots, y^d) \in \mathbb{R}^d \mid r^j - \frac{1}{2} \leq y^j < r^j + \frac{1}{2}, \, j = 1, \ldots, d \right\}, \quad r = (r^1, \ldots, r^d) \in \mathbb{Z}^d. \tag{2.26}$$

Define (cf. Eq. 2.22)

$$\mathcal{Y}_0 \equiv \bigcup_{N \in \mathbb{N}} U_N \subset \mathcal{Y}, \tag{2.27}$$

and the corresponding σ-field

$$\sigma[\mathcal{Y}_0] \equiv \{ B \cap \mathcal{Y}_0 \mid B \in \sigma[\mathcal{Y}] \}. \tag{2.28}$$

By Eqs. 2.22 and 2.23, we restrict ν (originally defined on $\mathcal{Y}$) to $\mathcal{Y}_0 \subset \mathcal{Y}$, and denote the restriction by the same notation ν. Then, we can define the probability space

$$(\mathcal{Y}_0, \sigma[\mathcal{Y}_0], \nu). \tag{2.29}$$

Subsequently, we shall interpret the probability measure ν on the configuration space $\mathcal{Y}_0$ to a probability measure on a subset of the space of Radon measures on $\mathbb{R}^d$. We note that each $\mathbb{Y} \in \mathcal{Y}$ can be identified with z, an element of positive integer valued Radon measures on $\mathbb{R}^d$, such that (cf. [69])

$$z = \sum_{i=1}^{\infty} m_i \delta_{y_i}, \quad y_i \in \mathbb{R}^d, \quad i \in \mathbb{N}, \tag{2.30}$$

for given $\mathbb{Y} \in \mathcal{Y}$ with $\{y_i\}_{i \in \mathbb{N}} \equiv \{y \in \mathbb{R}^d \mid \mathbb{Y}(y) \neq 0\}$ and $m_i = \mathbb{Y}(y_i)$, where δ_{y_i} denotes the Dirac measure at the point $y_i \in \mathbb{R}^d$. Define

$$\tilde{\mathcal{Y}}_0 \equiv \left\{ z \mid z \text{ corresponds with an } \mathbb{Y} \in Y_0 \text{ by } (2.30) \right\}, \tag{2.31}$$

$$\sigma[\tilde{\mathcal{Y}}_0] \equiv \left\{ \left\{ z \mid z \text{ corresponds with an } \mathbb{Y} \in B, \text{ by } (2.30) \right\} \,\middle|\, B \in \sigma[\mathcal{Y}_0] \right\}, \tag{2.32}$$

and

$$\tilde{\nu}(\tilde{B}) \equiv \nu(B), \quad \text{for } \tilde{B} = \left\{ z \mid z \text{ corresponds with an } \mathbb{Y} \in B, \text{ by } (2.30) \right\} \in \sigma[\tilde{\mathcal{Y}}_0]. \tag{2.33}$$

Then, from Eq. 2.29, through Eqs. 2.30–2.33, we can define the probability space (on a *subset* of the space of Radon measures (cf., e.g., [51], also as a general reference cf. [68]) such that

$$(\tilde{\mathcal{Y}}_0, \sigma[\tilde{\mathcal{Y}}_0], \tilde{\nu}). \tag{2.34}$$

Next, we embed $\tilde{\mathcal{Y}}_0$ defined by Eq. 2.31 into a Hilbert space, and interpret the present random field $(\tilde{\mathcal{Y}}_0, \sigma[\tilde{\mathcal{Y}}_0], \tilde{\nu})$ to be the one on which we can apply Theorems 1, 2 and 3. To this end, for the present consideration, we modify the Hilbert-Schmidt operator and the corresponding nuclear space defined through Eqs. 1.19–1.32 as follows: Let

$$\mathcal{H}_0 \equiv L^2(\mathbb{R}^d \to \mathbb{R}; \lambda), \quad \text{with } \lambda \text{ the Lebesgue measure on } \mathbb{R}^d, \tag{2.35}$$

and

$$\tilde{H} \equiv (|x|^2 + 1)^{d+1} (-\Delta + 1)^{\frac{d+1}{2}} (|x|^2 + 1)^{d+1}, \tag{2.36}$$

$$\tilde{H}^{-1} \equiv (|x|^2 + 1)^{-(d+1)} (-\Delta + 1)^{-\frac{d+1}{2}} (|x|^2 + 1)^{-(d+1)}, \tag{2.37}$$

$\tilde{\mathcal{H}}_n$ and $\tilde{\mathcal{H}}_{-n}$ be the completion of $\mathcal{S} \equiv \mathcal{S}(\mathbb{R}^d \to \mathbb{R})$, the space of Schwartz's rapidly decreasing functions, equipped with the norms corresponding to the inner products $(\cdot, \cdot)_n$ and $(\cdot, \cdot)_{-n}$ respectively such that

$$(f, g)_n \equiv (\tilde{H}^n f, \tilde{H}^n g)_{\mathcal{H}_0}, \qquad f, g \in \mathcal{S}, \tag{2.38}$$

$$(f, g)_{-n} \equiv ((\tilde{H}^{-1})^n f, (\tilde{H}^{-1})^n g)_{\mathcal{H}_0}, \qquad f, g \in \mathcal{S}. \tag{2.39}$$

