An L^p -Comparison, $p \in (1, \infty)$, on the Finite Differences of a Discrete Harmonic Function at the Boundary of a Discrete Box

Tuan Anh Nguyen¹

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Abstract

It is well-known that for a harmonic function u defined on the unit ball of the d -dimensional Euclidean space, $d \geq 2$, the tangential and normal component of the gradient ∇u on the sphere are comparable by means of the L^p -norms, $p \in (1, \infty)$, up to multiplicative constants that depend only on d, p . This paper formulates and proves a discrete analogue of this result for discrete harmonic functions defined on a discrete box on the d -dimensional lattice with multiplicative constants that do not depend on the size of the box.

Keywords Discrete harmonic function · Discrete boundary problems · Discrete Fourier multiplier theorem · Discrete Poisson kernel

Mathematics Subject Classification (2010) 65N22 · 35J25

1 Introduction

This paper formulates and proves a discrete analogue of a classical result in the continuum setting which states that the tangential and normal component of the gradient of a harmonic function on the boundary of a domain are comparable by means of L^p -norms, $p \in (1, \infty)$. For convenience we give a simplified version of this result in Theorem 1.1 below. For complete formulations and proofs we refer the reader to Maergoiz [1] (see, e.g., Theorems 1 and 2), Mikhlin [2] (see §44 p. 208), and Bella, Fehrman, and Otto [3] (see Lemma 4). This result can be viewed as a *stability estimate for harmonic extensions* of given Dirichlet or Neumann boundary conditions. Thus, it plays an important role in the proof of a Liouville theorem for a class of elliptic equations with degenerate random coefficient fields (see formulas (40) and (41) in [3]) where the so-called idea *perturbing around the homogenized coefficients* is realized by *harmonic extensions* of given boundary conditions. The discrete analogue that we want to show here can be applied to prove a Liouville theorem for the ran-

✉ Tuan Anh Nguyen
tuan.nguyen@uni-due.de

¹ Faculty of Mathematics, University of Duisburg-Essen, 45117, Essen, Germany

dom conductance model under degenerate conditions, which is the discrete analogue of [3] (see the paragraph below Lemma 4 in [3] and the PhD thesis of the author [4] (e.g., Section III.2.3 for an outline).

Theorem 1.1 For every $d \in \mathbb{N}$, $x \in \mathbb{R}^d$ denote by $\|x\|$ the Euclidean norm of x . Denote by $\int d\sigma$ the usual surface integral. For every $d \in \mathbb{N}$, $r \in (0, \infty)$ let $\mathbb{B}_r^d, \bar{\mathbb{B}}_r^d, \partial\mathbb{B}_r^d$ be the sets given by

$$\mathbb{B}_r^d = \{x \in \mathbb{R}^d : \|x\|_{\mathbb{R}^d} < r\}, \quad \bar{\mathbb{B}}_r^d = \{x \in \mathbb{R}^d : \|x\|_{\mathbb{R}^d} \leq r\}, \quad \partial\mathbb{B}_r^d = \{x \in \mathbb{R}^d : \|x\|_{\mathbb{R}^d} = r\}. \quad (1.1)$$

For every $d \in \mathbb{N} \cap [2, \infty)$, $r \in (0, \infty)$, $u \in C^\infty(\partial\mathbb{B}_r^d)$ let $\partial u : \partial\mathbb{B}_r^d \rightarrow \mathbb{R}^d$ be the gradient of u , let $\partial_\tau u : \partial\mathbb{B}_r^d \rightarrow \mathbb{R}^d$ be the tangential component of ∂u , and let $\partial_\nu u : \partial\mathbb{B}_r^d \rightarrow \mathbb{R}^d$ be the normal component of ∂u , i.e., it holds for all $x \in \partial\mathbb{B}_r^d$ that $(\partial_\tau u)(x)$ is the orthogonal projection of $\partial u(x)$ onto the tangential space of $\partial\mathbb{B}_r^d$ at x and $(\partial u)(x) = (\partial_\tau u)(x) + (\partial_\nu u)(x)$. Then there exists a function $C : ([2, \infty) \cap \mathbb{N}) \times (1, \infty) \rightarrow (0, \infty)$ such that for all $d \in \mathbb{N} \cap [2, \infty)$, $p \in (1, \infty)$, $r \in (0, \infty)$, $u : \bar{\mathbb{B}}_r^d \rightarrow \mathbb{R}$ with $u|_{\partial\mathbb{B}_r^d} \in C^\infty(\partial\mathbb{B}_r^d)$ and $\forall x \in \partial\mathbb{B}_r^d : \sum_{i=1}^d (\partial_{ii}^2 u)(x) = 0$ it holds that

$$\frac{1}{C(d, p)} \int_{\partial\mathbb{B}_r^d} \|\partial_\tau u\|^p d\sigma \leq \int_{\partial\mathbb{B}_r^d} \|\partial_\nu u\|^p d\sigma \leq C(d, p) \int_{\partial\mathbb{B}_r^d} \|\partial_\tau u\|^p d\sigma. \quad (1.2)$$

In order to formulate the discrete analogue of Theorem 1.1 let us introduce our notation. For the rest of this paper we always use the notation given in Setting 1.2 below.

