Uniqueness for an Obstacle Problem Arising from Logistic-Type Equations with Fractional Laplacian

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Abstract

We prove a uniqueness theorem for the obstacle problem for linear equations involving the fractional Laplacian with zero Dirichlet exterior condition. The problem under consideration arises as the limit of some logistic-type equations. Our result extends (and slightly strengthens) the known corresponding results for the classical Laplace operator with zero boundary condition. Our proof, as compared with the known proof for the classical Laplace operator, is entirely new, and is based on the probabilistic potential theory. Its advantage is that it may be applied to a wide class of integro-differential operators.

Keywords Dirichlet fractional Laplacian · Obstacle problem · Logistic equation · Intrinsic ultracontractivity

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1 Introduction

Let $D \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain, $D_0 \subset D$ be a bounded *Dirichlet regular* domain (with respect to the fractional Laplacian). For $\alpha \in (0, 2)$, we denote by $(\Delta^{\alpha/2})|_D$ the Dirichlet fractional Laplacian on D with zero exterior condition (see Section 2 for details). In case $\alpha = 2$, by $(\Delta^{\alpha/2})|_D$ we mean the classical Laplace operator Δ_D on D with zero boundary condition. In the present paper, we prove a uniqueness result for the following obstacle problem:

$$\begin{cases} \max \{ -(\Delta^{\alpha/2})|_D u - au, u - \mathbb{I}_{D \setminus \overline{D}_0} \} = 0, \\ u > 0, \quad \text{on } D, \end{cases} \quad (1.1)$$

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where a is a positive constant, and

$$\mathbb{I}_{D \setminus \overline{D}_0}(x) = \begin{cases} 1, & x \in D \setminus \overline{D}_0, \\ +\infty, & x \in \overline{D}_0. \end{cases} \quad (1.2)$$

Problem of this type arises in the study of asymptotic behaviour, as $p \rightarrow \infty$, of logistic type equations (see [10, 27] for the case $\alpha = 2$ and [20] for equations with $\alpha \in (0, 2)$). Problem (1.1) with $\alpha = 2$ also arises as the limit of some predator-prey models (see [8, 11]).

From [10] (in case $\alpha = 2$) and [20] (in case $\alpha \in (0, 2)$) we know that Eq. 1.1 has a solution if and only if $a \in [\lambda_1^D, \lambda_1^{D_0})$, where $\lambda_1^D, \lambda_1^{D_0}$ stand for the first eigenvalue of $-(\Delta^{\alpha/2})|_D$ and $-(\Delta^{\alpha/2})|_{D_0}$, respectively. In the present paper, we concentrate on the uniqueness of solutions to Eq. 1.1. This problem turns out to be difficult, and it has been carefully investigated in [9] in case $\alpha = 2$. Suppose that D_0, D are smooth and $\overline{D}_0 \subset D$. The main result of [9] states that Eq. 1.1 has at most one solution if $a \in (\lambda_1^D, \lambda_1^{D_0})$. This result is proved by using an equivalent free boundary formulation of Eq. 1.1 and tools from the theory of variational inequalities and harmonic functions.

The method used in [9] seems to be suitable only for $\alpha = 2$. To deal with nonlocal operators, we propose a new method. It allows us to prove that if $a \in (\lambda_1^D, \lambda_1^{D_0})$, then for any $\alpha \in (0, 2]$ there exists at most one solution to Eq. 1.1. This generalizes the result of [9] to nonlocal operators but also slightly strengthens the known uniqueness result for $\alpha = 2$ because we assume that $D_0 \subset D$ and not that $\overline{D}_0 \subset D$ as in [9]. Moreover, we consider less regular than in [9] domains D_0, D (see comments in [9, Remark (i)]).

In the present paper, we use a definition of a solution to Eq. 1.1, which is equivalent to, but different from that of [9]. Let $\mathcal{E}_D$ denote the Dirichlet form associated with the operator $(\Delta^{\alpha/2})|_D$ (see Section 2). By a solution to Eq. 1.1 we mean a strictly positive function $u \in H_0^{\alpha/2}(D)$ (closure of $C_c^\infty(D)$ in $H^{\alpha/2}(\mathbb{R}^d)$) having the property that $u \leq \mathbb{I}_{D \setminus \overline{D}_0}$ m -a.e. and such that for some bounded smooth positive Borel measure ν on D (called the reaction measure for u) we have

$$\mathcal{E}_D(u, v) = a(u, v) - \int_D \tilde{v} \, d\nu, \quad v \in H_0^{\alpha/2}(D),$$

where $(\cdot, \cdot)$ stands for the usual inner product in $L^2(D; m)$ and $\tilde{v}$ is a quasi-continuous m -version of v (m stands for the Lebesgue measure on $\mathbb{R}^d$). Moreover, we require that u satisfies the minimality condition, which says that for every quasi-continuous η such that $u \leq \eta \leq \mathbb{I}_{D \setminus \overline{D}_0}$ m -a.e. we have

$$\int_D (\eta - \tilde{u}) \, d\nu = 0,$$

where $\tilde{u}$ is a quasi-continuous m -version of u . In other words, u is a solution of the following equation

$$-(\Delta^{\alpha/2})|_D u = au - \nu,$$

where ν is a positive measure which acts only when u touches the barrier. Our proof of uniqueness is based on the following two crucial observations. The first one is that if u is a solution to Eq. 1.1, then

$$u \cdot \nu = \nu,$$

so in fact,

$$-(\Delta^{\alpha/2})|_D u = au - u \cdot \nu.$$

Equivalently,

$$(-(\Delta^{\alpha/2})|_D + \nu)u = au.$$

This shows that any solution to Eq. 1.1 is in fact an eigenfunction for the operator $-(\Delta^{\alpha/2})|_D + v$, i.e. the operator $-(\Delta^{\alpha/2})|_D$ perturbed by the smooth bounded measure v . It is well known that this operator is a self-adjoint positive definite operator on $L^2(D; m)$ generating a Markov C_0 -semigroup of contractions on $L^2(D; m)$. The second crucial observation is that v has a compact support in D . This allows us to prove, by using some results of Hansen [16], that the Green function G_D for $-(\Delta^{\alpha/2})|_D$ is comparable with the Green function G_D^v for $-(\Delta^{\alpha/2})|_D + v$. These two facts, when combined with the sub and supersolutions method (generalized in the present paper to the case of our obstacle problem), give the uniqueness result.

2 Potential Theory for Fractional Laplacian on Bounded Domain

For fixed $x \in \mathbb{R}^d$ and $r > 0$, we denote $B(x, r) = \{y \in \mathbb{R}^d : |y - x| < r\}$. In the whole paper, we assume that D is an open bounded Lipschitz domain in $\mathbb{R}^d$ with Lipschitz character (r_0, λ) , $r_0, \lambda > 0$, i.e. open nonempty subset of $\mathbb{R}^d$ such that: for every $x \in \partial D$ there exists a function $\Gamma_x : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ such that

$$|\Gamma_x(a) - \Gamma_x(b)| \leq \lambda|a - b|, \quad a, b \in \mathbb{R}^{d-1},$$

and there exists an orthogonal coordinate system S_x such that if $y = (y_1, y_2, \dots, y_d)$ in S_x , then

$$B(x, r_0) \cap D = B(x, r_0) \cap \{y : y_d > \Gamma_x(y_1, y_2, \dots, y_{d-1})\}.$$

We denote by m the Lebesgue measure on $\mathbb{R}^d$. By $\mathcal{B}(\mathbb{R}^d)$ (resp. $\mathcal{B}(D)$), we denote the set of Borel measurable functions on $\mathbb{R}^d$ (on D) with values in $\mathbb{R} \cup \{-\infty\} \cup \{\infty\}$. $\mathcal{B}_b(D)$ is the subset of $\mathcal{B}(D)$ consisting of all bounded functions.

2.1 Dirichlet Fractional Laplacian

Let $\alpha \in (0, 2)$. To define the Dirichlet fractional Laplacian on D , we first set

$$\mathfrak{D}(\Delta^{\alpha/2}) = \{u \in L^2(\mathbb{R}^d; m) : \int_{\mathbb{R}^d} |x|^{2\alpha} |\hat{u}(x)|^2 m(dx) < \infty\},$$

and for $u \in \mathfrak{D}(\Delta^{\alpha/2})$, we set

$$\widehat{\Delta^{\frac{\alpha}{2}} u}(x) = |x|^\alpha \hat{u}(x), \quad x \in \mathbb{R}^d,$$

where $\hat{u}$ stands for the Fourier transform of u . Let us consider the form $(\mathcal{E}, \mathfrak{D}(\mathcal{E}))$ defined as

$$\mathcal{E}(u, v) = \int_{\mathbb{R}^d} \hat{u}(x) \hat{v}(x) |x|^\alpha m(dx), \quad u, v \in \mathfrak{D}(\mathcal{E}),$$

where

$$\mathfrak{D}(\mathcal{E}) = H^{\alpha/2}(\mathbb{R}^d) := \{u \in L^2(\mathbb{R}^d; m) : \int_{\mathbb{R}^d} |x|^\alpha |\hat{u}(x)|^2 m(dx) < \infty\}.$$

