Potential Analysis (2023) 59:1107–1116
https://doi.org/10.1007/s11118-022-09995-8

Sharp Estimates for the Gaussian Torsional Rigidity with Robin Boundary Conditions

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Received: 13 October 2021 / Accepted: 3 March 2022 / Published online: 21 April 2022

Abstract

In this paper we provide a comparison result between the solutions to the torsion problem for the Hermite operator with Robin boundary conditions and the one of a suitable symmetrized problem.

Keywords Torsional rigidity · Hermite operator · Robin boundary conditions

Mathematics Subject Classification (2010) 35J25 · 35B45

1 Introduction

Let Ω be a smooth and possibly unbounded domain of $\mathbb{R}^n$ and let ν be the unit outer normal to $\partial\Omega$. In this paper we consider the following torsion problem for the Hermite operator with Robin boundary conditions

$$\begin{cases} -\operatorname{div}(\phi(x)\nabla u) = \phi(x) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where β is a positive parameter and $\phi(x)$ denotes the density of the normalized Gaussian measure in $\mathbb{R}^n$, that is

$$\phi(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \exp\left(-\frac{|x|^2}{2}\right).$$

The interest in the study of the Hermite operator relies on its applications in various fields. Just to mention a few, it enters in the description of the harmonic oscillator in quantum mechanics (see, e.g., [4] and the references therein). It attracts attention from probabilists too. Indeed, as well known, the Hermite operator is the generator of the Ornstein-Uhlenbeck semigroup (see, e.g., [3] and the references therein).

As we will recall in the next section, suitable weighted embedding trace theorems hold true if Ω is sufficiently smooth. Therefore, classical arguments ensure that problem (1.1) has a unique positive solution u. Furthermore, u is a minimizer of the following functional

$$\frac{1}{T_\phi(\Omega)} = \inf_{w \in H^1(\Omega,\phi)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla w|^2 \phi(x)\, dx + \beta \int_{\partial\Omega} w^2(x)\phi(x)\, d\mathcal{H}^{n-1}}{\left(\displaystyle\int_\Omega w(x)\phi(x)\, dx\right)^2}, \tag{1.2}$$

where $H^1(\Omega, \phi)$ is the weighted Sobolev space naturally associated to problem (1.1) (see Section 2 for the definitions and properties). Note that, as a straightforward computation shows, it holds that

$$T_\phi(\Omega) = \|u\|_{L^1(\Omega,\phi)} := \int_\Omega |u(x)|\phi(x)\, dx.$$

The aim of this paper, is to prove an isoperimetric inequality for $T_\phi(\Omega)$ by means of the so-called Gaussian symmetrization. This will be achieved by comparing the solution to problem (1.1) with that to the following one

$$\begin{cases} -\operatorname{div}(\phi(x)\nabla v) = \phi(x) & \text{in } \Omega^\# \\ \dfrac{\partial v}{\partial \nu} + \beta v = 0 & \text{on } \partial\Omega^\#, \end{cases} \tag{1.3}$$

where $\Omega^\# := \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_1 > \lambda\}$ and λ is such that

$$|\Omega|_\phi := \int_\Omega \phi(x)\, dx = |\Omega^\#|_\phi = h(\lambda),$$

with

$$h(\lambda) := \frac{1}{\sqrt{2\pi}} \int_\lambda^{+\infty} \exp\left(-\frac{t^2}{2}\right) dt. \tag{1.4}$$

Our main result is the following

Theorem 1.1 *Let $\Omega \in \mathcal{G}$, see Definition 2.1, Let u and v be the solutions to problems* (1.1) *and* (1.3)*, respectively. Then the following comparison result holds*

$$\|u\|_{L^1(\Omega,\phi)} \le \|v\|_{L^1(\Omega^\#,\phi)}. \tag{1.5}$$

In other words, among all sufficiently smooth sets of $\mathbb{R}^n$, having prescribed Gaussian measure, the half-spaces maximize $T_\phi(\Omega)$. Note that inequality (1.5) provides a sharp and explicit estimate for $\|u\|_{L^1(\Omega,\phi)}$, since it is elementary to derive the exact form of v

$$v(x) = v(x_1) = C(\lambda, \beta) + \int_\lambda^{x_1} \exp\left(\frac{r^2}{2}\right) \int_r^{+\infty} \exp\left(-\frac{t^2}{2}\right) dt\, dr, \tag{1.6}$$

where

$$C(\lambda, \beta) = \frac{1}{\beta} \exp\left(\frac{\lambda^2}{2}\right) \int_{\lambda}^{+\infty} \exp\left(-\frac{t^2}{2}\right) dt.$$