Setting 1.2 (Notation for the whole paper) Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. For every $d \in \mathbb{N}$ let $\mathbf{e}_i^d$, $i \in [1, d] \cap \mathbb{Z}$, be the standard d -dimensional basis vectors and let E_d be the set of all *oriented nearest neighbour edges* (for short: *edges*) on the d -dimensional lattice, i.e., the set given by $E_d = \{(x, y) : (x, y \in \mathbb{Z}^d) \wedge (\sum_{i=1}^d |x_i - y_i| = 1)\}$. For every $d \in \mathbb{N}$, $A \subset \mathbb{Z}^d$, $u : A \rightarrow \mathbb{K}$ let $\Delta u : \{x \in \mathbb{Z}^d : \forall i \in [1, d] \cap \mathbb{Z} : (x + \mathbf{e}_i^d \in A) \wedge (x - \mathbf{e}_i^d \in A)\} \rightarrow \mathbb{K}$ be the *discrete Laplacian* of u , i.e., the function which satisfies for all $x \in A$ with $\forall i \in [1, d] \cap \mathbb{Z} : (x + \mathbf{e}_i^d \in A) \wedge (x - \mathbf{e}_i^d \in A)$ that

$$(\Delta u)(x) = \left[\sum_{i=1}^d u(x + \mathbf{e}_i^d) + u(x - \mathbf{e}_i^d) \right] - 2du(x) \quad (1.3)$$

and let $\nabla u : (A \times A) \cap E_d \rightarrow \mathbb{K}$ be the function which satisfies that for all $x, y \in A$ with $(x, y) \in E_d$ it holds that $\nabla_{(x,y)} u = u(y) - u(x)$. For every $d \in \mathbb{N} \cap [2, \infty)$, $p \in [1, \infty]$, every finite set A , and every function $f : A \rightarrow \mathbb{K}$ let $\|f\|_{L^p(A)} \in [0, \infty)$ be the real number given by

$$\|f\|_{L^p(A)}^p = \begin{cases} \sum_{x \in A} |f(x)|^p & : p \in [1, \infty) \\ \max_{x \in A} |f(x)| & : p = \infty. \end{cases} \quad (1.4)$$

Note that in Setting 1.2 above the arguments of ∇u are edges. We will also introduce another notation for discrete derivatives which are functions of vertices (see Section 3). However, to formulate the main result let us temporarily use the notation in Setting 1.3 below.

Setting 1.3 For every $d, N \in \mathbb{N} \cap [2, \infty)$ let $E_{d,N}^\tau, E_{d,N}^v \subseteq E_d$ be the sets of edges which satisfy that

$$E_{d,N}^\tau = \left\{ (x, y) \in E_d : x, y \in ([0, N]^d) \setminus ((0, N)^d) \right\}, \quad (1.5)$$

$$E_{d,N}^v = \left\{ (x, y) \in E_d : x \in ([0, N]^d) \setminus ((0, N)^d) \text{ and } y \in [1, N-1]^d \right\}, \quad (1.6)$$

let $V_{d,N}^\tau \subseteq \mathbb{Z}^d$ be the set of vertices given by $V_{d,N}^\tau = \mathbb{Z}^d \cap ([0, N]^d) \setminus ((0, N)^d)$, let $\mathcal{D}_{d,N}$ be the set of functions $v: V_{d,N}^\tau \rightarrow \mathbb{R}$, let $\mathcal{N}_{d,N}$ be the set of functions $v: E_{d,N}^v \rightarrow \mathbb{R}$ with $\sum_{e \in E_{d,N}^v} v(e) = 0$, and let $\mathbb{Q}_{d,N}$ be the set of functions $u: ([0, N] \cap \mathbb{Z})^d \rightarrow \mathbb{R}$ which satisfy that $\forall x \in ([1, N-1] \cap \mathbb{Z})^d: (\Delta u)(x) = 0$.

