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A BMO-Type Characterization of Higher Order Sobolev Spaces

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Abstract
We obtain a new characterization of the higher Sobolev space $W^{m,p}(\mathbb{R}^n)$, $m \in \mathbb{N}$ and $p \in (1, +\infty)$ and of the space BV^m, the space of functions of higher order bounded variation. The characterizations are in term of BMO-type seminorms. The results unify and substantially extend previous results in Fusco et al. (ESAIM Control Optim. Calc Var., **24**(2), 835–847 2018) and Farroni et al. (J. Funct. Anal., **278**(9), 108451 2020).

Keywords Higher order Sobolev spaces · Higher order bounded variation · BMO-type seminorms

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1 Introduction

Let $W^{m,p}_{\text{loc}}(\mathbb{R}^n)$ ($m \in \mathbb{N}$, $1 \leq p < \infty$), denote the Sobolev space of functions belonging to $L^p_{\text{loc}}(\mathbb{R}^n)$ whose distribution derivatives up to order m belong to $L^p_{\text{loc}}(\mathbb{R}^n)$.

In [3], the Authors studied a characterization of $W^{m,p}$ based on J. Bourgain, H. Brezis and P. Mironescu's approach introduced in [5] (see also [7]). In particular they prove that if $f \in W^{m-1,p}(\Omega)$, $1 < p < \infty$ and Ω is a smooth bounded domain in $\mathbb{R}^n$ then f belongs to $W^{m,p}(\Omega)$ if and only if,

$$\liminf_{\varepsilon \to 0} \int_\Omega \int_\Omega \frac{|R^{m-1} f(x, y)|^p}{|x - y|^{mp}} \rho_\varepsilon(|x - y|)\, dx\, dy < \infty \tag{1}$$

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where ρ_ε, with $\varepsilon > 0$, are radial mollifiers and $R^{m-1} f$ is the Taylor $(m-1)$ remainder of f. For $p = 1$, the condition (1) describes BV^m.

Here we say that a $W^{m-1,1}(\Omega)$ is of m-th order bounded variation BV^m if its m-th order partial derivatives in the sense of distributions are finite Radon measures. Spaces of this kind have been studied in [10] as applications in mathematical imaging in the setting of isotropic and anisotropic variants of the TV-model (see also [13]).

Another characterization of $W^{m,p}$, $1 < p < \infty$, (BV^m for $p = 1$) formulated in terms of the m-th differences has been presented in [4].

In this article we are concerned with a characterization of $W^{m,p}$ $1 < p < \infty$, (BV^m for $p = 1$) as the limit of certain BMO–type seminorms similar to the one introduced by J. Bourgain, H. Brezis, P. Mironescu in [6].

In [15] the Authors showed that a function $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ belongs to the Sobolev space $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$, $1 < p < +\infty$, if and only if

$$\lim_{\varepsilon \to 0^+} K(\varepsilon, 1, p) < +\infty$$

where

$$K_\varepsilon(f, 1, p) := \varepsilon^{n-p} \sup_{\mathcal{G}_\varepsilon} \sum_{Q' \in \mathcal{G}_\varepsilon} \fint_{Q'} \left| f(x) - \fint_{Q'} f \right|^p dx\,, \tag{2}$$

and the supremum on the right hand side is taken over all families $\mathcal{G}_\varepsilon$ of disjoint ε-cubes $Q' = Q'(x_0, \varepsilon)$ of side length ε, centered in x_0, with arbitrary orientation. Moreover, if $f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ and $p \geq 1$ then

$$\lim_{\varepsilon \to 0^+} K_\varepsilon(f, 1, p) = \gamma(n, p) \int_{\mathbb{R}^n} |\nabla f|^p\, dx \tag{3}$$

where

$$\gamma(n, p) := \max_{\nu \in \mathbb{S}^{n-1}} \int_Q |x \cdot \nu|^p\, dx \tag{4}$$

where $Q = \left(-\frac{1}{2}, \frac{1}{2}\right)^n$.

Following some ideas in [1], an analogous representation formula is obtained for the total variation of SBV functions in [14] (see also [11]). For related results see also [9, 12].

Here, given a function $f \in W^{m-1,p}_{\mathrm{loc}}(\mathbb{R}^n)$, $p \geq 1$, for any $\varepsilon > 0$, we consider

$$K_\varepsilon(f, m, p) := \varepsilon^{n-mp} \sup_{\mathcal{G}_\varepsilon} \sum_{Q' \in \mathcal{G}_\varepsilon} \fint_{Q'} \left| f(x) - P^{m-1}_{Q'}[f](x) \right|^p dx\,,$$

where the families $\mathcal{G}_\varepsilon$ are as above and $P^{m-1}_{Q'}[f]$ is the polynomial of degree $m-1$ centered at x_0, given by

$$P^{m-1}_{Q'}[f](x) = \sum_{|\alpha| \leq m-1} (x - x_0)^\alpha \fint_{Q'} \left(D^\alpha f\right)(s)\, ds. \tag{5}$$

In particular, for $m = 1$ and $m = 2$ we have:

$$P^0_{Q'}[f](x) = \fint_{Q'} f; \qquad P^1_{Q'}[f](x) = \fint_{Q'} f + \sum_{i=1}^n (x_i - x_{0_i}) \left(\fint_{Q'} \frac{\partial f}{\partial y_i}(y)\, dy \right).$$

Our main Theorem reads as follows:

Theorem 1 *Let $p > 1$ and $f \in W^{m-1,p}_{loc}(\mathbb{R}^n)$, then*

$$|\nabla^m f| \in L^p_{loc}(\mathbb{R}^n) \iff \liminf_{\varepsilon \to 0} K_\varepsilon(f, m, p) < \infty. \tag{6}$$

Moreover, if $f \in W^{m,p}_{loc}(\mathbb{R}^n)$ *and* $p \geq 1$ *we have also*

$$\lim_{\varepsilon \to 0} K_\varepsilon(f, m, p) = \beta(n, m, p) \int_{\mathbb{R}^n} |\nabla^m f|^p \, dx. \tag{7}$$

The constant in Eq. 7 is given by

$$\beta(n, m, p) := \max_{\nu \in \mathbb{S}^{N-1}} \left(\frac{1}{m!}\right)^p \int_Q \left|\nu \cdot x^m - \int_Q \nu \cdot y^m \, dy\right|^p dx. \tag{8}$$

where $N = n^m$ and we refer to Section 2 for the notation.

Note that this Theorem is exactly an extension of Theorem 2.2 in [15] to the higher order case; indeed, in the case $m = 1$, since $\int_Q x \cdot \nu \, dx = 0$, the constant $\beta(n, 1, p)$ coincides with the one defined in Eq. 4.

A drawback of the formula Eq. 7 is that one does not recover the function in BV^m. However, we are able to show that it is possible to characterize the functions in $BV^m(\mathbb{R}^n)$ as the functions $f \in W^{m-1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ such that $\limsup_{\varepsilon \to 0} K_\varepsilon(f, m, 1) < +\infty$.

2 Notation and Preliminaries

We denote by $Q = \left(-\frac{1}{2}, \frac{1}{2}\right)^n \subset \mathbb{R}^n$ the unit cube with faces parallel to coordinate axes in $\mathbb{R}^n$. For any $z \in \mathbb{R}^n$ and $\varepsilon > 0$ we denote by $Q_\varepsilon(z) = z + \varepsilon Q$ the cube of sidelenght ε centered in z.