Now let us briefly describe how our result is inserted in the literature. In [1] and [6] the authors investigate the analogous issue for the classical Laplace operator. In particular, in [6], the authors obtain an isoperimetric inequality for the Robin torsional rigidity in a wider context, by studying a family of Faber-Krahn inequalities. They prove that the Robin Laplace torsional rigidity is maximum on balls among all bounded and Lipschitz domains, once the Lebesgue measure is fixed. Their proof, unlike the one used for the Dirichlet boundary conditions, does not make use of any symmetrization techniques, rather, it is based on reflection arguments. Recently, in [1], see also [2], the authors obtain the same isoperimetric inequality via a "Talenti type comparison result". Note that the result contained in [1] are quite surprising. Indeed, as well known, the Talenti's technique is designed for problems whose solution has level sets that do not touch the boundary of the domain where the problem is posed. A phenomenon that tipically occurs when Robin boundary conditions are imposed. In this paper, because of the structure of the differential operator we are considering, in place of the more common Schwarz symmetrization, we use the Gauss symmetrization. A procedure that transforms a positive function into a new one having as super level sets half-spaces whose Gauss measure is the same as the original function.

The structure of the paper is the following. In Section 2 we fix some notation and we recall some results that we will use in the paper. The third Section contains the proof of our main result.

2 Notation and Preliminaries

Let A be any Lebesgue measurable set of $\mathbb{R}^n$. The Gaussian perimeter of A is

$$P_\phi(A) = \begin{cases} \displaystyle\int_{\partial A} \phi(x)\, d\mathcal{H}^{n-1} & \text{if } \partial A \text{ is } (n-1)-\text{rectifiable} \\ +\infty & \text{otherwise,} \end{cases}$$

where $d\mathcal{H}^{n-1}$ denotes the $(n-1)-$dimensional Hausdorff measure in $\mathbb{R}^n$.

While the Gaussian measure of A is given by

$$|A|_\phi = \int_A \phi(x)\, dx \in [0, 1]\,. \tag{2.1}$$

The celebrated Gaussian isoperimetric inequality (see [5, 11] and [10]) states that among all Lebesgue measurable sets in $\mathbb{R}^n$, with prescribed Gaussian measure, the half-spaces minimize the Gaussian perimeter. Furthermore the isoperimetric set is unique, clearly, up to a rotation with respect to the origin (see [7] and [8]).

The isoperimetric function in the Gauss space, $I(s)$, is

$$I : s \in [0, 1] \to \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(h^{-1}(s))^2}{2}\right), \tag{2.2}$$

where h^{-1} is the inverse function of h, defined in (1.4). Note, indeed, that the Gaussian perimeter of any half-space of Gaussian measure s is equal to $I(s)$. The isoperimetric property of the half-spaces can finally be stated as follows.

Theorem 2.1 *If $\Omega \subset \mathbb{R}^n$ is any Lebesgue measurable set it holds that*

$$P_\phi(\Omega) \geq P_\phi(\Omega^\#) = I\left(|\Omega|_\phi\right),$$

where equality holds, if and only if, Ω is equivalent to an half-space.

Let $\Omega \subset \mathbb{R}^n$ be an open connected set. We will denote by $L^2(\Omega, \phi)$ the set of all real measurable functions defined in Ω such that

$$\|u\|^2_{L^2(\Omega,\phi)} := \int_\Omega u^2(x)\phi(x)dx < +\infty.$$

For our future purposes we need also to introduce the following weighted Sobolev space

$$H^1(\Omega, \phi) := \left\{u \in W^{1,1}_{\mathrm{loc}}(\Omega) : (u, |\nabla u|) \in L^2(\Omega, \phi) \times L^2(\Omega, \phi)\right\},$$

endowed with the norm

$$\|u\|_{H^1(\Omega,\phi)} = \|u\|_{L^2(\Omega,\phi)} + \|\nabla u\|_{L^2(\Omega,\phi)}.$$

In the sequel of the paper, we need to introduce the following family of sets.