In Setting 1.3 above for $d \in [2, \infty) \cap \mathbb{Z}$ we consider boxes on the d -dimensional lattice instead of balls on the d -dimensional Euclidean space. Figure 1 illustrates the sets $V_{d,N}^\tau, E_{d,N}^\tau, E_{d,N}^v$ for $d = 2, N = 10$: $V_{d,N}^\tau$ contains all points on the boundary of the box (red), $E_{d,N}^\tau$ contains all edges with two endpoints on the boundary (red), and $E_{d,N}^v$ contains all edges which are perpendicular to the boundary with one endpoint on the boundary and the other inside the box (blue). For fixed $d, N \in \mathbb{N} \cap [2, \infty)$ the set $V_{d,N}^\tau$ can be viewed as a *discrete boundary* of the box $([0, N] \cap \mathbb{Z})^d$, a function $v: V_{d,N}^\tau \rightarrow \mathbb{R}$ is thus a *discrete Dirichlet condition*, an edge $e \in E_{d,N}^\tau$ can be viewed as a tangential vector, an edge $e \in E_{d,N}^v$ can be viewed as a normal vector, and a function $v: E_{d,N}^v \rightarrow \mathbb{R}$ with $\sum_{e \in E_{d,N}^v} v(e) = 0$ is thus a *discrete Neumann condition*. Here, the vanishing mean is a necessary condition for the Neumann problem to have a solution, which also holds in the continuum setting. Finally, the set $\mathbb{Q}_{d,N}$ is the set of functions which is defined on the box $([0, N] \cap \mathbb{Z})^d$ and harmonic in the interior $([1, N-1] \cap \mathbb{Z})^d$ of the box.

Theorem 1.4 (Main result) *Assume Setting 1.3. Then there exists a function $C: ([2, \infty) \cap \mathbb{N}) \times (1, \infty) \rightarrow (0, \infty)$, there exists a unique family of linear operators*

$$\{\Phi_{d,N}^D: \mathcal{D}_{d,N} \rightarrow \mathbb{Q}_{d,N}: d, N \in [2, \infty) \cap \mathbb{N}\}, \quad (1.7)$$

and there exists a family of linear operators

$$\{\Phi_{d,N}^N: \mathcal{N}_{d,N} \rightarrow \mathbb{Q}_{d,N}: d, N \in [2, \infty) \cap \mathbb{N}\} \quad (1.8)$$

Fig. 1 Tangential and normal edges of a two dimensional box

such that

- i) it holds for all $d, N \in [2, \infty) \cap \mathbb{N}$, $f \in \mathcal{D}_{d,N}$, $x \in V_{d,N}^\tau$ that $(\Phi_{d,N}^\mathcal{D} f)(x) = f(x)$,
- ii) it holds for all $d, N \in [2, \infty) \cap \mathbb{N}$, $p \in (1, \infty)$, $f \in \mathcal{D}_{d,N}$ that

$$\|\nabla(\Phi_{d,N}^\mathcal{D} f)\|_{L^p(E_{d,N}^v)} \leq C(d, p) \|\nabla f\|_{L^p(E_{d,N}^\tau)}, \quad (1.9)$$

- iii) it holds for all $d, N \in [2, \infty) \cap \mathbb{N}$, $e \in E_{d,N}^v$, $g \in \mathcal{N}_{d,N}$ that $\nabla_e(\Phi_{d,N}^\mathcal{N} g) = g(e)$, and
- iv) it holds for all $d, N \in [2, \infty) \cap \mathbb{N}$, $p \in (1, \infty)$, $g \in \mathcal{N}_{d,N}$ that

$$\|\nabla(\Phi_{d,N}^\mathcal{N} g)\|_{L^p(E_{d,N}^\tau)} \leq C(d, p) \|g\|_{L^p(E_{d,N}^v)}. \quad (1.10)$$

Item (i) in Theorem 1.4 above implies that for every $d, N \in [2, \infty) \cap \mathbb{N}$, $f \in \mathcal{D}_{d,N}$ the function $\Phi_{d,N}^\mathcal{D} f$ is the solution $u: ([0, N] \cap \mathbb{Z})^d \rightarrow \mathbb{R}$ to the *discrete Dirichlet problem*

$$\begin{cases} \forall x \in ([1, N-1] \cap \mathbb{Z})^d: & (\Delta u)(x) = 0, \\ \forall x \in V_{d,N}^\tau: & u(x) = f(x) \end{cases} \quad (1.11)$$

and the family $(\Phi_{d,N}^\mathcal{D})_{d,N \in [2, \infty) \cap \mathbb{N}}$ therefore exists uniquely as in the statement of Theorem 1.4. Next, Item (iii) in Theorem 1.4 above implies that for every $d, N \in [2, \infty) \cap \mathbb{N}$, $g \in \mathcal{N}_{d,N}$ the function $\Phi_{d,N}^\mathcal{N} g$ is a solution $u: ([0, N] \cap \mathbb{Z})^d \rightarrow \mathbb{R}$ to the *discrete Neumann problem*

$$\begin{cases} \forall x \in ([1, N-1] \cap \mathbb{Z})^d: & (\Delta u)(x) = 0, \\ \forall e \in E_{d,N}^v: & \nabla_e u = g(e). \end{cases} \quad (1.12)$$