For $m, n \geq 1$, we denote by $N_j = n^{m-j}$ for $j = 0, \ldots, m$. Given $\nu \in \mathbb{R}^{N_0}$ we denotes its components by $\nu_{i_1,\ldots,i_k,\ldots,i_m}$ with $i_k = 1 \ldots n$. Taking $x \in \mathbb{R}^n$, $x = (x_{i_k})_{i_k \in \{1,\ldots,n\}}$ we define the product $\nu \cdot x$ as the element of $\mathbb{R}^{N_1}$ given by

$$(\nu \cdot x)_{i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_m} = \sum_{i_k=1}^{n} \nu_{i_1,i_2,\ldots,i_m} x_{i_k}.$$

The product of $\nu \in R^{N_0}$ and m times the vector $x \in \mathbb{R}^n$, $\nu \cdot x \cdot x \cdot \cdots \cdot x$ is an element of $\mathbb{R}^{N_m} = \mathbb{R}$ and it is denoted for brevity by $\nu \cdot x^m$.

For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\alpha_i \geq 0$ and a point $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, we denote by

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

the monomial of degree $|\alpha| = \sum_{i=1}^n \alpha_i$.

In the same way,

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

is a weak partial derivative of order $|\alpha|$.

Sometimes, we use the convention that $D^0 u = u$. Moreover, let $\nabla^m u$ be a vector with the components $D^\alpha u$, $|\alpha| = m$.

2.1 The Sobolev Space $W^{m,p}$

Definition 1 Let $\Omega \subset \mathbb{R}^n$ be an open set, let $m \in \mathbb{N}$, and let $1 \leq p < \infty$. The Sobolev space $W^{m,p}(\Omega)$ is the space of all functions $u \in L^p(\Omega)$ which admit $\alpha-$th weak derivative $D^\alpha u$ in $L^p(\Omega)$ for every $\alpha \in \mathbb{N}^n$ with $1 \leq |\alpha| \leq m$.

The space $W^{m,p}(\Omega)$ is endowed with the norm

$$\|u\|_{W^{m,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{1\leq|\alpha|\leq m} \|D^\alpha u\|_{L^p(\Omega)}$$

Definition 2 Let $\Omega \subset \mathbb{R}^n$ be an open set, let $m \in \mathbb{N}$, and let $1 \leq p < \infty$. The homogeneous Sobolev space $\dot{W}^{m,p}(\Omega)$ is the space of all functions $u \in L^1_{loc}(\Omega)$ whose $\alpha-$th weak derivative $D^\alpha u$ belongs to $L^p(\Omega)$ for every $\alpha \in \mathbb{N}^n$ with $|\alpha| = m$.

Note that the inclusion

$$W^{m,p}(\Omega) \subseteq \dot{W}^{m,p}(\Omega)$$

holds. Moreover, as a consequence of Poincarè's inequality for sufficiently regular domains of finite measure the spaces $\dot{W}^{m,p}(\Omega)$ and $W^{m,p}(\Omega)$ actually coincide.

The space $\dot{W}^{m,p}(\Omega)$ is equipped with the seminorm

$$|u|_{\dot{W}^{m,p}(\Omega)} = \|\nabla^m u\|_{L^p(\Omega)}.$$

Sometimes we will also use the equivalent seminorm $u \mapsto \sum_{|\alpha|=m} \|D^\alpha u\|_{L^p(\Omega)}$.

The equivalence of the norm permit to have a useful density result as in [17, Remark 11.28]. Indeed, if $u \in \dot{W}^{m,p}(\Omega)$ then for every $\sigma > 0$ there exists $v \in C^\infty(\Omega) \cap \dot{W}^{m,p}(\Omega)$ such that $\|u - v\|_{W^{m,p}(\Omega)} \leq \sigma$.

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and let $E \subset \Omega$ be a Lebesgue measurable set with finite positive measure. Let $1 \leq p \leq +\infty$ and let $m \in \mathbb{N}$. Then, for every $u \in W^{m,p}(\Omega)$, there exists a polynomial $P_E^{m-1}[u]$ of degree $m-1$ such that for every multi-index $\alpha \in \mathbb{R}^n$ with $0 \leq |\alpha| \leq m-1$ (see [17, Exercise 13.26]),

$$\int_E \left(D^\alpha u(x) - D^\alpha P_E^{m-1}[u](x)\right)\, dx = 0. \tag{9}$$

Theorem 2 (Poincarè inequality in $W^{m,p}$ [17, Theorem 13.27]) *Let $m \in \mathbb{N}$, let $1 \leq p < +\infty$ and let $\Omega \subset \mathbb{R}^n$ be an open, bounded and convex set. Then there exists a positive constant $C = C(m, n, p, \Omega) > 0$ such that,*

$$\sum_{k=0}^{m-1} \|\nabla^k(u - P_\Omega^{m-1}[u])\|_{L^p(\Omega)} \leq C\|\nabla^m u\|_{L^p(\Omega)},$$

for every $u \in W^{m,p}$ and for every $k = 0, \dots, m-1$.

Notice that for $m = 1$ the previous Theorem is the classical Poincarè inequality and the polynomial $P_\Omega[u]$ is the mean of u over Ω. In particular, if $u \in W^{m,p}(Q')$ with $Q' = Q'(x_0, \varepsilon)$, then there exists a unique polynomial $P_{Q'}^{m-1}[u]$ of degree $m-1$ such that Eq. 9 holds and there exists a constant $C = C(n, m, p)$ such that

$$\int_{Q'} |u - P_{Q'}^{m-1}[u]|^p \leq C\varepsilon^{mp} \int_{Q'} |\nabla^m u|^p. \tag{10}$$

Next, we consider the Sobolev–Gagliardo–Nirenberg's embedding in $W^{m,p}$(see Lemma 2.1 in [18]).

Let $n > mp$, $1 \leq p < \frac{n}{m}$. Let $u \in W^{m,p}(Q')$ with $Q' = Q'(x_0, \varepsilon)$. Then there exists a unique polynomial $P_{Q'}^{m-1}[u]$ of degree $m-1$ such that Eq. 9 holds and there exists a constant $C = C(n, m, p)$ such that

$$\left(\fint_{Q'} |u - P_{Q'}^{m-1}[u]|^{p^\star}\right)^{\frac{1}{p^\star}} \leq C \left(\frac{1}{\varepsilon^{n-mp}} \int_{Q'} |\nabla^m u|^p\right)^{\frac{1}{p}} \tag{11}$$

where $p^\star = \frac{np}{n-mp}$.

Moreover, the following easy properties of $P_\Omega[u]$ holds:

- Linearity:

$$P_\Omega[u](x) + P_\Omega[v](x) = P_\Omega[u + v](x).$$

- Scaling:

$$P_{\varepsilon\Omega}[u](\varepsilon x) = P_\Omega[u_\varepsilon](x),$$

  where $u_\varepsilon(x) := u(\varepsilon x)$.

We write

$$T_y^m u(x) = \sum_{|\alpha| \leq m} D^\alpha u(y) \frac{(x - y)^\alpha}{\alpha!}$$

for the Taylor polynomial of order m and

$$R^m u(x, y) = u(x) - T_y^m u(x)$$

for the Taylor remainder of order m.

2.2 Functions of Higher-Order Bounded Variation

Let $\Omega \subset \mathbb{R}^n$ be an open set. A function $u \in L^1(\Omega)$ is of bounded variation (for short $u \in BV(\Omega)$) if u has a distributional gradient in form of a Radon measure of finite total mass and write

$$|\nabla u|(\Omega) = \sup\left\{u \ \operatorname{div}\varphi : \varphi \in C_0^1(\Omega), \|\varphi\|_{L^\infty} \leq 1\right\}.$$

We define

$$BV^m(\Omega) = \{u \in W^{m-1,1}(\Omega), \ \nabla^{m-1}u \in BV(\Omega, S^{m-1}(\mathbb{R}))\}$$

the space of (real valued) functions of m-th order bounded variation, i.e. the set of all functions, whose distributional gradients up to order $m-1$ are represented through 1-integrable tensor-valued functions and whose m-th distributional gradient is a tensor-valued Radon measure of finite total variation. Here $S^k(\mathbb{R}^n)$ denotes the set of all symmetric tensors of order k with real components, which is naturally isomorphic to the set of all k-linear symmetric maps $(\mathbb{R}^n)^k \to \mathbb{R}$ (see [10]).