Definition 2.1 A Lipschitz domain Ω of $\mathbb{R}^n$ is in $\mathcal{G}$ if $|\Omega|_\phi \in (0, 1)$ and the following conditions are fulfilled:

(i) $H^1(\Omega, \phi)$ is compactly embedded in $L^2(\Omega, \phi)$.

(ii) The trace operator T

$$T : u \in H^1(\Omega, \phi) \to u|_{\partial\Omega} \in L^2(\partial\Omega, \phi),$$

is well defined;

(iii) The trace operator defined in the previous point is compact from $H^1(\Omega, \phi)$ onto $L^2(\partial\Omega, \phi)$.

In (ii) and (iii) the functional space $L^2(\partial\Omega, \phi)$ is endowed with the norm

$$\|u\|^2_{L^2(\partial\Omega,\phi)} = \int_{\partial\Omega} u^2(x)\phi(x)d\mathcal{H}^{n-1}.$$

We stress that $\mathcal{G}$ is non empty (see for instance Remark 2.1 in [9]).

Finally, we recall the following version of Gronwall's Lemma.

Lemma 2.1 *Let $\xi(\tau)$ be a continuously differentiable function satisfying, for some constant $C \geq 0$ the following differential inequality*

$$\tau\xi'(\tau) \leq \xi(\tau) + C \quad \textit{for all } \tau \geq \tau_0 > 0.$$

Then

$$\xi(\tau) \leq \tau\frac{\xi(\tau_0) + C}{\tau_0} - C \quad \textit{for all } \tau \geq \tau_0. \tag{2.3}$$

$$\xi'(\tau) \leq \frac{\xi(\tau_0) + C}{\tau_0} \quad \textit{for all } \tau \geq \tau_0. \tag{2.4}$$

3 Proof of the Main Result

In this section, u and v will denote the solutions to problems (1.1) and (1.3), respectively. In order to prove our isoperimetric inequality for $T_\phi(\Omega)$, we need the following auxiliary result which may have independent interest.

Lemma 3.1 *The following inequalities hold true*

$$0 \le u_m \le v_m, \tag{3.1}$$

where

$$u_m := \inf_{\Omega} u, \quad v_m := \min_{\overline{\Omega^\#}} v.$$

Proof In order to prove the first inequality in (3.1), we use $u^- := \max\{0, -u\}$ as test function in (1.1), obtaining

$$-\int_\Omega |\nabla u^-|^2 \phi(x)\,dx - \beta \int_{\partial\Omega} (u^-)^2(x)\phi(x)\,d\mathcal{H}^{n-1} = \int_\Omega u^-(x)\phi(x)\,dx.$$

Hence $u^- = 0$ a.e. in Ω.

Concerning the second inequality in (3.1), we observe that the function $v(x) = v(x_1)$ defined in (1.6) is increasing. Therefore it achieves its minimum v_m on $\partial\Omega^\#$. Let u be the solution to the problem (1.1), then

$$\begin{aligned} v_m\, P_\phi(\Omega^\#) &= \int_{\partial\Omega^\#} v(x)\,\phi(x)\,d\mathcal{H}^{n-1} = -\tfrac{1}{\beta}\int_{\partial\Omega^\#} \tfrac{\partial u}{\partial \nu}\phi(x)\,d\mathcal{H}^{n-1} \\ &= -\tfrac{1}{\beta}\int_{\Omega^\#} \operatorname{div}(\phi(x)\nabla u)\,dx = \tfrac{1}{\beta}|\Omega^\#|_\phi = \tfrac{1}{\beta}|\Omega|_\phi \\ &= \int_{\partial\Omega} u(x)\,\phi(x)d\mathcal{H}^{n-1} \ge u_m P_\phi(\Omega) \ge u_m P_\phi(\Omega^\#), \end{aligned} \tag{3.2}$$

where last inequality follows from the weighted isoperimetric inequality (2.1). The claim is hence proven. □

In the sequel the following notation will be in force.