Note that there is no full statement on the uniqueness of the Neumann problem (1.12). More precisely, the uniqueness of the Neumann problem (1.12) only holds up to a constant on $([1, N-1] \cap \mathbb{Z})^d$, i.e., if $\forall e \in E_{d,N}^v: g(e) = 0$ and if u is a solution to Eq. 1.12, then the restriction of u on $([1, N-1] \cap \mathbb{Z})^d$ is a constant function. In addition, note that for fixed $d, N \in [2, \infty) \cap \mathbb{N}$, $g \in E_{d,N}^v$ there exists no real numbers $C \in (0, \infty)$ such that for every solution u to Eq. 1.12 it holds that $\|\nabla u\|_{L^p(E_{d,N}^\tau)} \leq C \|g\|_{L^p(E_{d,N}^v)}$. Indeed, e.g., in the case $d = 2$ we can freely change the value of u at the four corners of the rectangle in Fig. 1 to make $\|\nabla u\|_{L^p(E_{d,N}^\tau)}$ arbitrary large without damaging the fact that u is a solution to Eq. 1.12. Consequently, it is impossible to make any claims on the uniqueness of the family $(\Phi_{d,N}^\mathcal{N})_{d,N \in [2, \infty) \cap \mathbb{N}}$ in the statement of Theorem 1.4.

Next, let us give a brief and rough explanation why Theorem 1.4 is useful for the idea of using *harmonic extensions* in the proof of the Liouville theorem in [4]. Let u be a function defined on the box in Fig. 1. We keep the Dirichlet condition of u at boundary points and replace the values of u at other points by an extension that is harmonic in the interior of the box. This will clearly erase the Neumann condition of u . However, Theorem 1.4 claims that the new Neumann condition can still be bounded by the remaining Dirichlet condition.

Discrete Laplacian and discrete harmonic functions are interesting topics that date back to 1920s (see, e.g., the fundamental works by Lewy, Friedrichs, and Courant [5], Heilbronn [6], Duffin [7]). Discrete boundary problems have been widely studied in numerical analysis, e.g., to approximate the continuum solutions (see, e.g., the classical work by Stummel [8] and for further references see, e.g., G\"urlebeck and Hommel [9–11], who studied Dirichlet and Neumann boundary problems on general two-dimensional discretized domains using difference potentials, and the references therein).

Although discrete and continuum objects often have many similar properties, it is not always trivial to adapt things from the continuum case to the discrete case and vice versa.

To the best of the author's knowledge, there exists no result in the discrete case which deals with the bounds Eqs. 1.9 and 1.10, while L^p -comparisons, $p \in (1, \infty)$, between the tangential and non-tangential components of harmonic functions on Lipschitz and C^1 -domains and related topics have been studied by several papers, e.g., in chronological order: Mikhlin [2], Maergoiz [1], Calderon, Calderon, Fabes, Jodeit, and Rivière [12], Fabes, Jodeit, and Rivière [13], Jerison and Kenig [14], Verchota [15], Dahlberg and Kenig [16], Mitrea and Mitrea [17]. The main issue in the discrete case is to show that the functions C in Eqs. 1.9 and 1.10 do not depend on the size N of the discrete box while in the continuum case this is not an issue due to a simple scaling argument. In fact, for Eq. 1.2 we only need to consider $r = 1$.

The proof of Theorem 1.4 that we represent here essentially mimics the proof of Lemma 4 in Bella, Fehrman, and Otto [3] who formulate and prove Theorem 1.1 with balls replaced by boxes in the continuum case. We separate the proof into several steps and organize the paper as follows. Section 2 formulates and proves a discrete counterpart of inequality (88) in [3], which was shown by using the continuum Poisson kernels. In order to adapt this idea to the discrete case we use a result in Lawler and Limic [18] to *approximate the discrete Poisson kernels by the continuum Poisson kernels*. Estimates by means of the *Marcinkiewicz multiplier theorem*, e.g., inequalities (78), (79), (82), and (99) in [3] are adapted in Section 3 which focuses on *discrete harmonic functions on haft spaces with periodic boundary conditions*. In order to avoid many tedious calculations with higher derivatives of the multipliers we apply Cauchy's integral formula. In addition, with some elementary arguments, Section 3.4 provides a result of independent interest that the author has not found in the literature. Finally, Section 4 applies the results obtained in Sections 2 and 3 to prove the main result, Theorem 1.4. As Bella, Fehrman, and Otto [3] we call estimate (1.9) *the Dirichlet case* and estimate Eq. 1.10 *the Neumann case* and prove them separately. The main techniques here are basically to adapt two ideas learnt from [3] to the discrete case: i) returning to the case of periodic boundary conditions by using *even and odd reflections* and ii) reducing to the case of haft spaces. Another interesting application of even and odd reflections and the discrete Marcinkiewicz multiplier theorem is to prove L^p -estimates for discrete Poisson equations (see Section 2.5.2 in Jovanović and Süli [19]).