By [14, Example 1.4.1], $(\mathcal{E}, H^{\alpha/2}(\mathbb{R}^d))$ is a regular symmetric Dirichlet form on $L^2(\mathbb{R}^d; m)$. When $u \in C_c^\infty(\mathbb{R}^d)$ the more explicit formula for $-\Delta^{\alpha/2}u$ is known (see e.g. [14, Example 1.4.1]):

$$-\Delta^{\alpha/2}u(x) = c_{d,\alpha} \lim_{r \searrow 0} \int_{\mathbb{R}^d \setminus B(x,r)} \frac{u(x) - u(y)}{|x - y|^{d+\alpha}} dy, \quad x \in \mathbb{R}^d.$$

The capacity $\text{Cap}: 2^{\mathbb{R}^d} \rightarrow \mathbb{R}^+ \cup \{\infty\}$ associated with $(\mathcal{E}, H^{\alpha/2}(\mathbb{R}^d))$ is defined as follows: for an arbitrary open set $U \subset \mathbb{R}^d$, we set

$$\text{Cap}(U) = \inf\{\mathcal{E}(u, u), u \in H^{\alpha/2}(\mathbb{R}^d), u \geq \mathbf{1}_U \text{ m-a.e.}\}, \quad (2.1)$$

and then, for an arbitrary $B \subset \mathbb{R}^d$, we set

$$\text{Cap}(B) = \inf\{\text{Cap}(U) : U \supset B, U \subset \mathbb{R}^d, U \text{ open}\}. \quad (2.2)$$

We say that some property holds $\mathcal{E}$ -quasi everywhere ($\mathcal{E}$ -q.e. in abbreviation) if it holds except possibly a set of capacity Cap zero. Such sets shall be called $\mathcal{E}$ -exceptional.

Recall that a function u on $\mathbb{R}^d$ is called $\mathcal{E}$ -quasi-continuous if for every $\varepsilon > 0$ there exists a closed set $F_\varepsilon \subset \mathbb{R}^d$ such that $\text{Cap}(\mathbb{R}^d \setminus F_\varepsilon) \leq \varepsilon$ and $u|_{F_\varepsilon}$ is continuous. By [14, Theorem 2.1.3], each $u \in H^{\alpha/2}(\mathbb{R}^d)$ admits an $\mathcal{E}$ -quasi-continuous m -version. In what follows for given function $u \in \mathcal{B}(\mathbb{R}^d)$, we denote by $\tilde{u}$ its $\mathcal{E}$ -quasi-continuous m -version (whenever it exists).

We denote by $(\mathcal{E}_D, \mathfrak{D}(\mathcal{E}_D))$ the part of $(\mathcal{E}, H^{\alpha/2}(\mathbb{R}^d))$ on D . Recall that

$$\mathfrak{D}(\mathcal{E}_D) = \{u \in H^{\alpha/2}(\mathbb{R}^d) : \tilde{u} = 0, \text{ q.e. on } \mathbb{R}^d \setminus D\}, \quad \mathcal{E}_D(u, v) = \mathcal{E}(u, v), \quad u, v \in \mathfrak{D}(\mathcal{E}_D).$$

By [14, Theorem 4.4.3], $(\mathcal{E}_D, \mathfrak{D}(\mathcal{E}_D))$ is a regular symmetric Dirichlet form on $L^2(D; m)$. Therefore, by [14, Sections 1.3, 1.4], there exists a unique self-adjoint nonpositive definite operator $(A_D, \mathfrak{D}(A_D))$ on $L^2(D; m)$ such that

$$\mathfrak{D}(A_D) \subset \mathfrak{D}(\mathcal{E}_D), \quad \mathcal{E}_D(u, v) = (-A_D u, v), \quad u \in \mathfrak{D}(A_D), v \in \mathfrak{D}(\mathcal{E}_D).$$

The operator $(A_D, \mathfrak{D}(A_D))$ is called the Dirichlet fractional Laplacian. We put

$$(\Delta^{\alpha/2})|_D := A_D.$$

By [14, Theorem 4.4.3], $C_c^\infty(D)$ is a dense subspace of $\mathfrak{D}(\mathcal{E}_D)$ in the norm determined by $\mathcal{E}$. Therefore $\mathfrak{D}(\mathcal{E}_D) = H_0^{\alpha/2}(D)$ ($\tilde{H}^{\alpha/2}(D)$ in the notation of [24, page 77]). For $u \in C_c^\infty(D)$,

$$(\Delta^{\alpha/2})|_D u(x) = \Delta^{\alpha/2} u(x) - \kappa_D(x) u(x), \quad x \in D,$$

where

$$\kappa_D(x) = c_{d,\alpha} \int_{\mathbb{R}^d \setminus D} |x - y|^{-d-\alpha} dy, \quad x \in D.$$

It is worth noting here that $A_D \neq -(\Delta_D)^{\alpha/2}$, where Δ_D is the Laplace operator with zero Dirichlet boundary condition on D . The latter operator is called the fractional Dirichlet Laplacian.

By replacing $(\mathcal{E}, H^{\alpha/2}(\mathbb{R}^d))$ by $(\mathcal{E}_D, H_0^{\alpha/2}(D))$ and $\mathbb{R}^d$ by D in Eqs. 2.1 and 2.2, we define the capacity Cap_D associated with $\mathcal{E}_D$, and then we define the notions of $\mathcal{E}_D$ -exceptional sets and $\mathcal{E}_D$ -quasi-continuity. By [14, Theorem 4.4.4], Cap and Cap_D are strongly equivalent on D . Therefore, without ambiguity, we may write q.e., exceptional or quasi-continuous instead of $\mathcal{E}$ -q.e., $\mathcal{E}_D$ -q.e. $\mathcal{E}$ -exceptional, $\mathcal{E}_D$ -exceptional or $\mathcal{E}$ -quasi-continuous, $\mathcal{E}_D$ -quasi-continuous.

2.2 Green Functions and Transition Functions

We denote by $(J_\beta)_{\beta>0}$, $(T_t)_{t\geq 0}$ the resolvent and the C_0 -semigroup of contractions generated by $A := \Delta^{\alpha/2}$, respectively, and by $(J_\beta^D)_{\beta>0}$, $(T_t^D)_{t\geq 0}$ the resolvent and the C_0 -semigroup of contractions generated by $(\Delta^{\alpha/2})|_D$, respectively.

Let ∂ be a one-point compactification of D . Let $\mathcal{D}$ denote the set of all functions $\omega : [0, \infty) \rightarrow \mathbb{R}^d \cup \{\partial\}$, that are right continuous and possess the left limits for all $t \geq 0$

(cádlágs), and have the property that if $\omega(t) = \partial$, then $\omega(s) = \partial$, $s \geq t$. We equip $\mathcal{D}$ with the Skorokhod topology, see e.g. Section 12 of [1]. Define the *canonical process*: $X : \mathcal{D} \rightarrow \mathcal{D}$, $X_t(\omega) := \omega(t)$, $\omega \in \mathcal{D}$, *shift operator*: $\theta_s : \mathcal{D} \rightarrow \mathcal{D}$, $\theta_s(\omega)(t) = \omega(t + s)$, and *life time*: $\zeta : \mathcal{D} \rightarrow [0, \infty]$, $\zeta(\omega) := \inf\{t > 0 : X_t(\omega) = \partial\}$. It is well known that there exists a rotation invariant α -stable Lévy process $\mathbb{X} = (X, (P_x)_{x \in \mathbb{R}^d}, (\mathcal{F}_t)_{t \geq 0})$ on $\mathbb{R}^d$ such that for every positive $f \in \mathcal{B}(\mathbb{R}^d)$,

$$T_t f(x) = \mathbb{E}_x f(X_t) := \int_{\mathcal{D}} f(X_t(\omega)) dP_x(\omega), \quad x \in \mathbb{R}^d, t \geq 0, \quad m\text{-a.e.} \quad (2.3)$$

We let for $\beta > 0$ and $f \in \mathcal{B}^+(\mathbb{R}^d)$

$$R_\beta f(x) = \mathbb{E}_x \int_0^\infty e^{-\beta t} f(X_t) dt, \quad x \in \mathbb{R}^d.$$

In the whole paper we adopt the convention that any function f defined on a subset of $\mathbb{R}^d$ equals zero at ∂ . By [14, Theorem 4.4.2], there exists a Hunt process $\mathbb{X}^D = (X, (P_x^D)_{x \in D \cup \{\partial\}}, (\mathcal{F}_t)_{t \geq 0})$ on $D \cup \{\partial\}$ such that

$$T_t^D f(x) = \mathbb{E}_x^D f(X_t) := \int_{\mathcal{D}} f(X_t(\omega)) dP_x^D(\omega), \quad x \in D, t \geq 0, \quad m\text{-a.e.} \quad (2.4)$$

Here $(\mathcal{F}_t)_{t \geq 0}$ is a filtration (non-decreasing sequence of σ -fields of subsets of Ω). By [14, Theorem 4.4.1, Theorem 4.4.2], for all $t \geq 0$ and positive $f \in \mathcal{B}(D)$,

$$P_t^D f(x) := \mathbb{E}_x^D f(X_t) = \mathbb{E}_x[f(X_t)\mathbf{1}_{\{t < \tau_D\}}], \quad x \in D, \quad (2.5)$$

where

$$\tau_D = \inf\{t > 0 : X_t \in \mathbb{R}^d \setminus D\}.$$

Hence

$$R_\beta^D f(x) := \mathbb{E}_x^D \int_0^\infty e^{-\beta t} f(X_t) dt = \mathbb{E}_x \int_0^{\tau_D} e^{-\beta t} f(X_t) dt, \quad x \in D. \quad (2.6)$$

