Coercive Inequalities and U-Bounds on Step-Two Carnot Groups

Esther Bou Dagher¹ · Bogusław Zegarliński¹

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Abstract

We prove Poincaré and Log^β -Sobolev inequalities for a class of probability measures on step-two Carnot groups.

Keywords Poincaré inequality · Logarithmic-Sobolev inequality · Carnot groups · Sub-gradient · Probability measures · Kaplan norm

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1 Introduction

Although the question of obtaining coercive inequalities such as the Poincaré or the Logarithmic Sobolev inequalities for a probability measure on a metric measure space has been a subject of numerous works, the literature on this topic in the setup of Carnot groups is scarce.

In [14], L. Gross obtained the following Logarithmic Sobolev inequality:

$$\int_{\mathbb{R}^n} f^2 \log \left(\frac{f^2}{\int_{\mathbb{R}^n} f^2 d\mu} \right) d\mu \leq 2 \int_{\mathbb{R}^n} |\nabla f|^2 d\mu, \quad (1.1)$$

where ∇ is the standard gradient on $\mathbb{R}^n$ and $d\mu = \frac{e^{-\frac{|x|^2}{2}}}{Z} d\lambda$ is the Gaussian measure.

In a setup of a more general metric space, a natural question would be to try to find similar inequalities with different measures of the form $d\mu = \frac{e^{-U(d)}}{Z} d\lambda$, where U is a function of a metric d , and where the Euclidean gradient is replaced by a more general sub-gradient in $\mathbb{R}^n$.

✉ Esther Bou Dagher
esther.bou-dagher17@imperial.ac.uk

Bogusław Zegarliński
b.zegarlinski@imperial.ac.uk

¹ Department of Mathematics, Imperial College London, 180 Queen's Gate, London SW7 2AZ, UK

Aside from their theoretical importance, such inequalities are needed because of their applications, some of which will be discussed briefly. L. Gross also pointed out [14] the importance of the inequality (1.1) in the sense that it can be extended to infinite dimensions with additional useful results. (See also works: [5, 6, 15, 28, 30, 31, 35].) He proved that if $\mathcal{L}$ is the non-positive self-adjoint operator on $L^2(\mu)$ such that

$$(-\mathcal{L}f, f)_{L^2(\mu)} = \int_{\mathbb{R}^n} |\nabla f|^2 d\mu,$$

then (1.1) is equivalent to the fact that the semigroup $P_t = e^{t\mathcal{L}}$ generated by $\mathcal{L}$ is hypercontractive: i.e. for $q(t) \leq 1 + (q-1)e^{2t}$ with $q > 1$, we have $\|P_t f\|_{q(t)} \leq \|f\|_q$ for all $f \in L^q(\mu)$. [14]

In [1], D. Bakry and M. Emery extended the Logarithmic Sobolev inequality for a larger class of probability measures defined on Riemannian manifolds under an important Curvature-Dimension condition. More generally, if (Ω, F, μ) is a probability space, and $\mathcal{L}$ is a non-positive self-adjoint operator acting on $L^2(\mu)$, we say that the measure μ satisfies a Logarithmic Sobolev inequality if there is a constant c such that, for $f \in D(\mathcal{L})$,

$$\int f^2 \log \frac{f^2}{\int f^2 d\mu} d\mu \leq c \int f (-\mathcal{L}f) d\mu,$$

In this general setting, the connection between this inequality and the property of hypercontractivity was shown in [14].

Another generalisation, the so-called q -Logarithmic Sobolev inequality, in the setting of a metric measure space, was obtained by S. Bobkov and M. Ledoux in [4], in the form:

$$\int f^q \log \frac{f^q}{\int f^q d\mu} d\mu \leq c \int |\nabla f|^q d\mu,$$

where $q \in (1, 2]$. Here, on a metric space, the magnitude of the gradient is defined by

$$|\nabla f|(x) = \limsup_{d(x,y) \rightarrow 0} \frac{|f(x) - f(y)|}{d(x,y)}.$$

In [5], S. Bobkov and B. Zegarliński showed that the q -Logarithmic Sobolev inequality is better than the classical $q = 2$ inequality in the sense that one gets a stronger decay of tail estimates. In addition, when the space is finite, and under weak conditions, they proved that the corresponding semigroup P_t is ultracontractive i.e.

$$\|P_t f\|_\infty \leq \|f\|_p$$

for all $t \geq 0$ and $p \in [1, \infty)$.

We point out, that in [21], M. Ledoux made a connection between the Logarithmic Sobolev inequality and the isoperimetric problem. (See also: [1, 3, 18])

The important q -Poincaré inequality

$$\int \left| f - \int f d\mu \right|^q d\mu \leq c \int |\nabla f|^q d\mu$$

can be obtained from the q -Logarithmic Sobolev inequality by simply replacing f by $1 + \varepsilon f$ in that inequality, and letting $\varepsilon \rightarrow 0$.

In this paper, our primary interest is to prove the existence of coercive inequalities for different measures in the setting of step-two nilpotent Lie groups, whose tangent space at every point is spanned by a family of degenerate and non-commuting vector fields

$\{X_i, i \in \mathcal{R}, 1 < |\mathcal{R}| < \infty\}$, where $|\mathcal{R}|$ is the cardinality of the index set $\mathcal{R}$. These inequalities, when satisfied, give us information about the spectra of the associated generators of the form

$$\mathcal{L} = \sum_{i \in \mathcal{R}} X_i^2 \quad (1.2)$$

where $|\mathcal{R}|$ is strictly less than the dimension of the space. (See also [13] and references therein)

Thus, by Hörmander's result in [7], the sub-Laplacian (1.2) is hypoelliptic; in other words, every distributional solution to $\mathcal{L}u = f$ is of class C^∞ whenever f is of class C^∞ .

We point out that, according to [33], if we have a uniqueness of solution in the space of square integrable functions for the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t} = \mathcal{L}u \\ u|_{t=0} = f, \end{cases}$$

then the solution of the heat equation will be given by $u = P_t f$.

In the setting of step-two nilpotent Lie groups, since the Laplacian is of Hörmander type and has some degeneracy, D. Bakry and M. Emery's Curvature-Dimension condition in [1] will no longer hold true. In [16], a method of studying coercive inequalities on general metric spaces that does not require a bound on the curvature of space was developed. Working on a general metric space equipped with non-commuting vector fields $\{X_1, \dots, X_n\}$, their method is based on U-bounds, which are inequalities of the form:

$$\int f^q \mathcal{U}(d) d\mu \leq C \int |\nabla f|^q d\mu + D \int f^q d\mu$$

where $d\mu = \frac{e^{-U(d)}}{Z} d\lambda$ is a probability measure, $U(d)$ and $\mathcal{U}(d)$ are functions having a suitable growth at infinity, λ is a natural measure like the Lebesgue measure for instance (which is the Haar measure for nilpotent Lie groups), d is a metric related to the gradient $\nabla = (X_1, \dots, X_n)$, and $q \in (1, \infty)$.

It is worth mentioning that in the setting of nilpotent Lie groups, heat kernel estimates were studied to get a variety of coercive inequalities ([2, 11, 22–27, 29, 34]). In our setting, we study coercive inequalities involving sub-gradients and probability measures on the group which is a difficult and much less explored subject. An approach, pioneered in [16], was used by J. Inglis to get Poincaré inequality in the setting of the Heisenberg-type group with measure as a function of Kaplan distance [17] and by M. Chatzakou et al. to get Poincaré inequality in the setting of the Engel-type group with a measure as a function of some homegenous norm [10].

In Section 2, we define the step-two Carnot group, and we introduce a family of homogeneous norms N , which generalise the Kaplan norm on Heisenberg-type groups. Section 3 contains the main theorem, which is a proof of a U-Bound of the form

$$\int \frac{g'(N)}{N^2} |f|^q d\mu \leq C \int |\nabla f|^q d\mu + D \int |f|^q d\mu,$$

where $g(N)$ satisfies some growth conditions. In Section 4, we apply this U-bound together with some results of [16] to get the q-Poincaré inequality with $q \geq 2$ for the measures

$d\mu = \frac{e^{-g(N)}}{Z} d\lambda$. This generalises the result by J. Inglis [17] who, in the setting of the Heisenberg-type group, proved the q-Poincaré inequality for the measure $d\mu = \frac{e^{-\alpha \tilde{N}^p}}{Z} d\lambda$,

where $p \geq 2$, $\alpha > 0$, q is the finite index conjugate to p , and with $\tilde{N}$ the Kaplan norm. (Note that the examples studied in [16] do not include measures weighted by more general functions of the Kaplan norm.) In Section 5, we extend J. Inglis et al.'s Theorem 2.1 [18] who proved a Log^β -Sobolev, $\beta \in (0, 1)$ inequality in the context of the Heisenberg group. Recall that for density defined with a smooth homogenous norm, $\beta = 1$ is not allowed ([16]). We extend the corresponding results to a ϕ -Logarithmic Sobolev inequality, where ϕ is concave, on step-two Carnot groups. Finally, we utilise the U-Bound to get to a Log^β -Sobolev inequality for $d\mu = \frac{e^{-\alpha N^p}}{Z} d\lambda$, where $p \geq 4$, $q \geq 2$, and $0 < \beta \leq \frac{p-3}{p}$, indicating also no-go zone of parameters where the corresponding inequality fails.

We were able to use the methods of this paper to obtain similar results in the setting of the Higher-Dimensional Anisotropic Heisenberg group [8].

2 Setup

Carnot groups are geodesic metric spaces that appear in many mathematical contexts like harmonic analysis in the study of hypoelliptic differential operators ([1, 33]) and in geometric measure theory (see extensive reference list in the survey paper [4]).