Using the techniques introduced here we can easily verify Theorem 1.5 below, which is not a surprising result and which could be important for several applications. The proof of Theorem 1.5 is left to the reader.

Theorem 1.5 *Assume Setting 1.3 and for every $d, N \in [2, \infty) \cap \mathbb{N}$ let $E_{d,N}$ be the set of edges given by $E_{d,N} = \{(x, y) \in E_d : \frac{1}{2}(x + y) \in [0, N]^d\}$. Then there exist a function $C : ([2, \infty) \cap \mathbb{N}) \times (1, \infty) \rightarrow (0, \infty)$ and a unique family of linear operators $\{\Phi_{d,N}^D : \mathcal{D}_{d,N} \rightarrow \mathbb{Q}_{d,N} : d, N \in [2, \infty) \cap \mathbb{N}\}$ such that for all $d, N \in [2, \infty) \cap \mathbb{N}$, $p \in (1, \infty)$, $f \in \mathcal{D}_{d,N}$, $x \in V_{d,N}^\tau$ it holds that $(\Phi_{d,N}^D f)(x) = f(x)$, $\|\Phi_{d,N}^D f\|_{L^p([0,N]^d \cap \mathbb{Z}^d)}^p \leq C(d, p)N \|f\|_{L^p(E_{d,N}^\tau)}^p$, and $\|\nabla(\Phi_{d,N}^D f)\|_{L^p(E_{d,N})}^p \leq C(d, p)N \|\nabla f\|_{L^p(E_{d,N}^\tau)}^p$.*

Note that the functions C in Theorems 1.4 and 1.5 may depend exponentially on the dimension: these results, as finite difference method in general, may not be quite useful for high-dimensional applications (the so-called *curse of dimensionality*).

For convenience, throughout this paper, the arguments here are often compared with that in the continuum case in [3]. However, since there are several differences between the discrete case and the continuum case, this paper is organized so that the reader can easily start from scratch.

Our notation will be defined clearly in the formulation of each result. In addition, remember that throughout this paper we always use the notation in Setting 1.2 above and the usual conventions in Setting 1.6 below.

Setting 1.6 (Conventions) Denote by $\mathbf{i}$ the imaginary unit. Denote by $\Re(z)$ and $\Im(z)$ the real and imaginary part of $z \in \mathbb{K}$, respectively, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Write $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $d \in \mathbb{N}$, $x, y \in \mathbb{K}^d$, $i \in [1, d] \cap \mathbb{Z}$ denote by x_i the i -th coordinate of x (if no confusion can arise), denote by $x \cdot y$ the standard scalar product of x and y , i.e., $x \cdot y = \sum_{i=1}^d x_i \bar{y}_i$, and denote by $|x|_\infty$ the maximum norm of x , i.e., $|x|_\infty = \max_{i=1}^d |x_i|$. For every set A denote by $|A|$ the cardinality of A . Partial derivatives will be denoted by ∂_i , $\frac{\partial}{\partial t_i}$, $\frac{\partial}{\partial \xi_i}$. When applying a result we often use a phrase like 'Lemma 3.8 with $d \leftarrow d - 1$ ' that should be read as 'Lemma 3.8 applied with d (in the notation of Lemma 3.8) replaced by $d - 1$ (in the current notation)' and we often omit a trivial replacement to lighten the notation, e.g., we rarely write, e.g., 'Lemma 3.28 with $d \leftarrow d$ '.

2 Potential-Theoretic Results for Harmonic Functions on Haft Spaces

2.1 Main Result

In this section we essentially prove Corollary 2.2 below, which formulates a discrete analogue of inequality (88) in Bella, Fehrman, and Otto [3]. We basically follow the proof in [3]. However, to make the argument more illustrative we introduce a simple random walk in Setting 2.3. Lemmas 2.7 and 2.11 are discrete counterparts of inequality (92) and (93) in [3]. Combining Lemmas 2.7 and 2.11 with a Marcinkiewicz-type interpolation argument we obtain Corollary 2.12. Approximating the discrete Poisson kernels by the continuum counterparts we obtain Lemma 2.10. This and Corollary 2.12 imply Corollary 2.2.