It is well known (see [14, Exercise 4.2.1, Lemma 4.2.4]) that there is $G_\beta \in \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d)$ (called the β -Green function) such that for every $f \in \mathcal{B}^+(\mathbb{R}^d)$,

$$R_\beta f(x) = \int_{\mathbb{R}^d} G_\beta(x, y) f(y) dy, \quad x \in \mathbb{R}^d.$$

Similarly, there is $G_{D,\beta} \in \mathcal{B}(D) \times \mathcal{B}(D)$ (called the β -Green function for D) such that for every $f \in \mathcal{B}^+(D)$,

$$R_\beta^D f(x) = \int_D G_{D,\beta}(x, y) f(y) dy, \quad x \in D.$$

Given a positive Borel measure μ on D , we set

$$R_\beta^D \mu(x) = \int_D G_{D,\beta}(x, y) \mu(dy), \quad x \in D.$$

Note that $R_\beta^D f = J_\beta^D f$ m -a.e. for every $f \in L^2(D; m)$. Moreover, by [14, Theorem 4.2.3], $J_\beta^D f \in H_0^{\alpha/2}(D)$ and $\widetilde{J_\beta^D f} = R_\beta^D f$ q.e. for $f \in L^2(D; m)$. By [14, Exercise 4.2.1], there exists a transition function $p_D : \mathbb{R}^+ \times D \times D \rightarrow \mathbb{R}^+$ such that for every $f \in \mathcal{B}^+(D)$,

$$P_t^D f(x) = \int_D p_D(t, x, y) f(y) dy \quad x \in D, t > 0.$$

It is well known (see e.g. [15]) that $p_D(t, x, y)$ is finite and strictly positive for $x, y \in D, t > 0$. Given a positive Borel measure μ on D , we set

$$P_t^D \mu(x) = \int_D p_D(t, x, y) \mu(dy), \quad x \in D, t > 0.$$

By [14, Theorem 4.2.3], $T_t^D f \in H_0^{\alpha/2}(D)$ and $\widetilde{T_t^D f} = P_t^D f$ q.e. for all $f \in L^2(D; m)$ and $t > 0$. Moreover,

$$G_{D,\beta}(x, y) = \int_0^\infty e^{-\beta t} p_D(t, x, y) dt, \quad x, y \in D.$$

It follows that we can set

$$G_D(x, y) := \sup_{\beta > 0} G_{D,\beta}(x, y) = \lim_{\beta \searrow 0} G_{D,\beta}(x, y), \quad x, y \in D.$$

It is well known that $(\Delta^{\alpha/2})|_D : \mathfrak{D}((\Delta^{\alpha/2})|_D) \rightarrow L^2(D; m)$ is a bijection. From the above definition of G_D it follows that for every $f \in L^2(D; m)$,

$$J^D f := (-\Delta^{\alpha/2})|_D^{-1} f = \int_D G_D(\cdot, y) f(y) dy \quad m\text{-a.e. on } D.$$

By [14, Lemma 2.2.11], $J^D f \in H_0^{\alpha/2}(D)$ and $\widetilde{J^D f} = R^D f$ q.e. for every $f \in L^2(D; m)$. For a positive Borel measure μ on D , we set

$$R^D \mu(x) = \int_D G_D(x, y) \mu(dy), \quad x \in D.$$

Recall that a positive Borel measurable function u on D is called excessive if

$$\sup_{t > 0} P_t^D f(x) = f(x), \quad x \in D.$$

By [14, Lemma 4.2.4], for every positive Borel measure μ , $R^D \mu$ is an excessive function.

2.3 Smooth Measures

An increasing sequence $\{F_n\}$ of closed subsets of $\mathbb{R}^d$ (resp. D) is called a generalized $\mathcal{E}$ -nest (resp. generalized $\mathcal{E}_D$ -nest) if for every compact $K \subset \mathbb{R}^d$ (resp. $K \subset D$), $\text{Cap}(K \setminus F_n) \rightarrow 0$ (resp. $\text{Cap}_D(K \setminus F_n) \rightarrow 0$) as $n \rightarrow \infty$. A Borel signed measure μ on $\mathbb{R}^d$ (resp. D) is called $\mathcal{E}$ -smooth (resp. $\mathcal{E}_D$ -smooth) if it charges no set of capacity Cap (resp. Cap_D) zero and there exists a generalized $\mathcal{E}$ -nest (resp. $\mathcal{E}_D$ -nest) such that $|\mu|(F_n) < \infty, n \geq 1$, where $|\mu|$ denotes the variation of μ . Note that if a Borel measure μ on D is bounded, then it is $\mathcal{E}$ -smooth if and only if it is $\mathcal{E}_D$ -smooth. This follows from the fact that Cap and Cap_D are equivalent.

We denote by $\mathcal{M}_0$ (resp. $\mathcal{M}_0(D)$) the set of all $\mathcal{E}$ -smooth (resp. $\mathcal{E}_D$ -smooth) measures on $\mathbb{R}^d$ (resp. D). We also set

$$\mathbb{M}_0(D) = \{\mu \in \mathcal{M}_0(D) : R^D |\mu| < \infty \text{ q.e.}\}, \quad M_{0,b}(D) = \{\mu \in \mathcal{M}_0(D) : |\mu|(D) < \infty\}.$$

By [21, Proposition 5.13],

$$\mathcal{M}_{0,b}(D) \subset \mathbb{M}_0(D). \quad (2.7)$$

Also note that if μ is a Borel measure such that $R^D |\mu| < \infty$ on D , then $\mu \in \mathcal{M}_0(D)$ (see [14, Exercise 4.2.2]). We say that an $\mathcal{E}_D$ -smooth measure μ belongs to the class $S_0(D)$

(called the class of measures of finite energy integral) if there exists $c > 0$ such that

$$\int_D |\tilde{u}| d|\mu| \leq c\sqrt{\mathcal{E}_D(u, u)}, \quad u \in H_0^{\alpha/2}(D).$$

By [14, Theorem 2.2.2], for any positive $\mu \in S_0(D)$ and $\eta \in H_0^{\alpha/2}(D)$,

$$\mathcal{E}_D(R^D\mu, \eta) = \int_D \tilde{\eta} d\mu. \quad (2.8)$$

2.4 Probabilistic Potential Theory

Recall that a bounded subset D of $\mathbb{R}^d$ is called *Dirichlet regular* (with respect to the fractional Laplacian) if for every $x \in \mathbb{R}^d \setminus D$,

$$P_x(\tau_D > 0) = 0.$$

By [2, Example VII.3.4.3], each bounded Lipschitz domain is Dirichlet regular. From this and [5] it follows that $(P_t^D)_{t \geq 0}$ is doubly Feller. This means that it is strongly Feller, i.e. $P_t^D f \in C_b(D)$ for every $f \in \mathcal{B}_b(D)$, and it is Fellerian, i.e. $P_t^D f \in C_0(D)$ for every $f \in C_0(D)$.

By [14, Theorem 5.1.4], there is a one-to-one correspondence between positive $\mathcal{E}_D$ -smooth measures and positive continuous additive functionals (PCAFs) of $\mathbb{X}^D$ (see [14, Section 5.1]). By A^μ we denote the unique PCAF of $\mathbb{X}^D$ associated with $\mu \in \mathcal{M}_0(D)$. By [14, Theorem 5.1.3] and Eq. 2.6,

$$\mathbb{E}_x^D \int_0^\infty f(X_r) dA_r^\mu = \mathbb{E}_x \int_0^{\tau_D} f(X_r) dA_r^\mu = \int_D G_D(x, y) f(y) \mu(dy) = R^D(f \cdot \mu)(x) \quad (2.9)$$

for q.e. $x \in D$. For a signed measure $\mu \in \mathcal{M}_0(D)$ having a decomposition $\mu = \mu^+ - \mu^-$ into a positive and negative part, we set $A^\mu = A^{\mu^+} - A^{\mu^-}$. Note also that if $\mu \in \mathcal{M}_{0,b}^+(D)$ and $\sup_{x \in D} R^D\mu < \infty$, then there exists a strict PCAF A^μ of $\mathbb{X}^D$ such that Eq. 2.9 holds for every $x \in D$ (see [14, Theorem 5.1.6]).

Recall that a càdlàg process M adapted to $(\mathcal{F}_t)_{t \geq 0}$ is called a martingale additive functional (MAF) of $\mathbb{X}^D$ iff $M_0 = 0$, M is an additive functional, $\mathbb{E}_x^D |M_t| < \infty$ q.e. $t \geq 0$, and $\mathbb{E}_x^D M_t = 0$ q.e. $t \geq 0$. A MAF M of $\mathbb{X}^D$ is called uniformly integrable iff for q.e. $x \in D$, and every stopping time $\alpha \leq \tau_D$,

$$\mathbb{E}_x^D |M_\alpha| < \infty, \quad \mathbb{E}_x^D (M_{\tau_D} | \mathcal{F}_\alpha) = M_\alpha, \quad P_x^D\text{-a.s.}$$

The following very useful result will be frequently used in the paper.