It becomes a Banach space with the norm

$$\|u\|_{BV^m(\Omega)} = \|u\|_{W^{m-1,1}(\Omega)} + |\nabla^m u|(\Omega).$$

Here the total variation of $\nabla^{m-1}u$ is denoted by $|\nabla^m u|(\Omega)$ and defined by

$$|\nabla^m u|(\Omega) = \sup\left(\sum_{\alpha_1,\dots,\alpha_m=1}^{n} \int_\Omega D_{\alpha_1,\dots,\alpha_{m-1}} u \cdot \partial_{\alpha_m}\varphi_{\alpha_1,\dots,\alpha_m}\, dx\right),$$

where the supremum is taken over all $\varphi \in C_0^1(\Omega, \mathbb{R}^n)$ with $\|\varphi\|_\infty = 1$.

Obviously, $W^{m,1}(\Omega)$ is a subspace of $BV^m(\Omega)$.

The definition of BV^m generalizes that of the classical space of functions of bounded variation and many results about BV can be obtained in BV^m similarly (see [16]). We recall a higher-order variant of the famous Poincaré inequality, which will be useful throughout the sequel:

Theorem 3 (Poincarè inequality in BV^m [13, Lemma 2.2]) *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded subset with Lipschitz boundary, $m \in \mathbb{N}$, $1 \leq p < \infty$. Then there exist a constant $C > 0$, depending only on Ω, m and n such that for all $u \in BV^m(\Omega)$*

$$\|u\|_{BV^m(\Omega)} \leq C|\nabla^m u|(\Omega).$$

In particular, the following version of Poincare's inequality holds.

Let $u \in BV^m(Q')$ with $Q' = Q'(x_0, \varepsilon)$, then there exists a unique polynomial $P^{m-1}_{Q'}[u]$ of degree $m-1$ such that Eq. 9 holds and there exists a constant $C = C(n, m)$ such that

$$\int_{Q'} |u - P^{m-1}_{Q'}[u]| \leq C\varepsilon^m |\nabla^m u|(Q') \tag{12}$$

By the nature of its definition, the space BV^m inherits the Poincare-Wirtinger inequality which can be proved exactly as the corresponding first order result.

Let $n > m$, $u \in BV^m(Q')$ with $Q' = Q'(x_0, \varepsilon)$. Then there exists a unique polynomial $P^{m-1}_{Q'}[u]$ of degree $m-1$ such that Eq. 9 holds and there exists a constant $C = C(n, m)$ such that

$$\left(\fint_{Q'} |u - P^{m-1}_{Q'}[u]|^{\frac{n}{n-m}}\right)^{\frac{n-m}{n}} \leq C\frac{1}{\varepsilon^{n-m}}|\nabla^m u|(Q'). \tag{13}$$

We end this subsection with a higher- order variant of the compactness result in BV (Theorem 3.23 in [2]).

Proposition 1 (Compactness result in BV^m [16, Lemma 2.1]) *Let $\Omega \subset \mathbb{R}^n$ be open and bounded with Lipschitz boundary, and let $(u_k)_{k=1}^{\infty}$ be a sequence of BV^m functions such that*

$$\|u_k\|_{BV^m(\Omega)} \leq M$$

for some constant $M > 0$. Then there is a subsequence $(u_{k_l})_{l=1}^{\infty}$ and a function $u \in BV^m(\Omega)$ such that

$$\|u - u_{k_l}\|_{W^{m-1,1}(\Omega)} \to 0 \quad \text{for } l \to \infty \quad \text{and} \quad \|u\|_{BV^m(\Omega)} \leq M.$$

2.3 Other Useful Inequalities

The following tools will be useful in the sequel.

Given $\delta \in (0, 1)$, from the convexity of the function $t \to |t|^p$ we get for every $a, b \in \mathbb{R}$

$$|a+b|^p = \left|\frac{1}{(1+\delta)}(1+\delta)a + \frac{\delta}{1+\delta}\frac{1+\delta}{\delta}b\right|^p \leq (1+\delta)^p|a|^p + \frac{(1+\delta)^p}{\delta^p}|b|^p \tag{14}$$

Taking into account Eq. 14, we also obtain the following pointwise inequality

$$|a-b|^p \geq \frac{1}{(1+\delta)^p}|a|^p - \frac{1}{\delta^p}|b|^p \tag{15}$$

for every $a, b \in \mathbb{R}$. Given $\xi, \eta \in \mathbb{R}^n$ it holds

$$\left||\xi|^p - |\eta|^p\right| \leq p\,(|\xi| + |\eta|)^{p-1}\,|\xi - \eta| \tag{16}$$

and, given $\xi, \eta \in \mathbb{R}^n \setminus \{0\}$ it holds

$$\left|\frac{\xi}{|\xi|} - \frac{\eta}{|\eta|}\right| \leq 2\frac{|\xi - \eta|}{|\xi|}. \tag{17}$$

2.4 The Local Version of the Functional $K_\varepsilon(f, m, p)$

We define the following local counterpart of Eq. 2 which will be use in Step 3 of proof of Theorem 1

$$K_\varepsilon(f, m, p, \Omega) = \varepsilon^{n-mp} \sup_{\mathcal{G}_\varepsilon} \sum_{Q' \in \mathcal{G}_\varepsilon} \fint_{Q'} \left| f(x) - P^{m-1}_{Q'}[f](x) \right|^p dx\,, \tag{18}$$

where the supremum on the right hand side is taken over all families $\mathcal{G}_\varepsilon$ of disjoint open cubes of sidelenght ε and arbitrary orientation contained in Ω.

This quantity is strictly related to the L^p norm of $\nabla^m f$. Indeed, for $p < \frac{n}{m}$ with $p^\star = \frac{np}{n-mp}$, by using Hölder inequality, we have

$$\|f\|_{L^p(Q)} \leq \|f\|_{L^{p^\star}(Q)} \tag{19}$$

Thus, there exists a constant C depending only on Q, m, p such that for $Q' = \varepsilon Q + x_0$, by Eqs. 19 and 11, we get

$$\varepsilon^{n-mp} \fint_{Q'} |f(x) - P^{m-1}_{Q'}[f]|^p\, dx \leq C \int_{Q'} |\nabla^m f|^p. \tag{20}$$

Summing over all sets Q' in $\mathcal{G}_\varepsilon$, we obtain

$$\varepsilon^{n-mp} \sum_{Q' \in \mathcal{G}_\varepsilon} \fint_{Q'} |f(x) - P^{m-1}_{Q'}[f]|^p\, dx \leq C \|\nabla^m f\|^p_{L^p(\Omega)}$$

and therefore

$$K_\varepsilon(f, m, p, \Omega) \leq C \|\nabla^m f\|^p_{L^p(\Omega)}.$$

We conclude this subsection, by observing that if $\bar{\nu} \in \mathbb{S}^{N-1}$ is a vector maximizing the integral in Eq. 8, $x_0 \in \mathbb{R}^n$ and $Q_\eta(x_0)$ is a cube of side length η with center in x_0 then

$$\frac{1}{(m!)^p} \int_{Q_\eta(x_0)} \left| (x - x_0)^m \cdot \bar{\nu} - \int_{Q_\eta(x_0)} (y - x_0)^m \cdot \bar{\nu}\, dy \right|^p dx = \beta(n, m, p) \cdot \eta^{n+mp}. \tag{21}$$

3 The Case $m = 2$

In this section we deal with the case $m = 2$. In this case it is easier to make some explicit computations. Moreover we give an estimates on the constant $\beta(n, 2, p)$ in terms of the Laplacian of the function $f \in W^{2,p}$.