Then, through arguments analogous to those performed in Section 1 and in the previous examples (with obvious adequate modifications of notations and notions) we have the *continuous* inclusion

$$\tilde{\mathcal{H}}_3 \subset \tilde{\mathcal{H}}_2 \subset \tilde{\mathcal{H}}_1 \subset \mathcal{H}_0 = L^2 \subset \tilde{\mathcal{H}}_{-1} \subset \tilde{\mathcal{H}}_{-2} \subset \tilde{\mathcal{H}}_{-3}, \tag{2.40}$$

and by the self-adjoint extension of H^{-1} on $\mathcal{S}$, for each domain $\tilde{\mathcal{H}}_n$, $n \in \mathbb{Z}$ (setting $\tilde{\mathcal{H}}_0 = \mathcal{H}_0$), we also have the Hilbert-Schmidt operator $\tilde{H}^{-1}$.

Let $\{\tilde{\varphi}_i\}_{i\in\mathbb{N}}$ be the orthonormal base of the Hilbert space $\mathcal{H}_0$ (cf. Eqs. 1.29–1.32) such that

$$\tilde{H}^{-1}\tilde{\varphi}_i = \tilde{\lambda}_i \tilde{\varphi}_i, \qquad i \in \mathbb{N}, \tag{2.41}$$

where $\{\tilde{\lambda}_i\}_{i\in\mathbb{N}}$ is the family of the corresponding eigenvalues, that satisfies

$$0 < \cdots < \tilde{\lambda}_2 < \tilde{\lambda}_1 \leq 1, \qquad \{\tilde{\lambda}_i\}_{i\in\mathbb{N}} \in l^2. \tag{2.42}$$

Through the preparations (2.35)–(2.42) above, we see that for any $\mathbb{Y} \in Y_0$ the corresponding Radon measure $z \in \tilde{\mathcal{Y}}_0$ defined by (2.30) satisfies

$$z \in \tilde{\mathcal{H}}_{-1}. \tag{2.43}$$

Equivalently, we are able to show that the following Lemma 2.1 holds:

Lemma 2.1 *For the subset of Radon measures $\tilde{\mathcal{Y}}_0$ defined by* Eq. 2.31*, it holds that*

$$\tilde{\mathcal{Y}}_0 \subset \tilde{\mathcal{H}}_{-1}.$$

The proof of Lemma 2.1 is given in Appendix A.

Moreover, the following Lemma 2.2 holds.

Lemma 2.2 *For the σ-field $\sigma[\tilde{\mathcal{Y}}_0]$ defined by* Eq. 2.32 *and the Borel field $\mathcal{B}(\tilde{\mathcal{H}}_{-r})$ of the Hilbert space $\tilde{\mathcal{H}}_{-r}$, it holds that $\sigma[\tilde{\mathcal{Y}}_0] \supset \big(\mathcal{B}(\tilde{\mathcal{H}}_{-r}) \cap \tilde{\mathcal{Y}}_0\big)$, for $r \geq 1$.*

The proof of Lemma 2.2 is also given in the Appendix A.

By Lemmas 2.1 and 2.2, the *probability measure* of the classical infinite particle system $\tilde{\nu}$ on $(\tilde{\mathcal{Y}}_0, \sigma[\tilde{\mathcal{Y}}_0])$ (cf. Eq. 2.34) can be extended, for each $r \geq 1$, to a *Borel probability measure* ν_r on $(\tilde{\mathcal{H}}_{-r}, \mathcal{B}(\tilde{\mathcal{H}}_{-r}))$ as follows:

$$\nu_r(B) = \tilde{\nu}(B \cap \tilde{\mathcal{Y}}_0), \qquad B \in \mathcal{B}(\tilde{\mathcal{H}}_{-r}). \tag{2.44}$$

For the subsequent discussion, we take $r = 3$, and consider the corresponding extended random field $(\tilde{\mathcal{H}}_{-3}, \mathcal{B}(\tilde{H}_{-3}), \nu_3)$ to $(\tilde{\mathcal{Y}}_0, \sigma[\tilde{\mathcal{Y}}_0], \tilde{\nu})$ (cf. Eq. 2.34).

By making use of Lemma 2.2, we shall proceed to the application of Theorems 1, 2 and 3 to the random field $(\tilde{H}_{-3}, \mathcal{B}(\tilde{\mathcal{H}}_{-3}), \nu_3)$. From Eq. 2.40, 2.41 and 2.42 (cf. Eqs. 1.29–1.34), we see that for $k = 0, 1, 2, 3$ with $\tilde{\mathcal{H}}_{-0} = \tilde{\mathcal{H}}_0 = \mathcal{H}$,

$$\{(\tilde{\lambda}_i)^k \tilde{\varphi}_i\}_{i\in\mathbb{N}} \qquad \text{is an O.N.B. of } \tilde{\mathcal{H}}_k, \tag{2.45}$$

$$\{(\tilde{\lambda}_i)^{-k} \tilde{\varphi}_i\}_{i\in\mathbb{N}} \qquad \text{is an O.N.B. of } \tilde{\mathcal{H}}_{-k},, \tag{2.46}$$

and we can define an isometric isomorphism τ between $\tilde{\mathcal{H}}_{-3}$ and $l^2(\tilde{\lambda}^6)$ such that