For $t \ge 0$ we denote by

$$U_t = \{x \in \Omega : u(x) > t\}, \ \partial U_t^{int} = \partial U_t \cap \Omega, \ \partial U_t^{ext} = \partial U_t \cap \partial\Omega,$$

and by

$$\mu(t) = |U_t|_\phi \ \text{ and } \ P_u(t) = P_\phi(U_t). \tag{3.3}$$

Analogously if $t \ge 0$ we denote by

$$V_t = \{x \in \Omega : v(x) > t\}, \ \varphi(t) = |V_t|_\phi \ \text{ and } \ P_v(t) = P_\phi(V_t). \tag{3.4}$$

Remark 3.1 An immediate consequence of Proposition 3.1 is the following inequality

$$\mu(t) \le \varphi(t) = |\Omega|_\phi \quad \forall t \in [0, v_m]. \tag{3.5}$$

In order to prove our main results we need some further lemmata.

Lemma 3.2 *For a.e. $t \ge 0$ it holds*

$$\frac{1}{2\pi}\exp\left(-\left(h^{-1}(\mu(t))\right)^2\right) \le \mu(t)\left((-\mu'(t)) + \frac{1}{\beta}\int_{\partial U_t^{ext}} \frac{\phi(x)}{u(x)}\,d\mathcal{H}^{n-1}\right), \tag{3.6}$$

while

$$\frac{1}{2\pi}\exp\left(-\left(h^{-1}(\varphi(t))\right)^2\right)=\varphi(t)\left((-\varphi'(t))+\frac{1}{\beta}\int_{\partial V_t}\frac{\phi(x)}{v(x)}\,d\mathcal{H}^{n-1}\right), \tag{3.7}$$

where h is defined in (1.4).

Proof Sard's Lemma ensures that U_t is a regular level set, for almost every $t\geq 0$. Then it holds

$$\begin{aligned}P_u^2(t)&=\left(\int_{\partial U_t}\phi(x)\,d\mathcal{H}^{n-1}\right)^2=\left(\int_{\partial U_t}\left(\frac{\phi(x)}{\left|\frac{\partial u}{\partial \nu}\right|}\right)^{\frac12}\left(\phi(x)\left|\frac{\partial u}{\partial \nu}\right|\right)^{\frac12}d\mathcal{H}^{n-1}\right)^2\\&\leq\left(\int_{\partial U_t}\frac{\phi(x)}{\left|\frac{\partial u}{\partial \nu}\right|}\,d\mathcal{H}^{n-1}\right)\left(\int_{\partial U_t}\phi(x)\left|\frac{\partial u}{\partial \nu}\right|d\mathcal{H}^{n-1}\right)=\mu(t)\left(\int_{\partial U_t}\frac{\phi(x)}{\left|\frac{\partial u}{\partial \nu}\right|}\,d\mathcal{H}^{n-1}\right)\\&=\mu(t)\left(\int_{\partial U_t^{int}}\frac{\phi(x)}{\left|\frac{\partial u}{\partial \nu}\right|}\,d\mathcal{H}^{n-1}+\int_{\partial U_t^{ext}}\frac{\phi(x)}{\left|\frac{\partial u}{\partial \nu}\right|}\,d\mathcal{H}^{n-1}\right)\\&=\mu(t)\left((-\mu'(t))+\frac{1}{\beta}\int_{\partial U_t^{ext}}\frac{\phi(x)}{u(x)}\,d\mathcal{H}^{n-1}\right).\end{aligned} \tag{3.8}$$

The Gaussian isoperimetric inequality (2.1) gives

$$P_u^2(t)\geq\frac{1}{2\pi}\exp\left(-\left(h^{-1}(\mu(t))\right)^2\right), \tag{3.9}$$

where h is defined in (1.4). Inequalities (3.9) and (3.8) finally imply

$$\frac{1}{2\pi}\exp\left(-\left(h^{-1}(\mu(t))\right)^2\right)\leq\mu(t)\left((-\mu'(t))+\frac{1}{\beta}\int_{\partial U_t^{ext}}\frac{\phi(x)}{u(x)}\,d\mathcal{H}^{n-1}\right), \tag{3.10}$$

which is inequality (3.6). Clearly, repeating the same arguments for the function v, we get equality (3.7) in place of inequality (3.10). □

The following result allows to handle the right hand side in (3.10).