Lemma 2.1 Assume that $u \in \mathcal{B}(D)$ and $\mu \in \mathbb{M}_0(D)$. Then

$$u(x) = R^D\mu(x) \quad (2.10)$$

for q.e. $x \in D$ if and only if there exists a uniformly integrable MAF M of $\mathbb{X}^D$ such that for q.e. $x \in D$,

$$u(X_t) = \int_t^{\tau_D} dA_r^\mu - \int_t^{\tau_D} dM_r, \quad t \in [0, \tau_D], \quad P_x\text{-a.s.}, \quad (2.11)$$

Moreover, if $\sup_{x \in D} R^D|\mu|(x) < \infty$, then “for q.e. $x \in D$ ” may be replaced by “for every $x \in D$ ” with A^μ being a strict PCAF of $\mathbb{X}^D$.

Proof First suppose that Eq. 2.11 is satisfied. Taking the expectation with respect to P_x of both sides of Eq. 2.11 with $t = 0$, and then using (2.9) with $f = 1$ we get (2.10). Now suppose that Eq. 2.10 holds. Let N be an exceptional set such that Eq. 2.10 holds for $x \in D \setminus N$. By [14, Theorem 4.1.1], we may assume that $P_x^D(X_t \notin N, t \geq 0) = 1$ for every $x \in D \setminus N$. Hence, by Eq. 2.10, additivity of A^μ , and strong Markov property, for every $x \in D \setminus N$ and every stopping time $\sigma \leq \tau_D$, we have

$$u(X_\sigma) = \mathbb{E}_{X_\sigma}^D \int_0^\infty dA_r^\mu = \mathbb{E}_x^D \left(\int_\sigma^\infty dA_r^\mu | \mathcal{F}_\sigma \right) = \mathbb{E}_x^D \left(\int_0^\infty dA_r^\mu | \mathcal{F}_\sigma \right) - A_\sigma^\mu \quad P_x^D\text{-a.s.} \quad (2.12)$$

Set

$$M_t = u(X_t) - u(X_0) + A_t^\mu, \quad t \geq 0. \quad (2.13)$$

By quasi-continuity of u and [14, Theorem 4.6.1], M is a càdlàg process. Clearly, M is an additive functional and $M_0 = 0$. By Eq. 2.12, M is a uniformly integrable martingale on $[0, \tau_D]$ under measure P_x^D . Equivalently, by Eq. 2.5, M is a uniformly integrable martingale on $[0, \tau_D]$ under measure P_x for $x \in D \setminus N$. If $\sup_{x \in D} R^D |\mu| < \infty$, then the above argument holds true with $N = \emptyset$ with one exception that instead of [14, Theorem 4.6.1] one need to apply [3, Theorem III.5.7]. $\square$

2.5 Feynman-Kac Formula

For a positive measure $\mu \in \mathcal{M}_{0,b}(D)$ such that $\sup_{x \in D} R^D \mu(x) < \infty$, we define the perturbation of the form $(\mathcal{E}_D, H_0^{\alpha/2}(D))$ by μ as follows:

$$\mathfrak{D}(\mathcal{E}_D^\mu) = \{u \in H_0^{\alpha/2}(D) : \int_D \tilde{u}^2 d\mu < \infty\}, \quad \mathcal{E}_D^\mu(u, v) = \mathcal{E}_D(u, v) + \int_D \tilde{u} \tilde{v} d\mu.$$

By [14, Theorem 6.1.2], $(\mathcal{E}_D^\mu, \mathfrak{D}(\mathcal{E}_D^\mu))$ is a regular symmetric Dirichlet form on $L^2(D; m)$. Let $(T_t^{D,\mu})_{t \geq 0}$ be the semigroup generated by $\mathcal{E}_D^\mu$ and A_D^μ be its generator. We let

$$-(\Delta^{\alpha/2})|_D + \mu := -A_D^\mu.$$

By [14] there exists a Hunt process $\mathbb{X}^{D,\mu} = (X, (P_x^{D,\mu})_{x \in D \cup \{\partial\}}, (\mathcal{F}_t)_{t \geq 0})$ on $D \cup \{\partial\}$ such that

$$T_t^{D,\mu} f(x) = \mathbb{E}_x^{D,\mu} f(X_t) := \int_D f(X_t(\omega)) dP_x^{D,\mu}(\omega), \quad x \in D, t \geq 0, \quad m\text{-a.e.} \quad (2.14)$$

We denote by G_D^μ the Green function for $-(\Delta^{\alpha/2})|_D + \mu$ (see [14, Exercise 6.1.1]). By [14, Theorem 6.1.1], we have the Feynman-Kac representation formula

$$\mathbb{E}_x^{D,\mu} f(X_t) = \mathbb{E}_x[e^{-A_t^\mu} f(X_t) \mathbf{1}_{\{t < \tau_D\}}], \quad x \in D, t \geq 0. \quad (2.15)$$

3 Integral Supersolutions of the Obstacle Problem

Let $h : D \rightarrow \mathbb{R} \cup \{\infty\}$ be a measurable strictly positive function and $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Consider the following obstacle problem:

$$\begin{cases} \max \{ -(\Delta^{\alpha/2})|_D u - f(u), u - h \} = 0, \\ u > 0 \quad \text{on } D. \end{cases} \quad (3.1)$$

In this section, we give a definition of an *integral solution* to Eq. 3.1. We also recall widely used in the literature definition of *weak solutions* via variational inequalities. We show that

both definitions are equivalent under regularity assumptions on u . However, we prefer to work on integral solutions. The advantage of this notion is that its formulation does not require any regularity from solutions (merely integrability). Moreover, by using integral solutions, we may apply probabilistic potential theory, which is very useful in many reasonings. We then show that minimum of two *integral supersolutions* of Eq. 3.1 is again an integral supersolution. This property shall be one of the crucial ingredients in the proof of uniqueness of Eq. 1.1.

Definition 3.1 We say that a quasi-continuous function $u \in L^1(D; m)$ is an *integral solution* to Eq. 3.1 if there exists a positive $\nu \in \mathbb{M}_0(D)$ (we call it the reaction measure for u) such that

- (a) $0 < u \leq h$ m -a.e. and $R^D|f(u)| < \infty$ q.e.,
- (b) For q.e. $x \in D$,

$$u(x) = R^D f(u)(x) - R^D \nu(x),$$

- (c) For any quasi-continuous function η on D such that $u \leq \eta \leq h$ m -a.e.,

$$\int_D (\eta - u) d\nu = 0.$$

Remark 3.2 (i) Take any quasi-continuous function $\hat{h}$ such that $u \leq \hat{h} \leq h$ m -a.e. Observe that u is an integral solution to Eq. 3.1 if and only if u is an integral solution to Eq. 3.1 with h replaced by $\hat{h}$.

- (ii) If $h \in H_0^{\alpha/2}(D)$ is quasi-continuous, then u is an integral solution to Eq. 3.1 if and only if (a), (b) are satisfied and $\int_D (h - u) d\nu = 0$.
- (iii) Recall that by [14, Lemma 2.1.4], if $u \leq \eta$, m -a.e. and u, η are quasi-continuous, then $u \leq \eta$ q.e.
- (iv) Note that by [14, Theorem 4.6.1, Theorem A.2.7], functions $R^D f^+(u)$, $R^D f^-(u)$, $R^D \nu$ are quasi-continuous as all the mentioned functions are excessive and finite q.e. So, if (b) holds m -a.e., then by (iii), (b) holds q.e. Clearly, $R^D f^+(u)$, $R^D f^-(u)$ does not depend on the version of u . Therefore, if we assume in the Definition 3.1 that u is merely a measurable function such that (a), (c) hold, and (b) holds m -a.e., then there exists a version $\tilde{u}$ of u such that $\tilde{u}$ is quasi-continuous and (b) holds q.e. with u replaced by $\tilde{u}$: this version is given by $\tilde{u}(x) := R^D f(u)(x) - R^D \nu(x)$ if $R^D|f|(u)(x) + R^D \nu(x) < \infty$ and zero otherwise. Clearly, $\tilde{u} = R^D f(\tilde{u}) - R^D \nu$ q.e.

Recall that $(\cdot, \cdot)$ stands for the usual inner product in $L^2(D; m)$.

Definition 3.3 We say that a quasi-continuous function $u \in H_0^{\alpha/2}(D)$ is a *weak solution* to Eq. 3.1 if

- (a) $0 < u \leq h$ m -a.e. and $f(u) \in L^2(D; m)$,
- (b) For every $\eta \in H_0^{\alpha/2}(D)$ such that $\eta \leq h$ m -a.e.,

$$\mathcal{E}_D(u, \eta - u) \geq (f(u), \eta - u).$$

Proposition 3.4 Let $u \in H_0^{\alpha/2}(D)$ and $f(u) \in L^2(D; m)$. Then u is an integral solution to Eq. 3.1 iff u is a weak solution to Eq. 3.1.