$$\tau \ : \ \tilde{\mathcal{H}}_{-3} \ni f \longmapsto (\tilde{\lambda}_1^{-3}a_1, \tilde{\lambda}_2^{-3}a_2, \dots) \in l^2(\tilde{\lambda}^6), \quad \text{with } a_i \equiv (f, \tilde{\lambda}_i^{-3}\tilde{\varphi}_i)_{-3}, \ i \in \mathbb{N}, \tag{2.47}$$

where the inner product $(\cdot, \cdot)_{-3}$ is defined, not by Eq. 1.24, but by Eq. 2.39. Then, through ν_p defined by Lemma 2.2 and the mapping τ, we can define a Borel probability measure μ on $l^2(\tilde{\lambda}^6)$ as follows:

$$\mu(B) \equiv \nu_3 \circ \tau^{-1}(B) \qquad \text{for } B \in \mathcal{B}(l^2(\tilde{\lambda}^6)). \tag{2.48}$$

Now, by setting $S = l^2(\tilde{\lambda}^6)$ in Theorems 1, 2 and 3, for each $\alpha \in (0, 1]$, we have an $l^2(\tilde{\lambda}^6)$-valued Hunt process $\mathbb{M} \equiv (\Omega, F, (X_t)_{t\geq 0}, (P_{\mathbf{x}})_{\mathbf{x}\in S_\triangle})$, associated to the non-local Dirichlet form $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ on $L^2(S, \mu)$. We can then define an $\tilde{\mathcal{H}}_{-3}$-valued Hunt process $(Y_t)_{t\geq 0}$ (cf. Eqs. 1.42 and 1.43) such that

$$Y_t = \begin{cases} \sum_{i\in\mathbb{N}} X_i(t)\tilde{\varphi}_i, & X_i(t) \neq \Delta \\ \Delta', & X_i(t) = \Delta, \end{cases} \tag{2.49}$$

where Δ' is a point adjoint to $\tilde{\mathcal{H}}_{-3}$ (the cemetery).

Through the above discussion we have proven the following:

Theorem 11 *Let $d \in \mathbb{N}$ be given. Let $(\tilde{\mathcal{Y}}_0, \sigma[\tilde{\mathcal{Y}}_0], \tilde{\nu})$ be the probability space on the space of non negative integer valued Radon measures defined by* Eq. 2.34*, which is a replesentation of $(\mathcal{Y}_0, \sigma[\mathcal{Y}_0], \nu)$ defined by* Eq. 2.29*, the probabilty space of the infinite particle system on the configuration space $\mathcal{Y}_0$ (see* Eqs. 2.21 and 2.27*).*

Define the probability measure ν_3 by Eq. 2.44*, by setting $r = 3$, and let $(\tilde{\mathcal{H}}_{-3}, \mathcal{B}(\tilde{\mathcal{H}}_{-3}), \nu_3)$ be the probability space on the Hilbert space $\tilde{\mathcal{H}}_{-3} \supset \tilde{\mathcal{Y}}_0$, defined through* Eqs. 2.35–2.40*, which is an extension of $(\tilde{\mathcal{Y}}_0, \sigma[\tilde{\mathcal{Y}}_0], \tilde{\nu})$.*

Then by setting μ as the probabilty measure on the abstract sequence space $l^2(\tilde{\lambda}^6)$ such that $\mu(B) \equiv \nu_3 \circ \tau^{-1}(B)$, for $B \in \mathcal{B}(l^2(\tilde{\lambda}^6))$, with τ defined through Eqs. 2.41 and 2.47*, for each $\alpha \in (0, 1]$, the non-local Dirichlet form $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ on $L^2(l^2(\tilde{\lambda}^6), \mu)$ is well defined, and there corresponds an $l^2(\tilde{\lambda}^6)$-valued Hunt process $\mathbb{M} \equiv (\Omega, \mathcal{F}, (X_t)_{t\geq 0}, (P_{\mathbf{x}})_{\mathbf{x}\in S_\triangle})$ with $S = l^2(\tilde{\lambda}^6)$ exists.*

Moreover, by defining $(Y_t)_{t\geq 0}$ by Eq. 2.49*, then Y_t is an $\tilde{\mathcal{H}}_{-3}$-valued Hunt process that is a stochastic quantization with respect to the non-local Dirichlet form $(\tilde{\mathcal{E}}_{(\alpha)}, \mathcal{D}(\tilde{\mathcal{E}}_{(\alpha)}))$ on $L^2(\tilde{\mathcal{H}}_{-3}, \nu_3)$, that is defined through $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$, by making use of τ. In particular $(Y_t)_{t\geq 0}$ is a non-local Markov process with invariant measure ν_3.*

Remark 2.1 i) These considerations performed in Example 3 are adapted to all the cases $\alpha \in (0, 1]$, but, if we restrict our discussions to the case where $\alpha = 1$, then we are able to take $S = l^2(\tilde{\lambda}^4)$, and have an $l^2(\tilde{\lambda}^4)$-valued Hunt process $\mathbb{M} \equiv (\Omega, \mathcal{F}, (X_t)_{t\geq 0}, (P_{\mathbf{x}})_{\mathbf{x}\in S_\triangle})$, associated to the non-local Dirichlet form $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ on $L^2(S, \mu)$, and then we can define an $\tilde{\mathcal{H}}_{-2}$-valued Hunt process $(Y_t)_{t\geq 0}$ (cf. Eq. 2.49) through the same discussion as in Example 0.

ii) In order to consider other jump type Markov processes, which are natural analogues of the diffusion process, with invariant measure $\tilde{\nu}$ defined by Eqs. 2.33 and 2.34, constructed through the local type Dirichlet form defined by [69], where the present $\tilde{\nu}$ was denoted by μ, we should define, analogous to [69], a corresponding non-local type Dirichlet forms by making use of a *system of density distributions* (cf. Assumption 1 and Remark 1 of [69]).