Lemma 3.3 *Let v_m be the minimum of v. For almost every $t\geq v_m$ it holds*

$$\int_0^\tau t\left(\int_{\partial U_t^{ext}}\frac{\phi(x)}{u(x)}\,d\mathcal{H}^{n-1}\right)dt\leq\frac{|\Omega|_\phi}{2\beta}, \tag{3.11}$$

and

$$\int_0^\tau t\left(\int_{\partial V_t\cap\partial\Omega^\#}\frac{\phi(x)}{v(x)}\,d\mathcal{H}^{n-1}\right)dt=\frac{|\Omega|_\phi}{2\beta}. \tag{3.12}$$

Proof Fubini's Theorem yields

$$\begin{aligned}\int_{\partial\Omega}\phi(x)\left(\int_0^{u(x)}\frac{t}{u(x)}\,dt\right)d\mathcal{H}^{n-1}&=\int_{\partial\Omega}\phi(x)\left(\int_0^\infty\frac{t}{u(x)}\chi_{\{u>t\}}\,dt\right)d\mathcal{H}^{n-1}\\&=\int_0^\infty t\left(\int_{\partial U_t^{ext}}\frac{\phi(x)}{u(x)}\,d\mathcal{H}^{n-1}\right)dt,\end{aligned} \tag{3.13}$$

where χ stands for the characteristic function. Since u is the solution to problem (1.1) it holds

$$\int_{\partial\Omega}\phi(x)\left(\int_0^{u(x)}\frac{t}{u(x)}\,dt\right)d\mathcal{H}^{n-1}=\tfrac{1}{2}\int_{\partial\Omega}\phi(x)u(x)\,d\mathcal{H}^{n-1}= -\tfrac{1}{2\beta}\int_{\partial\Omega}\phi(x)\frac{\partial u}{\partial\nu}\,d\mathcal{H}^{n-1}=\frac{|\Omega|_\phi}{2\beta}. \tag{3.14}$$

Observing that

$$\int_0^\tau t\left(\int_{\partial U_t^{ext}}\frac{\phi(x)}{u(x)}\,d\mathcal{H}^{n-1}\right)dt\le\int_0^\infty t\left(\int_{\partial U_t^{ext}}\frac{\phi(x)}{u(x)}\,d\mathcal{H}^{n-1}\right)dt,$$

from (3.3) and (3.14) we get (3.11). On the other hand, by repeating the same arguments, we get (3.12). Note that, for all $\tau\ge v_m$ the following equality holds true

$$\int_0^\tau t\left(\int_{\partial V_t\cap\partial\Omega^\#}\frac{\phi(x)}{v(x)}\,d\mathcal{H}^{n-1}\right)dt=\int_0^\infty t\left(\int_{\partial V_t\cap\partial\Omega^\#}\frac{\phi(x)}{v(x)}\,d\mathcal{H}^{n-1}\right)dt=\frac{|\Omega|_\phi}{2\beta},$$

since $\partial V_t\cap\partial\Omega^\#=\emptyset$ for any $\tau>v_m$. □

Now we can prove our main result.