We prove the following

Proposition 2 *Let $f \in W^{2,p}$ and $\beta(n, 2, p)$ as in Eq. 8. Then the following estimate from below holds true*

$$\beta(n, 2, p) \geq C_{n,p} |\Delta f(0)|^p. \tag{22}$$

First, by virtue of Eq. 9, it is possible to characterize $P_\Omega[u]$ for $m = 2$. Fixed $x_0 \in \Omega$, a generic polynomial of degree 1 centered in x_0 is given by

$$P^1_\Omega[u](x) = \langle a, x - x_0 \rangle + b, \qquad a \in \mathbb{R}^n, b \in \mathbb{R}.$$

By Eq. 9 with $|\alpha| = 0$, we have

$$b|\Omega| = \int_\Omega (u(x) - \langle a, x - x_0 \rangle)\, dx$$

which implies

$$b = \fint_{\Omega} (u(x) - \langle a, x - x_0 \rangle)\, dx.$$

Moreover, for every $i = 1, \ldots, n$, again Eq. 9 for $|\alpha| = 1$ gives

$$a_i = \fint_{\Omega} \frac{\partial u}{\partial x_i}(x)\, dx$$

and we write

$$a = \fint_{\Omega} \nabla u(x)\, dx.$$

Then the polynomial $P^1_{\Omega}(u)$ is

$$P^1_{\Omega}[u](x) = \fint_{\Omega} \left(u(y) - \langle \fint_{\Omega} \nabla u, y \rangle \right) dy + \langle \fint_{\Omega} \nabla u(y)\, dy, x - x_0 \rangle \tag{23}$$

where, with a slight abuse of notation, we mean

$$\langle \fint_{\Omega} \nabla u(y)\, dy, x - x_0 \rangle = \sum_{j=1}^{n} (x_i - x_{0_i}) \fint_{\Omega} \frac{\partial u}{\partial y_i}(y)\, dy.$$

Remark 1 We observe that if Ω is symmetric with respect to x_0, the polynomial $P^1_{\Omega}[u]$ has a simpler form, indeed

$$\fint_{\Omega} \langle \fint_{\Omega} \nabla u, y \rangle\, dy = 0,$$

and then

$$P^1_{\Omega}[u](x) = \fint_{\Omega} u(y)\, dy + \langle \fint_{\Omega} \nabla u(y)\, dy, x - x_0 \rangle \tag{24}$$

Proof of Proposition 5 We observe that when $m = 2$, $p \geq 1$, Eq. 8 reads as

$$\beta(n, 2, p) := \max_{\nu \in \mathbb{S}^{n^2-1}} \frac{1}{4} \int_Q \left| \nu \cdot x^2 - \int_Q \nu \cdot y^2\, dy \right|^p dx. \tag{25}$$

In this case $\nu \cdot x^2$ can equivalently be write as

$$\langle Ax, x \rangle$$

where $A \in \mathcal{M}(n)$ is a matrix $n \times n$ and $\langle \cdot, \cdot \rangle$ denote the usual scalar product in $\mathbb{R}^n$.

It is worth to remark that

$$\beta(n, 2, p) \geq \frac{1}{2^p} \fint_Q \left| \langle \nabla^2 f(0)x, x \rangle - \fint_Q \langle \nabla^2 f(0)y, y \rangle \right|^p dx\, . \tag{26}$$

Firstly we observe that denoting by e_i the canonical basis of $\mathbb{R}^n$, by $O \in \mathcal{O}(n)$ an orthogonal matrix and by $\mathcal{R} \in SO(n)$ a rotation around the origin taking $O^{-1}(Q)$ into Q we have

$$\int_{O^{-1}(Q)} y_i^2\, dy = \int_{O^{-1}(Q)} (y \cdot e_i)^2\, dy = \int_{R \circ O^{-1}(Q)} (Rw \cdot e_i)^2\, dw = \int_Q (w \cdot R^{-1} e_i)^2\, dy = \frac{1}{12}$$

Moreover, given $A \in \mathcal{S}(n)$ a symmetric matrix there exist $O \in \mathcal{O}(n)$ and $D \in \mathcal{D}(n)$ such that $A = ODO^{-1}$. Thus we have

$$\begin{aligned} \textstyle\int_Q \langle Az, z \rangle\, dz &= \textstyle\int_Q \langle (ODO^{-1})z, z \rangle\, dz = \int_Q \langle (DO^{-1})z, O^{-1}z \rangle\, dz = \int_{O^{-1}(Q)} \langle Dy, y \rangle\, dy \\ &= \textstyle\int_{O^{-1}(Q)} \sum_{i=1}^n \lambda_i y_i^2\, dy = \sum_{i=1}^n \lambda_i \int_{O^{-1}(Q)} y_i^2\, dy = \frac{1}{12} \sum_{i=1}^n \lambda_i \end{aligned} \tag{27}$$

Then we can estimate from below $\beta(n, 2, p)$ using Eqs. 26 and 27, proving Eq. 22. Indeed, setting $\nabla^2 f(0) = A$ we have

$$\int_Q \langle Ax, x\rangle = \frac{\Delta f(0)}{12}.$$

Moreover setting $\overline{y} = \min y_i$, we have

$$\begin{aligned}\frac{1}{2^p}\int_Q \left|\langle Ax, x\rangle - \int_Q \langle Ay, y\rangle\right|^p dx &= \frac{1}{2^p}\int_Q \left|\langle (DO^{-1})x, O^{-1}x\rangle - \frac{\Delta f(0)}{12}\right|^p dx \\ = \frac{1}{2^p}\int_Q \left|\sum \lambda_i y_i^2 - \frac{\Delta f(0)}{12}\right|^p dx &\geq \frac{1}{2^p}\int_{O^{-1}(Q)} \left|\sum \lambda_i \overline{y}^2 - \frac{\Delta f(0)}{12}\right|^p dx \\ &= \frac{1}{2^p}|\Delta f(0)|^p \int_{O^{-1}(Q)} \left|\overline{y} - \frac{1}{12}\right|^p dx = C_{n,p}|\Delta f(0)|^p. \quad (28)\end{aligned}$$

□

4 A Characterization of $W^{m,p}$

Proof of Theorem 1 We divide the proof in three steps, proving first the limsup and liminf inequalities in Eq. 7 and then the validity of Eq. 6.

As a starting point we fix a bounded open set $\Omega \subset \mathbb{R}^n$ and $f \in W^{m,p}(\Omega)$. Given $\sigma > 0$, there exists a function $g \in C_c^\infty(\Omega)$ such that $\|f - g\|_{W^{m,p}(\Omega)} < \sigma$ and we choose $\varepsilon > 0$ such that

$$|\nabla^m g(x) - \nabla^m g(y)| \leq \sigma, \qquad \forall x, y, \ \ |x - y| \leq \frac{\sqrt{n}\varepsilon}{2} \quad (29)$$

Let us take now a family $\mathcal{G}_\varepsilon$ of disjoint open cubes Q' of side length ε and arbitrary orientation and let us denote by $\mathcal{G}'_\varepsilon$ the subfamily of $\mathcal{G}_\varepsilon$ made by all cubes $Q' \in \mathcal{G}_\varepsilon$ such that $Q' \subset \Omega$.