These would be different from the non-local type Dirichlet forms discussed in the present paper, since they would involve the mentioned system of density distributions.

Remark 2.2 (On advantages of *non-local* Dirichlet forms, on the related SPDE, on the space cut-off)
i) Let us consider a toy model of 1-dimensional *non-local* stochastic quantiztaon as follows: Let the state space $S \equiv \{-\frac{1}{2}, \frac{1}{2}\} \subset \mathbb{R}^1$. Suppose that we are given some $p \in [0, 1]$, and a probabilty measure μ on S such that $\mu(\{\frac{1}{2}\}) = p$, $\mu(\{-\frac{1}{2}\}) = 1 - p$. Take $f, g \in \mathcal{S}(\mathbb{R} \to \mathbb{R})$, and denote $u_1 = f(\frac{1}{2})$, $u_2 = f(-\frac{1}{2})$, $v_1 = g(\frac{1}{2})$ and $v_2 = g(-\frac{1}{2})$. Define an inner product $(f, g)_{L^2(\mu)}$ such that $(f, g)_{L^2(\mu)} \equiv \int_{\mathbb{R}} f(x)\, g(x)\, \mu(dx) = u_1 v_1 p + u_2 v_2 (1 - p)$.

Then, $L^2(\mu)$ is the space of real valued bounded measurable functions on $\mathbb{R}$. On $L^2(\mu)$, define a closable non-local Markovian symmetric form $\mathcal{E}_{(\alpha)}$, $\alpha \in (0, 2)$ such that

$$\begin{aligned}
\mathcal{E}_{(\alpha)}(f, g) &\equiv \int_{x \neq y} \frac{(f(x) - f(y))\,(g(x) - g(y))}{|x - y|^{1+\alpha}} \mu(dx)\, \mu(dy) \\
&= (u_1 - u_2)(v_1 - v_2)p(1 - p) + (u_2 - u_1)(v_2 - v_1)(1 - p)p \\
&= \left(A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right)_{L^2(\mu)} = \left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, A \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right)_{L^2(\mu)},
\end{aligned}$$

where $A = 2 \begin{pmatrix} 1 - p & p - 1 \\ -p & p \end{pmatrix}$. The closed extension of $\mathcal{E}_{(\alpha)}$, denoted by the same notation, is a Dirichlet form with domain $\mathcal{D}(\mathcal{E}_{(\alpha)}) = L^2(\mu)$. Then, A is the generator of Markovian semi group e^{-At}, $t \geq 0$ such that

$$e^{-At} = \sum_{n=0}^{\infty} \frac{(-1)^n t^n 2^n}{n!} \begin{pmatrix} 1 - p & p - 1 \\ -p & p \end{pmatrix}^n = \begin{pmatrix} p & 1 - p \\ p & 1 - p \end{pmatrix} + e^{-2t} \begin{pmatrix} 1 - p & p - 1 \\ -p & p \end{pmatrix}. \tag{2.50}$$

Denote the right hand side of Eq. 2.50, the matrix, by M_t, $t \geq 0$, then it holds that

$$(p, 1 - p)M_t = (p, 1 - p).$$

Thus M_t, and hence e^{-At} defines a continuous time conservative Markov process with invariant measure μ, which is a non-local stochastic quantization of μ. Obviously, for $L^2(\mu)$ it is impossible to formulate a corresponding local Dirichlet form. Roughly speaking, for a given random field μ, in order to consider the stochastic quantizations through the arguments by means of the local Dirichlet forms, we have to suppose a sufficient regularity for μ.

By keeping the above description in mind, let us compare the main theorems given in [16] and [24]. For this we use the same notations used in [16] and [24] without giving their definitions. By making use of a directional derivative, to define a local Dirichlet form on $L^2(E; \mu)$, with E a locally convex Hausdorff topological vector space which is in addition Souslinean, and a Borel positive mesure μ, we have to firstly consider the so-called μ-admissibility (see Definition 2.1 and Theorem 2.2 of [24]. By Theorem 2.2 of [24], it is shown that for each $k \in E \setminus \{0\}$, the μ-admissibility corresonding to k is completly characterized by means of $\rho_k : E_k \times \mathcal{B}(\mathbb{R}) \to [0, 1]$, a kernel function, and ν_k, the image measure of μ under the projection $\pi_k : E \to E_k$ with $E = E_k \oplus k\mathbb{R}$, such that

$$\int_E u(z)\mu(dz) = \int_{E_k} \int_{\mathbb{R}} u(x+sk)\rho(x, ds)\nu_k(dx), \quad \text{for all Borel function} \quad u : E \to [0, \infty).$$