Proof of Theorem 1.1 We first observe that the function

$$F(s):=\frac{\exp\left(-\left(h^{-1}(s)\right)^2\right)}{s^2} \tag{3.15}$$

is strictly decreasing $\forall s\in(0,1)$, where h is the function defined in (1.4). More precisely we are going to show that

$$F'(s)<0\quad\forall s\in(0,1). \tag{3.16}$$

A straightforward computation gives

$$\begin{aligned}F'(s)&=-2\frac{\exp\left(-\left(h^{-1}(s)\right)^2\right)}{s^3}+\frac{1}{s^2}\left[-2\frac{h^{-1}(s)}{h'(h^{-1}(s))}\right]\exp\left(-\left(h^{-1}(s)\right)^2\right)\\
&=-2\frac{\exp\left(-\left(h^{-1}(s)\right)^2\right)}{s^3}+\frac{1}{s^2}\left[2\frac{h^{-1}(s)}{\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(h^{-1}(s))^2}{2}\right)}\right]\exp\left(-\left(h^{-1}(s)\right)^2\right)\\
&=-\frac{2}{s^2}\left[\frac{\exp\left(-\left(h^{-1}(s)\right)^2\right)}{s}-\sqrt{2\pi}h^{-1}(s)\exp\left(-\frac{\left(h^{-1}(s)\right)^2}{2}\right)\right].\end{aligned}$$

Therefore $F'(s)<0$ if and only if

$$\frac{1}{s}\exp\left(-\frac{\left(h^{-1}(s)\right)^2}{2}\right)-\sqrt{2\pi}h^{-1}(s)>0,\quad\forall s\in(0,1).$$

Setting $t:=h^{-1}(s)$, the last inequality is equivalent to the following one

$$\Psi(t):=\exp\left(-\frac{t^2}{2}\right)-t\int_t^{+\infty}\exp\left(-\frac{\sigma^2}{2}\right)d\sigma>0,\quad\forall t\in\mathbb{R}.$$

Clearly it holds that

$$\Psi(t)>0\quad\forall t\in(-\infty,0].$$

On the other hand

$$\Psi'(t) = -\int_t^{+\infty} \exp\left(-\frac{\sigma^2}{2}\right) d\sigma < 0 \quad \forall t \in \mathbb{R}.$$

Hence we get the claim (3.16) if we show that

$$\lim_{t\to+\infty} \Psi(t) = 0.$$

This is easily verified since on one hand

$$\lim_{t\to+\infty} \exp\left(-\frac{t^2}{2}\right) = 0$$

on the other L'Hôpital's rule ensures that

$$\lim_{t\to+\infty} t\int_t^{+\infty} \exp\left(-\frac{\sigma^2}{2}\right) d\sigma = \lim_{t\to+\infty} \frac{t^2}{\exp\left(\frac{t^2}{2}\right)} = 0.$$

In order to prove (1.5), we first multiply each side of inequality (3.6) by $t\,\mu(t)\exp\left(\left(h^{-1}(\mu(t))\right)^2\right)$ obtaining

$$\begin{aligned}\frac{1}{2\pi} t\,\mu(t) \; &\le t\,(\mu(t))^2 \exp\left(\left(h^{-1}(\mu(t))\right)^2\right)(-\mu'(t)) + \\ &+ \frac{1}{\beta} t\,(\mu(t))^2 \exp\left(\left(h^{-1}(\mu(t))\right)^2\right)\int_{\partial U_t^{ext}} \frac{\phi(x)}{u(x)}\, d\mathcal{H}^{n-1}. \end{aligned} \tag{3.17}$$

We then integrate between 0 and τ such inequality, obtaining, $\forall \tau \ge v_m$,

$$\begin{aligned}\frac{1}{2\pi}\int_0^\tau t\,\mu(t)\,dt \le &\int_0^\tau t\,(\mu(t))^2 \exp\left(\left(h^{-1}(\mu(t))\right)^2\right)(-\mu'(t))\,dt + \\ &+ \frac{1}{\beta}\int_0^\tau t(\mu(t))^2 \exp\left(\left(h^{-1}(\mu(t))\right)^2\right)\int_{\partial U_t^{ext}} \frac{\phi(x)}{u(x)}\, d\mathcal{H}^{n-1} dt. \end{aligned} \tag{3.18}$$