Step1 (limsup inequality) We are going to show that

$$\limsup_{\varepsilon\to 0^+} K_\varepsilon(f, m, p) \leq \beta(n, m, p) \int_{\mathbb{R}^n} |\nabla^m f|^p \, dx.$$

We may assume, without loss of generality, that $|\nabla^m f| \in L^p(\Omega)$. Using Eq. 14 and the linearity of $P^{m-1}_{Q'}[f]$, for any $Q' \in \mathcal{G}'_\varepsilon$ we have:

$$⨍_{Q'} \left|f - P^{m-1}_{Q'}[f]\right|^p dx \leq (1+\delta)^p ⨍_{Q'} \left|g - P^{m-1}_{Q'}[g]\right|^p dx + M_\delta ⨍_{Q'} \left|(f-g) - P^{m-1}_{Q'}[f-g]\right|^p dx \quad (30)$$

where $M_\delta = (1+\delta)^p/\delta^p$.

We recall the notation in Section 2, so denoting by x_0 the center of the cube Q' and for all $x \in Q'$ we write

$$g(x) = T^m_{x_0} g(x) + R^m g(x, x_0),$$

where $|R^m g(x, y)| < (n^{\frac{m}{2}}\sigma\varepsilon^m)/2^m = C_1\sigma\varepsilon^m$.

We now estimate the two terms in Eq. 30. Let us focus on the first addendum: using again Eq. 14 we have

$$\begin{aligned}
&\fint_{Q'} \left|g - P^{m-1}_{Q'}[g]\right|^p dx \\
&= \fint_{Q'} \left|\frac{1}{m!}\nabla^m g(x_0)\cdot(x-x_0)^m + R^m g(x,x_0) - \left[\fint_{Q'} \frac{1}{m!}\nabla^m g(x_0)\cdot(y-x_0)^m\,dy + \fint_{Q'} R^m g(y,x_0)\,dy\right]\right|^p dx \\
&\le (1+\delta)^p \tfrac{1}{(m!)^p} \fint_{Q'} \left|\nabla^m g(x_0)\cdot(x-x_0)^m - \fint_{Q'} \nabla^m g(x_0)\cdot(y-x_0)^m\,dy\right|^p dx + 2^p M_\delta \fint_{Q'} \left|R^m g(x,x_0)\right|^p dx \\
&\le (1+\delta)^p \beta(n,m,p)\varepsilon^{mp}|\nabla^m g(x_0)|^p + C_2 M_\delta \sigma^p \varepsilon^{mp}.
\end{aligned} \tag{31}$$

Moreover, applying again Eqs. 14 and 29 we have

$$|\nabla^m g(x_0)|^p \le (1+\delta)^p \fint_{Q'} \left|\nabla^m g(x)\right|^p dx + C_3 M_\delta \sigma^p.$$

Hence

$$\fint_{Q'} \left|g - P^{m-1}_{Q'}[g]\right|^p dx \le \beta(n,m,p)(1+\delta)^{2p}\varepsilon^{mp} \fint_{Q'} |\nabla^m g(x)|^p\,dx + C_4 M_\delta \varepsilon^{mp}\sigma^p. \tag{32}$$

Let us focus now on the second addendum in Eq. 30. By Poincaré inequality in $W^{m,p}$ (see Theorem 2), we have

$$\fint_{Q'} \left|(f-g) - P^{m-1}_{Q'}[f-g]\right|^p dx \le C_p \varepsilon^{mp-n} \int_{Q'} |\nabla^m (f-g)|^p\,dx \tag{33}$$

where C_p is the Poincaré constant for cubes.

Observe now that $\sharp(\mathcal{G}'_\varepsilon) \le \varepsilon^{-n}|\Omega|$ and set $\Omega_\varepsilon = \{x \in \Omega \,:\, \mathrm{dist}(x, \partial\Omega) > \varepsilon\sqrt{n}\}$. Using Eqs. 30, 32 and 33 we have

$$\begin{aligned}
&\varepsilon^{n-mp} \sum_{Q'\in\mathcal{G}_\varepsilon} \fint_{Q'} \left|f - P^{m-1}_{Q'}[f]\right|^p dx \\
&\le \varepsilon^{n-mp} \sum_{Q'\in\mathcal{G}'_\varepsilon} \fint_{Q'} \left|f - P^{m-1}_{Q'}[f]\right|^p dx + C_6 \sum_{Q'\in\mathcal{G}_\varepsilon\setminus\mathcal{G}'_\varepsilon} \int_{Q'} |\nabla^m f|^p \\
&\le (1+\delta)^p \varepsilon^{n-mp} \sum_{Q'\in\mathcal{G}'_\varepsilon} \fint_{Q'} \left|g - P^{m-1}_{Q'}[g]\right|^p dx + C_p M_\delta \int_\Omega |\nabla^m (f-g)|^p + C_6 \int_{\mathbb{R}^n\setminus\Omega_\varepsilon} |\nabla^m f|^p dx \\
&\le (1+\delta)^{3p}\beta(n,m,p) \sum_{Q'\in\mathcal{G}'_\varepsilon} \int_{Q'} |\nabla^m g(x)|^p\,dx + C_4 M_\delta \varepsilon^n \sigma^p + C_p M_\delta \sigma^p + C_6 \int_{\mathbb{R}^n\setminus\Omega_\varepsilon} |\nabla^m f|^p dx \\
&\le (1+\delta)^{3p}\beta(n,m,p) \int_\Omega |\nabla^m f(x)|^p\,dx + C_4 M_\delta \varepsilon^n \sigma^p + C_p M_\delta \sigma^p + C_6 \int_{\mathbb{R}^n\setminus\Omega_\varepsilon} |\nabla^m f|^p dx
\end{aligned} \tag{34}$$

where the constants depend only on n, p and $|\Omega|$. Then, taking the supremum over all the families of cubes $\mathcal{G}_\varepsilon$, and then letting first $\varepsilon \to 0^+$, $\sigma \to 0$, $\delta \to 0$ and $\Omega \uparrow \mathbb{R}^n$ we conclude.

Step2 (liminf inequality) We fix $\Omega \subset \mathbb{R}^n$, we assume again that $f \in W^{m,p}_{loc}(\Omega)$ and we fix $\sigma > 0$ and $g \in C^\infty_c(\Omega)$ as in the previous Step. We prove that

$$\liminf_{\varepsilon\to 0^+} K_\varepsilon(f,m,p) \ge \beta(n,m,p) \int_{\mathbb{R}^n} |\nabla^m f|^p\,dx. \tag{35}$$

So, for $\eta \in (0,1)$ we consider the set

$$U_\eta = \{x \in \Omega \,:\, |\nabla^m g(x)| > \eta\}$$

With a clever use of Lemma 2.95 of [2] (as in Proposition 3.6 of [14]) it is possible to find k sufficiently small pairwise disjoint open sets $S_j \subset \mathbb{S}^{N-1}$ covering $\mathbb{S}^{N-1}$. Precisely,

$$\bigcup_{j=1}^{k} \bar{S}_j = \mathbb{S}^{N-1}$$

$$\text{diam } S_j < \eta \quad \text{for all } j = 1...k$$

$$\left| \bigcup_{j=1}^{k} \left\{ x \in U_\eta : \frac{\nabla^m g(x)}{|\nabla^m g(x)|} \in \partial S_j \right\} \right| = 0.$$

For all $j = 1, \ldots, k$ we denote

$$A_j = \left\{ x \in U_\eta \ : \ \frac{\nabla^m g(x)}{|\nabla^m g(x)|} \in S_j \right\},$$

which are open sets with the property

$$\left| U_\eta \setminus \bigcup_{j=1}^{k} A_j \right| = 0. \tag{36}$$

For $\varepsilon > 0$ we consider the family $\mathcal{F}_\varepsilon$ of all open cubes with faces parallel to the coordinate planes, side length ε, centered at all points of the form εv, with $v \in \mathbb{Z}^n$. Then for all $j = 1, \ldots, k$ we choose $M_j \in S_j$ and we denote by $R_j \in SO(n)$ a rotation that takes e_1 into M_j.