Proof Let u be an integral solution to Eq. 3.1. Since $f(u) \in L^2(D; m)$, $R^D f(u) \in \mathcal{D}((\Delta^{\alpha/2})|_D) \subset H_0^{\alpha/2}(D)$. Thus, $v \in S_0(D)$. Therefore, by Eq. 2.8 and (c) of Definition 3.1, for every $\eta \in H_0^{\alpha/2}(D)$ such that $\eta \leq h$ m -a.e.,

$$\mathcal{E}_D(u, \eta - u) = \mathcal{E}_D(R^D f(u) - R^D v, \eta - u) = (f(u), \eta - u) - \int_D (\tilde{\eta} - \tilde{u}) dv \geq (f(u), \eta - u).$$

Now, let u be a weak solution to Eq. 3.1. By [23, Theorem 5.2, Chapter 3] $u_n \rightarrow u$ weakly in $H_0^{\alpha/2}(D)$, where $u_n \in H_0^{\alpha/2}(D)$ is a unique solution to the following variational equality

$$\mathcal{E}_D(u_n, \eta) = (f(u), \eta) - (n(u_n - h)^+, \eta), \quad \eta \in H_0^{\alpha/2}(D).$$

Clearly, $u_n = R^D f(u) - R^D v_n$, m -a.e., where $v_n = n(u_n - h)^+ \cdot m$. By [19, Theorem 3.8], $u_n \searrow w$, where w is an integral solution to

$$\max \{ -(\Delta^{\alpha/2})|_D w - f(u), w - h \} = 0. \quad (3.2)$$

Since $u = w$, m -a.e., we get the desired result (cf. Remark 3.2(iv)). $\square$

Definition 3.5 We say that a quasi-continuous function $u \in L^1(D; m)$ is an *integral supersolution* (resp. *integral subsolution*) to Eq. 3.1 if there exists a positive measure $\nu \in \mathbb{M}_0(D)$ and a positive (resp. negative) measure $\mu \in \mathbb{M}_0(D)$ such that conditions (a) and (c) of Definition 3.1 are satisfied, and moreover, for q.e. $x \in \mathbb{R}^d$,

$$u(x) = R^D f(u)(x) + R^D \mu(x) - R^D \nu(x).$$

Proposition 3.6 If u is an integral supersolution and an integral subsolution to Eq. 3.1, then u is an integral solution to Eq. 3.1.

Proof By assumptions and Definition 3.5, there exist positive measures $\nu_1, \nu_2, \mu_1, \mu_2 \in \mathbb{M}_0(D)$ such that

$$u(x) = R^D f(u)(x) + R^D \mu_1(x) - R^D \nu_1(x),$$

$$u(x) = R^D f(u)(x) - R^D \mu_2(x) - R^D \nu_2(x),$$

and $\int_D (\eta - u) dv_1 = \int_D (\eta - u) dv_2 = 0$ for every quasi-continuous η on D such that $u \leq \eta \leq h$ m -a.e. Thus,

$$R^D \nu_1(x) - R^D \nu_2(x) = R^D \mu_1(x) + R^D \mu_2(x)$$

for q.e. $x \in D$. From this, we conclude that $\nu_1 - \nu_2 = \mu_1 + \mu_2$. Therefore, there exist positive $\alpha, \beta, \gamma \in \mathcal{B}(D)$ such that $\alpha + \beta = 1, \gamma \leq 1$ and

$$\nu_2 = \gamma \cdot \nu_1, \quad \mu_1 = \alpha(1 - \gamma) \cdot \nu_1, \quad \mu_2 = \beta(1 - \gamma) \cdot \nu_1.$$

Consequently, for q.e. $x \in D$,

$$u(x) = R^D f(u)(x) - R^D ((\beta + \alpha\gamma) \cdot \nu_1)(x).$$

Since $\int_D (\eta - u)(\beta + \alpha\gamma) dv_1 = 0$ for every quasi-continuous η on D such that $u \leq \eta \leq h$ m -a.e., we see that u is an integral solution to Eq. 3.1. $\square$

Proposition 3.7 Let u_1, u_2 be integral supersolutions to Eq. 3.1. Then $u_1 \wedge u_2$ is an integral supersolution to Eq. 3.1.

Proof By the definition of an integral supersolution to Eq. 3.1 and Lemma 2.1, for q.e. $x \in D$, we have

$$u_i(X_t) = \int_t^{\tau_D} dA_r^{\mu_i} + \int_t^{\tau_D} f(u(X_r)) dr - \int_t^{\tau_D} dA_r^{v_i} - \int_t^{\tau_D} dM_r^i, \quad t \in [0, \tau_D], \quad P_x\text{-a.s.},$$

$i = 1, 2$, for some positive $v_1, v_2, \mu_1, \mu_2 \in \mathbb{M}_0(D)$ and some uniformly integrable MAFs M^1, M^2 of $\mathbb{X}^D$. By the Tanaka-Meyer formula (see, e.g., [25, IV.Theorem 70]) applied to $u_1(X) - (u_1(X) - u_2(X))^+ = u_1(X) \wedge u_2(X)$, there exists an increasing càdlàg process C with $C_0 = 0$ such that for q.e. $x \in D$,

$$\begin{aligned} (u_1 \wedge u_2)(X_t) &= (u_1 \wedge u_2)(x) - C_t \\ &\quad - \int_0^t \mathbf{1}_{\{u_1(X_r) > u_2(X_r)\}} f(u_2(X_r)) dr - \int_0^t \mathbf{1}_{\{u_1(X_r) > u_2(X_r)\}} (dA_r^{\mu_2} - dA_r^{v_2}) \\ &\quad - \int_0^t \mathbf{1}_{\{u_2(X_r) \geq u_1(X_r)\}} f(u_1(X_r)) dr - \int_0^t \mathbf{1}_{\{u_2(X_r) \geq u_1(X_r)\}} (dA_r^{\mu_1} - dA_r^{v_1}) \\ &\quad + \int_0^t \mathbf{1}_{\{u_1(X_{r-}) > u_2(X_{r-})\}} dM_r^2 + \int_0^t \mathbf{1}_{\{u_2(X_{r-}) \geq u_1(X_{r-})\}} dM_r^1 \\ &= (u_1 \wedge u_2)(x) - C_t - \int_0^t f((u_1 \wedge u_2)(X_r)) dr \\ &\quad - \int_0^t \mathbf{1}_{\{u_1(X_r) > u_2(X_r)\}} dA_r^{\mu_2} + \int_0^t \mathbf{1}_{\{u_2(X_r) \geq u_1(X_r)\}} dA_r^{\mu_1} \\ &\quad + \int_0^t \mathbf{1}_{\{u_1(X_r) > u_2(X_r)\}} dA_r^{v_2} + \int_0^t \mathbf{1}_{\{u_2(X_r) \geq u_1(X_r)\}} dA_r^{v_1} \\ &\quad + \int_0^t \mathbf{1}_{\{u_1(X_{r-}) > u_2(X_{r-})\}} dM_r^2 + \int_0^t \mathbf{1}_{\{u_2(X_{r-}) \geq u_1(X_{r-})\}} dM_r^1, \end{aligned} \quad (3.3)$$

$t \in [0, \tau_D]$, P_x -a.s. From the above formula, we get, in particular, that C is a positive AF of $\mathbb{X}^D$. By [14, Theorem A.3.16] there exists a positive AF C^p which is the dual predictable projection of C under measure P_x^D for q.e. $x \in D$. Since $\mathbb{X}^D$ is a Hunt process it is, by the very definition, quasi-left continuous, so it has only *totally inaccessible jumps*. Therefore, since $u_1 \wedge u_2$ is quasi-continuous, process $(u_1 \wedge u_2)(X)$ has only totally inaccessible jumps under measure P_x^D for q.e. $x \in D$ (see [14, Theorem 4.2.2]). Moreover, by [6, Proposition 2, Proposition 4] every local $(\mathcal{F}_t)_{t \geq 0}$ -martingale has only totally inaccessible jumps. By the definition of dual predictable projection, $C^p - C$ is an $(\mathcal{F}_t)_{t \geq 0}$ -martingale. Therefore, by Eq. 3.3, C^p has only totally inaccessible jumps under measure P_x^D for q.e. $x \in D$. However, C^p is predictable. Consequently, C^p is continuous. So, C^p is a PCAF of $\mathbb{X}^D$. Hence $C^p = A^\beta$ for some positive $\beta \in \mathbb{M}_0(D)$ (cf. Section 2.4). Define $\mu = \mathbf{1}_{u_1 > u_2} \cdot \mu_2 + \mathbf{1}_{u_2 \geq u_1} \cdot \mu_1 + \beta$ and $\nu = \mathbf{1}_{u_1 > u_2} \cdot v_2 + \mathbf{1}_{u_2 \geq u_1} \cdot v_1$. By Lemma 2.1, for q.e. $x \in D$,

$$(u_1 \wedge u_2)(x) = R^D f(u_1 \wedge u_2)(x) + R^D \mu(x) - R^D \nu(x). \quad (3.4)$$

Let η be a quasi-continuous function such that $u_1 \wedge u_2 \leq \eta \leq h$ m -a.e. Observe that

$$\begin{aligned} \int_D (\eta - u_1 \wedge u_2) d\nu &= \int_{u_1 > u_2} (\eta - u_2) dv_2 + \int_{u_1 \leq u_2} (\eta - u_1) dv_1 \\ &\leq \int_{u_1 > u_2} (\eta \vee u_2 - u_2) dv_2 + \int_{u_1 \leq u_2} (\eta \vee u_1 - u_1) dv_1 \\ &\leq \int_D (\eta \vee u_2 - u_2) dv_2 + \int_D (\eta \vee u_1 - u_1) dv_1. \end{aligned}$$

Clearly, $u_2 \leq \eta \vee u_2 \leq h$ m -a.e., and $u_1 \leq \eta \vee u_1 \leq h$ m -a.e. So, by condition (c) of Definition 3.1 applied to (u_1, v_1) and (u_2, v_2) , we get

$$\int_D (\eta \vee u_2 - u_2) dv_2 + \int_D (\eta \vee u_1 - u_1) dv_1 = 0.$$

Thus, $\int_D (\eta - u_1 \wedge u_2) dv = 0$. This combined with Eq. 3.4 implies that $u_1 \wedge u_2$ is an integral supersolution to Eq. 3.1. $\square$

