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An Area Theorem for Joint Harmonic Functions on the Product of Homogeneous Trees

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Abstract

For harmonic functions v on the disc, it has been known for a long time that non-tangential boundedness a.e.is equivalent to finiteness a.e. of the integral of the area function of v (Lusin area theorem). This result also hold for functions that are non-tangentially bounded only in a measurable subset of the boundary, and has been extended to rank-one hyperbolic spaces, and also to infinite trees (homogeneous or not). No equivalent of the Lusin area theorem is known on higher rank symmetric spaces, with the exception of the degenerate higher rank case given by the cartesian product of rank-one hyperbolic spaces. Indeed, for products of two discs, an area theorem for jointly harmonic functions was proved by M.P. and P. Malliavin, who introduced a new area function; non-tangential boundedness a.e. is a sufficient condition, but not necessary, for the finiteness of this area integral. Their result was later extended to general products of rank-one hyperbolic spaces by Korányi and Putz. Here we prove an area theorem for jointly harmonic functions on the product of a finite number of infinite homogeneous trees; for the sake of simplicity, we give the proofs for the product of two trees. This could be the first step to an area theorem for Bruhat–Tits affine buildings, thereby shedding light on the higher rank continuous set-up.

Keywords Lusin area integral · Local Fatou theorem · Trees

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1 Introduction

The boundary behaviour of harmonic functions on the upper half plane or the unit disc is well understood for admissible (that is, non-tangential) convergence. A function f harmonic on the disc or the half-space is non-tangentially bounded almost everywhere if and only if its area function $A(f)(\omega) = \int_{\Gamma_\alpha(\omega)} \|\nabla f\|^2 dm$ (where $\Gamma_\alpha(\omega)$ is a Stoltz domain at the boundary point ω and m is Lebesgue measure) is finite for almost every ω. This is called the Lusin area theorem ([4, 13]; see also [7, 15] and references therein).

This equivalence has been established for rank-one symmetric spacess in [9] and for infinite homogeneous trees in [1, 6, 11] and, with a more probabilistic argument, in [14]. The aim of this paper is to prove an area theorem for product of trees, of the type: existence of admissible limits $\Rightarrow$ finiteness of the area function almost everywhere. Our approach adapts to this discrete environment ideas developed in [10, 12] for the product of discs and of rank-one symmetric spaces (see also [8]). The argument overcomes several complications arising from the use of discrete difference equations instead of the classical identities for the Green function and the Laplacian, and extends the results of [1] valid for one tree. The converse implication, finiteness of this area function on a product $\Rightarrow$ existence of admissible limits almost everywhere, cannot hold in full generality in a cartesian product (see Proposition 2.1; for hints on variants of the area theorem suitable to prove the converse implication see Remark 2.2). Thereby we extend the Lusin area theorem proved in [1] for one tree to the product of finitely many trees; for simplicity, we restrict attention to the product of two trees of the same homogeneity. We consider jointly harmonic functions that are non-tangentially bounded (almost) everywhere.

The product of two discs was studied in [12], where the Lusin area theorem was proved for jointly harmonic functions h non-tangentially bounded only locally, and then extended to the product of finitely many rank-one symmetric spaces in [10]. The argument of [10, 12] estimates the area integral via the Green formula, that holds on bi-discs of finite radius; then a very complicated computation, that makes use of suitable mollifiers, is used to restrict attention to the portion of the bi-discs contained in a truncated admissible domain where h is bounded. This local approach is extremely difficult in a discrete setting, where we cannot use derivatives of the mollifiers; we shall consider the local version of our theorem in a future paper.

The motivation of this work is the following. The area theorem has never been stated for higher rank symmetric spaces except in the degenerate case of the product of rank-one spaces like half-planes or discs. An appropriate expression of the area function for non-degenerate higher rank is therefore unknown. It was observed by A. Korányi that it should be easier to find this expression in the combinatorial setting of higher rank buildings of Bruhat-Tits. The degenerate case of higher-rank buildings is the product of homogeneous or semi-homogeneous trees, and the present work is the first step towards this goal; the next step should be the environment of rank-2 affine buildings.

2 Preliminaries on Harmonic Functions on Trees and Statement of the Main Theorems

2.1 Homogeneous Trees

We adopt most of the terminology of [6]: here is a review. A *tree* T is a connected, simply connected, locally finite graph. With abuse of notation we shall also write T for the set of vertices of the tree. We suppose that T is *homogeneous*, that is, every vertex of T belongs exactly to $q+1$ edges, where $q \geq 2$ is a constant, called the *degree*. For $x, y \in T$ we write $x \sim y$ if x, y are neighbours. For any $x, y \in T$ there exist a unique $n \in \mathbb{N}$ and a unique minimal finite sequence $(z_0, \ldots, z_n)$ of distinct vertices such that $z_0 = x$, $z_n = y$ and $z_k \sim z_{k+1}$ for all $k < n$; this sequence is called the *geodesic* path from x to y. The integer n is denoted by $d(x, y)$; d is a metric on T. We fix a reference vertex $o \in T$; this induces a partial ordering in T: $x \leqslant y$ if x belongs to the geodesic path from o to y. Every $x \in T$, $x \neq o$, has exactly one neighbor closer to o, called the *predecessor* of x. For $x \in T$, the *length* $|x|$ is defined as $|x| = d(o, x)$. For any vertex x and any integer $k \leqslant |x|$, x_k is the vertex of length k in the geodesic from o to x.

Definition 1 The nearest neighbor average operator P on functions f on the vertices of T is

$$Pf(x) = \frac{1}{q+1} \sum_{y:\, d(x,y)=1} f(y)\,.$$

The Laplace operator associated with P is $\triangle f = Pf - f$.

Definition 2 A function $f : T \to \mathbb{R}$ is harmonic if $\triangle f(x) = 0$ for every $x \in T$. We shall say that f is harmonic at x if $\triangle f(x) = 0$ for the vertex x. A function is harmonic on a subset of T f it is harmonic at every vertex therein.