Note that in this way, denoting by x' the center of the cube $Q' \in \mathcal{F}_\varepsilon$, we have (see Eq. 21),

$$\frac{1}{(m!)^p} \int_{R_j(Q')} \left| (x - x')^m \cdot \bar{v} - \int_{R_j(Q')} (y - x')^m \cdot \bar{v}\, dy \right|^p dx = \beta(n, m, p) \cdot \varepsilon^{n+mp}.$$

For all $j = 1, \ldots, k$ we denote by $R_j(Q_{h,j})$, $Q_{h,j} \in \mathcal{F}_\varepsilon$, $h = 1, \ldots, m_j$, the elements of $\mathcal{G}_\varepsilon$ contained in A_j. By Eq. 36 there exists $\varepsilon(\sigma, \eta)$ such that if $\varepsilon < \varepsilon(\sigma, \eta)$ then

$$\left| U_\eta \setminus \bigcup_{j=1}^{k} \bigcup_{h=1}^{m_j} \mathcal{R}_j(Q_{h,j}) \right| \leq \eta^p.$$

We denote by $x_{h,j}$ the center of the cube $\mathcal{R}_j(Q_{h,j})$ and we argue as in Step 1. Indeed we have

$$\begin{aligned}
&\fint_{R_j(Q_{h,j})} \left| g - P^{m-1}_{R_j(Q_{h,j})}[g] \right|^p dx \\
&\geq \tfrac{1}{(1+\delta)^p} \tfrac{1}{(m!)^p} \fint_{R_j(Q_{h,j})} \left| \nabla^m g(x_{h,j}) \cdot (x - x_{h,j})^m - \fint_{R_j(Q_{h,j})} \nabla^m g(x_{h,j}) \cdot (x - x_{h,j})^m \right|^p dx \\
&\quad - \tfrac{2^p}{\delta^p} \fint_{R_j(Q_{h,j})} |R^m g(x, x_{h,j})|^p \, dx \\
&\geq \tfrac{1}{(1+\delta)^{2p}} \tfrac{|\nabla^m g(x_{h,j})|^p}{(m!)^p} \fint_{R_j(Q_{h,j})} \left| M_j \cdot (x - x_{h,j})^m - \fint_{R_j(Q_{h,j})} M_j \cdot (x - x_{h,j})^m \right|^p dx \\
&\quad - \tfrac{2^p}{\delta^p} \tfrac{|\nabla^m g(x_{h,j})|^p}{(m!)^p} \fint_{R_j(Q_{h,j})} \left| (\nabla^m g(x_{h,j}) - M_j) \cdot (x - x_{h,j})^m - \fint_{R_j(Q_{h,j})} (\nabla^m g(x_{h,j}) - M_j) \cdot (x - x_{h,j})^m \right|^p dx \\
&\quad - C_7 \tfrac{\sigma^p \varepsilon^{mp}}{\delta^p} \\
&\geq \tfrac{\varepsilon^{mp} \beta(n,m,p) |\nabla^m g(x_{h,j})|^p}{(1+\delta)^{2p}} - \tfrac{C_8 \eta^p \varepsilon^{mp}}{\delta^p} \|\nabla^m g\|^p_{L^\infty} - C_7 \tfrac{\sigma^p \varepsilon^{mp}}{\delta^p}.
\end{aligned} \tag{37}$$

Now, adding on j and h the previous inequality, recalling Eq. 36, we have

$$\begin{aligned}
&\varepsilon^{n-mp} \sum_{R_j(Q_{h,j})\in\mathcal{G}'_\varepsilon} \fint_{R_j(Q_{h,j})} \left|g - P^{m-1}_{Q'}[g]\right|^p dx \\
&\geq \varepsilon^{n-mp} \sum_{j=1}^{k} \sum_{h=1}^{m_j} \tfrac{\varepsilon^{mp}\beta(n,m,p)|\nabla^m g(x_{h,j})|^p}{(1+\delta)^{2p}} - \tfrac{C_8\eta^p\varepsilon^{mp}}{\delta^p}\|\nabla^m g\|^p_{L^\infty} - C_7\tfrac{\sigma^p\varepsilon^{mp}}{\delta^p} \\
&\geq \tfrac{\beta(n,m,p)}{(1+\delta)^{3p}} \sum_{j=1}^{k}\sum_{h=1}^{m_j} \int_{R_j(Q_{h,j})} |\nabla^m g|^p - \tfrac{C_8\eta^p\varepsilon^n}{\delta^p}\|\nabla^m g\|^p_{L^\infty} - C_7\tfrac{\sigma^p\varepsilon^n}{\delta^p} \\
&\geq \tfrac{\beta(n,m,p)}{(1+\delta)^{3p}} \int_\Omega |\nabla^m g|^p - \tfrac{C\eta^p}{\delta^p}(1+\|\nabla^m g\|^p_{L^\infty}) - C\tfrac{\sigma^p}{\delta^p} \\
&\geq \tfrac{\beta(n,m,p)}{(1+\delta)^{4p}} \int_\Omega |\nabla^m f|^p - \tfrac{C\eta^p}{\delta^p}(1+\|\nabla^m g\|^p_{L^\infty}) - C\tfrac{\sigma^p}{\delta^p},
\end{aligned}$$

where the constants may change from line to line and depend only on p, n and $|\Omega|$. We conclude choosing η small enough and consequently ε small,

$$\begin{aligned}
&\varepsilon^{n-mp} \sum_{Q'\in\mathcal{G}_\varepsilon} \fint_{Q'} \left|f - P^{m-1}_{Q'}[f]\right|^p dx \\
&\qquad\geq \tfrac{1}{(1+\delta)^p}\varepsilon^{n-mp}\sum_{Q'\in\mathcal{G}_\varepsilon}\fint_{Q'}\left|g - P^{m-1}_{Q'}[g]\right|^p dx - \frac{1}{\delta^p}\int_\Omega |\nabla^m(f-g)|^p \\
&\qquad\geq \tfrac{\beta(n,m,p)}{(1+\delta)^{5p}} \int_\Omega |\nabla^m f|^p - \tfrac{C\sigma^p}{\delta^p},
\end{aligned}$$

where again C may change from line to line and depend on p, n and $|\Omega|$. To conclude we take the supremum over all the families $\mathcal{G}_\varepsilon$ and let first $\varepsilon \to 0$, $\sigma \to 0$, $\delta \to 0$ and $\Omega \uparrow \mathbb{R}^n$, proving Eq. 35.

Step3 (proof of Eq. 6) Now let $p > 1$, $f \in W^{m-1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ and $\liminf_{\varepsilon\to 0} K_\varepsilon(f, m, p) < \infty$. We fix $\sigma > 0$, $\Omega \subset \mathbb{R}^n$ and observe that there exist $r > 0$ and a finite family of pairwise disjoint open cubes $Q(x_i, r)$ such that

$$\left|\Omega \setminus \bigcup_{i=1}^{m} Q(x_i, r)\right| < \sigma. \tag{38}$$

$$|\nabla^m f(x) - \nabla^m f(y)| < \sigma \tag{39}$$

Moreover we fix $0 < \varepsilon < r$ and we set $f_\varepsilon(x) = (\varrho_\varepsilon * f)(x)$, where ϱ is a standard mollifier with compact support in the unit cube Q and $\varrho_\varepsilon(x) = \varepsilon^{-n}\varrho(x/\varepsilon)$.