In [16] we inherit the concept of the deconposition of μ by meams of ρ_k and ν_k, as an abstract way, and define the non-local Dirichlet forms on weighted l^p spaces (see Eq. 1.7 in the present paper). Roughly speaking, $\mu(dy_i \mid \sigma_{i^c})$ in Eq. 1.7 corresponds to ρ_k (more precisely see Remark 2 of [16]). For the cases of local Dirichlet forms on infinite dimensional topological vector spaces, to certify the conditions under which the μ-admissibility holds, we need fine informations on the measure μ (see Theorems 2.2, 2.4 and Corollary 2.5 in [24], also for the finite dimensional case cf. Section II-2-a) of [54], on the other hand for the cases of non-local Dirichlet forms on weighted l^p spaces, a sufficient conditions under which we can define quasi-regular Dirichlet forms becomes simple (see Theorem 1, and Eqs. 1.11 and 1.12 of Theorem 2 in the present paper, see [16] for more general situations). (Naturally, even a finite dimensional model, the above model set in the remark does not satisfy the conditions given in Section II-2-a) of [54]).

In [3, 4, 18, 41–43], the local stochastic quantizations corresponding to Φ_3^4 Euclidean field are considered by not passing through the arguments of the Dirichlet form. By using the directional derivatives which were adopted in [22–24], because of a less regularity of Euclidean Φ_3^4 measure, it would be impossible to define a local Dirichlet form for the Euclidean Φ_3^4 field (for a related discussion, cf. [20], and cf. [30, 70, 71] where a local Dirichlet form reduced from the solution of the SPDE given by [43] is discussed). On the other hand, in [16] by making use of the non-local Dirichlet form, a non-local stochastic quantization of Φ_3^4 field is considered. In short, non-local stochastic quantizations could be considered also for random fields possessing less regularities (see the above paragraph).

ii) Let $\mathbb{M} \equiv \big(\Omega, \mathcal{F}, (X_t)_{t\geq 0}, (P_{\mathbf{x}})_{\mathbf{x}\in S_\triangle}\big)$ be the Hunt process defined through Theorem 3. By a direct application of Theorem VI-2.5 of [54] we see that for $u \in \mathcal{D}(\mathcal{E}_{(\alpha)})$, there exists a unique martingale additive functional of finite energy (MAF) $M^{[u]}$ and a continuous additive functional of zero energy (CAF's zero energy) $N^{[u]}$ such that

$$A^{[u]} = M^{[u]} + N^{[u]}, \tag{2.51}$$

where

$$A^{[u]} \equiv (A_t^{[u]})_{t\geq 0}, \qquad A_t^{[u]} = \tilde{u}(X_t) - \tilde{u}(X_0),$$

with $\tilde{u}$ an $\mathcal{E}_{(\alpha)}$-quasi continuous μ-version of $u \in \mathcal{D}(\mathcal{E}_{(\alpha)})$. Since, Examples 0, 1, 2 and 3 are considered through Theorem 3, every Hunt process appearing in Theorems 5, 6, 7, 8, 9, 10 and 11 admits the decomposition formula (2.51) with X_t being substituted by Y_t.

In order to consider the martingale problems (cf., e.g., [39] and [55]) corresponding to the decomposition given by Eq. 2.51, and then to give the explicit expressions by means of SPDEs to the Hunt process $\mathbb{M}$, by setting some additional assumptions for the probability measure μ (cf. Eq. 1.3), e.g., a uniform regularity of its density function, we have to pass through the analogous discussions performed by, e.g., [39] and [5] for the finite dimensional vector spaces. The considerations of this concern are postponed to future works.

Since the *strictly quasi regular* Dirichlet forms $(\mathcal{E}(\alpha), \mathcal{D}(\mathcal{E}_{(\alpha)}))$ considered in this paper satisfy $1 \in \mathcal{D}(\mathcal{E}_{(\alpha)})$ (see the proofs of Theorem 2 and 3 of [16]), and the state spaces in considerations are separable Hilbert spaces, the Hunt process $\mathbb{M}$ associates with the *strictly quasi regular* Dirichlet forms discussed in Examples 0, 1, 2 and 3 are *conservative*, i.e., Eq. 2.51 are time gloval processes (see Proposition V.2.15 and the discriptions in pages 160, 161 of [54]).

iii) In order to apply Theorem 2 on the strictly quasi regular Dirichlet forms $(\mathcal{E}(\alpha), \mathcal{D}(\mathcal{E}_{(\alpha)}))$ to each concrete Euclidean random field μ on a topological vector space, we need the support property of μ. In Examples 1 and 2, by setting the assumptions of the space cut-off to the potential terms (exponential, trigonometric and $P(\phi)_2$), by applying Theorem 4 (Bochner-Minlos Theorem), we can get clear support properties such that $\text{support}[\mu] \subset \mathcal{H}_{-n}, n \geq 1$, and then we can apply Theorem 2 to these examples. In [16] the non-local stochastic quantization of Φ_3^4 has been considered, where for the random field μ we are able to use the support property that is guaranteed by the lattice approximations of μ provided by [31].