Proposition 3.8 *Assume that f is nondecreasing. Let $\underline{u}$ (resp. $\bar{u}$) be a bounded integral subsolution (resp. supersolution) to Eq. 3.1 and $\underline{u} \leq \bar{u}$ q.e. Then there exists an integral solution u to Eq. 3.1 such that $\underline{u} \leq u \leq \bar{u}$ q.e.*

Proof Let $u_0 = \underline{u}$. We first show that for each $n \geq 1$ there exists an integral solution u_n to the problem

$$\begin{cases} \max \{ -(\Delta^{\alpha/2})_D v - f(u_{n-1}), v - h \} = 0, \\ v > 0 \quad \text{on } D, \end{cases} \quad (3.5)$$

and

$$u_n \leq u_{n+1}, \text{ q.e.}, \quad v_n \leq v_{n+1} \quad \underline{u} \leq u_n \leq \bar{u}, \text{ q.e.} \quad n \geq 1,$$

where v_n is the reaction measure for u_n . Indeed, the existence of u_1 follows from [19, Theorem 3.8]. By [19, Proposition 3.12], $\underline{u} \leq u_1 \leq \bar{u}$ q.e. In particular, $R^D |f(u_1)| < \infty$ q.e. Hence, by [19, Theorem 3.8] again, there exists an integral solution u_2 to Eq. 3.5, and by [19, Proposition 3.12] again, $\underline{u} \leq u_2 \leq \bar{u}$ q.e. and $u_1 \leq u_2$ q.e. Continuing in this fashion, we get the existence of $\{u_n\}$ having the desired properties. Moreover, by [19, Proposition 3.12], $v_n \leq v_{n+1}$, $n \geq 1$. Let $u = \sup_{n \geq 1} u_n$, and

$$v(B) := \lim_n v_n(B), \quad B \in \mathcal{B}(D).$$

Observe that

$$R^D v_n \leq R^D f(u_{n-1}) \leq R^D |f(\bar{u})| + R^D |f(\underline{u})|, \quad \text{q.e.}$$

Therefore, by [21, Lemma 5.4],

$$\|v_n\|_{TV} \leq \|f(\underline{u})\|_{L^1} + \|f(\bar{u})\|_{L^1}, \quad n \geq 1.$$

The right-hand side is finite since $\underline{u}, \bar{u}$ are bounded and D is bounded. By the Vitali-Hahn-Saks theorem v is a bounded Borel measure and

$$\int_D f dv_n \rightarrow \int_D f dv, \quad f \in L^1(D; v). \quad (3.6)$$

By Eq. 2.7, $R^D v < \infty$ q.e., and $R^D |f(\bar{u})| + R^D |f(\underline{u})| < \infty$ q.e. So, by Eq. 3.6 and the Lebesgue dominated convergence theorem,

$$R^D v_n \rightarrow R^D v, \quad R^D f(u_{n-1}) \rightarrow R^D f(u), \quad \text{q.e.}$$

Thus,

$$u = R^D f(u) - R^D v, \quad \text{q.e.}$$

By Remark 3.2(iv), u is quasi-continuous. Clearly, $u \geq h$, m -a.e. Let η be an arbitrary quasi-continuous function such that $u \leq \eta \leq h$ m -a.e. Then, by the minimality condition (c) of Definition 3.1 applied to (u_n, v_n) , and by Eq. 3.6,

$$0 = \int_D (\eta - u_n) dv_n \geq \int_D (\eta - u) dv_n \rightarrow \int_D (\eta - u) dv \geq 0.$$

Thus, u is an integral solution to Eq. 3.1. $\square$

4 Uniqueness Result

As in Sections 2 and 3, we assume that D is a bounded Lipschitz domain in $\mathbb{R}^d$ ($d \geq 2$) and $D_0 \subset D$ is a bounded Dirichlet regular domain. We assume that $a > \lambda_1^D$, where λ_1^D is the first eigenvalue for the operator $-(\Delta^{\alpha/2})|_D$. By φ_1^D we denote the ground state for $-(\Delta^{\alpha/2})|_D$, i.e. a unique strictly positive function $\varphi_1^D \in H_0^{\alpha/2}(D)$ such that $\|\varphi_1^D\|_{L^2} = 1$ and

$$\mathcal{E}_D(\varphi_1^D, \eta) = \lambda_1^D(\varphi_1^D, \eta), \quad \eta \in D(\mathcal{E}_D).$$

It is well known that $\lambda_1^D > 0$, and by the regularity of D , $\varphi_1^D \in C_0(D)$.

To prove a uniqueness result for Eq. 1.1, we shall need some regularity results for integral solutions to Eq. 1.1, and the result which compare the Green function G_D for $(\Delta^{\alpha/2})|_D$ with the Green function G_D^v for $(\Delta^{\alpha/2})|_D - v$.

It is well known (see e.g. [15]) that the semigroup $(P_t^D)_{t>0}$ is intrinsically ultracontractive, which implies that for every $t > 0$ there exist constants $c_1(t), c_2(t) > 0$ such that

$$c_1(t)\varphi_1^D(x)\varphi_1^D(y) \leq p_D(t, x, y) \leq c_2(t)\varphi_1^D(x)\varphi_1^D(y), \quad x, y \in D. \quad (4.1)$$

Proposition 4.1 *Let u be an integral solution to Eq. 1.1 and v be the reaction measure for u . Then $u \cdot v = v$.*

Proof Set $w = R^D u$, $w_n = R^D(u \wedge n)$, and

$$h(x) = 1 + a\rho(x), \quad \rho(x) = \mathbb{E}_x \int_0^{\tau_{D_0}} u(X_r) dr, \quad x \in D. \quad (4.2)$$

Clearly, $w_n \in H_0^{\alpha/2}(D)$ since $u \wedge n \in L^2(D; m)$. By Dynkin's formula (see [14, (4.4.2)]),

$$\rho(x) + \mathbb{E}_x[w(X_{\tau_{D_0}})] = R^D u(x), \quad x \in D. \quad (4.3)$$

By [14, Lemma 4.3.1], $\mathbb{E}_x[w_n(X_{\tau_{D_0}})]$ is an excessive function for $n \geq 1$. So, by [3, Proposition II.2.2], $\mathbb{E}_x[w(X_{\tau_{D_0}})]$ is an excessive function. By the definition of an integral solution, $u \in L^1(D; m)$. Therefore, by Eq. 2.7, $R^D u$ is finite q.e. So, by Eq. 4.3, $\mathbb{E}_x[w(X_{\tau_{D_0}})]$ is finite q.e. Consequently, ρ as a difference of excessive functions finite q.e. is quasi-continuous (see comments in Remark 3.2(iv)). Thus, h is quasi-continuous. By regularity of D_0 , $\rho(x) = 0$, $x \in D \setminus D_0$. Moreover,

$$\rho(x) = \int_D G_{D_0}(x, y)u(y) m(dy) > 0, \quad x \in D_0. \quad (4.4)$$

By Dynkin's formula again

$$u(x) = \mathbb{E}_x u(X_{\tau_{D_0}}) + a\mathbb{E}_x \int_0^{\tau_{D_0}} u(X_r) dr - \mathbb{E}_x \int_0^{\tau_{D_0}} dA_r^v \leq 1 + a\rho(x) = h(x), \quad \text{q.e.}$$

From this and the fact that $\rho = 0$ on $D \setminus D_0$, we get that the n -th power of h satisfies

$$u \leq h^n \leq \mathbb{I}_{D \setminus \overline{D_0}}, \quad n \geq 1, \quad \text{q.e.}$$

Hence, by the definition of an integral solution to the obstacle problem,

$$\int_D (h^n - u) dv = 0, \quad n \geq 1. \quad (4.5)$$

By Eq. 4.4, for every $x \in D_0$, $h^n(x) \nearrow \infty$ as $n \rightarrow \infty$. It follows from Eq. 4.5 that $\text{supp}[v] \subset D \setminus D_0$. By this and Eq. 4.5 again, for every positive $\eta \in C_c(D)$ we have

$$0 = \int_D \eta(h - u) dv = \int_{D \setminus D_0} \eta(h - u) dv = \int_{D \setminus D_0} \eta(1 - u) dv = \int_D \eta(1 - u) dv,$$

which implies the desired result. $\square$

Proposition 4.2 *If u is an integral solution to Eq. 1.1, then*

- (i) $u(x) \leq c\|u\|_{L^1(D;m)}\varphi_1^D(x)$ for q.e. $x \in D$.
- (ii) $u \in C_0(D)$.
- (iii) *The reaction measure v for u is bounded and $\|v\|_{TV} \leq a\|u\|_{L^1(D;m)}$.*