Setting 2.1 For every $L \in \mathbb{N}$, $d \in [2, \infty) \cap \mathbb{N}$ let $\mathbb{I}_L$ be the set given by $\mathbb{I}_L = [-L+1, L] \cap \mathbb{Z}$ and let $\mathbb{H}_{d,L,\geq 0}$ be the set of all bounded functions $u: \mathbb{Z}^{d-1} \times \mathbb{N}_0 \rightarrow \mathbb{R}$ with the properties that

1. it holds for all $x \in \mathbb{Z}^{d-1} \times \mathbb{N}_0$, $i \in [1, d-1] \cap \mathbb{Z}$ that $u(x) = u(x + 2L\mathbf{e}_i^d)$ and
2. it holds for all $x \in \mathbb{Z}^{d-1} \times \mathbb{N}$ that $(\Delta u)(x) = 0$.

Corollary 2.2 Assume Setting 2.1. Then there exists $C: ([2, \infty) \cap \mathbb{N}) \times (1, \infty) \times (0, \infty) \rightarrow (0, \infty)$ such that for all $d \in [2, \infty) \cap \mathbb{N}$, $p \in (1, \infty)$, $\bar{r} \in (0, \infty)$, $L \in \mathbb{N}$, $N \in (0, L\bar{r}) \cap \mathbb{N}$, $u \in \mathbb{H}_{d,L,\geq 0}$ it holds that $\|u\|_{L^p(\{0\} \times \mathbb{I}_L^{d-2} \times ([1,N] \cap \mathbb{Z}))} \leq C(d, p, \bar{r}) \|u\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})}$.

2.2 Results Which Directly Follow from the Simple Random Walk Representation

Throughout this section we use the notation given in Setting 2.3 below. Due to the Riesz-Thorin interpolation argument for Corollary 2.6 we have to consider the function u in Setting 2.3 as a complex-valued function. For other results we only need to replace $\mathbb{K}$ by $\mathbb{R}$.

Setting 2.3 (Simple random walks) Let $d \in [2, \infty) \cap \mathbb{N}$ be fixed, let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space with expectation denoted by $\mathbb{E}$, let $X_n: \Omega \rightarrow \mathbb{Z}^d$, $n \in \mathbb{N}$, be independent random variables which satisfy for all $n \in \mathbb{N}$, $i \in [1, d] \cap \mathbb{Z}$ that $\mathbb{P}(X_n = \mathbf{e}_i^d) = \mathbb{P}(X_n = -\mathbf{e}_i^d) = \frac{1}{2d}$, and let $S_n: \Omega \rightarrow \mathbb{Z}^d$, $n \in \mathbb{N}_0$, $T: \Omega \rightarrow \mathbb{N}_0$ be the random variables which

satisfy for all $n \in \mathbb{N}$ that

$$S_n = \sum_{j=1}^n X_j \quad \text{and} \quad T = \inf \left\{ j \in \mathbb{N}_0 : S_j \in \mathbb{Z}^{d-1} \times \{0\} \right\}. \quad (2.1)$$

Lemma 2.4 Assume Settings 2.1 and 2.3 and let $L \in \mathbb{N}$, $u \in \mathbb{H}_{d,L,\geq 0}$. Then

1. it holds for all $(x, y) \in \mathbb{Z}^{d-1} \times \mathbb{N}_0$ that $u(x, y) = \mathbb{E}[u(S_T + (x, 0)) | S_0 = (0, y)]$ and
2. it holds for all $y \in \mathbb{N}_0$ that $\sum_{x \in \mathbb{I}_L^{d-1}} u(x, y) = \sum_{x \in \mathbb{I}_L^{d-1}} u(x, 0)$.

The following proof relies on martingale theory. For an elementary proof see Appendix A.2.

Proof of Lemma 2.4 The assumption that $\forall (x, y) \in \mathbb{Z}^{d-1} \times \mathbb{N}$: $(\Delta u)(x, y) = 0$ and the assumption that u is bounded demonstrate for all $x \in \mathbb{Z}^{d-1}$ that $(u(S_n + (x, 0)))_{n \in \mathbb{N}_0}$ is a bounded martingale. The optional stopping theorem proves that

$$u(x, y) = \mathbb{E}[u(S_0 + (x, 0)) | S_0 = (0, y)] = \mathbb{E}[u(S_T + (x, 0)) | S_0 = (0, y)]. \quad (2.2)$$