Appendix A

Proof of Lemma 2.1 From Eqs. 2.22 and 2.23, the assumption for the original measure ν, Lemma 2.1 can be proven as follows: It suffices to show that Eq. 2.43 holds for any z that corresponds with an $\mathbb{Y} \in U_N \subset \mathcal{Y}_0$ for some $N \in \mathbb{N}$ (see Eq. 2.27).

$$\text{For } \mathbb{Y} \in U_N \subset \mathcal{Y}_0 \ \textit{with} \ N \in \mathbb{N}, \qquad \textit{let} \quad z = \sum\nolimits_{i=1}^{\infty} m_i \delta_{y_i} \in \tilde{\mathcal{Y}}_0, \tag{A.1}$$

the Radon measure corresponding with $\mathbb{Y}$ through Eq. 2.30. Then, for any test function $\varphi \in \mathcal{S}$ (denoting by $< z, \varphi >$ the dualization between the distribution z and the test function φ), we have

$$\begin{aligned}
| < z, \varphi > | &= \left| \sum_{i=1}^{\infty} m_i \varphi(y_i) \right| = \left| \sum_{l=0}^{\infty} \left(\sum_{y_i \in Q_r : |r| = l} m_i \varphi(y_i) \right) \right| \\
&\leq \sum_{l=0}^{\infty} \left(\sum_{|r|=l} n(\mathbb{Y}, r) \left(\sup_{y \in Q_r} |\varphi(y)| \right) \right) \leq \sum_{l=0}^{\infty} \left(\sum_{|r|=l} (n^2(\mathbb{Y}, r)) \left(\sup_{y \in Q_r} |\varphi(y)| \right) \right) \\
&\leq \sum_{l=0}^{\infty} \left(\sum_{|r|=l} (n^2(\mathbb{Y}, r)) (\sup_{y \in Q_r} |(|y|^2 + 1)^{d+1} \varphi(y)|) \right) \\
&\leq \sum_{l=0}^{\infty} \left(\sum_{|r|=l} (n^2(\mathbb{Y}, r)) \left(\int_{Q_r} ((-\Delta + 1)^{\frac{d+1}{2}} (|y|^2 + 1)^{d+1} \varphi(y))^2 dy \right)^{\frac{1}{2}} \right) \\
&\leq \sum_{l=0}^{\infty} (\sum_{|r|=l}) (n^2(\mathbb{Y}, r)) \left(\int_{Q_r} ((|y|^2 + 1)^{-(d+1)})^2 dy \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{Q_r} ((|y|^2 + 1)^{d+1} ((-\Delta + 1)^{\frac{d+1}{2}} (|y|^2 + 1)^{d+1} \varphi(y))^2 dy)^{\frac{1}{2}} \right), \\
&\leq \sum_{l=0}^{\infty} \left(\sum_{|r|=l} (n^2(\mathbb{Y}, r)) \left(\int_{Q_r} ((|y|^2 + 1)^{-(d+1)})^2 dy \right)^{\frac{1}{2}} \right) \|\varphi\|_{\tilde{\mathcal{H}}_1}.
\end{aligned} \tag{A.2}$$

In the above deductions, to get the third inequality we have applied the Sobolev's embedding theorem (cf., e.g., [56]) that gives for the Sobolev space $W^{m,2}$ with $m = [\frac{d}{2}] + 1$, that $W^{m,2} \subset C_b(\mathbb{R}^d)$ (cf. the explanation given below (A.10), where $C_b(\mathbb{R}^d)$ denotes the space of *real* valued bounded continuous functions on $\mathbb{R}^d$. Since, for $r \in \mathbb{Z}^d$ by denoting $|r| = l$, for some $C < \infty$ it holds that

$$\int_{Q_r} \left((|y|^2+1)^{-(d+1)}\right)^2 dt \le C(l^2+1)^{-\frac{3d+5}{2}}, \tag{A.3}$$

by this together with Eq. A.1, we can evaluate the right hand side of Eq. A.2. We consequently see that the following holds for some constants $C_1, C_2, C_3 < \infty$ (only C_3 depends on N):

$$\begin{aligned} | < z, \varphi > | &= C_1 \left\{ \sum_{l=1}^{\infty} n^2(\mathbb{Y}, r)(2l+1)^{-d}(l^2+1)^{-\frac{3d+5}{4}+\frac{d}{2}} + N^2 \right\} \|\varphi\|_{\tilde{\mathcal{H}}_1} \\ &\le C_2 \left\{ \sum_{l=1}^{\infty} N^2 (l+1)^{-\frac{d}{2}-\frac{5}{2}} + N^2 \right\} \|\varphi\|_{\tilde{\mathcal{H}}_1} \le C_3 \|\varphi\|_{\tilde{\mathcal{H}}_1}, \qquad \forall \varphi \in \tilde{\mathcal{H}}_1. \end{aligned} \tag{A.4}$$

Equation A.4 shows that

$$z \in \tilde{\mathcal{H}}_1^* = \tilde{\mathcal{H}}_{-1}, \tag{A.5}$$

for any z that corresponds with an $\mathbb{Y} \in U_N \subset \mathcal{Y}_0$ for some $N \in \mathbb{N}$. Since $N \in \mathbb{N}$ is arbitrary, the proof of Eq. 2.43 is completed. □

Proof of Lemma 2.2 Let $r \ge 1$. We shall show that

$$\sigma[\tilde{\mathcal{Y}}_0] \supset \left(\mathcal{B}(\tilde{\mathcal{H}}_{-r}) \cap \tilde{\mathcal{Y}}_0\right). \tag{A.6}$$