Proof By the definition of an integral solution to Eq. 1.1 and Proposition 4.1,

$$u(x) = a\mathbb{E}_x \int_0^{\tau_D} u(X_r) dr - \mathbb{E}_x \int_0^{\tau_D} u(X_r) dA_r^v$$

for q.e. $x \in D$. Using Lemma 2.1 and the integration by part formula applied to the product $e^{at-A_t^v} u(X_t)$ yields

$$u(x) = e^{at}\mathbb{E}_x e^{-A_t^v} u(X_t), \quad t \geq 0,$$

for q.e. $x \in D$. Therefore, by the ultracontractivity of $(P_t^D)_{t \geq 0}$, for every $t > 0$,

$$u(x) \leq e^{at} \int_D p_D(t, x, y) u(y) dy \leq c_t e^{at} \varphi_1^D(x) \|u\|_{L^1(D;m)}$$

for q.e. $x \in D$, which proves (i). To prove (ii), consider the function h defined by Eq. 4.2. By regularity of the set D_0 and [5], $(P_t^{D_0})_{t \geq 0}$ is doubly Feller (cf. Section 2.4). Therefore, by (i) $\rho \in C(D_0)$. By regularity of D_0 , in fact $\rho \in C_0(D_0)$. Thus, $h \in C(\overline{D})$. By the proof of Proposition 4.1,

$$u \leq h \leq \mathbb{I}_{D \setminus \overline{D_0}}, \quad \text{q.e.}$$

So, by Remark 3.2(i), u is an integral solution to Eq. 1.1 with $\mathbb{I}_{D \setminus \overline{D_0}}$ replaced by h . Therefore, $u \in C_0(D)$ by [28, Theorem 1], which proves (ii). By the definition of an integral solution to Eq. 1.1,

$$\mathbb{E}_x \int_0^{\tau_D} dA_r^v \leq a\mathbb{E}_x \int_0^{\tau_D} u(X_r) dr \quad (4.6)$$

for q.e. $x \in D$. From Eq. 4.6 and [21, Lemma 5.4] we get (iii). $\square$

Proposition 4.3 *Let u be an integral solution to Eq. 1.1 and v be its reaction measure. Then*

- (i) $\text{supp}[v]$ is a compact subset of D .
- (ii) $\sup_{x \in D} R^D v(x) < \infty$.

Proof By Proposition 4.1, $\text{supp}[v] \subset \{u = 1\}$, which when combined with Proposition 4.2(ii) implies (i). Assertion (ii) follows easily from Eq. 4.6, Proposition 4.2 and Eq. 2.9. $\square$

Proposition 4.4 *Let u be an integral solution to Eq. 1.1 and let v be its reaction measure. Then $u \in H_0^{\alpha/2}(D)$, $v \in S_0(D)$ and for every $\eta \in H_0^{\alpha/2}(D)$,*

$$\mathcal{E}_D(u, \eta) = a(u, \eta) - \int_D \tilde{\eta} dv. \quad (4.7)$$

Proof By Proposition 4.2, $u \in L^1(D; m)$ and $v \in \mathcal{M}_{0,b}(D)$. Therefore, by [22, Theorem 3.5], u is a renormalized solution to (1.1) in the sense defined in [22]. By the definition of a renormalized solution, $u \wedge k \in H_0^{\alpha/2}(D)$, $k \geq 1$, which when combined with Proposition 4.2(i) implies that $u \in H_0^{\alpha/2}(D)$. Moreover, by the definition of a renormalized solution, there exists a sequence $v_k \subset \mathcal{M}_{0,b}(D)$ such that $\|v_k\|_{TV} \rightarrow 0$ and for every bounded $\eta \in H_0^{\alpha/2}(D)$,

$$\mathcal{E}_D(u \wedge k, \eta) + \int_D \tilde{\eta} dv = a(u, \eta) + \int_D \tilde{\eta} dv_k.$$

Letting $k \rightarrow \infty$ yields (4.7) for every bounded $\eta \in H_0^{\alpha/2}(D)$. Applying now a simple approximation argument, we get that $v \in S_0(D)$, and Eq. 4.7 holds for any $\eta \in H_0^{\alpha/2}(D)$. $\square$

Let w be a strictly positive excessive function with respect to $(P_t^D)_{t \geq 0}$. We say that G_D has w -triangle property iff there exists $C > 0$ such that

$$G_D(x, z)G_D(z, y) \leq CG_D(x, y) \max\left(\frac{w(z)}{w(x)}G_D(x, z), \frac{w(z)}{w(y)}G_D(z, y)\right), \quad x, y, z \in D. \quad (4.8)$$

The above notion was introduced in [16]. Observe that if we set

$$\rho(x, y) := \frac{w(x)w(y)}{G_D(x, y)}, \quad x, y \in D,$$

then Eq. 4.8 is equivalent to

$$\rho(x, y) \leq C \max(\rho(x, z), \rho(z, y)), \quad x, y, z \in D. \quad (4.9)$$

Clearly, $\rho(x, y) = 0$ if and only if $x = y$, and $\rho(x, y) = \rho(y, x)$. Therefore, w -triangle property is equivalent to the statement that ρ is a quasi-metric on D .

Recall here that at the beginning of Section 2, we introduced $\lambda > 0$ and $r_0 > 0$. We shall show that G_D has w -triangle property for Lipschitz domains, where $w = \phi$ and

$$\phi(x) = \min(G_D(x, x_0), c_{d,\alpha}(r_0/4)^{\alpha-d}), \quad x \in D$$

for a fixed $x_0 \in D$. Here $c_{d,\alpha} = \frac{\Gamma((d-\alpha)/2)}{2^\alpha \pi^{d/2} |\Gamma(\alpha/2)|}$. Set $\kappa = 1/2\sqrt{1+\lambda^2}$, and fix $x_1 \in D$ such that $|x_0 - x_1| = r_0/4$. We let $\delta_D(x) := \text{dist}(x, \partial D)$, $x \in D$.

Proposition 4.5 *Green function G_D has ϕ -triangle property.*

Proof Let $x, y, z, z_1, z_2 \in D$. By [17, Theorem 1],

$$c_1 \frac{\phi(x)\phi(z_i)}{\phi^2(A_i)} \leq G_D(x, z_i) \leq c_2 \frac{\phi(x)\phi(z_i)}{\phi^2(A_i)}, \quad (4.10)$$

for $A_i \in \mathfrak{B}(x, z_i)$, $i = 1, 2$, where c_1, c_2 depend only on D, d, λ, α . Here, for $r_i := \delta_D(x) \vee \delta_D(z_i) \vee |x - z_i| \leq r_0/32$,

$$\mathfrak{B}(x, z_i) = \{A \in D : B(A, \kappa r_i) \subset D \cap B(x, 3r_i) \cap B(z_i, 3r_i)\},$$

and $\mathfrak{B}(x, z_i) = \{x_i\}$ for $r_i > r_0/32$. Suppose that $|x - z| \geq |y - z|$. Set $z_1 := y, z_2 := z$. By Eq. 4.10,

$$\frac{G_D(x, z)}{G_D(x, y)} = \frac{G_D(x, z_2)}{G_D(x, z_1)} \leq \frac{c_2 \phi(z_2)}{c_1 \phi(z_1)} \left(\frac{\phi(A_1)}{\phi(A_2)}\right)^2 = \frac{c_2 \phi(z)}{c_1 \phi(y)} \left(\frac{\phi(A_1)}{\phi(A_2)}\right)^2.$$

Observe that $|x - z_1| \leq 2|x - z_2|$. Therefore, by [17, Lemma 13], there exists $C_1 > 0$ such that $\frac{\phi(A_1)}{\phi(A_2)} \leq C_1$, where C_1 depends only on α , d and D . Consequently, there exists $C > 0$ such that

$$\frac{G_D(x, z)}{G_D(x, y)} \leq C \frac{\phi(z)}{\phi(y)}, \quad x, y, z \in D; |x - z| \geq |y - z|.$$

Equivalently,

$$\rho(x, y) \leq C\rho(x, z), \quad x, y, z \in D; |x - z| \geq |y - z|.$$

Hence we get easily Eq. 4.9. $\square$

Let G denote the Green function for $\mathbb{R}^d$ (and the operator $\Delta^{\alpha/2}$). It is well known that there is $c > 0$ such that

$$G(x, y) = \frac{c}{|x - y|^{d-\alpha}}, \quad x, y \in \mathbb{R}^d.$$

Lemma 4.6 Assume that $\mu \in \mathcal{M}_{0,b}(D)$ is positive, $K := \text{supp}[\mu]$ is a compact subset of D and $\sup_{x \in D} R^D \mu(x) < \infty$. Then $G_D^\mu \sim G_D$, i.e. there exist $c_1, c_2 > 0$ such that $c_1 G_D^\mu \leq G_D \leq c_2 G_D^\mu$ on $D \times D$.