We will be working in the setting of $\mathbb{G}$, a step-two Carnot group, i.e. a group isomorphic to $\mathbb{R}^{n+m}$ with the group law

$$(x, z) \circ (x', z') = \left(x_i + x'_i, z_j + z'_j + \frac{1}{2} \langle \Lambda^{(j)} x, x' \rangle_{\mathbb{R}^n} \right)_{i=1, \dots, n; j=1, \dots, m}$$

for $x, x' \in \mathbb{R}^n$, $z, z' \in \mathbb{R}^m$, where the matrices $\Lambda^{(j)}$ are $n \times n$ skew-symmetric and linearly independent and $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ stands for the inner product on $\mathbb{R}^n$. One can verify that $(\mathbb{R}^{n+m}, \circ)$ is a Lie group whose identity is the origin and where the inverse is given by $(x, z)^{-1} = -(x, z)$.

The dilation

$$\delta_\lambda : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^{n+m}, \quad \delta_\lambda(x, z) = (\lambda x, \lambda^2 z)$$

is an automorphism of $(\mathbb{R}^{n+m}, \circ)$ for any $\lambda > 0$. Then, $\mathbb{G} = (\mathbb{R}^{n+m}, \circ, \delta_\lambda)$ is a homogeneous Lie group.

The Jacobian matrix at $(0, 0)$ of the left translation $\tau_{(x,z)}$ i.e the map

$$\mathbb{G} \ni (x', z') \rightarrow \tau_{(x,z)}((x', z')) := (x, z) \circ (x', z') \in \mathbb{G}$$

for fixed $(x, z) \in \mathbb{G}$ takes the following form

$$\mathcal{J}_{\tau_{(x,z)}}(0, 0) = \left(\begin{array}{ccc|c} \mathbb{I}_n & & & 0_{n \times m} \\ \frac{1}{2} \sum_{l=1}^n \Lambda_{1l}^{(1)} x_l & \cdots & \frac{1}{2} \sum_{l=1}^n \Lambda_{nl}^{(1)} x_l & \\ \vdots & \cdots & \vdots & \\ \frac{1}{2} \sum_{l=1}^n \Lambda_{1l}^{(m)} x_l & \cdots & \frac{1}{2} \sum_{l=1}^n \Lambda_{nl}^{(m)} x_l & \mathbb{I}_m \end{array} \right).$$

Then, the Jacobian basis of $\mathfrak{g}$, the Lie algebra of $\mathbb{G}$, is given by

$$X_i = \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{k=1}^m \sum_{l=1}^n \Lambda_{il}^{(k)} x_l \frac{\partial}{\partial z_k} \quad \text{and} \quad Z_j = \frac{\partial}{\partial z_j},$$

for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$.

Let $\nabla \equiv (X_i)_{i=1,\dots,n}$ and $\Delta \equiv \sum_{i=1,\dots,n} X_i^2$ denote the associated sub-gradient and sub-Laplacian, respectively. We consider the following smooth homogeneous norm on $\mathbb{G}$

$$N \equiv \left(|x|^4 + a|z|^2\right)^{\frac{1}{4}}$$

with $a \in (0, \infty)$.

The motivation behind choosing such a norm is that in the setting of the Heisenberg-type groups (where we assume in addition that $\Lambda^{(j)}$ are orthogonal matrices and that $\Lambda^{(i)}\Lambda^{(j)} = -\Lambda^{(j)}\Lambda^{(i)}$ for every $i, j \in \{1, \dots, m\}$ with $i \neq j$), $N \equiv \left(|x|^4 + 16|z|^2\right)^{\frac{1}{4}}$ is the Kaplan norm which arises from the fundamental solution of the sub-Laplacian. In other words, $\Delta N^{2-n-2m} = 0$ in $\mathbb{G} \setminus \{0\}$. Recall that J. Inglis, in [17], proved the q-Poincaré inequality in the setting of the Heisenberg-type group for the measure $d\mu = \frac{e^{-\alpha N^p}}{Z} d\lambda$, where λ is the Lebesgue measure, $\alpha > 0$, $p \geq 2$, q is the finite index conjugate to p , and $N \equiv \left(|x|^4 + 16|z|^2\right)^{\frac{1}{4}}$. We extend, giving a simpler proof, the result of J. Inglis by obtaining a q-Poincaré inequality in the setting of step-two Carnot groups for the measures $d\mu = \frac{e^{-g(N)}}{Z} d\lambda$, with $\frac{g'(N)}{N^2}$ an increasing function, $q \geq 2$, and where $N \equiv \left(|x|^4 + a|z|^2\right)^{\frac{1}{4}}$.

Our first key result this paper is obtaining the following U-Bound (Section 3):

$$\int \frac{g'(N)}{N^2} |f|^q d\mu \leq C \int |\nabla f|^q d\mu + D \int |f|^q d\mu, \quad (2.1)$$

under certain growth conditions for $g(N)$. This U-bound is a useful tool to get a q-Poincaré inequality (Section 4) and a Log ^{β} -Sobolev inequality (Section 5) for $q \geq 2$. We expect that this U-bound can be used to extend those coercive inequalities to (non-product) measures in an infinite dimensional setting. [36]

3 U-Bound

Theorem 1 *Let $N = \left(|x|^4 + a|z|^2\right)^{\frac{1}{4}}$ with $a \in (0, \infty)$, and let $g : [0, \infty) \rightarrow [0, \infty)$ be a differentiable increasing function such that $g''(N) \leq g'(N)^3 N^3$ on $\{N \geq 1\}$. Let $d\mu = \frac{e^{-g(N)}}{Z} d\lambda$ be a probability measure, where Z is the normalization constant. Then, given $q \geq 2$,*

$$\int \frac{g'(N)}{N^2} |f|^q d\mu \leq C \int |\nabla f|^q d\mu + D \int |f|^q d\mu$$

holds for all locally Lipschitz functions f , supported outside the unit ball $\{N < 1\}$, with C and D positive constants depending on q and independent of f .

The main tools we use are Hardy's inequality (see [32] and references therein), which we will prove, and the Euclidean Coarea formula (see [12]). We also use the following properties of a smooth norm N proven in the [Appendix](#).

Lemma 2 *There exist constants $A, C \in (0, \infty)$*

$$A \frac{|x|^2}{N^2} \leq |\nabla N|^2 \leq C \frac{|x|^2}{N^2} \quad (3.1)$$

and there exists a constant $B \in (0, \infty)$ such that

$$|\Delta N| \leq B \frac{|x|^2}{N^3} \quad (3.2)$$

and

$$\frac{x}{|x|} \cdot \nabla N = \frac{|x|^3}{N^3}. \quad (3.3)$$

We note that the horizontal tangent space $\mathbb{G}$ is identified with $\mathbb{R}^n$, and the function x in (3.3) corresponds to horizontal vector fields $\sum_i x_i X_i$.

Proof of Theorem 1 First, we prove the result for $q = 2$:

We note that using integration by parts, one gets

$$\begin{aligned} \int (\nabla N) \cdot (\nabla f) e^{-g(N)} d\lambda &= - \int \nabla \cdot (\nabla N e^{-g(N)}) f d\lambda \\ &= - \int \Delta N f e^{-g(N)} d\lambda + \int |\nabla N|^2 f g'(N) e^{-g(N)} d\lambda. \end{aligned}$$

Next, using (3.1) and (3.2),

$$\int (\nabla N) \cdot (\nabla f) e^{-g(N)} d\lambda \geq -B \int \frac{|x|^2}{N^3} f e^{-g(N)} d\lambda + A \int \frac{|x|^2}{N^2} f g'(N) e^{-g(N)} d\lambda.$$

Replacing f by $\frac{f^2}{|x|^2}$:

$$\int (\nabla N) \cdot \left(\nabla \left(\frac{f^2}{|x|^2} \right) \right) e^{-g(N)} d\lambda \geq \int f^2 \left(\frac{A g'(N)}{N^2} - \frac{B}{N^3} \right) e^{-g(N)} d\lambda. \quad (3.4)$$

As for the left-hand side of (3.4),

$$\begin{aligned} \int (\nabla N) \cdot \left(\nabla \left(\frac{f^2}{|x|^2} \right) \right) e^{-g(N)} d\lambda &= \int (\nabla N) \cdot \left[2f \frac{\nabla f}{|x|^2} - \frac{2f^2 \nabla |x|}{|x|^3} \right] e^{-g(N)} d\lambda \\ &= \int \frac{2f}{|x|^2} \nabla N \cdot \nabla f e^{-g(N)} d\lambda - 2 \int f^2 \frac{\nabla N \cdot x}{|x|^4} e^{-g(N)} d\lambda \end{aligned}$$

Using the calculation of $\nabla N \cdot x$, from (3.3), we get:

$$\begin{aligned} &= \int \frac{2f}{|x|^2} \nabla N \cdot \nabla f e^{-g(N)} d\lambda - 2 \int \frac{f^2}{N^3} e^{-g(N)} d\lambda \\ &\leq \int \frac{2f}{|x|^2} \nabla N \cdot \nabla f e^{-g(N)} d\lambda. \end{aligned}$$

Combining with (3.4),

$$\begin{aligned} \int f^2 \left(\frac{Ag'(N)}{N^2} - \frac{B}{N^3} \right) e^{-g(N)} d\lambda &\leq \int (\nabla N) \cdot \left(\nabla \left(\frac{f^2}{|x|^2} \right) \right) e^{-g(N)} d\lambda \\ &\leq 2 \left| \int \frac{f}{|x|^2} \nabla N \cdot \nabla f e^{-g(N)} d\lambda \right| \\ &\leq 2 \int \frac{f}{|x|^2} |\nabla N| |\nabla f| e^{-g(N)} d\lambda \end{aligned}$$

using (3.1),

$$\leq 2\sqrt{C} \int \frac{|f|}{N|x|} |\nabla f| e^{-g(N)} d\lambda$$

Let $E = \{(x, z) : |x| \geq \frac{1}{\sqrt{g'(N)}}\}$ and $F = \{(x, z) : |x| < \frac{1}{\sqrt{g'(N)}}\}$.