2.2 Restricted Non-Tangential Convergence and the Area Function on a Tree

Let Ω be the set of infinite (one-sided) geodesics starting at o. In analogy with the previous notation, for $\omega \in \Omega$ and $n \in \mathbb{N}$, ω_n is the vertex of length n in the geodesic ω. For $x \in T$ the *arc* $U(x) \subset \Omega$ generated by x is the set $U(x) = \left\{\omega \in \Omega \colon x = \omega_{|x|}\right\}$. The sets $U(\omega_n)$, $n \in \mathbb{N}$, form an open base at $\omega \in \Omega$. Equipped with this topology Ω is compact and totally disconnected. Moreover, let $S(x) = \{y \in T : x \leqslant y\}$. Then the sets $S(x) \cup U(x) \colon x \in T$ generate a compact topology on $T \cup \Omega$.

Denote by Λ the set of all the oriented edges in T (i.e., ordered pairs of neighbours). For $\sigma \in \Lambda$ denote by $b(\sigma)$ the beginning vertex of σ and by $e(\sigma)$ the ending vertex: $\sigma = (b(\sigma), e(\sigma))$. The choice of the reference vertex $o \in T$ gives rise to a positive orientation on edges: an edge σ is *positively oriented* if $b(\sigma)$ is the predecessor of $e(\sigma)$ with respect to o.

The beginning and ending vertices induce two maps $b : \Lambda \to T$ and $e : \Lambda \to T$ defined as above. These maps induce two different liftings to T, $f \circ b$ and $f \circ e$, of any $f : \Lambda \to \mathbb{R}$.

We shall henceforth write $b(\sigma) = x$.

Definition 3 For any function $f \colon T \to \mathbb{R}$, the gradient $\nabla f : \Lambda \to \mathbb{R}$ is

$$\nabla f(\sigma) = f(e(\sigma)) - f(b(\sigma)).$$

For $x \in T$, let $\Lambda(x) = \{\sigma \in \Lambda : b(\sigma) = x\}$ be the *star* of x, and let

$$\|\nabla f\|^2(x) = \frac{1}{q+1} \sum_{\sigma \in \Lambda(x)} |\nabla f(\sigma)|^2. \tag{2.1}$$

For $x \in T$ and $\omega \in \Omega$ we consider the distance $d(x, \omega) = \min_{j \in \mathbb{N}} d(x, \omega_j)$.

Definition 4 Let $\alpha \geq 0$ be an integer. The tube $\Gamma_\alpha(\omega)$ around the geodesic $\omega \in \Omega$ is

$$\Gamma_\alpha(\omega) = \{x \in T : d(x, \omega) \leqslant \alpha\}\,.$$

Definition 5 The area function of f on T is the function on Ω defined by

$$A_\alpha f(\omega) = \left(\sum_{x \in \Gamma_\alpha(\omega)} \|\nabla f\|^2(x) \right)^{\frac{1}{2}} .$$

Definition 6 (admissible regions) Let E be a measurable subset of Ω and $\alpha \in \mathbb{N}$, or more generally let $\alpha = \alpha(\omega)$ be a measurable function on E. For simplicity let us fix a reference vertex $o \in T$. We define the *admissible region* (or *Stoltz domain*) $W_\alpha(E) = \bigcup_{\omega \in E} \Gamma_\alpha(\omega)$, that is, the set of vertices $x \in T$ whose distance from the geodesic ray from x to some $\omega \in E$ is at most $\alpha(\omega)$.

For every positive integer n the family $\{U(x)\colon |x| = n\}$ is a partition of Ω into $(q + 1)\, q^{n-1}$ open and closed sets. We define the o-isotropic measure ν on the algebra of sets generated by the sets $U(x)$, by

$$\nu\,(U(x)) = \frac{q}{q+1}\, q^{-|x|}. \tag{2.2}$$

The measure ν extends to a regular Borel probability measure on Ω, called *harmonic measure*,. This is the hitting distribution of the random walk on Ω starting at o induced by P.

2.3 Product of Trees and Statement of the Area Theorem

Let now T_1, T_2 be homogeneous trees with reference vertices o_1, o_2 respectively, $\boldsymbol{T} = T_1 \times T_2$, $\boldsymbol{\Omega} = \Omega_1 \times \Omega_2$, $E \subset \boldsymbol{\Omega}$ measurable and let $\alpha\colon E \to \mathbb{N}$ be a measurable function (we shall see in Section 2.4 that, without loss of generality, α may be assumed to be a constant integer; for simplicity, it is convenient to regard α as a constant throughout this paper). For every $\boldsymbol{\omega} \in \boldsymbol{\Omega}$, the tube (or bi-cone) Γ_α of width α is

$$\Gamma_\alpha(\boldsymbol{\omega}) = \Gamma_\alpha(\omega_1) \times \Gamma_\alpha(\omega_2).$$

The admissible region of width α over a subset $E \subset \boldsymbol{\Omega}$ is

$$W_\alpha(E) = \bigcup_{\boldsymbol{\omega} \in E} \Gamma_\alpha(\boldsymbol{\omega}).$$

Definition 7 When applied to the first or second variable of functions f defined on $\boldsymbol{T}$, the Laplacian is denoted by $\triangle_j$, $j = 1, 2$. A function f on $T_1 \times T_2$ is called *bi-harmonic* (or jointly harmonic) if $\triangle_1 f = \triangle_2 f = 0$. Without loss of generality, we restrict attention to real valued bi-harmonic functions.

By abuse of notation, we shall denote again by ν the product measure $\nu_1 \times \nu_2$ on $\boldsymbol{\Omega}$ of the harmonic measures in each tree.

Definition 8 We have defined the gradient on a tree (Definition 3), hence also the gradients on each variable of the product of two trees, denoted by $\|\nabla_1 f\|^2(x_1, x_2)$ and $\|\nabla_2 f\|^2(x_1, x_2)$. Now we extend the definition to the bi-gradient:

$$\|\nabla_{12} f\|^2(x_1, x_2) = \frac{1}{(q+1)^2} \sum_{\sigma_1 \in \Lambda(x_1)} \sum_{\sigma_2 \in \Lambda(x_2)} |\nabla_1 \nabla_2 f(\sigma_1, \sigma_2)|^2.$$

When there is no danger of confusion we shall write f_1, f_2, f_{12} instead of $\nabla_1 f$, $\nabla_2 f$, $\nabla_{12} f$, respectively.