Note that inequality (3.16) ensures that the function

$$s^2 \exp\left(\left(h^{-1}(s)\right)^2\right) = \frac{1}{F(s)},$$

is strictly increasing in (0, 1). Therefore inequality (3.18) together with Lemma 3.3, implies

$$\begin{aligned}\frac{1}{2\pi}\int_0^\tau t\,\mu(t)\,dt \; \le &\int_0^\tau t\,(\mu(t))^2 \exp\left(\left(h^{-1}(\mu(t))\right)^2\right)(-\mu'(t))\,dt + \\ &+ \frac{\exp\left(\left(h^{-1}(|\Omega|_\phi)\right)^2\right)|\Omega|_\phi^3}{2\beta^2}, \quad \forall \tau \ge v_m. \end{aligned} \tag{3.19}$$

Let us define the following function

$$H(l) = \int_0^l s^2 \exp\left(\left(h^{-1}(s)\right)^2\right) ds.$$

Integrating by parts both sides in inequality (3.19), we get

$$\tau \int_0^\tau \frac{\mu(t)}{2\pi}\, dt + \tau H(\mu(\tau))\, dt \le \int_0^\tau \int_0^t \frac{\mu(r)}{2\pi}\, dr\, dt + \int_0^\tau H(\mu(t))\, dt + \\ + \frac{\exp\left(\left(h^{-1}(|\Omega|_\phi)\right)^2\right)|\Omega|_\phi^3}{2\beta^2}, \quad \forall \tau \ge v_m. \tag{3.20}$$

Lemma 2.1, ensures that, $\forall \tau \ge v_m$ it holds

$$\int_0^\tau \frac{\mu(t)}{2\pi}\, dt + H(\mu(\tau))\, dt \le \frac{1}{v_m}\left\{\int_0^{v_m}\int_0^t \frac{\mu(r)}{2\pi}\, dr\, dt + \int_0^{v_m} H(\mu(t))\, dt + \frac{\exp\left(\left(h^{-1}(|\Omega|_\phi)\right)^2\right)|\Omega|_\phi^3}{2\beta^2}\right\}. \tag{3.21}$$

Repeating the same procedure for the solution to the problem (1.3), we obtain the following equality

$$\int_0^\tau \frac{\varphi(t)}{2\pi}\, dt + H(\varphi(\tau))\, dt = \frac{1}{v_m}\left\{\int_0^{v_m}\int_0^t \frac{\varphi(r)}{2\pi}\, dr\, dt + \int_0^{v_m} H(\varphi(t))\, dt + \frac{\exp\left(\left(h^{-1}(|\Omega|_\phi)\right)^2\right)|\Omega|_\phi^3}{2\beta^2}\right\}. \tag{3.22}$$

Taking into account of (3.5), we can rewrite (3.22) as follows

$$\int_0^\tau \frac{\varphi(t)}{2\pi}\, dt + H(\varphi(\tau))\, dt = \frac{v_m|\Omega|_\phi}{4\pi} + H(|\Omega|_\phi) + \frac{\exp\left(\left(h^{-1}(|\Omega|_\phi)\right)^2\right)|\Omega|_\phi^3}{2v_m\beta^2}. \tag{3.23}$$

Then we can compare (3.20) and (3.21) obtaining

$$\int_0^\tau \frac{\mu(t)}{2\pi}\, dt + H(\mu(\tau)) \le \int_0^\tau \frac{\varphi(t)}{2\pi}\, dt + H(\varphi(\tau)), \quad \forall \tau \ge v_m. \tag{3.24}$$

Passing to the limit as $\tau \to +\infty$ in (3.24) we get

$$\int_0^{+\infty} \mu(t)\, dt \le \int_0^{+\infty} \varphi(t)\, dt,$$

i.e. the claim. □

Funding Open access funding provided by Università degli Studi di Napoli Federico II within the CRUI-CARE Agreement. This work has been partially supported by the PRIN project 2017JPCAPN (Italy) grant: "Qualitative and quantitative aspects of nonlinear PDEs", by FRA 2020 "Optimization problems in Geometric-functional inequalities and nonlinear PDE's" (OPtImIzE) and by GNAMPA of INdAM.

Availability of data and material Not applicable.

Code Availability Not applicable.

Declarations

Competing interests On behalf of all authors, the corresponding author declares that there are no financial or non-financial interests that are directly or indirectly related to the work submitted for publication.

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