Applying Cauchy's inequality with ϵ , where ϵ is a fixed constant to be chosen later,
 $ab \leq \frac{\epsilon a^2}{2} + \frac{b^2}{2\epsilon}$ with $a = \frac{|f|}{N|x|} e^{-\frac{g(N)}{2}}$ and $b = \sqrt{C} |\nabla f| e^{-\frac{g(N)}{2}}$,
 to obtain

$$\begin{aligned} \int f^2 \left(\frac{Ag'(N)}{N^2} - \frac{B}{N^3} \right) e^{-g(N)} d\lambda &\leq \epsilon \int \frac{|f|^2}{N^2|x|^2} e^{-g(N)} d\lambda + \frac{C}{\epsilon} \int |\nabla f|^2 e^{-g(N)} d\lambda \\ &= \epsilon \int_F \frac{|f|^2}{N^2|x|^2} e^{-g(N)} d\lambda + \epsilon \int_E \frac{|f|^2}{N^2|x|^2} e^{-g(N)} d\lambda \\ &\quad + \frac{C}{\epsilon} \int |\nabla f|^2 e^{-g(N)} d\lambda \\ &\leq \epsilon \int_F \frac{|f e^{-\frac{g(N)}{2}}|^2}{N^2|x|^2} d\lambda + \epsilon \int \frac{g'(N)|f|^2}{N^2} e^{-g(N)} d\lambda \\ &\quad + \frac{C}{\epsilon} \int |\nabla f|^2 e^{-g(N)} d\lambda, \end{aligned} \quad (3.5)$$

where (3.5) is the consequence of $E = \{(x, z) : |x| \geq \frac{1}{\sqrt{g'(N)}}\}$.

The aim now is to estimate the first term on the right-hand side of (3.5). We use the following integration by parts formula:

$$\int_{F_r} (X_i u) v d\lambda = - \int_{F_r} (X_i v) u d\lambda + \int_{\partial F_r} u v < X_i I, \frac{\nabla \phi}{|\nabla \phi|} >_{\mathbb{R}^{n+m}} dH^{n+m-1}, \quad (3.6)$$

for $F_r := \{g \in \mathbb{G} : \phi(g) < r\}$ and $I(y) = (y_1, y_2, \dots, y_{n+m})^T$ the identity map on $\mathbb{R}^{n+m}$. Now consider $F_r = \left\{ |x| \sqrt{g'(N)} < r \right\}$, where $1 < r < 2$. Integrating by parts using (3.6),

$$\begin{aligned}
 \epsilon \int_{F_r} \frac{|f e^{\frac{-g(N)}{2}}|^2}{N^2 |x|^2} d\lambda &= \frac{\epsilon}{n-2} \int_{F_r} \frac{|f e^{\frac{-g(N)}{2}}|^2}{N^2} \nabla \cdot \left(\frac{x}{|x|^2} \right) d\lambda \\
 &= -\frac{\epsilon}{n-2} \int_{F_r} \nabla \left(\left| \frac{f e^{\frac{-g(N)}{2}}}{N} \right|^2 \right) \cdot \frac{x}{|x|^2} d\lambda \\
 &\quad + \frac{\epsilon}{n-2} \int_{\partial F_r} \frac{f^2 e^{-g(N)}}{N^2 |x|^2} \sum_{j=1}^n \frac{x_j \langle X_j I, \nabla_{euc}(|x| \sqrt{g'(N)}) \rangle_{\mathbb{R}^{n+m}}}{|\nabla_{euc}(|x| \sqrt{g'(N)})|} dH^{n+m-1} \\
 &= -\frac{2\epsilon}{n-2} \int_{F_r} \frac{f e^{\frac{-g(N)}{2}}}{N} \nabla \left(\frac{f e^{\frac{-g(N)}{2}}}{N} \right) \cdot \frac{x}{|x|^2} d\lambda \\
 &\quad + \frac{\epsilon}{n-2} \int_{\partial F_r} \frac{f^2 e^{-g(N)}}{N^2 |x|^2} \sum_{j=1}^n \frac{x_j \langle X_j I, \nabla_{euc}(|x| \sqrt{g'(N)}) \rangle_{\mathbb{R}^{n+m}}}{|\nabla_{euc}(|x| \sqrt{g'(N)})|} dH^{n+m-1} \\
 &\leq \frac{\epsilon}{2} \int_{F_r} \frac{|f e^{\frac{-g(N)}{2}}|^2}{N^2 |x|^2} d\lambda + \frac{2\epsilon}{(n-2)^2} \int_{F_r} \left| \nabla \left(\frac{f e^{\frac{-g(N)}{2}}}{N} \right) \right|^2 d\lambda \\
 &\quad + \frac{\epsilon}{n-2} \int_{\partial F_r} \frac{f^2 e^{-g(N)}}{N^2 |x|^2} \sum_{j=1}^n \frac{x_j \langle X_j I, \nabla_{euc}(|x| \sqrt{g'(N)}) \rangle_{\mathbb{R}^{n+m}}}{|\nabla_{euc}(|x| \sqrt{g'(N)})|} dH^{n+m-1},
 \end{aligned}$$

where in the last step we used Cauchy's inequality. Subtracting on both sides of the last inequality by $\frac{\epsilon}{2} \int_{F_r} \frac{|f e^{\frac{-g(N)}{2}}|^2}{N^2 |x|^2} d\lambda$, and using the fact that $1 < r < 2$, we get:

$$\begin{aligned}
 \epsilon \int_{F_1} \frac{|f e^{\frac{-g(N)}{2}}|^2}{N^2 |x|^2} d\lambda &\leq \epsilon \int_{F_r} \frac{|f e^{\frac{-g(N)}{2}}|^2}{N^2 |x|^2} d\lambda \\
 &\leq \frac{4\epsilon}{(n-2)^2} \int_{F_r} \left| \nabla \left(\frac{f e^{\frac{-g(N)}{2}}}{N} \right) \right|^2 d\lambda \\
 &\quad + \frac{2\epsilon}{n-2} \int_{\partial F_r} \frac{f^2 e^{-g(N)}}{N^2 |x|^2} \sum_{j=1}^n \frac{x_j \langle X_j I, \nabla_{euc}(|x| \sqrt{g'(N)}) \rangle_{\mathbb{R}^{n+m}}}{|\nabla_{euc}(|x| \sqrt{g'(N)})|} dH^{n+m-1} \\
 &\leq \frac{4\epsilon}{(n-2)^2} \int_{F_2} \left| \nabla \left(\frac{f e^{\frac{-g(N)}{2}}}{N} \right) \right|^2 d\lambda \\
 &\quad + \frac{2\epsilon}{n-2} \int_{\partial F_r} \frac{f^2 e^{-g(N)}}{N^2 |x|^2} \sum_{j=1}^n \frac{x_j \langle X_j I, \nabla_{euc}(|x| \sqrt{g'(N)}) \rangle_{\mathbb{R}^{n+m}}}{|\nabla_{euc}(|x| \sqrt{g'(N)})|} dH^{n+m-1}
 \end{aligned}$$

Integrating both sides of the inequality from $r = 1$ to $r = 2$, we get:

$$\begin{aligned} & \epsilon \int_1^2 \int_{F_1} \frac{|f e^{\frac{-g(N)}{2}}|^2}{N^2 |x|^2} d\lambda dr \\ & \leq \frac{4\epsilon}{(n-2)^2} \int_1^2 \int_{F_2} \left| \nabla \left(\frac{f e^{\frac{-g(N)}{2}}}{N} \right) \right|^2 d\lambda dr \\ & \quad + \frac{2\epsilon}{n-2} \int_1^2 \int_{\partial F_r} \frac{f^2 e^{-g(N)}}{N^2 |x|^2} \sum_{j=1}^n \frac{x_j \langle X_j I, \nabla_{euc} (|x| \sqrt{g'(N)}) \rangle_{\mathbb{R}^{n+m}}}{\left| \nabla_{euc} (|x| \sqrt{g'(N)}) \right|} dH^{n+m-1} dr \end{aligned}$$

To recover the full measure in the boundary term, we use the Euclidean Coarea formula ([12], page 247):

$$\begin{aligned} & \epsilon \int_{F_1} \frac{|f e^{\frac{-g(N)}{2}}|^2}{N^2 |x|^2} d\lambda \\ & \leq \frac{4\epsilon}{(n-2)^2} \int_{F_2} \left| \nabla \left(\frac{f e^{\frac{-g(N)}{2}}}{N} \right) \right|^2 d\lambda \\ & \quad + \frac{2\epsilon}{n-2} \int_{\{1 < |x| \sqrt{g'(N)} < 2\}} \frac{f^2 e^{-g(N)}}{N^2 |x|^2} \sum_{j=1}^n x_j \langle X_j I, \nabla_{euc} (|x| \sqrt{g'(N)}) \rangle_{\mathbb{R}^{n+m}} d\lambda \quad (3.7) \end{aligned}$$

It remains to compute the right hand side of (3.7). The first term,

$$\begin{aligned} \hat{A} &= \frac{4\epsilon}{(n-2)^2} \int_{F_2} \left| \nabla \left(\frac{f e^{\frac{-g(N)}{2}}}{N} \right) \right|^2 d\lambda \\ &= \frac{4\epsilon}{(n-2)^2} \int_{F_2} \left| \nabla f \frac{e^{-\frac{g(N)}{2}}}{N} - f \frac{g'(N)}{2} \frac{\nabla N e^{-\frac{g(N)}{2}}}{N} - \frac{f \nabla N e^{-\frac{g(N)}{2}}}{N^2} \right|^2 d\lambda \\ &\leq \frac{16\epsilon}{(n-2)^2} \int_{F_2} \frac{|\nabla f|^2}{N^2} e^{-g(N)} d\lambda + \frac{4\epsilon}{(n-2)^2} \int_{F_2} f^2 g'(N)^2 \frac{|\nabla N|^2}{N^2} e^{-g(N)} d\lambda \\ &\quad + \frac{16\epsilon}{(n-2)^2} \int_{F_2} f^2 \frac{|\nabla N|^2}{N^4} e^{-g(N)} d\lambda \end{aligned}$$