Similarly, for every function $h(\sigma_1, \sigma_2)$, we define

$$\|h\|^2(x_1, x_2) = \frac{1}{(q+1)^2} \sum_{\sigma_1 \in \Lambda(x_1)} \sum_{\sigma_2 \in \Lambda(x_2)} |h(\sigma_1, \sigma_2)|^2.$$

Our goal is to extend to the context of the cartesian product of two trees the following theorem, that is a particular case of the results of [1]: *If f is a bounded harmonic function f on T, for every fixed $\alpha \geq 0$, $A_\alpha f(\omega) < \infty$ for almost every $\omega \in \Omega$.*

Our extension to $\boldsymbol{T}$, inspired by statements valid on products of disks or half-spaces [3, 8, 12] or of symmetric spaces [10], is the following:

Lusin Area Theorem for bi-harmonic functions *Let v a bi-harmonic function on $\boldsymbol{T}$ that is non-tangentially bounded almost everywhere, that is, for almost all $\boldsymbol{\omega}$ there exists $\alpha \in \mathbb{N}$ such that*

$$\sup_{\boldsymbol{x} \in \Gamma_\alpha(\boldsymbol{\omega})} |v(\boldsymbol{x})| < \infty.$$

Then, for every $\epsilon > 0$, there is a subset $E \subset \boldsymbol{\Omega}$ such that $\nu(\boldsymbol{\Omega} \setminus E) < \epsilon$ and for almost every $\boldsymbol{\omega} \in E$ the area sum

$$\sum_{\boldsymbol{x} \in \Gamma_\alpha(\boldsymbol{\omega})} \left(\|v_{12}\|^2 + \|v_1\|^2 \|v_2\|^2 \right)(\boldsymbol{x}) \tag{2.3}$$

is finite.

If v is uniformly bounded, then the area sum is finite ν-almost everywhere.

Proposition 2.1 *Let v be a function on $\boldsymbol{T} = T_1 \times T_2$. If v is constant on one of its two variables and unbounded in the other, the area function vanishes identically, hence the conclusion of the Main Theorem holds trivially, although v is unbounded.*

Remark 2.2 Despite of the previous Proposition, some form of an inverse Lusin area theorem for the bi-disc is known, due to J. Brossard, who showed in [2] that for a bi-harmonic function $u(x_1, x_2)$, the finiteness over a bi-cone Γ of the area integral introduced by P. and M.P. Malliavin in [12] is actually equivalent to the existence of a finite limit when (x_1, x_2) tends to the boundary inside any smaller bi-cone Γ' of the function $(x_1, x_2) \mapsto u(x_1, x_2) - u(x_1^0, x_2) - u(x_1, x_2^0) + u(x_1^0, x_2^0)$, where (x_1^0, x_2^0) is any fixed point in Γ'. This fact, and a stimulating presentation due to R.F. Gundy [8], suggest that, for a bounded bi-harmonic function on a bi-tree, non-tangential convergence and finiteness of the area sum should be almost everywhere equivalent provided that the area sum of [12] is replaced by a suitably larger one. This will be the subject of a future paper.

2.4 Reduction to Uniformly Bounded Bi-Harmonic Functions

We claim that, for every bi-harmonic function v on $\boldsymbol{T}$ that is non-tangentially bounded almost everywhere and every $\epsilon > 0$, there is a constant $\alpha \geqslant 0$ and a subset $E \subset \boldsymbol{\Omega}$ with $\nu(\boldsymbol{\Omega} \setminus E) < \epsilon$ such that v is bounded in $W_\alpha(E)$.

Indeed, observe that we are assuming that, for almost all $\boldsymbol{\omega}$,

$$\sup_{\boldsymbol{x} \in \Gamma_\alpha(\boldsymbol{\omega})} |v(\boldsymbol{x})| < +\infty. \tag{2.4}$$

This means that, up to a null set, $\boldsymbol{\Omega}$ is the union of subsets of the type

$$\{\boldsymbol{\omega} \in E : |v(\boldsymbol{x})| < M \text{ for all } \boldsymbol{x} \in \Gamma_\alpha(\boldsymbol{\omega}) \text{ and for some } \alpha, M \in \mathbb{N}\}.$$

It follows that, given any non-negative integer n, there exists a closed subset $E_n \subset \mathbf{\Omega}$ with $\nu(\mathbf{\Omega} \setminus E_n) < \frac{1}{n}$ and constants α_n, M_n such that $|f| \leqslant M_n$ on $W_{\alpha_n}(E_n)$. This proves the claim.

If v is uniformly bounded, then $E = \mathbf{\Omega}$; hence the last statement of the Theorem follows from the first.

3 Potential Theory on a Homogeneous Tree

3.1 Green Kernel

Definition 9 Let us regard the nearest-neighbor isotropic transition operator P of Definition 1 as a kernel: $Pf(x) = \sum_{y \in T} P(x, y) f(y)$. Then its operator powers are given by $P^{(n+1)}(x, y) = \sum_{t \in T} P(x, t) P^{(n)}(t, y)$. The Green kernel is defined as

$$G(x, y) = \sum_{n \geqslant 0} P^{(n)}(x, y). \tag{3.1}$$

The Green function g singular at the vertex $o \in T$ is defined as $g(x) = G(o, x)$, and it is easy to see [5, (4.49)] that

$$g(x) = \frac{q}{q-1}\, q^{-|x|}. \tag{3.2}$$

The Green function on $T_1 \times T_2$ is now $g = g^{(1)} \otimes g^{(2)}$, that is $g(x_1, x_2) = g^{(1)}(x_1)\, g^{(2)}(x_2)$. Observe that this Green function $g(\boldsymbol{x})$ is still proportional to the product harmonic measure $\nu(U(\boldsymbol{x})) = (\nu_1 \times \nu_2)(U(\boldsymbol{x}))$ except at $(\{o_1\} \times T_2) \cup (T_1 \times \{o_2\})$.

Remark 3.1 The Green function at a vertex x in a homogeneous tree is a multiple of the reciprocal of the number of vertices of length $|x|$ except for $x = o$. Indeed, for $x \neq o$, $g(x) = \dfrac{q+1}{q-1}\, \nu(U(x))$.