Since $\tilde{\mathcal{H}}_{-r}$ is a separable Hilbert space, and hence, it is a *Souslin space*, and also since the dual space of $\tilde{\mathcal{H}}_{-r}$ is $\tilde{\mathcal{H}}_r$ (see Eq. 2.40), it holds that (cf. e.g., [27])

$$\mathcal{B}(\tilde{\mathcal{H}}_{-r}) = \sigma[\tilde{\mathcal{H}}_r] \equiv \text{the } \sigma - \text{field generated by } \{z \,|\, z \in \tilde{\mathcal{H}}_{-r}, \varphi(z) < t\},\ t \in \mathbb{R},\ \varphi \in \tilde{\mathcal{H}}_r, \tag{A.7}$$

where $\varphi(z) =< z, \varphi >$ denotes the dualization between the distribution $z \in \tilde{\mathcal{H}}_{-r}$ and the test function $\varphi \in \tilde{\mathcal{H}}_r$ (cf. Eq. A.2). To see that Eq. A.6 holds, by Eq. A.7 it suffices to show that

$$\left(\{z \,|\, z \in \tilde{\mathcal{H}}_{-r},\ \varphi(z) < t\} \cap \tilde{\mathcal{Y}}_0\right) \in \sigma[\tilde{\mathcal{Y}}_0], \quad \text{for any} \quad \varphi \in \tilde{\mathcal{H}}_r \quad \text{and} \quad \forall t \in \mathbb{R}. \tag{A.8}$$

By Lemma 2.1, since $\tilde{\mathcal{H}}_{-r} \supset \tilde{\mathcal{Y}}_0$, and it holds that

$$\left(\{z \,|\, z \in \tilde{\mathcal{H}}_{-r},\ \varphi(z) < t\} \cap \tilde{\mathcal{Y}}_0\right) = \{z \,|\, z \in \tilde{\mathcal{H}}_{-r} \cap \tilde{\mathcal{Y}}_0,\ \varphi(z) < t\} = \{z \,|\, z \in \tilde{\mathcal{Y}}_0,\ \varphi(z) < t\},$$

we see that Eq. A.8 is equivalent to the following:

$$\{z \,|\, z \in \tilde{\mathcal{Y}}_0,\ \varphi(z) < t\} \in \sigma[\tilde{\mathcal{Y}}_0], \quad \text{for any} \quad \varphi \in \tilde{\mathcal{H}}_r \quad \text{and} \quad \forall t \in \mathbb{R}. \tag{A.9}$$

On the other hand, by Eq. 2.38, from the Sobolev's embedding theorem (cf., e.g., Th. 3.15 in [56]),since $C_b(\mathbb{R}^d \to \mathbb{R}) \subset W^{r(d+1),2}(\mathbb{R}^d)$, it holds that

$$\tilde{C} \equiv \{(|x|^2+1)^{-r(d+1)}\psi \,|\, \psi \in C_b(\mathbb{R}^d \to \mathbb{R})\} \supset \tilde{\mathcal{H}}_r, \tag{A.10}$$

where C_b denotes the space of real valued bounded continuous functions, and $W^{r(d+1),2}$ denotes the Sobolev space defined., e.g., by Def. 2.9 of [56], where the notation such that $\mathcal{E}^{r(d+1)}_{L^2} = W^{r(d+1),2}$ is adopted. Thus, by Eq. A.10, in order to prove Eq. A.9, that is equivalent to Eq. A.8, it suffices to show that

$$\{z \,|\, z \in \tilde{\mathcal{Y}}_0,\ \varphi(z) < t\} \in \sigma[\tilde{\mathcal{Y}}_0], \qquad \forall \varphi \in \tilde{C}, \quad \forall t \in \mathbb{R}. \tag{A.11}$$

For Eq. A.11, we used the fact that $\tilde{C}$ can be taken as the dual space of $\tilde{\mathcal{Y}}_0$, which is included in the proof of Lemma 2.1 (cf. Eq. A.2), but is easily seen as follows: By Eq. A.10, for $\varphi = (|x|^2+1)^{-r(d+1)}\psi \in \tilde{C}$ with $\psi \in C_b$ and any $z \in \tilde{Y}_0$, it holds that

$$\begin{aligned}|\varphi(z)| &= |\sum_{i=1}^{\infty} m_i \varphi(y_i)| = |\sum_{l=0}^{\infty}(\sum_{y_i \in Q_t : |t|=l} m_i \varphi(y_i))| \\ &\le \sum_{l=0}^{\infty}(\sum_{|t|=l} n(\mathbb{Y},t)(\sup_{y\in Q_t}|\varphi(y)|)) \le \|\psi\|_{L^\infty} \sum_{l=0}^{\infty}\sum_{|t|=l}(n^2(\mathbb{Y},t))(\sup_{y\in Q_t}(|y|^2+1)^{-r(d+1)}) \\ &\le \|\psi\|_{L^\infty} \sum_{l=1}^{\infty}\left(\sum_{|t|=l}(n^2(\mathbb{Y},t))\left(t^2-\frac{1}{2}\right)^{-r(d+1)}\right) + n^2(\mathbb{Y},0) < \infty, \end{aligned} \tag{A.12}$$

where the last equality follows from Eqs. 2.24 and 2.27.