Using (3.1) and taking into consideration that $N > 1$ and on F_2 , $|\nabla N|^2 \leq \frac{C|x|^2}{N^2} \leq \frac{4C}{N^2 g'(N)}$,

$$\hat{A} \leq \tilde{C} \int_{F_2} |\nabla f|^2 e^{-g(N)} d\lambda + \tilde{D} \int_{F_2} f^2 e^{-g(N)} d\lambda + \frac{16\epsilon C}{(n-2)^2} \int_{F_2} f^2 \frac{g'(N)}{N^4} e^{-g(N)} d\lambda$$

We do not worry about the third term in this inequality since it is dominated by $\int f^2 \frac{g'(N)}{N^2} e^{-g(N)} d\lambda$ for $N > 1$. For the second term of (3.7),

$$\hat{B} = \frac{2\epsilon}{n-2} \int_{\{1 < |x| \sqrt{g'(N)} < 2\}} \frac{f^2 e^{-g(N)}}{N^2 |x|^2} \sum_{j=1}^n x_j \langle X_j I, \nabla_{euc} (|x| \sqrt{g'(N)}) \rangle_{\mathbb{R}^{n+m}} d\lambda.$$

For e_i the standard Euclidean basis on $\mathbb{R}^{n+m}$,

$$X_j I \cdot e_i = \begin{cases} 0 & \text{for } i \neq j \text{ and } i \leq n \\ 1 & \text{for } i = j \text{ and } i \leq n \\ \frac{1}{2} \sum_{l=1}^n \Lambda_{jl}^{(i)} x_l & \text{for } n+1 \leq i \leq n+m. \end{cases}$$

$$\nabla_{\text{euc}} \left(|x| \sqrt{g'(N)} \right) \cdot e_i = \begin{cases} \frac{x_i \sqrt{g'(N)}}{|x|} + \frac{|x|^3 g''(N) x_i}{2 \sqrt{g'(N)} N^3} & \text{for } i = j \text{ and } i \leq n \\ \frac{a |x| g''(N) z_i}{4 \sqrt{g'(N)} N^3} & \text{for } n+1 \leq i \leq n+m \end{cases}$$

Taking the dot product and summing,

$$\begin{aligned} & \sum_{j=1}^n x_j \langle X_j I, \nabla_{\text{euc}} \left(|x| \sqrt{g'(N)} \right) \rangle_{\mathbb{R}^{n+m}} \\ &= |x| \sqrt{g'(N)} + \frac{|x|^5 g''(N)}{2 \sqrt{g'(N)} N^3} + \sum_{j=1}^n x_j \sum_{i=n+1}^{n+m} \left(\frac{a |x| g''(N)}{8 \sqrt{g'(N)} N^3} \right) z_i \sum_{l=1}^n \Lambda_{jl}^{(i)} x_l \\ &= |x| \sqrt{g'(N)} + \frac{|x|^5 g''(N)}{2 \sqrt{g'(N)} N^3} + \left(\frac{a |x| g''(N)}{8 \sqrt{g'(N)} N^3} \right) \sum_{i=n+1}^{n+m} z_i \sum_{j=1}^n \sum_{l=1}^n \Lambda_{jl}^{(i)} x_l x_j \\ &= |x| \sqrt{g'(N)} + \frac{|x|^5 g''(N)}{2 \sqrt{g'(N)} N^3}, \end{aligned}$$

where $\sum_{j=1}^n \sum_{l=1}^n \Lambda_{jl}^{(i)} x_l x_j = 0$ since $\Lambda_{jl}^{(i)}$ is skew symmetric.

Therefore, replacing,

$$\begin{aligned} \hat{B} &= \frac{2\epsilon}{n-2} \int_{\{1 < |x| \sqrt{g'(N)} < 2\}} \frac{f^2 e^{-g(N)}}{N^2 |x|^2} \sum_{j=1}^n x_j \langle X_j I, \nabla_{\text{euc}} \left(|x| \sqrt{g'(N)} \right) \rangle_{\mathbb{R}^{n+m}} d\lambda \\ &= \frac{2\epsilon}{n-2} \int_{\{1 < |x| \sqrt{g'(N)} < 2\}} \frac{f^2 \sqrt{g'(N)}}{N^2 |x|} e^{-g(N)} d\lambda \\ &\quad + \frac{2\epsilon}{n-2} \int_{\{1 < |x| \sqrt{g'(N)} < 2\}} \frac{f^2 |x|^3 g''(N)}{2 N^5 \sqrt{g'(N)}} e^{-g(N)} d\lambda. \end{aligned}$$

Using the fact that we are integrating over $\{1 < |x| \sqrt{g'(N)} < 2\}$,

$$\hat{B} \leq \frac{2\epsilon}{n-2} \int_{\{1 < |x| \sqrt{g'(N)} < 2\}} \frac{f^2 g'(N)}{N^2} e^{-g(N)} d\lambda + \frac{8\epsilon}{n-2} \int_{\{1 < |x| \sqrt{g'(N)} < 2\}} \frac{f^2 g''(N)}{N^5 g'(N)^2} e^{-g(N)} d\lambda.$$

Using the condition of the theorem that $g''(N) \leq g'(N)^3 N^3$, we get

$$\hat{B} \leq \frac{10\epsilon}{n-2} \int_{\{1 < |x| \sqrt{g'(N)} < 2\}} \frac{f^2 g'(N)}{N^2} e^{-g(N)} d\lambda.$$

Inserting bounds on $\hat{A}$ and $\hat{B}$ in (3.7), we get:

$$\begin{aligned} \epsilon \int_{F_1} \frac{|f e^{-\frac{g(N)}{2}}|^2}{N^2 |x|^2} d\lambda &\leq \tilde{C} \int_{F_2} |\nabla f|^2 e^{-g(N)} d\lambda + \tilde{D} \int_{F_2} f^2 e^{-g(N)} d\lambda \\ &\quad + \frac{16\epsilon C}{(n-2)^2} \int_{F_2} f^2 \frac{g'(N)}{N^4} e^{-g(N)} d\lambda \\ &\quad + \frac{10\epsilon}{n-2} \int_{\{1 < |x| \sqrt{g'(N)} < 2\}} \frac{f^2 g'(N)}{N^2} e^{-g(N)} d\lambda. \end{aligned}$$

Using this last bound to estimate (3.5), we get:

$$\begin{aligned} &\int f^2 \left(\frac{Ag'(N)}{N^2} - \frac{B}{N^3} \right) e^{-g(N)} d\lambda \\ &\leq \tilde{C} \int |\nabla f|^2 e^{-g(N)} d\lambda + \tilde{D} \int f^2 e^{-g(N)} d\lambda \\ &\quad + \frac{16\epsilon C}{(n-2)^2} \int f^2 \frac{g'(N)}{N^4} e^{-g(N)} d\lambda + \left(\frac{10\epsilon}{n-2} + \epsilon \right) \int \frac{f^2 g'(N)}{N^2} e^{-g(N)} d\lambda. \end{aligned}$$

Since f is supported on $\{N > 1\}$, the terms $\int f^2 \left(\frac{B}{N^3} \right) e^{-g(N)} d\lambda$ and $\frac{16\epsilon C}{(n-2)^2} \int f^2 \frac{g'(N)}{N^4} e^{-g(N)} d\lambda$ are of lower order than the term $\int f^2 \left(\frac{Ag'(N)}{N^2} \right) e^{-g(N)} d\lambda$ on the left-hand side.

So, choosing $\left(\frac{10\epsilon}{n-2} + \epsilon \right) < A$ we get

$$\int f^2 \left(\frac{g'(N)}{N^2} \right) e^{-g(N)} d\lambda \leq C \int |\nabla f|^2 e^{-g(N)} d\lambda + D \int f^2 e^{-g(N)} d\lambda.$$

Secondly, for $q > 2$, replacing $|f|$ by $|f|^{\frac{q}{2}}$, we get:

$$\int \frac{g'(N)}{N^2} |f|^q d\mu \leq C \int \left| \nabla |f|^{\frac{q}{2}} \right|^2 d\mu + D \int |f|^q d\mu. \quad (3.8)$$

Calculating,

$$\begin{aligned} \int \left| \nabla |f|^{\frac{q}{2}} \right|^2 d\mu &= \int \left| \frac{q}{2} |f|^{\frac{q-2}{2}} (\operatorname{sgn}(f)) \nabla f \right|^2 d\mu \\ &\leq \int \frac{q^2}{4} |f|^{q-2} |\nabla f|^2 d\mu. \end{aligned}$$

Remark We note that at this point we get the inequality which implies the necessary and sufficient condition for exponential decay in $\mathbb{L}_p$ as described in [31].

Using Hölder's inequality,

$$\begin{aligned} \int \left| \nabla |f|^{\frac{q}{2}} \right|^2 d\mu &\leq \int \frac{q^2}{4} |f|^{q-2} |\nabla f|^2 d\mu \leq \frac{q^2}{4} \left(\int |f|^q d\mu \right)^{\frac{q-2}{q}} \left(\int |\nabla f|^q d\mu \right)^{\frac{2}{q}} \\ &\leq \frac{q(q-2)}{4} \int |f|^q d\mu + \frac{q}{2} \int |\nabla f|^q d\mu. \end{aligned} \quad (3.9)$$

Where the last inequality uses $ab \leq \frac{(q-2)}{q} a^{\frac{q}{q-2}} + \frac{2}{q} b^{\frac{q}{2}}$, with $a = \left(\int |f|^q d\mu \right)^{\frac{q-2}{q}}$ and $b = \left(\int |\nabla f|^q d\mu \right)^{\frac{2}{q}}$.