Remark 3.2 It follows immediately from Eq. 3.1 that the Green function singular at o introduced in (3.2) is harmonic outside of $\{o\}$, and is the resolvent of $\triangle$:

$$\triangle g = -\delta_o\,. \tag{3.3}$$

Moreover, it is clear that the sign of $\nabla g(\sigma)$ is negative if and only if σ is positively oriented, and

$$|\nabla g(\sigma)| = \begin{cases} q^{-|x|} = \dfrac{q-1}{q}\, g(x) & \text{if } x = b(\sigma) \neq o, \\[2ex] 1 = \dfrac{q-1}{q}\, g(o) & \text{if } b(\sigma) = o, \end{cases} \tag{3.4}$$

The following is easy to prove (see [1, Lemma 4.4]):

Lemma 3.3 *If $E \subset \Omega$ is measurable, $f \geqslant 0$ on $W_\alpha(E)$, g is the Green function singular at o and $\sum_{W_\alpha(E)} fg < \infty$, then for almost every $\omega \in E$*

$$\sum_{\Gamma_\alpha(\omega)} f < \infty.$$

Example 3.4 For a vertex x in a tree T and $\omega \in \Omega$, the *Poisson kernel* normalized at o in a tree T is the harmonic function $K(x,\omega)$ defined as follows. Let $N(x,\omega) \in \mathbb{N}$ be the *bifurcation index*, that is the number of edges in common between the finite geodesic path $[o,x]$ and the infinite geodesic ω starting at o; similarly, let $N(\omega,\omega')$ be the number of edges in common between the geodesics ω and ω'. Then it is easy to see [5] that $K(x,\omega) := \lim_{y\to\omega} G(o,x)/G(o,y)$ exists for every x and ω and

$$K(x,\omega) = q^{2N(x,\omega)-|x|}. \tag{3.5}$$

It is clear that $\lim_{x\to\omega} K(x,\omega') = 0$ for every $\omega \neq \omega'$. It follows from Eq. 3.5 that $A_\alpha(\omega) = \sum_{x\in\Gamma_\alpha(\omega)} \|\nabla K(x,\omega')\|^2$ is finite for $\omega \neq \omega'$, and the sum of the series diverges (as $q^{2N(\omega,\omega')}$) for $\omega \to \omega'$: this confirms the one-dimensional Area Theorem of [1].

Now consider $\boldsymbol{T} = T_1 \times T_2$ and the joint harmonic function $\boldsymbol{K}(\boldsymbol{x},\boldsymbol{\omega}) = K^{(1)} \otimes K^{(2)} = K^{(1)}(x_1,\omega_1)\, K^{(2)}(x_2,\omega_2)$. The reader can easily verify that $\nabla_{12}\boldsymbol{K} = \nabla_1 K^{(1)} \otimes \nabla_2 K^{(2)}$, and that, for $i, j = 1, 2$ with $i \neq j$, one has $\nabla_i \boldsymbol{K} = (\nabla_i K_i) \otimes K_j$. Hence $\|\nabla_{12}\boldsymbol{K}\|^2 = \|\nabla_1 K^{(1)}\|^2 \otimes \|\nabla_2 K^{(2)}\|^2$, and $\|\nabla_1 \boldsymbol{K}\|^2 \|\nabla_2 \boldsymbol{K}\|^2 = \|\nabla_1 K^{(1)}\|^2 \otimes \|\nabla_2 K^{(2)}\|^2 \boldsymbol{K}^2$. Therefore the area function becomes $\|\nabla_1 K^{(1)}\|^2 \otimes \|\nabla_2 K^{(2)}\|^2 (1+\boldsymbol{K}^2)$. Now an easy computation shows that, if α is bounded, the area sum $\sum_{\Gamma_\alpha(\boldsymbol{\omega})} \|\nabla_1 K^{(1)}\|^2 \otimes \|\nabla_2 K^{(2)}\|^2 (1+\boldsymbol{K}^2)$ is finite at every $\boldsymbol{\omega} = (\omega_1,\omega_2)$ such that $\omega_i \neq \omega_i'$ for $i = 1, 2$, and for all $\boldsymbol{\omega} \neq \boldsymbol{\omega}'$, i.e., $\omega_1 \neq \omega_1'$ or $\omega_2 \neq \omega_2'$, then for $0 < c_- \leqslant c_+$ the *restricted area sum*

$$\sum_{\boldsymbol{x}\in\Gamma_\alpha(\boldsymbol{\omega}),\; c_-<|x_1|/|x_2|<c_+} \left(\|\nabla_{12}\boldsymbol{K}\|^2 + \|\nabla_1 \boldsymbol{K}\|^2 \|\nabla_2 \boldsymbol{K}\|^2 \right)(\boldsymbol{x})$$

is finite. In particular, the (unrestricted) area sum is finite almost everywhere. It actually follows from Eq. 3.5 that the assumption that α be bounded is unnecessary; anyway it is not restrictive for our results, in view of the uniformization procedure of Section 2.4.

3.2 Gradient and Laplacian

For all functions f, g on the homogeneous tree $T = T_q$ let us introduce the bilinear form

$$\langle \nabla f,\, \nabla g\rangle(x) = \frac{1}{q+1} \sum_{\sigma\in\Lambda(x)} \nabla f(\sigma)\, \nabla g(\sigma). \tag{3.6}$$

Therefore

$$\|\nabla f\|^2(x) = \langle \nabla f,\, \nabla f\rangle(x), \tag{3.7}$$

If h and k are functions on the edges of T, the bilinear form $\frac{1}{q+1} \sum_{\sigma\in\Lambda(x)} h(\sigma)\, k(\sigma)$, for each fixed x, is a (real) inner product that we denote by $\langle h, k\rangle$; its associated norm satisfies

$$\|h\|^2(x) = \frac{1}{q+1} \sum_{\sigma\in\Lambda(x)} |h(\sigma)|^2. \tag{3.8}$$

In particular, the norm in Eq. 2.1 is of this type. In what follows we shall consider this inner product on two copies T_1 and T_2 of the tree T: for clarity, we shall denote the bilinear forms on each of the two copies by $\langle\cdot\,,\cdot\rangle_1$ and $\langle\cdot\,,\cdot\rangle_2$, respectively.

Definition 10 If f is defined on the vertices of a tree T, let

$$Mf(\sigma) = \frac{f(e(\sigma)) + f(b(\sigma))}{2}\,.$$

Let us write $x = b(\sigma)$ and note that

$$2Mf(\sigma) = \nabla f(\sigma) + 2f(x)\,. \tag{3.9}$$

The following useful facts follow immediately from the definition of $\triangle$ (Definition 1) and of ∇ (Definition 3).