For every $Q(x_i, r)$ we consider a family $\mathcal{H}_\varepsilon$ of pairwise disjoint cubes $Q_j = z_j + \varepsilon Q \subset Q(x_i, r)$, for $j = 1, \ldots, k$.

We compute now

$$\begin{aligned}
|\nabla^m f_\varepsilon(z_j)|^p &= \left|\int_{\mathbb{R}^n} f(y)\nabla^m\rho_\varepsilon(z_j - y)\,dy\right|^p = \left|\int_{\mathbb{R}^n}\left(f(y) - P^{m-1}_{Q_j}[f](y)\right)\nabla^m\rho_\varepsilon(z_j - y)\,dy\right|^p \\
&\leq \varepsilon^{(-m-n)p+np-n}\int_{Q_j}\left|f(y) - P^{m-1}_{Q_j}[f](y)\right|^p dy = \varepsilon^{-mp}\fint_{Q_j}\left|f(y) - P^{m-1}_{Q_j}[f](y)\right|^p dy.
\end{aligned}$$

noindent Moreover, by Eqs. 29 and 14, we have

$$|\nabla^m f_\varepsilon(z_j)|^p \geq \frac{1}{1+\delta}\varepsilon^{-n}\int_{Q_j}|\nabla^m f_\varepsilon(x)|^p\,dx - \frac{C}{\delta^p}\sigma^p$$

Then

$$\frac{1}{1+\delta}\int_{Q_j}|\nabla^m f_\varepsilon(x)|^p\,dx \leq \varepsilon^{n-mp}\fint_{Q_j}\left|f(y) - P^{m-1}_{Q_j}[f](y)\right|^p dy + \frac{C}{\delta^p}\sigma^p\varepsilon^n.$$

Summing up all the cubes in $\mathcal{H}_\varepsilon$, we obtain

$$
\begin{aligned}
&\frac{1}{1+\delta}\int_{Q(x_i,r)} |\nabla^m f_\varepsilon(x)|^p\,dx \\
&\leq \frac{1}{1+\delta}\sum_{j=1}^k \int_{Q_j} |\nabla^m f_\varepsilon(x)|^p\,dx \leq \varepsilon^{n-mp}\sum_{j=1}^k \fint_{Q_j} \left|f(y)-P^{m-1}_{Q_j}[f](y)\right|^p dy + \frac{C}{\delta^p}\sigma^p\varepsilon^n \\
&\leq \varepsilon^{n-mp}\sum_{j=1}^k \fint_{Q_j} \left|f(y)-P^{m-1}_{Q_j}[f](y)\right|^p dy + \frac{C}{\delta^p}\sigma^p r^n, \quad (40)
\end{aligned}
$$

where the last inequality follows since $k\varepsilon^n \leq r^n$. Taking the supremum with respect to all families $\mathcal{H}_\varepsilon$ and the liminf with respect to ε, we have

$$
\frac{1}{1+\delta}\int_{Q(x_i,r)} |\nabla^m f(x)|^p\,dx \leq \liminf_{\varepsilon\to 0} K_\varepsilon(f,m,p,Q(x_i,r)) + \frac{C}{\delta^p}\sigma^p r^n.
$$

Summing up with respect to i and using Eq. 38 we have

$$
\frac{1}{1+\delta}\int_{\Omega} |\nabla^m f(x)|^p\,dx \leq \liminf_{\varepsilon\to 0} K_\varepsilon(f,m,p,\Omega) + \frac{C}{\delta^p}\sigma^p|\Omega|.
$$

Letting $\sigma\to 0$, $\delta\to 0$ and $\Omega\uparrow\mathbb{R}^n$, we conclude. □

Remark 2 We observe that Theorem 7 hold also in an open set Ω with the same proof replacing $K_\varepsilon(f,m,p)$ by the quantity $K_\varepsilon(f,m,p,\Omega)$ defined in Eq. 18.

Corollary 1 *Let $p>1$, $n>mp$, $p^\star=\frac{np}{n-mp}$, $\Omega\subset\mathbb{R}^n$ and $\mathcal{G}_\varepsilon$ a pairwise disjoint family of translations Q' of εQ contained in Ω. Then, the following three statements are equivalent:*

i) $f\in W^{m,p}(\Omega)$;

ii)

$$
\sup_{\mathcal{G}_\varepsilon}\sum_{Q'\in\mathcal{G}_\varepsilon}\varepsilon^{n-mp}\fint_{Q'}\left|f-P^{m-1}_{Q'}[f]\right|^p < +\infty;
$$

iii)

$$
\sup_{\mathcal{G}_\varepsilon}\sum_{Q'\in\mathcal{G}_\varepsilon}\|f-P^{m-1}_{Q'}[f]\|^p_{L^{p^\star}(Q')} < +\infty
$$

.

Proof In this proof the constant C may change from line to line.

We prove that $iii)\Rightarrow ii)$. By Hölder's inequality it holds

$$
\varepsilon^{n-mp}\fint_{Q'}\left|f-P^{m-1}_{Q'}[f]\right|^p dx \leq \frac{\varepsilon^{n-mp}}{\varepsilon^n}\left(\int_{Q'}\left|f-P^{m-1}_{Q'}[f]\right|^{\frac{np}{n-mp}}\right)^{\frac{n-mp}{n}}|Q'|^{\frac{mp}{n}} = \left\|f-P^{m-1}_{Q'}[f]\right\|^p_{L^{p^\star}(Q')}. \quad (41)
$$

Summing over all sets Q' in $\mathcal{G}_\varepsilon$ and passing to the supremum, we conclude.

We prove that $i)\Rightarrow iii)$. Using the Sobolev-Gagliardo-Nirenberg inequality Eq. 11, we obtain that there exists a constant $C=C(n,m,p)$ such that

$$
\left\|f-P^{m-1}_{Q'}[f]\right\|_{L^{p^\star}(Q')} \leq C\left\|\nabla^m f\right\|_{L^p}. \quad (42)
$$

Summing over Q' in $\mathcal{G}_\varepsilon$ and passing to the supremum over all families G_ε the proof is completed.

The equivalence $i) \Leftrightarrow ii)$ is proved in [8]. □

5 A Characterization of Higher Order Bounded Variation

In this section we deal with the case $p = 1$. This case is not included in Theorem 1 since Eq. 6 hold only for $p > 1$.

The case $m = 1$ was treated in [15]. They proved that (see Proposition 2.4 of [15]) if $f \in L^1_{loc}(\mathbb{R}^n)$ then

$$f \in BV(\mathbb{R}^n) \quad \Longleftrightarrow \quad \liminf_{\varepsilon \to 0} K_\varepsilon(f, 1, 1) < +\infty \tag{43}$$

Precisely, they prove that for $f \in L^1_{loc}(\mathbb{R}^n)$ it holds

$$\frac{1}{4}|\nabla f|(\mathbb{R}^n) \leq \liminf_{\varepsilon \to 0^+} K_\varepsilon(f, 1, 1) \leq \limsup_{\varepsilon \to 0^+} K_\varepsilon(f, 1, 1) \leq \frac{1}{2}|\nabla f|(\mathbb{R}^n),$$

where the total variation of f in $\Omega \subset \mathbb{R}^n$, possibly equal to $+\infty$, is defined by setting

$$|\nabla f|(\Omega) := \sup \left\{ \int_\Omega f(x) \operatorname{div} \varphi(x)\, dx \quad : \varphi \in C^1_c(\Omega),\ \|\varphi\|_\infty \leq 1 \right\}$$

We prove a similar characterization for the case $m > 1$. Now an equivalence similar to Eq. 43 involve the space $BV^m(\mathbb{R}^n)$ of functions of m-th order bounded variation (see Section 2).