Using the inequalities (3.8) and (3.9), we get,

$$\begin{aligned} \int \frac{g'(N)}{N^2} |f|^q d\mu &\leq C \int \left| \nabla |f|^{\frac{q}{2}} \right|^2 d\mu + D \int |f|^q d\mu \\ &\leq C' \int |\nabla f|^q d\mu + D' \int |f|^q d\mu. \end{aligned}$$

□

4 Poincaré Inequality

We now have the U-Bound (2.1) at our disposal and are ready to prove the q -Poincaré inequality using the method [16]:

Let λ be a measure satisfying the q -Poincaré inequality for every ball $B_R = \{x : N(x) < R\}$, i.e. there exists a constant $C_R \in (0, \infty)$ such that

$$\frac{1}{|B_R|} \int_{B_R} \left| f - \frac{1}{|B_R|} \int_{B_R} f \right|^q d\lambda \leq C_R \frac{1}{|B_R|} \int_{B_R} |\nabla f|^q d\lambda,$$

where $1 \leq q < \infty$.

Note that we have this Poincaré inequality on balls in the setting of Nilpotent lie groups thanks to J. Jerison's celebrated paper [19]. With this we can use the following result:

Theorem 3 (Hebisch, Zegarliński [16]) *Let μ be a probability measure on $\mathbb{R}^n$ which is absolutely continuous with respect to the measure λ and such that*

$$\int f^q \eta d\mu \leq C \int |\nabla f|^q d\mu + D \int f^q d\mu$$

with some non-negative function η and some constants $C, D \in (0, \infty)$ independent of a function f . If for any $L \in (0, \infty)$ there is a constant A_L such that $\frac{1}{A_L} \leq \frac{d\mu}{d\lambda} \leq A_L$ on the set $\{\eta < L\}$ and, for some $R \in (0, \infty)$ (depending on L), we have $\{\eta < L\} \subset B_R$, then μ satisfies the q -Poincaré inequality

$$\mu |f - \mu f|^q \leq c \mu |\nabla f|^q$$

with some constant $c \in (0, \infty)$ independent of f .

Note that the constants in the above Theorem depend on q . The role of η in Theorem 3 is played by $\frac{g'(N)}{N^2}$ from the U-Bound of Theorem 1. Hence, we get the following corollaries:

Corollary 4 *The Poincaré inequality for $q \geq 2$ holds for the measure $d\mu = \frac{\exp(-\cosh(N^k))}{Z} d\lambda$, where λ is the Lebesgue measure, and $k \geq \frac{3}{2}$ in the setting of the step-two Carnot group.*

Proof $g(N) = \cosh(N^k)$, so $g'(N) = kN^{k-1} \sinh(N^k)$, and $g''(N) = k(k-1)N^{k-2} \sinh(N^k) + k^2 N^{2k-2} \cosh(N^k)$.

First, on $\{N \geq 1\}$, $g''(N) \leq k^3 N^{3k} \sinh^3(N^k) = g'(N)^3 N^3$, so the condition of Theorem 1 is satisfied. Second,

$$\begin{aligned} \int \frac{g'(N)}{N^2} f^q d\mu &= \int f^q \left[k N^{k-3} \sinh(N^k) \right] d\mu \\ &= \int_{\{N < 1\}} f^q \left[k N^{k-3} \frac{e^{N^k} - e^{-N^k}}{2} \right] d\mu + \int_{\{N \geq 1\}} f^q \left[k N^{k-3} \frac{e^{N^k} - e^{-N^k}}{2} \right] d\mu \\ &\leq \int_{\{N < 1\}} f^q \left[k \frac{e - e^{-1}}{2} \right] d\mu + C' \int_{\{N \geq 1\}} |\nabla f|^q d\mu + D' \int_{\{N \geq 1\}} |f|^q d\mu \\ &\leq C \int |\nabla f|^q d\mu + D \int |f|^q d\mu. \end{aligned}$$

Thus, the conditions of Theorem 3 are satisfied for $\eta = k N^{k-3} \sinh(N^k)$, and $k \geq \frac{3}{2}$. So, the Poincaré inequality holds for $q \geq 2$. $\square$

The following corollary was proven in the setting of the Heisenberg-type group for the Kaplan norm $N = (|x|^4 + 16|z|^2)^{\frac{1}{4}}$ by J. Inglis, Theorem 4.5.5 of [17].

In the setting of the step-two Carnot group, we obtain a generalised version for a similar homogeneous norm $N = (|x|^4 + a|z|^2)^{\frac{1}{4}}$.

Corollary 5 *The Poincaré inequality for $q \geq 2$ holds for the measure $d\mu = \frac{\exp(-N^k)}{Z} d\lambda$, where λ is the Lebesgue measure, and $k \geq 4$ in the setting of the step-two Carnot group.*

Proof Let $g(N) = N^k$, so $g'(N) = k N^{k-1}$, and $g''(N) = k(k-1) N^{k-2}$. First, on $\{N \geq 1\}$, $g''(N) \leq k^3 N^{3k} = g'(N)^3 N^3$, so the condition of Theorem 1 is satisfied. Secondly,

$$\begin{aligned} \int \frac{g'(N)}{N^2} f^q d\mu &= \int f^q \left[k N^{k-3} \right] d\mu \\ &= \int_{\{N < 1\}} f^q \left[k N^{k-3} \right] d\mu + \int_{\{N \geq 1\}} f^q \left[k N^{k-3} \right] d\mu \\ &\leq \int_{\{N < 1\}} k f^q d\mu + C' \int_{\{N \geq 1\}} |\nabla f|^q d\mu + D' \int_{\{N \geq 1\}} |f|^q d\mu \\ &\leq C \int |\nabla f|^q d\mu + D \int |f|^q d\mu. \end{aligned}$$

Thus, the conditions of Theorem 3 are satisfied for $\eta = k N^{k-3}$, and $k \geq 4$. So, the Poincaré inequality holds for $q \geq 2$. $\square$

The following corollary improves Corollary 5 in an interesting way. Namely, at a cost of a logarithmic factor, we now get the Poincaré inequality for polynomial growth of order $k \geq 3$.

Corollary 6 *The Poincaré inequality for $q \geq 2$ holds for the measure $d\mu = \frac{\exp(-N^k \log(N+1))}{Z} d\lambda$, where λ is the Lebesgue measure, and $k \geq 3$ in the setting of the step-two Carnot group.*

Proof Let $g(N) = N^k \log(N+1)$, so $g'(N) = kN^{k-1} \log(N+1) + \frac{N^k}{N+1}$, and

$$g''(N) = k(k-1)N^{k-2} \log(N+1) + \frac{2kN^{k-1}}{N+1} - \frac{N^k}{(N+1)^2}.$$

First, on $\{N \geq 1\}$, since $k \geq 3$, we have $k(k-1)N^{k-2} \log(N+1) \leq k^3 N^{3k} \log^3(N+1)$ and $\frac{2kN^{k-1}}{N+1} \leq \frac{3k^2 N^{3k+1} \log^2(N+1)}{N+1}$. Hence,

$$\begin{aligned} g''(N) &\leq k^3 N^{3k} \log^3(N+1) + \frac{N^{3k+3}}{(N+1)^3} + \frac{3k^2 N^{3k+1} \log^2(N+1)}{N+1} \\ &\quad + \frac{3kN^{3k+2} \log(N+1)}{(N+1)^2} = N^3 g'(N)^3, \end{aligned}$$

so the condition of Theorem 1 is satisfied. Secondly,

$$\begin{aligned} \int \frac{g'(N)}{N^2} f^q d\mu &= \int f^q \left[kN^{k-3} \log(N+1) + \frac{N^{k-2}}{N+1} \right] d\mu \\ &= \int_{\{N < 1\}} f^q \left[kN^{k-3} \log(N+1) + \frac{N^{k-2}}{N+1} \right] d\mu \\ &\quad + \int_{\{N \geq 1\}} f^q \left[kN^{k-3} \log(N+1) + \frac{N^{k-2}}{N+1} \right] d\mu \\ &\leq \int_{\{N < 1\}} (k \log(2) + 1) f^q d\mu + C' \int_{\{N \geq 1\}} |\nabla f|^q d\mu + D' \int_{\{N \geq 1\}} |f|^q d\mu \\ &\leq C \int |\nabla f|^q d\mu + D \int |f|^q d\mu. \end{aligned}$$

Thus, the conditions of Theorem 3 are satisfied for $\eta = kN^{k-3} \log(N+1) + \frac{N^{k-2}}{N+1}$, and $k \geq 3$. Hence, the Poincaré inequality holds for $q \geq 2$. $\square$

5 ϕ -Logarithmic Sobolev Inequality

After proving the q -Poincaré inequality for measures as a function of the homogeneous norm $N = (|x|^4 + a|z|^2)^{\frac{1}{4}}$, a natural question would be if one could obtain other coercive inequalities. J. Inglis et al.'s Theorem 2.1 [18] proved that, for the measure $d\mu = \frac{e^{-U} d\lambda}{Z}$, provided we have the U-Bound

$$\mu(|f|(|U|^\beta + |\nabla U|)) \leq A\mu|\nabla f| + B\mu|f|,$$

one obtains

$$\mu \left(|f| \left| \log \frac{|f|}{\mu|f|} \right|^\beta \right) \leq C\mu|\nabla f| + B\mu|f|.$$

We will first extend their theorem, and then we will use Theorem 1 to get more general coercive inequalities.