Corollary 3.5 *For every $f : T \to \mathbb{R}$,*

$$\triangle f(x) = \frac{1}{q+1} \sum_{\sigma \in \Lambda(x)} \nabla f(\sigma),$$

and, if f is bounded, then $|\nabla f| \leqslant 2\|f\|_\infty$.

3.3 The Green Formula in One Tree

Definition 11 (Boundary) The *boundary* ∂Q of a finite subset $Q \subset T$ is $\partial Q = \{\sigma \in \Lambda : b(\sigma) \in Q, e(\sigma) \notin Q\}$.

The Green formulas are well known in the continuous setup. In the discrete context of a tree, the following interesting analogue has been observed in [6].

Proposition 3.6 (The Green formula) *If f and h are functions on T and Q is a finite subset of T, then*

$$\sum_Q (h\triangle f - f\triangle h) = \frac{1}{q+1} \sum_{\partial Q} \big((h \circ b)\,\nabla f - (f \circ b)\,\nabla h\big).$$

4 Difference Operators on T and Proof the Area Theorem

4.1 Identities for Discrete Difference Operators in One Variable

The following statement is easily verified. The last identity in its part (i) follows from Eq. 3.9.

Lemma 4.1 *Let f, g be functions on the homogeneous tree T. Put, as usual, $x = b(\sigma)$ and let $\langle \nabla f,\ \nabla g\rangle(x)$ be as in Definition 3. Then:*

(i)

$$\nabla(fg)(\sigma) = f(x)\,\nabla g(\sigma) + g(x)\,\nabla f(\sigma) + \nabla f(\sigma)\,\nabla g(\sigma).$$

(ii)

$$\triangle(fg)(x) = f(x)\,\triangle g(x) + g(x)\,\triangle f(x) + \langle \nabla f,\ \nabla g\rangle(x)\,.$$

The following statements are immediate consequences of the definitions of gradient and mean:

Corollary 4.2 $\nabla f^2(\sigma) = 2Mf(\sigma)\nabla f(\sigma)$.

As a consequence we have the Green identities already observed in [6]:

Corollary 4.3 (i) *If f is harmonic,*

$$\triangle(f^2) = \|\nabla f\|^2 .$$

(ii) *If f and g are harmonic,*

$$\triangle(fg) = \langle \nabla f, \nabla g\rangle .$$

Corollary 4.4 *Let f and v be functions on T with v harmonic. Then*

$$\triangle(f v^2) = f\, \|\nabla v\|^2 + v^2\, \triangle f + \langle \nabla f, \nabla(v^2)\rangle.$$

Proof By Lemma 4.1 (i), $\triangle(f v^2) = f\triangle(v^2) + v^2\triangle f + \langle \nabla f, \nabla(v^2)\rangle$, and, by Corollary 4.3 (i), $\triangle(v^2) = \|\nabla v\|^2$, hence the statement. □

4.2 Identities for Discrete Difference Operators in Two Variables

Lemma 4.5 (i) *Let u be bi-harmonic on $\boldsymbol{T} = T_1 \times T_2$ and, as before, let us write $\nabla_2 u = u_2$ and $\nabla_{12} u = u_{12}$. Then*

$$\triangle_1\triangle_2 u^2 = \|u_{12}\|^2 .$$

(ii) *If u and v are bi-harmonic, then*

$$\triangle_1\langle u_2,\ v_2\rangle_2 = \langle u_{12},\ v_{12}\rangle_2 .$$

Proof Write $\boldsymbol{x} = (x_1, x_2) \in \boldsymbol{T}$. By of Corollary 4.3 (i), $\triangle_2 u^2 = \|u_2\|^2$ and by Definition 8

$$\triangle_1\|u_2\|^2(\boldsymbol{x}) = \frac{1}{(q+1)^2} \sum_{\sigma_1\in\Lambda(x_1),\, \sigma_2\in\Lambda(x_2)} (\nabla_1\nabla_2 u(\sigma_1, \sigma_2))^2 \equiv \|u_{12}\|^2(x_1, x_2) .$$

This yields (i).

Part (ii) follows from the same computation via Corollary 4.3 (i). □

Corollary 4.6 *Let u be a bi-harmonic function on $\boldsymbol{T}$. Then*

$$\tfrac{1}{2}\triangle_1\triangle_2 u^4 = \|u_1\|^2\, \|u_2\|^2 + u^2\, \|u_{12}\|^2 + < \nabla_1(u^2), \nabla_1(\|\nabla_2 u\|^2)\rangle_1 + \tfrac{1}{2}\, \triangle_1(\|\nabla_2(u^2)\|^2) .$$

Proof By Corollary 4.3 (ii),

$$\triangle_2(u^2) = \|u_2\|^2 . \tag{4.1}$$

By Corollary 4.3 (i)) and Lemma 4.1 (ii), $\triangle_2(u^4) = 2u^2\|\nabla_2 u\|^2 + \|\nabla_2(u^2)\|^2$. By applying the Laplacian on the first variable on both sides, one has, again by Lemma 4.1 (ii),

$$\triangle_1\triangle_2(u^4) = 2(\triangle_1(u^2))\|u_2\|^2 + 2u^2\triangle_1(\|u_2\|^2) + 2\langle\nabla_1(u^2), \nabla_1(\|u_2\|^2)\rangle_1 + \triangle_1(\|\nabla_2(u^2)\|^2) .$$

In the first term at the right-hand side, $\triangle_1(u^2) = \|\nabla_1 u\|^2$ by Corollary 4.3 (i). In the second term, $\triangle_1(\|u_2\|^2) = \|u_{12}\|^2$ by Lemma 4.5. This proves the statement. □

Remark 4.7 Since ∇_1 and ∇_2 operate on different variables (the components x_1 and x_2 of $\boldsymbol{x}$), they commute. The same is true of $\triangle_1$ and ∇_2, and for the same reason $\triangle_1$ commutes with M_2; indeed,

$$\triangle_1 M_2 f(x_1, \sigma_2) = \tfrac{1}{q+1} \textstyle\sum_{y_1\sim x_1} M_2 f(y_1, \sigma_2) - M_2 f(x_1, \sigma_2) = M_2\triangle_1 f(x_1, \sigma_2).$$