In addition, note that for $\varphi \in \tilde{C}$, by the decomposition such that $\varphi = \varphi_+ - \varphi_-$, where $\varphi_+(x) \equiv \max\{\varphi(x), 0\}$ and $\varphi_-(x) \equiv \max\{-\varphi(x), 0\}$, it holds that

$$\varphi_+, \ \varphi_- \in \tilde{C}. \tag{A.13}$$

Also, note that the following holds:

$$\begin{aligned}\{z \mid z \in \tilde{\mathcal{Y}}_0, \ \varphi(z) < t\} &= \{z \mid z \in \tilde{\mathcal{Y}}_0, \ \varphi_+(z) - \varphi_-(z) < t\} \\ &= \bigcup_{s\in\mathbb{Q}}\left(\{f \mid z \in \tilde{\mathcal{Y}}_0, \ \varphi_+(z) < t+s\} \cap \{z \mid z \in \tilde{\mathcal{Y}}_0, \ \varphi_-(z) > s\}\right),\end{aligned} \tag{A.14}$$

where $\mathbb{Q}$ denotes the field of rational numbers. Thus, since the right hand side of Eq. A.14 is a countable operation, from Eqs. A.13 and A.14, to prove Eq. A.11 it suffices to show that the following holds:

$$\{z \mid z \in \tilde{\mathcal{Y}}_0, \ \varphi(z) < t\} \in \sigma[\tilde{\mathcal{Y}}_0], \quad \text{for any } \varphi \in \tilde{C} \text{ such that } \varphi(x) \ge 0, x \in \mathbb{R}^d, \text{ and } \forall t \in \mathbb{R}. \tag{A.15}$$

To this end for $\varphi \in \tilde{C}$ with $\varphi(x) \ge 0$, $\forall x \in \mathbb{R}^d$, define $\varphi_n \in \tilde{C}$, $n \in \mathbb{N}$, that satisfy the following:

$$0 \le \varphi_n(x) \le \varphi_{n+1}(x) \le \varphi(x), \qquad \forall x \in \mathbb{R}^d, \quad \forall n \in \mathbb{N}, \tag{A.16}$$

$$\operatorname{supp}[\varphi_n] \subset \{x \mid x \in \mathbb{R}^d, \ |x| \le n\}, \tag{A.17}$$

$$\lim_{n\to\infty} \|\varphi_n - \varphi\|_{L^\infty} = 0, \tag{A.18}$$

then, since $z \in \tilde{\mathcal{Y}}_0$ is a *non-negative (integer)*-valued Radon measure on $\mathbb{R}^d$ (cf. Eq. 2.30), we can use an argument of a *monotonicity*, we have

$$\{z \mid z \in \tilde{\mathcal{Y}}_0, \ \varphi(z) < t\} = \bigcup_{n\in\mathbb{N}}\{z \mid z \in \tilde{\mathcal{Y}}_0, \ \varphi_n(z) < t\}. \tag{A.19}$$

For each $\varphi_n \in C_0(\mathbb{R}^d \to \mathbb{R}_+)$, there exists a sequence of *simple functions* $\{\varphi_{n,k}\}_{k\in\mathbb{N}}$ on $\mathcal{B}(\mathbb{R}^d)$, the Borel σ-field of $\mathbb{R}^d$, such that

$$0 \le \varphi_{n,k}(x) \le \varphi_{n,k+1}(x) \le \varphi_n(x), \qquad \forall x \in \mathbb{R}^d, \quad k \in \mathbb{N}, \tag{A.20}$$

$$\lim_{k\to\infty} \|\varphi_{n,k} - \varphi_n\|_{L^\infty} = 0, \tag{A.21}$$

where $C_0(\mathbb{R}^d \to \mathbb{R}_+)$ denotes the space of non-negative continuous functions on $\mathbb{R}^d$ with compact supports. Then, by Eqs. A.20, A.21 and again by the monotonicity of the sequence of sets, that follows from the *positivity* of $z \in \tilde{\mathcal{Y}}_0$, it holds that

$$\{z \,|\, z \in \tilde{\mathcal{Y}}_0,\ \varphi_n(z) < t\} = \bigcup_{k \in \mathbb{N}} \{z \,|\, z \in \tilde{\mathcal{Y}}_0,\ \varphi_{n,k}(z) < t\}. \tag{A.22}$$

By the definition of the σ-field $\sigma[\tilde{\mathcal{Y}}_0]$, provided through Eqs. 2.21, 2.28 and 2.32, since

$$\{z \,|\, z \in \tilde{\mathcal{Y}}_0,\ \varphi_{n,k}(z) < t\} \in \sigma[\tilde{\mathcal{Y}}_0], \qquad \forall n \in \mathbb{N}, \quad \forall k \in \mathbb{N},$$

from Eq. A.22 we have

$$\{z \,|\, z \in \tilde{\mathcal{Y}}_0,\ \varphi_n(z) < t\} \in \sigma[\tilde{\mathcal{Y}}_0], \qquad \forall n \in \mathbb{N},$$

and thus, from Eq. A.19, we see that Eq. A.15 holds. This complete the proof of Eq. A.6. □

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