Theorem 7 Let U be a locally Lipschitz function on $\mathbb{R}^N$ which is bounded below such that $Z = \int e^{-U} d\lambda < \infty$ and $d\mu = \frac{e^{-U}}{Z} d\lambda$. Let $\phi : [0, \infty) \rightarrow \mathbb{R}^+$ be a non-negative, non-decreasing, concave function such that $\phi(0) > 0$, and $\phi'(0) > 0$. Assume the following classical Sobolev inequality is satisfied:

$$\left(\int |f|^{q+\epsilon} d\lambda \right)^{\frac{q}{q+\epsilon}} \leq a \int |\nabla f|^q d\lambda + b \int |f|^q d\lambda$$

for some $a, b \in [0, \infty)$, and $\epsilon > 0$. Moreover, if for some $A, B \in [0, \infty)$, we have:

$$\mu(|f|^q(\phi(U) + |\nabla U|^q)) \leq A\mu|\nabla f|^q + B\mu|f|^q,$$

Then, there exists constants $C, D \in [0, \infty)$ such that:

$$\mu\left(|f|^q \phi\left(\left|\log \frac{|f|^q}{\mu|f|^q}\right|\right)\right) \leq C\mu|\nabla f|^q + D\mu|f|^q,$$

for all locally Lipschitz functions f .

Proof First of all, we remark that for a concave function ϕ as in our assumptions, we have

$$\phi(y) - \phi(x) \leq \phi'(0)(y - x). \quad (5.1)$$

Suppose first that $\int |f|^q d\mu = 1$, and let $E = \{x \in \mathbb{R}^N : \left|\log |f|^q\right| > U\}$.

$$\begin{aligned} & \int |f|^q \phi(|\log |f|^q|) d\mu \\ &= \int_E |f|^q \phi(|\log |f|^q|) d\mu + \int_{E^c} |f|^q \phi(|\log |f|^q|) d\mu \\ &= \int_E |f|^q [\phi(|\log |f|^q|) - \phi(U)] d\mu + \int_E |f|^q \phi(U) d\mu + \int_{E^c} |f|^q \phi(|\log |f|^q|) d\mu \\ &\leq \phi'(0) \int_E |f|^q (|\log |f|^q| - U) d\mu + \int_E |f|^q \phi(U) d\mu + \int_{E^c} |f|^q \phi(U) d\mu, \\ &\leq \phi'(0) \int_E |f|^q (|\log |f|^q|) d\mu + \int_E |f|^q \phi(U) d\mu + \int_{E^c} |f|^q \phi(U) d\mu, \end{aligned}$$

where the last inequality uses (5.1) on E , and uses the fact that ϕ is non-decreasing on E^c , hence, $\left|\log |f|^q\right| < U$. Let $E_1 = \{\log |f|^q > U\}$, $E_2 = \{\log |f|^q < -U\}$, and

$$c = \int_{E_1} |f|^q e^{-U} d\lambda \leq Z \text{ since } \int |f|^q d\mu = 1 \text{ and } d\mu = \frac{e^{-U}}{Z} d\lambda.$$

$$\begin{aligned} \int |f|^q \phi(|\log |f|^q|) d\mu &\leq \phi'(0) \int_{E_1} |f|^q \log |f|^q d\mu + \phi'(0) \int_{E_2} |f|^q \log \frac{1}{|f|^q} d\mu \\ &\quad + \int |f|^q \phi(U) d\mu \end{aligned}$$

$$\text{using } \log \frac{1}{|f|^q} \leq \frac{1}{|f|^q},$$

$$\leq \frac{c\phi'(0)q}{\epsilon Z} \int_{E_1} \frac{\left(|f|e^{-\frac{U}{q}}\right)^q}{c} \log \left(\left(|f|e^{-\frac{U}{q}}\right)^\epsilon\right) d\lambda + \phi'(0) \int_{E_2} 1 d\mu + \int |f|^q \phi(U) d\mu$$

Using Jensen's inequality $\int_{E_1} \frac{(|f|e^{-\frac{U}{q}})^q}{c} \log \left(\left(|f|e^{-\frac{U}{q}} \right)^\epsilon \right) d\lambda \leq \log \left(\int_{E_1} \frac{(|f|e^{-\frac{U}{q}})^{q+\epsilon}}{c} d\lambda \right),$

with measure $d\eta = \frac{(|f|e^{-\frac{U}{q}})^q}{c} d\lambda$, and concave function $\log(x)$,

$$\begin{aligned} &\leq \frac{c\phi'(0)(q+\epsilon)}{Z\epsilon} \log \left(\left(\int_{E_1} \frac{(|f|e^{-\frac{U}{q}})^{q+\epsilon}}{c} d\lambda \right)^{\frac{q}{q+\epsilon}} \right) + \phi'(0) \int_{E_2} 1 d\mu + \int |f|^q \phi(U) d\mu \\ &\leq \frac{\frac{\epsilon}{c^{q+\epsilon}} \phi'(0)(q+\epsilon)}{Z\epsilon} \left(\int \left(|f|e^{-\frac{U}{q}} \right)^{q+\epsilon} d\lambda \right)^{\frac{q}{q+\epsilon}} + d\phi'(0) + \int |f|^q \phi(U) d\mu, \end{aligned}$$

where the last inequality is true since U is bounded below and $d = \int 1 d\mu$. Using the classical Sobolev inequality,

$$\begin{aligned} &\leq a \int |f|^q d\mu + b \int \left| \nabla (f e^{-\frac{U}{q}}) \right|^q d\lambda + d\phi'(0) + \int |f|^q \phi(U) d\mu \\ &= a + d\phi'(0) + b \int \left| (\nabla f) e^{-\frac{U}{q}} - \frac{f}{q} (\nabla U) e^{-\frac{U}{q}} \right|^q d\lambda + \int |f|^q \phi(U) d\mu \\ &\leq a + d\phi'(0) + b2^{q-1} Z \int |\nabla f|^q d\mu + \frac{Zb2^{q-1}}{q^q} \int |f|^q |\nabla U|^q d\mu + \int |f|^q \phi(U) d\mu \\ &\leq a + d\phi'(0) + b2^{q-1} Z \int |\nabla f|^q d\mu + \max \left\{ \frac{Zb2^{q-1}}{q^q}, 1 \right\} \int |f|^q (|\nabla U|^q + \phi(U)) d\mu \end{aligned}$$

Using the U-bound in the Theorem's condition

$$\leq B + A \int |\nabla f|^q d\mu$$

Finally, replace $|f|^q$ by $\frac{|f|^q}{\mu|f|^q}$ to get the desired inequality. $\square$

Corollary 8 $\phi(x) = (1+x)^\beta$, for $\beta \in (0, 1]$ is non-negative, non-decreasing, and concave function satisfying $\phi(0) = 1 > 0$, and $\phi'(0) = \beta > 0$. Therefore, Theorem 7 applies, and

$$\mu \left(|f|^q \left| \log \left(\frac{|f|^q}{\mu|f|^q} \right) \right|^\beta \right) \leq \mu \left(|f|^q \left(1 + \left| \log \left(\frac{|f|^q}{\mu|f|^q} \right) \right|^\beta \right) \right) \leq C\mu |\nabla f|^q + D\mu |f|^q.$$

Corollary 9 Let $h^{(1)}(x) = \log(\alpha + x)$, where $\alpha > 1$. Define recursively

$h^{(n)}(x) = \log(\alpha + h^{(n-1)}(x))$. Then, for all $n \geq 1$, $h^{(n)}(x)$ satisfies the hypotheses on $\phi(x)$ from Theorem 7. Therefore, we obtain

$$\mu \left(|f|^q \log^{*(n)} \left(\frac{|f|^q}{\mu|f|^q} \right) \right) \leq \mu \left(|f|^q h^{(n)} \left(\frac{|f|^q}{\mu|f|^q} \right) \right) \leq C\mu |\nabla f|^q + D\mu |f|^q,$$

where $\log^{*(n)}$ is the positive part of $\log^{(n)}$.

Proof The proof proceeds by induction. For $n = 1$, $h^{(1)}(x) = \log(\alpha + x)$, so $h^{(1)}(x)' = \frac{1}{\alpha+x}$, and $h^{(1)}(x)'' = \frac{-1}{(\alpha+x)^2}$. $h^{(1)}(0) = \log(\alpha) > 0$, and $h^{(1)}(0)' = \frac{1}{\alpha} > 0$. In addition, $h^{(1)}(x)$ is non-negative, non-decreasing, and concave; hence the conditions of Theorem 7 are satisfied.

Assume it is true for $n = k$, prove it is true for $n = k+1$: $h^{(k+1)}(x) = \log(\alpha + h^{(k)}(x))$, so $h^{(k+1)}(x)' = \frac{h^{(k)}(x)'}{\alpha+h^{(k)}(x)}$, and $h^{(k+1)}(x)'' = \frac{h^{(k)}(x)''}{\alpha+h^{(k)}(x)} - \frac{h^{(k)}(x)'}{(\alpha+h^{(k)}(x))^2}$. The result follows directly. $\square$

Returning to the measure as a function of the homogeneous norm $N = (|x|^4 + a|z|^2)^{\frac{1}{4}}$, $d\mu = \frac{e^{-N^p}}{Z} d\lambda$, we will prove using Theorems 1 and 7, that the Log^β -Sobolev inequality ($0 < \beta \leq 1$) (Corollary 8) holds for $q \geq 2$, yet fails for $1 < q < \frac{2p\beta}{p-1}$. To start with, we will show why the Log^β -Sobolev inequality fails for $1 < q < \frac{2p\beta}{p-1}$. The proof uses the idea of Theorem 6.3 of [16].

Theorem 10 *Let $\mathbb{G}$ be a stratified group, and ϕ be a smooth homogeneous norm on $\mathbb{G} \setminus \{0\}$. For $\alpha > 0$, $p \geq 1$, let $d\mu = \frac{e^{-\alpha\phi^p}}{Z} d\lambda$, where Z is the normalization constant. The measure μ satisfies no Log^β -Sobolev inequality ($0 < \beta \leq 1$) for $1 < q < \frac{2p\beta}{p-1}$.*

Proof The proof is by contradiction. Denote by $d(g, h)$ the control distance between two points $g, h \in \mathbb{G}$, Q the homogeneous dimension of $\mathbb{G}$, and δ_t the dilation by $t > 0$. Let $g_0 \in \mathbb{G}$ be such that $(\nabla\phi)(g_0) = 0$ (by Theorem 6.2 of [16], where it is required that $\mathbb{G}$ is at least of step-two). For $t > 0$ put $r = t^{\frac{-p+1}{2}}$, and $f = \max \left[\min \left(2 - \frac{d(g, \delta_t(g_0))}{r}, 1 \right), 0 \right]$. Eventually, we will be taking $t \rightarrow \infty$.