Lemma 4.8 *Denote by M_i the mean operator of Definition 10 acting on the $i-$th variable. Then*

$$\nabla_1(\|u_2\|^2)(\sigma_1, x_2) = 2\langle M_1 u_2, u_{12}\rangle_2(\sigma_1, x_2).$$

Proof By the definition of norm of the gradient in Eq. 3.7 and Corollary 4.2,

$$\begin{aligned}\nabla_1(\|u_2\|^2)(\sigma_1, x_2) &= \frac{2}{q+1}\sum_{\sigma_2\in\Lambda(x_2)} M_1u_2(\sigma_1,\sigma_2)\, u_{12}(\sigma_1,\sigma_2)\\ &= 2\langle M_1u_2,\ u_{12}\rangle_2(\sigma_1, x_2).\end{aligned}$$

□

Lemma 4.9 *Let u be a bi-harmonic function on $\boldsymbol{T}$. Then*

$$\triangle_1\left(\|\nabla_2(u^2)\|^2\right) \geqslant 2\langle\nabla_2(u^2),\, \nabla_2(\|u_1\|^2)\rangle_2\,.$$

Proof By Eq. 3.7, Corollary 4.3 (i) and Lemma 4.1 (ii),

$$\begin{aligned}&\triangle_1\left(\|\nabla_2(u^2)\|^2\right)\\ &= \frac{1}{q+1}\sum_{\sigma_2\in\Lambda(x_2)} \triangle_1(2M_2u\, u_2)^2\\ &= \frac{1}{q+1}\sum_{\sigma_2}\Big[2(2M_2u\,u_2)\,\triangle_1(2M_2u\,u_2)+\langle\nabla_1(2M_2u\,u_2),\,\nabla_1(2M_2u\,u_2)\rangle_1\Big]\\ &= \frac{4}{q+1}\sum_{\sigma_2}\Big[2M_2u\,u_2\,\triangle_1(M_2u\,u_2) + \|\nabla_1(M_2u\,u_2)\|^2\Big]\end{aligned}$$

(the last step follows from Eq. 3.7).

Since u is harmonic with respect to its first component, $\triangle_1 M_2u = 0 = \triangle_1\, u_2$ by Remark 4.7. Then, by Lemma 4.1 (ii), the right-hand side becomes

$$\frac{4}{q+1}\sum_{\sigma_2}\Big[2M_2u\,u_2\langle\nabla_1(M_2u)\,u_{12}\rangle_1 + \|\nabla_1(M_2u\,u_2)\|^2\Big].$$

Since $\nabla_1(M_2u) = M_2u_1$, it follows from Lemma 4.8 with indices interchanged and Corollary 4.2 that

$$\triangle_1\left(\|\nabla_2(u^2)\|^2\right) = \tfrac{4}{q+1}\Big[\textstyle\sum_{\sigma_2} M_2u\,u_2\,\nabla_2(\|\nabla_1u\|^2) + \|\nabla_1(M_2u\,\nabla_2u)\|^2\Big].$$

Therefore

$$\begin{aligned}\triangle_1\left(\|\nabla_2(u^2)\|^2\right)(\boldsymbol{x}) &\geqslant \frac{4}{q+1}\sum_{\sigma_2\in\Lambda(x_2)} (M_2u)\,u_2\,\nabla_2(\|\nabla_1u\|^2)\\ &= \frac{2}{q+1}\sum_{\sigma_2}\nabla_2(u^2)\,\nabla_2\|u_1\|^2 = 2\langle\nabla_2(u^2),\,\nabla_2(\|u_1\|^2)\rangle_2\end{aligned}$$

(the last two identities follow again from Corollary 4.3 (i) and Eq. 3.6). □

The following inequality is a direct consequence of Corollary 4.6 and Lemma 4.9:

Corollary 4.10 *Let u be a bi-harmonic function on $\boldsymbol{T}$. Then*

$$\frac{1}{2}\triangle_1\triangle_2 u^4 \geqslant \|u_1\|^2\|u_2\|^2 + \langle\nabla_1(u^2), (\nabla_1(\|u_2\|^2)\rangle_1 + \langle\nabla_2(u^2), \nabla_2(\|u_2\|^2)\rangle_2 .$$

Lemma 4.11 *Let u be a function on $\boldsymbol{T}$ bounded by a constant $K > 0$ at some bi-vertex $\boldsymbol{x}$ and on its set of bi-neighbors (t_1, t_2): $d(t_i, x_i) \leqslant 1,\ i = 1, 2$. Then*

$$\begin{aligned}\big|\langle\nabla_1(u^2), \nabla_1(\|u_2\|^2)\rangle_1 \ + \ \langle\nabla_2(u^2), \nabla_2(\|u_2\|^2)\rangle_2\big|\,(\boldsymbol{x})\\ \leqslant 8K\left((2K+1)\|u_{12}\|^2 + \|u_1\|^2\|u_2\|^2\right)(\boldsymbol{x}).\end{aligned}$$

Proof Write, as usual, $\boldsymbol{x} = (x_1, x_2) = (b(\sigma_1), b(\sigma_2))$. By symmetry, it is enough to show that

$$\begin{aligned}\left|\left\langle\nabla_1(u^2), \nabla_1(\|u_2\|^2)\right\rangle_1\right|(x_1, x_2) &= \frac{1}{q+1}\sum_{\sigma_1\in\Lambda(x_1)}\left|\nabla_1(u^2)(\sigma_1, x_2)\,\nabla_1(\|u_2\|^2)(\sigma_1, x_2)\right|\\ &\leqslant 4K\left((2K+1)\|u_{12}\|^2 + \|u_1\|^2\|u_2\|^2\right)(x_1, x_2).\end{aligned}$$

By Corollary 4.3 (i), $\nabla_1(u^2) = 2M_1u\, u_1$, and by Lemma 4.8

$$\nabla_1(\|u_2\|^2) = \frac{2}{q+1}\sum_{\sigma_2\in\Lambda(x_2)}(M_1u_2)\,u_{12}$$