Proof Let $\bar{\mu}$ be an extension of μ to $\mathbb{R}^d$ defined as $\bar{\mu}(B) = \mu(D \cap B)$ for any Borel set $B \subset \mathbb{R}^d$. Let V be an open set such that $K \subset V \subset \bar{V} \subset D$, and let $r = \text{dist}(K, \partial V)$. By [4, Theorem 1.2], there is $c > 0$ such that

$$cG(x, y) \leq G_D(x, y), \quad x, y \in V.$$

From this and the assumptions of the lemma it follows that

$$\begin{aligned} \sup_{x \in D} R\bar{\mu}(x) &\leq \max\{\sup_{x \in V} R\bar{\mu}(x), \sup_{x \in D \setminus V} R\bar{\mu}(x)\} \\ &\leq \max\left\{c \sup_{x \in V} R^D \mu(x), \sup_{x \in D \setminus V} \int_K G(x, y) \mu(dy)\right\} \\ &\leq c \max\{\sup_{x \in D} R^D \mu(x), r^{\alpha-d} \|\mu\|_{TV}\}. \end{aligned} \quad (4.11)$$

Next, for all $x, y \in D$,

$$\begin{aligned} \int_D \frac{G(x, z)G(z, y)}{G(x, y)} \mu(dz) &= \int_D \frac{|x - y|^{d-\alpha}}{|x - z|^{d-\alpha}|z - y|^{d-\alpha}} \mu(dz) \\ &\leq 2^{d-\alpha} \int_D \frac{\max\{|x - z|^{d-\alpha}, |z - y|^{d-\alpha}\}}{|x - z|^{d-\alpha}|z - y|^{d-\alpha}} \mu(dz) \\ &\leq 2^{d-\alpha} R\bar{\mu}(x) + 2^{d-\alpha} R\bar{\mu}(y). \end{aligned} \quad (4.12)$$

By the 3G Theorem (see [18]),

$$\int_D \frac{G_D(x, z)G_D(z, y)}{G_D(x, y)} \mu(dz) \leq c \int_D \frac{G(x, z)G(z, y)}{G(x, y)} \mu(dz), \quad x, y \in D.$$

This when combined with Eqs. 4.11 and 4.12 shows that there exists $C > 0$ such that

$$\int_D G_D(x, z)G_D(z, y) \mu(dz) \leq CG_D(x, y), \quad x, y \in D. \quad (4.13)$$

From this we conclude that for every positive Borel measure ν on D ,

$$\int_D G_D(x, z) \left(\int_D G_D(z, y) \nu(dy) \right) \mu(dz) \leq C \int_D G_D(x, y) \nu(dy), \quad x, y \in D.$$

Equivalently,

$$\int_D G_D(x, z) R^D v(z) \mu(dz) \leq C R^D v(x), \quad x, y \in D. \quad (4.14)$$

It is well known (see [13, Theorem 17, page 230]) that each $(P_t^D)_{t \geq 0}$ -excessive function is an increasing limit of functions of the form $R^D v$ for some positive Borel measure v . Therefore from (4.14) it follows that for every excessive function e ,

$$\int_D G_D(x, z) e(z) \mu(dz) \leq C e(x), \quad x, y \in D.$$

Taking $e = \phi$ (it is excessive as a minimum of excessive functions), we get

$$\sup_{x \in D} \int_D \frac{\phi(z)}{\phi(x)} G_D(x, z) \mu(dz) < \infty.$$

From this, Proposition 4.5, and [16, Theorem 9.1] we get the desired result. $\square$

Now, we shall proceed to the proof of the main result, i.e. the uniqueness for integral/weak solutions to Eq. 1.1 (see Definition 3.1 and Proposition 3.8). Recall that $d \geq 2$, $\alpha \in (0, 2)$, D_0, D are bounded domains in $\mathbb{R}^d$ which are assumed to be Dirichlet regular (see the beginning of Section 2.4) and Lipschitz (see the introductory part of Section 2), respectively, and $D_0 \subset D$. Recall also that $\lambda_1^{D_0}, \lambda_1^D$ stand for the principal eigenvalues for $(\Delta^{\alpha/2})|_{D_0}, (\Delta^{\alpha/2})|_D$, respectively (see the beginning of Section 4).

Theorem 4.7 *Assume that $a \in (\lambda_1^D, \lambda_1^{D_0})$. Then there exists at most one integral solution to Eq. 1.1.*

Proof Let u_1, u_2 be two integral solutions to Eq. 1.1. We divide the proof into two steps.

Step 1. We shall show that without loss of generality we may assume that $u_1 \leq u_2$. Assume that whenever we know that w, v are integral solutions to Eq. 1.1 such that $w \leq v$, then $w = v$. By Proposition 3.8, $u := u_1 \wedge u_2$ is an integral supersolution to Eq. 1.1. It is clear that for a sufficiently small $c > 0$, $c\phi_1^D \leq \mathbb{I}_{D \setminus \overline{D_0}}$. Therefore, since

$$-(\Delta^{\alpha/2})|_D(c\phi_1^D) = \lambda_1^D c\phi_1^D = ac\phi_1^D - c(a - \lambda_1^D)\phi_1^D$$

and $a > \lambda_1^D$, we see that $c\phi_1^D$ is an integral subsolution to Eq. 1.1 (cf. Eq. 2.8). By the definition of an integral supersolution to Eq. 1.1, there exist positive measures $\mu, \nu \in \mathbb{M}_0(D)$ such that

$$u = aR^D u + R^D \mu - R^D \nu,$$

and ν is the reaction measure for u . By Proposition 4.1, the above equation may be equivalently rewritten as

$$u = aR^D u + R^D \mu - R^D(u \cdot \nu).$$

By Lemma 2.1 there exists a MAF M such that for any $x \in D$,

$$u(X_t) = \int_t^{\tau_D} au(X_r) dr + \int_t^{\tau_D} dA_r^\mu - \int_t^{\tau_D} u(X_r) dA_r^\nu - \int_t^{\tau_D} dM_r, \quad t \in [0, \tau_D], \quad P_x\text{-a.s.} \quad (4.15)$$

By the integration by parts formula applied to the product $e^{-A_t} u(X_t)$, we get

$$e^{-A_t^\nu} u(X_t) = \int_t^{\tau_D} ae^{-A_r^\nu} u(X_r) dr + \int_t^{\tau_D} e^{-A_r^\nu} dA_r^\mu - \int_t^{\tau_D} e^{-A_r^\nu} dM_r, \quad t \in [0, \tau_D], \quad (4.16)$$

P_X -a.s. Taking expectation $\mathbb{E}_x$ of both sides of the above equation with $t = 0$ yields

$$u(x) = \mathbb{E}_x \int_0^{\tau_D} a e^{-A_r^v} u(X_r) dr + \mathbb{E}_x \int_0^{\tau_D} e^{-A_r^v} dA_r^\mu, \quad x \in D.$$

Therefore, by Eq. 2.15,

$$u(x) = a \int_D G_D^v(x, y) u(y) dy + \int_D G_D^v(x, y) \mu(dy), \quad x \in D.$$

By Proposition 4.2, Proposition 4.3 and Lemma 4.6, $G_D^v \sim G_D$. This when combined with the above equation and (4.1) gives

$$u(x) \geq ac \int_D G_D(x, y) u(y) dy \geq acc_1 \varphi_1^D(x) \int_D \varphi_1^D(y) u(y) dy, \quad x \in D,$$

so $u \geq c\varphi_1^D$ for some $c > 0$. Hence, for a sufficiently small $c > 0$, $c\varphi_1^D$ is an integral subsolution to Eq. 1.1 such that $c\varphi_1^D \leq u$. By Proposition 3.8, there exists an integral solution v to Eq. 1.1 such that $c\varphi_1^D \leq v \leq u$. Hence $v \leq u_1$ and $v \leq u_2$. By the assumption of Step 1, $u_1 = v = u_2$.

Step 2. Assume that $u_1 \leq u_2$. Let ν_1, ν_2 be the reaction measures for u_1 and u_2 , respectively. Then, by Proposition 4.4,

$$\mathcal{E}_D(u_1, u_2) + \int_D u_2 d\nu_1 = a(u_1, u_2), \quad \mathcal{E}_D(u_2, u_1) + \int_D u_1 d\nu_2 = a(u_2, u_1).$$

Hence

$$\int_D u_2 d\nu_1 - \int_D u_1 d\nu_2 = 0.$$

From this and Proposition 4.1 we conclude that

$$\int_D u_1 u_2 (d\nu_1 - d\nu_2) = 0. \quad (4.17)$$

We can regard u_i as a solution to the following obstacle problem

$$\max \{ -(\Delta^{\alpha/2})|_D w_i - g_i, w_i - \mathbb{I}_{D \setminus \overline{D}_0} \} = 0,$$

where $g_i = au_i$, $i = 1, 2$. Since $u_1 \leq u_2$, we have $g_1 \leq g_2$. Applying [19, Proposition 3.12] yields $d\nu_1 \leq d\nu_2$. This when combined with (4.17) and the fact that u_1, u_2 are strictly positive implies that $\nu_1 = \nu_2$. Therefore, we have

$$(u_2 - u_1) = aR^D(u_2 - u_1).$$

Thus,

$$(u_2 - u_1)(x) = a \int_D G_D(x, y) (u_2 - u_1)(y) dy, \quad x \in D. \quad (4.18)$$

We have assumed that $u_2 - u_1 \geq 0$. Striving for a contradiction, suppose that $(u_2 - u_1)(x) > 0$ for some $x \in D$. Then continuity of u_1, u_2 and (4.18) would imply that $u_2 - u_1$ is strictly positive on D , in contradiction with the fact that $a > \lambda_1^D$. $\square$

Remark 4.8 All the results of the paper hold for $\alpha = 2$. In case $\alpha = 2$ the proofs run in the same way as in case $\alpha \in (0, 2)$, the only difference being in the proof of Proposition 4.5 and Lemma 4.6. In case $d \geq 3$, Proposition 4.5 follows from [26, Theorem 3.1], and in case $d = 2$ it follows from [16, Corollary 9.6] - we need however D to be finitely connected. As

to the proof of Lemma 4.6, in case $d = 2$, to get (4.13), we use [7, Theorem 6.24], and in case $d \geq 3$, we use [7, Theorem 6.5]. Instead of [4, Theorem 1.2], we use [7, Lemma 6.7].

Ultracontractivity of the semigroup generated by Δ_D follows from [12, Theorem 9.3]. That Lipschitz bounded domains are Dirichlet regular is well known (see, e.g., [2, page 350]).

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