Precisely, we prove the following

Proposition 3 *Let $f \in W^{m-1,1}_{loc}(\mathbb{R}^n)$. Then*

$$f \in BV^m(\mathbb{R}^n) \quad \Longleftrightarrow \quad \liminf_{\varepsilon \to 0} K_\varepsilon(f, m, 1) < +\infty$$

Moreover, there is a positive constants C, independent of f, such that

$$|\nabla^m f|(\mathbb{R}^n) \leq \liminf_{\varepsilon \to 0^+} K_\varepsilon(f, m, 1) \leq \limsup_{\varepsilon \to 0^+} K_\varepsilon(f, m, 1) \leq C|\nabla^m f|(\mathbb{R}^n). \tag{44}$$

Proof To prove the first inequality in Eq. 44 we argue as in Step 3 of Theorem 1. In particular, we have

$$\frac{1}{1+\delta} \int_{Q(x_i,r)} |\nabla^m f_\varepsilon(x)|\, dx \leq \varepsilon^{n-m} \sum_{j=1}^{k} \fint_{Q_j} \left| f(y) - P^{m-1}_{Q_j}[f](y) \right| dy + \frac{C}{\delta}\sigma r^n, \tag{45}$$

Taking the supremum with respect to all families $\mathcal{H}_\varepsilon$ and the liminf with respect to ε, we have

$$\frac{1}{1+\delta} \liminf_{\varepsilon \to 0} \int_{Q(x_i,r)} |\nabla^m f_\varepsilon(x)|\, dx \leq \liminf_{\varepsilon \to 0} K_\varepsilon(f, m, Q(x_i, r)) + \frac{C}{\delta}\sigma r^n.$$

By the compactness in BV^m (Proposition 4), we get

$$\frac{1}{1+\delta}|\nabla^m f|(Q(x_i, r))\, dx \leq \liminf_{\varepsilon \to 0} K_\varepsilon(f, m, Q(x_i, r)) + \frac{C}{\delta}\sigma r^n$$

Summing up with respect to i and using Eq. 38 we obtain

$$\frac{1}{1+\delta}\,|\nabla^m f|(\Omega)\,dx \leq \liminf_{\varepsilon\to 0} K_\varepsilon(f,m,\Omega) + \frac{C}{\delta}\sigma r^n|\Omega|$$

We conclude letting $\sigma \to 0$, $\delta \to 0$, $\Omega \uparrow \mathbb{R}^n$.

In order to prove the estimate from above in Eq. 44, it is is sufficient to apply the Poincare' inequality in BV^m (see Section 2). □

Corollary 2 *Let $n > m$, $1^\star = \frac{n}{n-m}$, $\Omega \subset \mathbb{R}^n$ and $\mathcal{G}_\varepsilon$ is any pairwise disjoint family of translations Q' of εQ contained in Ω. Then, the following three statements are equivalent:*

i) $f \in BV^m(\Omega)$;

ii)

$$\sup_{\mathcal{G}_\varepsilon} \sum_{Q'\in\mathcal{G}_\varepsilon} \varepsilon^{n-m} \fint_{Q'} \left|f - P^{m-1}_{Q'}[f]\right| < +\infty$$

iii)

$$\sup_{\mathcal{G}_\varepsilon} \sum_{Q'\in\mathcal{G}_\varepsilon} \|f - P^{m-1}_{Q'}[f]\|_{L^{1^\star}(Q')} < +\infty$$

Proof We prove that $iii) \Rightarrow ii)$. By Hölder's inequality it holds

$$\varepsilon^{n-m}\fint_{Q'} \left|f - P^{m-1}_{Q'}[f]\right| dx \leq \left|f - P^{m-1}_{Q'}[f]\right|_{L^{1^\star}(Q')}. \tag{46}$$

The conclusion follows by summing over all sets Q' in $\mathcal{G}_\varepsilon$.

We prove that $i) \Rightarrow iii)$. By using Eq. 12 there exists a constant $C = C(n.m)$ such that

$$\left\|f - P^{m-1}_{Q'}[f]\right\|_{L^{1^\star}(Q')} \leq C\,\|\nabla^m f\|_{L^p}\,(Q') \tag{47}$$

The conclusion follows again by summing over all sets Q' in $\mathcal{G}_\varepsilon$.

The equivalence $i) \Leftrightarrow ii)$ is proved in [8]. □

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Declarations

Conflict of Interests (There is no conflict of interest)

References

1. Ambrosio, L., Bourgain, J., Brezis, H., Figalli, A.: BMO-type norms related to the perimeter of sets. Comm. Pure Appl. Math. **69**, 1062–1086 (2016)
2. Ambrosio, L., Fusco, N., Pallara, D.: Functions of Bounded Variation and Free Discontinuity Problems. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York (2000)
3. Bojarski, B., Ihnatsyeva, L., Kinnunen, J.: How to recognize polinomials in higher order Sobolev spaces. Math. Scand. **112**, 161–181 (2013)
4. Borghol, R.: Some properties of Sobolev spaces. Asymptot. Anal. **51**(3-4), 303–318 (2007)
5. Bourgain, J., Brezis, H., Mironescu, P.: Another Look at Sobolev Spaces, Optimal Control and Partial Differential Equations, pp. 439–455. IOS Press, Amsterdam (2001)
6. Bourgain, J., Brezis, H., Mironescu, P.: A new function space and applications. J. EMS **17**, 2083–2101 (2015)
7. Brezis, H.: How to recognize constant functions. A connection with Sobolev spaces. (Russian) Uspekhi Mat. Nauk **57**(4(346)), 59–74 (2002). translation in Russian Math. Surveys **57**(4), 693-708 (2002)
8. Brudnyi, A., Brudnyi, Y.: On the Banach structure of multivariate BV spaces. Dissertationes Math. **548**, 52 (2020)
9. Di Fratta, G., Fiorenza, A.: BMO-type seminorms from Escher-type tessellations. J. Funct. Anal. **279**, 108556 (2020)
10. Demengel, F., Temam, R.: Convex functions of a measure and applications. Indiana University Mathematical Journal **33**(5), 673–709 (1984)
11. Farroni, F., Fusco, N., Guarino Lo Bianco, S., Schiattarella, R.: A formula for the anisotropic total variation of BV functions. J. Funct. Anal. **278**(9), 108451 (2020)
12. Farroni, F., Guarino Lo Bianco, S., Schiattarella, R.: BMO-type seminorms generating Sobolev functions. J. Math. Anal. Appl. **491**(1), 124298 (2020). 15pp
13. Fuchs, M., Muller, J.: A higher order TV-type variational problem related to the denoising and inpainting of images. Nonlinear Anal. **154**, 122–147 (2017)
14. Fusco, N., Moscariello, G., Sbordone, C.: A formula for the total variation of SBV functions. J. Funct. Anal. **270**(1), 419–446 (2016)
15. Fusco, N., Moscariello, G., Sbordone, C.: BMO-type seminorms and Sobolev functions. ESAIM Control Optim. Calc. Var. **24**(2), 835–847 (2018)
16. Jerrard, R.L., Soner, H.M.: Functions of bounded higher variation. Indiana Univ. Math. J. **51**, 645–677 (2002). https://doi.org/10.1512/iumj.2002.51.2229
17. Leoni, G.: A First Course in Sobolev Spaces, 2nd edn. Graduate Studies in Mathematics, vol. 181. American Mathematical Society, Providence (2017)
18. Marcus: Exceptional sets with respect to Lebesgue differentiation of functions in Sobolev spaces. Ann. Sc. Norm. Sup. Pisa, Cl. Sc. **1**(1–2), 113–130 (1974)