(both sides are functions of (σ_1, x_2)). Therefore

$$\frac{1}{q+1}\sum_{\sigma_1}\left|\nabla_1(u^2)\,\nabla_1(\|u_2\|^2)\right| = \frac{4}{(q+1)^2}\sum_{\sigma_1,\sigma_2}|(M_1u)u_1(M_1u_2)u_{12}|\,. \tag{4.2}$$

On the other hand, $|M_1u| \leqslant K$ and $M_1u_2(\sigma_1, \sigma_2) = u_{12}(\sigma_1, \sigma_2) + 2u_2(x_1, \sigma_2)$ by Eq. 3.9. So we estimate the last sum in Eq. 4.2 as follows.

$$\sum_{(\sigma_1,\sigma_2)\in\Lambda(x_1)\times\Lambda(x_2)}|(M_1u)u_1(M_1u_2)u_{12}| \leqslant K\sum_{\sigma_1,\sigma_2}|u_1|\,|u_{12}+2u_2|\,|u_{12}|$$

(here the summands M_1u and u_1 are functions of (σ_1, x_2), u_2 is a function of (x_1, σ_2) and the other factors in the sums are functions of (σ_1, σ_2), but the sums in both sides are functions of (x_1, x_2)). The sum in the right-hand side is bounded by the sum of two parts that we now estimate separately. The first, $\sum_{\sigma_1,\sigma_2}|u_1|\,|u_{12}|^2$, is easily estimated by Definition 8 and the obvious inequality $|u_1(\sigma_1, x_2)| \leqslant 2K$:

$$\sum_{(\sigma_1,\sigma_2)\in\Lambda(x_1)\times\Lambda(x_2)}|u_1|\,|u_{12}|^2 \leqslant 2K(q+1)^2\|u_{12}\|^2\,,$$

Now let us estimate the second part, $\sum_{\sigma_1,\sigma_2} |u_1|\,|2u_2|\,|u_{12}|$, by Schwarz inequality:

$$
\begin{aligned}
&\sum_{(\sigma_1,\sigma_2)\in\Lambda(x_1)\times\Lambda(x_2)} |u_1(\sigma_1,x_2)|\,|2u_2(\sigma_2,x_1)|\,|u_{12}(\sigma_1,\sigma_2)| \\
&\leqslant 2\left(\sum_{\sigma_1\in\Lambda(x_1)} |u_1(\sigma_1,x_2)|^2 \sum_{\sigma_2\in\Lambda(x_2)} |u_2(x_1,\sigma_2)|^2\right)^{\frac{1}{2}} \left(\sum_{(\sigma_1,\sigma_2)\in\Lambda(x_1)\times\Lambda(x_2)} |u_{12}(\sigma_1,\sigma_2)|^2\right)^{\frac{1}{2}} \\
&= 2(q+1)^2\|u_1\|\,\|u_2\|\,\|u_{12}\| \leqslant (q+1)^2\big(\|u_{12}\|^2 + \|u_1\|^2\|u_2\|^2\big)
\end{aligned}
$$

because $ab \leqslant \frac{1}{2}(a^2+b^2)$ for $a, b \geqslant 0$. The statement follows. ☐

Corollary 4.12 *Let u be a bi-harmonic function on $\boldsymbol{T}$ bounded by K at some $\boldsymbol{x}$ and its bi-neighbors. Then, at $\boldsymbol{x}$,*

(i)
$$(2-16K)\,\|u_1\|^2\|u_2\|^2 \leqslant \triangle_1\triangle_2 u^4 + 16\,(2K^2+K)\,\|u_{12}\|^2\,.$$

(ii)
$$(2-16K)\,(\|u_1\|^2\|u_2\|^2 + \|u_{12}\|^2) \leqslant \triangle_1\triangle_2 u^4 + (32K^2+2)\triangle_1\triangle_2 u^2\,.$$

Proof By Corollary 4.10 and Lemma 4.11 we have

$$\left|\triangle_1\triangle_2\, u^4 - 2\|u_1\|^2\|u_2\|^2\right| \leqslant (32K^2+16K)\|u_{12}\|^2 + 16K\,\|u_1\|^2\|u_2\|^2.$$

Hence

$$\triangle_1\triangle_2\, u^4 \geqslant -(32K^2+16K)\|u_{12}\|^2 - (16K-2)\|u_1\|^2\|u_2\|^2.$$

This proves (i). Part (ii) follows from this and Lemma 4.5 (i). ☐

The following result is a straightforward extension to the cartesian product $\boldsymbol{T}$ of Lemma 3.3.

Lemma 4.13 *If f is a non-negative function on $\boldsymbol{T}$ and $\sum_{\boldsymbol{x}\in\boldsymbol{T}} f(\boldsymbol{x})\,g(\boldsymbol{x}) < \infty$, then $\sum_{\boldsymbol{x}\in\Gamma_\alpha(\boldsymbol{\omega})} f(\boldsymbol{x}) < \infty$ for all $\alpha \geqslant 0$ and for almost every $\boldsymbol{\omega}\in\Omega$.*

For $j = 1, 2$ let us denote by $D_{j,n}$ the discs $\{x_j \in T_j : |x_j| \leqslant n\}$ and let $D_n = D_{1,n}\times D_{2,n}$. Let $g^{(j)}$ the Green function singular at o_j on T_j (Definition 3.2). It follows immediately from Remark 3.1) that

Lemma 4.14 *For $j = 1, 2$ the sums*

$$\sum_{b(\partial D_{j,n})} g^{(j)}(x), \qquad \sum_{\partial D_{j,n}} |\nabla_j g^{(j)}(\sigma)|,$$

are uniformly bounded with respect to r.

Lemma 4.15 *For any function f on $\boldsymbol{T}$ and any $n > 0$,*

$$\Big|\sum_{D_{j,n}} (\triangle_j f) g^{(j)}\Big| \leqslant C \sup_{D_{j,n}} \left\{|f| + \|f_j\|\right\}, \tag{4.3}$$

and

$$\Big|\sum_{D_n}(\triangle_1\triangle_2 f)\, g^{(1)}g^{(2)}\Big| \leqslant C \sup_{D_n}\{|f| + \|f_1\| + \|f_2\| + \|f_{12}\|\} \tag{4.4}$$

for some constant C independent of n and f.