On $B(\delta_t(g_0), 2r) = \{g \in \mathbb{G} : d(g, \delta_t(g_0)) \leq 2r\}$, by homogeneity of d with respect to the dilation δ_t , by Lemma 6.3 of [16], and by the fact that $(\nabla\phi)(g_0) = 0$, we have: By homogeneity of ϕ ,

$$\begin{aligned} |\phi(g) - \phi(\delta_t(g_0))| &= t \left| \phi\left(\delta_{\frac{1}{t}}(g)\right) - \phi(g_0) \right| \\ &\leq t \left(\sup_{B(g_0, \frac{2r}{t})} |\nabla^2 \phi| \right) d(\delta_{\frac{1}{t}}(g), g_0)^2 \end{aligned}$$

by homogeneity of d ,

$$\begin{aligned} &= \frac{1}{t} \left(\sup_{B(g_0, \frac{2r}{t})} |\nabla^2 \phi| \right) d(g, \delta_t(g_0))^2 \\ &\leq \frac{4r^2}{t} \left(\sup_{B(g_0, \frac{2r}{t})} |\nabla^2 \phi| \right) \\ &\leq c_1 r^2. \end{aligned}$$

The last inequality is true for a constant c_1 since $\frac{2r}{t} \rightarrow 0$ as $t \rightarrow \infty$, so the sup can be taken over a fixed ball of sufficiently small radius excluding the identity. Since $\phi(g) \approx \phi(\delta_t(g_0)) \approx t$,

$$|\phi(g)^p - \phi(\delta_t(g_0))^p| = p|\phi(g)^{p-1} - \phi(\delta_t(g_0))^{p-1}||\phi(g) - \phi(\delta_t(g_0))| \\ \leq c_2 t^{p-1} r^2 = c_2.$$

Consequently, the exponential factor in μ is comparable to a constant on the support of f . Also, $|\nabla f| \leq \frac{1}{r}$, and

$$\mu|f|^q \approx r^Q \left(e^{-\alpha\phi^p(\delta_t(g_0))} \right) \quad (5.2)$$

$$\log(\mu|f|^q) \approx -t^p \quad (5.3)$$

$$\mu|\nabla f|^q \approx r^{-q} r^Q e^{-\alpha\phi^p(\delta_t(g_0))} \quad (5.4)$$

Choose t large enough so that $r < \frac{2}{3}$. On $B(\delta_t(g_0), 2r)$, using (5.3),

$$\mu \left(|f|^q \left| \log \left(\frac{|f|^q}{\mu|f|^q} \right) \right|^\beta \right) \approx \mu(|f|^q t^{p\beta})$$

by (5.2)

$$\approx t^{p\beta} r^Q e^{-\alpha\phi^p(\delta_t(g_0))}.$$

Assuming we have β -logarithmic Sobolev inequality, and using (5.4), we get:

$$t^{p\beta} r^Q e^{-\alpha\phi^p(\delta_t(g_0))} \leq M r^{-q} r^Q e^{-\alpha\phi^p(\delta_t(g_0))}$$

since $r = t^{\frac{-p+1}{2}}$,

$$t^{p\beta} \leq M t^{-q(\frac{-p+1}{2})}.$$

For t large enough, we get a contradiction when $p\beta > \frac{q(p-1)}{2}$ i.e. for $q < \frac{2p\beta}{p-1}$. So, the measure μ satisfies no β -logarithmic Sobolev inequality for $1 < q < \frac{2p\beta}{p-1}$. $\square$

Now we prove that for $q \geq 2$, Log $^\beta$ -Sobolev inequality holds true for $d\mu = \frac{e^{-\alpha N^p}}{Z} d\lambda$, where $N = (|x|^4 + a|z|^2)^{\frac{1}{4}}$ and $0 < \beta \leq \frac{p-3}{p}$.

Theorem 11 *Let $\mathbb{G}$ be an step-two Carnot group. Consider the probability measure given by*

$$d\mu = \frac{e^{-g(N)}}{Z} d\lambda,$$

where Z is the normalization constant and $N = (|x|^4 + a|z|^2)^{\frac{1}{4}}$ with $a \in (0, \infty)$. Let $g : [0, \infty) \rightarrow [0, \infty)$ be a differentiable increasing function such that g' is increasing, $g(N) \leq \left(c \frac{g'(N)}{N^2} \right)^{\frac{1}{\beta}}$ for some $\beta > 0$, and $g''(N) < dg'(N)^2$ on $\{N \geq 1\}$, for some constants $c, d \in (0, \infty)$. Then

$$\mu \left(|f|^q \left| \log \left(\frac{|f|^q}{\mu|f|^q} \right) \right|^\beta \right) \leq C \mu|f|^q + D \mu|\nabla f|^q,$$

for C and D positive constants and for $q \geq 2$.

Proof On $\{N \geq 1\}$, $g''(N) < dg'(N)^2 \leq g'(N)^3 N^3$, so the condition of Theorem 1 is satisfied. Thus, on $\{N \geq 1\}$, we have the U-bound (2.1):

$$\mu \left(\frac{g'(N)}{N^2} |f|^q \right) \leq C\mu |\nabla f|^q + D\mu |f|^q.$$

By the condition $g(N) \leq \left(c \frac{g'(N)}{N^2} \right)^{\frac{1}{\beta}}$, we obtain $\phi(g(N)) = (1 + g(N))^\beta \leq \frac{g'(N)}{N^2}$ on $\{N \geq 1\}$. Hence, since $g(N)$ is increasing and using the U-bound, we have

$$\begin{aligned} \mu(\phi(g(N))|f|^q) &\leq \int_{\{N \geq 1\}} \left(\frac{g'(N)}{N^2} |f|^q \right) d\mu + \int_{\{N < 1\}} \phi(g(N)) |f|^q d\mu \\ &\leq \int_{\{N \geq 1\}} \left(\frac{g'(N)}{N^2} |f|^q \right) d\mu + \int_{\{N < 1\}} (1 + g(1))^\beta |f|^q d\mu \\ &\leq C\mu |\nabla f|^q + D\mu |f|^q. \end{aligned} \quad (5.5)$$

In order to use Theorem 7, it remains to prove:

$$\mu(|f|^q |\nabla g(N)|^q) \leq C\mu |f|^q + D\mu |\nabla f|^q. \quad (5.6)$$

On $\{N < 1\}$, since $g'(N)$ is increasing and using (3.1),

$$\int_{\{N < 1\}} (|f|^q |\nabla g(N)|^q) d\mu = \int_{\{N < 1\}} (|f|^q |g'(N) \nabla N|^q) d\mu \leq C^{\frac{q}{2}} \int_{\{N < 1\}} |f|^q |g'(1)|^q d\mu.$$

We now need to consider $\{N \geq 1\}$. For any horizontal vector field V , we can integrate by parts to get:

$$\begin{aligned} &\int_{\mathbb{G}} |f|^q (\nabla g(N) \cdot V - \nabla \cdot V) d\mu \\ &= \int_{\mathbb{G}} |f|^q \nabla g(N) \cdot V d\mu + \frac{1}{Z} \int_{\mathbb{G}} \nabla (|f|^q e^{-g(N)}) \cdot V d\lambda = \int_{\mathbb{G}} \nabla |f|^q \cdot V d\mu \\ &\leq \frac{\epsilon}{p} \int_{\mathbb{G}} |f|^q |V|^p d\mu + \frac{1}{\epsilon^{\frac{q}{p}}} q^{q-1} \int_{\mathbb{G}} \nabla |f|^q d\mu, \end{aligned} \quad (5.7)$$

where the last inequality uses $ab \leq \epsilon \frac{a^p}{p} + \frac{b^q}{\epsilon^{\frac{q}{p}} q}$, where $a = |f|^{q-1} |V|$, and $b = q |\nabla f|$.

Let $V = \nabla N \frac{|x|^{q-2}}{N^{q-2}} g'(N)^{q-1}$. Since $\nabla g(N) = g'(N) \nabla N$, then $\nabla g(N) \cdot V = |\nabla g(N)|^q$, which is the term on the left hand side of (5.6). Using the inequality (3.1) on the first term on the right hand side of (5.7) we get

$$\begin{aligned} \frac{\epsilon}{p} \int |f|^q |V|^p d\mu &= \frac{\epsilon}{p} \int |\nabla N|^p |f|^q \frac{|x|^{(q-2)p} g'(N)^q}{N^{(q-2)p}} d\mu \\ &\leq \frac{\epsilon C^{\frac{p}{2}}}{p} \int |f|^q \frac{|x|^q g'(N)^q}{N^q} d\mu \end{aligned}$$

which can be subtracted from the left hand side of (5.7) since by choosing ϵ small enough and noting that using (3.1), one has, on $\{N \geq 1\}$,

$$\nabla g(N) \cdot V = |\nabla N|^2 \frac{|x|^{q-2}}{N^{q-2}} g'(N)^q \geq A \frac{|x|^q}{N^q} g'(N)^q.$$

It remains to compute $\nabla \cdot V$. Using $|\Delta N| \leq B \frac{|x|^2}{N^3}$, (3.2), and $\frac{x}{|x|} \cdot \nabla N = \frac{|x|^3}{N^3}$, (3.3), we have

$$\begin{aligned} \nabla \cdot V &= \Delta N \frac{|x|^{q-2}}{N^{q-2}} g'(N)^{q-1} + (q-2) \frac{|x|^q g'(N)^{q-1}}{N^{q+1}} - (q-2) \frac{|\nabla N|^2 |x|^{q-2} g'(N)^{q-1}}{N^{q-1}} \\ &\quad + (q-1) g'(N)^{q-2} g''(N) \frac{|x|^{q-2} |\nabla N|^2}{N^{q-2}} \end{aligned}$$

and hence

$$\begin{aligned} |\nabla \cdot V| &\leq B \frac{|x|^q}{N^{q+1}} g'(N)^{q-1} + (q-2) \frac{|x|^q g'(N)^{q-1}}{N^{q+1}} \\ &\quad + (q-2) \frac{C |x|^q g'(N)^{q-1}}{N^{q+1}} + C(q-1) g'(N)^{q-2} g''(N) \frac{|x|^q}{N^q}. \end{aligned}$$

All terms can be absorbed by the first term in (5.7). Using (5.5) and (5.6), the condition of Theorem 7 is satisfied, and we obtain Log^β -Sobolev inequality:

$$\mu \left(|f|^q \left| \log \left(\frac{|f|^q}{\mu |f|^q} \right) \right|^\beta \right) \leq C \mu |f|^q + D \mu |\nabla f|^q$$

for C and D positive constants. $\square$

Corollary 12 Let $\mathbb{G}$ be a step-two Carnot group and $N = (|x|^4 + a|z|^2)^{\frac{1}{4}}$ with $a \in (0, \infty)$.