This shows Item (i). Furthermore, Eq. 2.2, linearity, and periodicity imply for all $y \in \mathbb{N}_0$ that

$$\sum_{x \in \mathbb{I}_L^{d-1}} u(x, y) = \mathbb{E} \left[\sum_{x \in \mathbb{I}_L^{d-1}} u(S_T + (x, 0)) \middle| S_0 = (0, y) \right] = \sum_{x \in \mathbb{I}_L^{d-1}} u(x, 0). \quad (2.3)$$

The proof of Lemma 2.4 is thus completed. $\square$

Lemma 2.5 Assume Setting 2.1 and let $d \in [2, \infty) \cap \mathbb{N}$, $L \in \mathbb{N}$, $u \in \mathbb{H}_{d,L,\geq 0}$. Then it holds that

$$\|u\|_{L^1(\mathbb{I}_L^{d-1} \times \{N\})} \leq \|u\|_{L^1(\mathbb{I}_L^{d-1} \times \{0\})} \quad \text{and} \quad \|u\|_{L^\infty(\mathbb{I}_L^{d-1} \times \{N\})} \leq \|u\|_{L^\infty(\mathbb{I}_L^{d-1} \times \{0\})}. \quad (2.4)$$

Proof of Lemma 2.5 Recall that we use the notation in Setting 2.3. The fact that $\mathbb{P}$ -almost surely it holds that $S_T \in \mathbb{Z}^{d-1} \times \{0\}$ and the assumption on periodicity, i.e., $\forall (x, y) \in \mathbb{Z}^{d-1} \times \mathbb{N}_0$, $i \in [1, d-1] \cap \mathbb{Z}$: $u(x, y) = u(x + 2L\mathbf{e}_i^{d-1}, y)$ imply that $\mathbb{P}$ -almost surely it holds that

$$\sum_{x \in \mathbb{I}_L^{d-1}} |u(S_T + (x, 0))| = \sum_{x \in \mathbb{I}_L^{d-1}} |u(x, 0)| \quad \text{and} \quad \max_{x \in \mathbb{I}_L^{d-1}} |u(S_T + (x, 0))| = \max_{x \in \mathbb{I}_L^{d-1}} |u(x, 0)|. \quad (2.5)$$

Furthermore, Lemma 2.4 shows for all $(x, y) \in \mathbb{Z}^d \times \mathbb{N}_0$ that

$$|u(x, y)| = \left| \mathbb{E}[u(S_T + (x, 0)) | S_0 = (0, y)] \right| \leq \mathbb{E}[|u(S_T + (x, 0))| | S_0 = (0, y)] \quad (2.6)$$

This and Eq. 2.5 establish that

$$\begin{aligned} \sum_{x \in \mathbb{I}_L^{d-1}} |u(x, N)| &\leq \sum_{x \in \mathbb{I}_L^{d-1}} \mathbb{E}[|u(S_T + (x, 0))| | S_0 = (0, N)] \\ &= \mathbb{E} \left[\sum_{x \in \mathbb{I}_L^{d-1}} |u(S_T + (x, 0))| \middle| S_0 = (0, N) \right] = \sum_{x \in \mathbb{I}_L^{d-1}} |u(x, 0)| \end{aligned} \quad (2.7)$$

and

$$\begin{aligned} \max_{x \in \mathbb{I}_L^{d-1}} |u(x, N)| &\leq \max_{x \in \mathbb{I}_L^{d-1}} \mathbb{E} \left[|u(S_T + (x, 0))| \middle| S_0 = (0, N) \right] \\ &\leq \mathbb{E} \left[\max_{x \in \mathbb{I}_L^{d-1}} |u(S_T + (x, 0))| \middle| S_0 = (0, N) \right] = \max_{x \in \mathbb{I}_L^{d-1}} |u(x, 0)| \end{aligned} \quad (2.8)$$

This completes the proof of Lemma 2.5. $\square$

Combining Lemma 2.5 with a Riesz-Thorin interpolation we obtain Corollary 2.6 below.

Corollary 2.6 Assume Setting 2.1 and let $d \in [2, \infty) \cap \mathbb{N}$, $L \in \mathbb{N}$, $u \in \mathbb{H}_{d,L,\geq 0}$, $p \in [1, \infty]$, $N \in \mathbb{N}_0$. Then it holds that $\|u\|_{L^p(\mathbb{I}_L^{d-1} \times \{N\})} \leq \|u\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})}$.