Proof By symmetry, it is enough to look at $j = 2$. Let us fix $x_1 \in T_1$ and set $f_{x_1}(x_2) = f(x_1, x_2)$. By Eq. 3.2 and the fact that $g^{(2)}$ is $\triangle_2$-harmonic in $T_2 \setminus \{o_2\}$,

$$\sum_{|x_2|\leqslant n} g^{(2)}(x_2)\,\triangle_2 f_{x_1}(x_2) = g^{(2)}(o_2)\triangle_2 f_{x_1}(o_2) + \sum_{0<|x_2|\leqslant n} g^{(2)}(x_2)\,\triangle_2 f_{x_1}(x_2)$$

$$= \frac{q}{q-1}\,\triangle_2 f_{x_1}(o_2) + \sum_{0<|x_2|\leqslant n}\Big(g^{(2)}(x_2)(\triangle_2 f_{x_1}(x_2) - f_{x_1}(x_2)\triangle_2 g^{(2)}(x_2))\Big).$$

By the Green formula (Proposition 3.6) the last term in the right-hand side is equal to

$$\frac{1}{q+1}\sum_{\sigma_2\in\partial(D_{2,n}\setminus\{o_2\})}\Big(g^{(2)}(b(\sigma_2))\nabla_2 f_{x_1}(\sigma_2) - f_{x_1}(b(\sigma_2))\nabla_2 g^{(2)}(\sigma_2)\Big)\,.$$

The boundary of the finite region $D_{2,n}\setminus\{o_2\}$ splits into its outward part $\partial D_{2,n}$ and its inward part $\{\sigma_2\colon e(\sigma_2) = o_2\}$. Let us now apply Eq. 3.2 and Eq. 3.4 and Corollary 3.5 with sign changed (because the edge σ_2 have the reversed orientation, pointing toward o_2 instead of away from it). The last expression becomes

$$\frac{1}{q+1}\sum_{\partial D_{2,n}}\Big(g^{(2)}(b(\sigma_2))\nabla_2 f_{x_1}(\sigma_2)\;-\;f_{x_1}(b(\sigma_2))\nabla_2 g^{(2)}(\sigma_2)\Big)$$

$$-\;\frac{1}{q-1}\,\triangle_2 f_{x_1}(o_2) - \frac{1}{q+1}\sum_{|x_2'|=1} f_{x_1}(x_2')\,.$$

Therefore

$$\begin{aligned}
&\sum_{x_2\in D_{2,n}} g^{(2)}(x_2)\,\triangle_2 f_{x_1}(x_2)\\
&= \triangle_2 f_{x_1}(o_2) - \frac{1}{q+1}\sum_{|x_2'|=1} f_{x_1}(x_2')\\
&\qquad + \frac{1}{q+1}\sum_{\sigma_2\in\partial D_{2,n}}\Big(g^{(2)}(b(\sigma_2))\nabla_2 f_{x_1}(\sigma_2) - f_{x_1}(b(\sigma_2))\nabla_2 g^{(2)}(\sigma_2)\Big)\\
&= -f_{x_1}(o_2) + \frac{1}{q+1}\sum_{\sigma_2\in\partial D_{2,n}}\Big(g^{(2)}(b(\sigma_2))\nabla_2 f_{x_1}(\sigma_2) - f_{x_1}(b(\sigma_2))\nabla_2 g^{(2)}(\sigma_2)\Big)\,. \qquad (4.5)
\end{aligned}$$

This formula together with Lemma 4.14 proves Eq. 4.3.

Now Eq. 4.4 follows from Fubini's theorem and two applications of Eq. 4.3. $\square$

Lemma 4.16 *Let v be a bounded bi-harmonic function, $\phi(v) = v^2$ or $\phi(v) = v^4$, and $W_\alpha(E)$ as in* Section 2.4. *Then*

$$\sum_{\boldsymbol{x}\in D_n\cap W_\alpha(E)} \triangle_1\triangle_2\phi(v(\boldsymbol{x}))\, g^{(1)}(x_1)\, g^{(2)}(x_2)$$

is uniformly bounded with respect to n.

Proof Let $f := \phi\circ v$ and $\sum_{D_n} fg := \sum_{\boldsymbol{x}\in D_n} \triangle_1\triangle_2 f(x_1,x_2)\, g^{(1)}(x_1)\, g^{(2)}(x_2)$. Since v is bounded by some $\xi > 0$ near $\boldsymbol{x} \in D_n \cap W_\alpha(E)$, f is bounded by $\zeta = \max\{\xi^2, \xi^4\}$, hence also all its gradients are bounded by 4ζ; therefore the statement follows from Eq. 4.4. □

4.3 Proof of the Area Theorem

It is immediately seen that, if the statement holds for v, then it holds for βv with $\beta\in\mathbb{R}$. Then, as v is bounded in the set $W_\alpha(E)$, we may assume $|v| < 1/8$ there, hence the constants at the left-hand side of the inequalities of Corollary 4.12 are strictly positive.

By Lemma 4.13, the statement means that $\sum_{W_\alpha(E)}(\|v_{12}\|^2 + \|v_1\|^2\,\|v_2\|^2)g < \infty$, or equivalently that $\sum_{D_n\cap W_\alpha(E)}(\|v_{12}\|^2 + \|v_1\|^2\,\|v_2\|^2)g$ is uniformly bounded over n.

We know by Lemma 4.16 that $\sum_{\boldsymbol{x}\in D_n\cap W_\alpha(E)} \triangle_1\triangle_2 v^2(\boldsymbol{x})\, g^{(1)}(x_1)\, g^{(2)}(x_2)$ is uniformly bounded with respect to n. Since $\triangle_1\triangle_2 v^2 = \|v_{12}\|^2$ by Lemma 4.5 (i), we see that $\sum_{D_n\cap W_\alpha(E)} \|v_{12}\|^2\, g$ is uniformly bounded. It remains the uniform boundedness of $\sum_{D_n\cap W_\alpha(E)} \|v_1\|^2\,\|v_2\|^2 g$. This follows from Corollary 4.12 (i) if we show the uniform boundedness of $\sum_{D_n\cap W_\alpha(E)} \triangle_1\triangle_2 v^4 g$ and $\sum_{D_n\cap W_\alpha(E)} \|v_{12}\|^2\, g$. We have already proved the latter, and the former follows from Lemma 4.16.

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