Let the probability measure be $d\mu = \frac{e^{-\beta N^p}}{Z} d\lambda$, where Z is the normalization constant.

Then, for $p \geq 4$ and $0 < \beta \leq \frac{p-3}{p}$,

$$\mu \left(|f|^q \left| \log \left(\frac{|f|^q}{\mu |f|^q} \right) \right|^\beta \right) \leq C \mu |f|^q + D \mu |\nabla f|^q,$$

for C and D positive constants and for $q \geq 2$.

Appendix: Proof of Lemma 2

Proof We first compute $\nabla N = (X_i N)_{i=1, \dots, n}$.

$$X_i N = N^{-3} \left(|x|^2 x_i + \frac{a}{4} \sum_{k=1}^m \sum_{l=1}^n \Lambda_{il}^{(k)} x_l z_k \right).$$

Therefore,

$$\begin{aligned}
 |\nabla N|^2 &= N^{-6} \left(|x|^6 + \frac{a}{2} \sum_{i=1}^n \sum_{k=1}^m \sum_{l=1}^n \Lambda_{il}^{(k)} |x|^2 x_i x_l z_k + \sum_{i=1}^n \frac{a^2}{16} \sum_{k,k'=1}^m \sum_{l,l'=1}^n \Lambda_{il}^{(k)} \Lambda_{il'}^{(k')} x_l x_{l'} z_k z_{k'} \right) \\
 &= N^{-6} \left(|x|^6 + \frac{a^2}{16} \sum_{i=1}^n \sum_{k,k'=1}^m \sum_{l,l'=1}^n \Lambda_{il}^{(k)} \Lambda_{il'}^{(k')} x_l x_{l'} z_k z_{k'} \right) \\
 &= \frac{|x|^2}{N^2} N^{-4} \left(|x|^4 + \frac{a^2}{16} \sum_{i=1}^n \sum_{k,k'=1}^m \sum_{l,l'=1}^n \Lambda_{il}^{(k)} \Lambda_{il'}^{(k')} \frac{x_l}{|x|} \frac{x_{l'}}{|x|} z_k z_{k'} \right),
 \end{aligned} \tag{1}$$

where we used that for each skew-symmetric matrix $\Lambda^{(k)}$, all $k \in \{1, \dots, m\}$, we have that

$$\sum_{l=1}^n \sum_{i=1}^n \Lambda_{il}^{(k)} x_l x_i = 0.$$

From (1), that with some constants $A, C \in (0, \infty)$, we have

$$A \frac{|x|^2}{N^2} \leq |\nabla N|^2 \leq C \frac{|x|^2}{N^2}.$$

By choosing $a \in (0, \infty)$ sufficiently small, we can ensure that $C \leq 1$. We note that using antisymmetry of matrices $\Lambda_{il}^{(k)}$ we get

$$\frac{x}{|x|} \cdot \nabla N = \sum_{i=1}^n \frac{x_i}{|x|} N^{-3} \left(|x|^2 x_i + \frac{a}{4} \sum_{k=1}^m \sum_{l=1}^n \Lambda_{il}^{(k)} x_l z_k \right)$$

Next, we compute

$$\begin{aligned}
 X_i^2 N &= \left(\frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{k=1}^m \sum_{l=1}^n \Lambda_{il}^{(k)} x_l \frac{\partial}{\partial z_k} \right) \left(N^{-3} \left(|x|^2 x_i + \frac{a}{4} \sum_{k=1}^m \sum_{l=1}^n \Lambda_{il}^{(k)} x_l z_k \right) \right) \\
 &= -3 \left(N^{-7} \left(|x|^2 x_i + \frac{a}{4} \sum_{k=1}^m \sum_{l=1}^n \Lambda_{il}^{(k)} x_l z_k \right)^2 \right) \\
 &\quad + \left(N^{-3} \left(|x|^2 + 2x_i^2 + \frac{a}{4} \sum_{k=1}^m \sum_{l=1}^n \Lambda_{il}^{(k)} \delta_{il} z_k \right) \right) \\
 &\quad + \left(N^{-3} \left(\frac{1}{2} \sum_{k=1}^m \sum_{l=1}^n \Lambda_{il}^{(k)} x_l \frac{a}{4} \sum_{k'=1}^m \sum_{l'=1}^n \Lambda_{il'}^{(k')} x_{l'} \delta_{kk'} \right) \right).
 \end{aligned}$$

Hence, we obtain

$$\begin{aligned}\Delta N &= \sum_{i=1}^n X_i^2 N \\ &= -3 \left(N^{-7} \left(|x|^6 + 2 \frac{a}{4} \sum_{k=1}^m \sum_{i=1}^n \sum_{l=1}^n \Lambda_{il}^{(k)} |x|^2 x_l x_l z_k \right) \right. \\ &\quad + \frac{a^2}{16} \sum_{k=1}^m \sum_{l=1}^n \sum_{k'=1}^m \sum_{l'=1}^n \sum_{i=1}^n \Lambda_{il}^{(k)} \Lambda_{il'}^{(k')} x_{l'} x_l z_k z_{k'} \left. \right) \\ &\quad + \left(N^{-3} \left((n+2)|x|^2 + \frac{a}{4} \sum_{k=1}^m \sum_{i=1}^n \sum_{l=1}^n \Lambda_{il}^{(k)} \delta_{il} z_k \right) \right) \\ &\quad + \left(N^{-3} \left(\frac{1}{2} \sum_{k=1}^m \sum_{l=1}^n \sum_{i=1}^n \Lambda_{il}^{(k)} x_l \frac{a}{4} \sum_{k'=1}^m \sum_{l'=1}^n \Lambda_{il'}^{(k')} x_{l'} \delta_{kk'} \right) \right),\end{aligned}$$

which after simplifications yields

$$\begin{aligned}\Delta N &= \sum_{i=1}^n X_i^2 N \\ &= -3 \left(N^{-7} \left(|x|^6 + \frac{a^2}{16} \sum_{k=1}^m \sum_{l=1}^n \sum_{k'=1}^m \sum_{l'=1}^n \sum_{i=1}^n \Lambda_{il}^{(k)} \Lambda_{il'}^{(k')} x_{l'} x_l z_k z_{k'} \right) \right) \\ &\quad + \left(N^{-3} \left((n+2)|x|^2 \right) \right) \\ &\quad + \left(N^{-3} \left(\frac{a}{8} \sum_{k=1}^m \sum_{l=1}^n \sum_{l'=1}^n \sum_{i=1}^n \Lambda_{il}^{(k)} \Lambda_{il'}^{(k)} x_{l'} x_l \right) \right).\end{aligned}$$

Thus, we get

$$\begin{aligned}\Delta N &= \frac{|x|^2}{N^3} \left[-3 \left(N^{-4} \left(|x|^4 + \frac{a^2}{16} \sum_{k=1}^m \sum_{l=1}^n \sum_{k'=1}^m \sum_{l'=1}^n \sum_{i=1}^n \Lambda_{il}^{(k)} \Lambda_{il'}^{(k')} \frac{x_l}{|x|} \frac{x_{l'}}{|x|} z_k z_{k'} \right) \right) \right. \\ &\quad \left. + \left(n+2 + \frac{a}{8} \sum_{k=1}^m \sum_{l=1}^n \sum_{l'=1}^n \sum_{i=1}^n \Lambda_{il}^{(k)} \Lambda_{il'}^{(k)} \frac{x_l}{|x|} \frac{x_{l'}}{|x|} \right) \right],\end{aligned}$$

which can be represented as follows

$$\begin{aligned}\Delta N &= (n-1) \frac{|x|^2}{N^3} \\ &\quad + \frac{|x|^2}{N^3} \left[-3 \left(N^{-4} \left(-a|z|^2 + \frac{a^2}{16} \sum_{k=1}^m \sum_{l=1}^n \sum_{k'=1}^m \sum_{l'=1}^n \sum_{i=1}^n \Lambda_{il}^{(k)} \Lambda_{il'}^{(k')} \frac{x_l}{|x|} \frac{x_{l'}}{|x|} z_k z_{k'} \right) \right) \right. \\ &\quad \left. + \frac{a}{8} \sum_{k=1}^m \sum_{l=1}^n \sum_{l'=1}^n \sum_{i=1}^n \Lambda_{il}^{(k)} \Lambda_{il'}^{(k)} \frac{x_l}{|x|} \frac{x_{l'}}{|x|} \right].\end{aligned}$$

Hence, there exists a constant $B \in (0, \infty)$ such that

$$|\Delta N| \leq B \frac{|x|^2}{N^3}.$$

Remark If $a > 0$ is small, $\Delta N \geq 0$. For large a , in some directions ΔN can be negative. $\square$

Data Availability Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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