Lemma 2.7 (L^∞ -estimate) Assume Setting 2.1 and let $d \in [2, \infty) \cap \mathbb{N}$, $L \in \mathbb{N}$, $u \in \mathbb{H}_{d,L,\geq 0}$, $p \in [1, \infty)$. Then it holds that

$$\max_{z \in \mathbb{N}_0} \left[\sum_{y \in \mathbb{I}_L^{d-2}} \left| u(0, y, z) - \frac{1}{|\mathbb{I}_L|} \sum_{x \in \mathbb{I}_L} u(x, y, z) \right|^p \right]^{1/p} \leq 2 \max_{x \in \mathbb{I}_L} \left[\sum_{y \in \mathbb{I}_N^{d-2}} |u(x, y, 0)|^p \right]^{1/p} \quad (2.9)$$

Proof of Lemma 2.7 Recall that we use the notation in Setting 2.3. First, observe that Lemma 2.4, Jensen's inequality, and linearity of $\mathbb{E}$ show that

$$\begin{aligned} \sup_{z \in \mathbb{N}_0} \left[\sum_{y \in \mathbb{I}_L^{d-2}} |u(0, y, z)|^p \right]^{1/p} &= \sup_{z \in \mathbb{N}_0} \left[\sum_{y \in \mathbb{I}_L^{d-2}} \left| \mathbb{E} \left[u(S_T + (0, y, 0)) \middle| S_0 = (0, 0, z) \right] \right|^p \right]^{1/p} \\ &\leq \sup_{z \in \mathbb{N}_0} \left[\mathbb{E} \left[\sum_{y \in \mathbb{I}_L^{d-2}} |u(S_T + (0, y, 0))|^p \middle| S_0 = (0, 0, z) \right] \right]^{1/p} \\ &\leq \max_{x \in \mathbb{I}_L} \left[\sum_{y \in \mathbb{I}_N^{d-2}} |u(x, y, 0)|^p \right]^{1/p}. \end{aligned} \quad (2.10)$$

Next, Jensen's inequality and Corollary 2.6 ensure that

$$\begin{aligned} \sup_{z \in \mathbb{N}_0} \left[\sum_{y \in \mathbb{I}_L^{d-2}} \left| \frac{1}{|\mathbb{I}_L|} \sum_{x \in \mathbb{I}_L} u(x, y, z) \right|^p \right]^{1/p} &\leq \sup_{z \in \mathbb{N}_0} \left[\frac{1}{|\mathbb{I}_L|} \sum_{x \in \mathbb{I}_L} \sum_{y \in \mathbb{I}_L^{d-2}} |u(x, y, z)|^p \right]^{1/p} \\ &\leq \left[\frac{1}{|\mathbb{I}_L|} \sum_{x \in \mathbb{I}_L} \sum_{y \in \mathbb{I}_L^{d-2}} |u(x, y, 0)|^p \right]^{1/p} \leq \max_{x \in \mathbb{I}_N} \left[\sum_{y \in \mathbb{I}_N^{d-2}} |u(x, y, 0)|^p \right]^{1/p}. \end{aligned} \quad (2.11)$$

Combining this, Eq. 2.10, and the triangle inequality completes the proof of Lemma 2.7. $\square$

Lemma 2.8 Assume Setting 2.1 and let $d \in [2, \infty) \cap \mathbb{N}$, $L \in \mathbb{N}$, $u \in \mathbb{H}_{d,L,\geq 0}$, $\bar{r} \in (0, \infty)$, $p \in [1, \infty)$, $N \in \mathbb{N}$ satisfy that $N/L \leq \bar{r}$. Then it holds that

$$\left[\sum_{y \in \mathbb{I}_L^{d-2}} \sum_{z=1}^N \left| \frac{1}{|\mathbb{I}_L|} \sum_{x \in \mathbb{I}_L} u(x, y, z) \right|^p \right]^{1/p} \leq (\bar{r}/2)^{1/p} \left[\sum_{x \in \mathbb{I}_L} \sum_{y \in \mathbb{I}_L^{d-2}} |u(x, y, 0)|^p \right]^{1/p}. \quad (2.12)$$

Proof of Lemma 2.8 Jensen's inequality and the assumption $N/L \leq \bar{r}$ ensure that

$$\begin{aligned} \sum_{y \in \mathbb{I}_L^{d-2}} \sum_{z=1}^N \left| \frac{1}{|\mathbb{I}_L|} \sum_{x \in \mathbb{I}_L} u(x, y, z) \right|^p &\leq \frac{1}{|\mathbb{I}_L|} \sum_{x \in \mathbb{I}_L} \sum_{y \in \mathbb{I}_L^{d-2}} \sum_{z=1}^N |u(x, y, z)|^p \\ &\leq \frac{N}{2L} \sum_{x \in \mathbb{I}_L} \sum_{y \in \mathbb{I}_L^{d-2}} |u(x, y, 0)|^p \\ &\leq \frac{\bar{r}}{2} \sum_{x \in \mathbb{I}_L} \sum_{y \in \mathbb{I}_L^{d-2}} |u(x, y, 0)|^p. \end{aligned} \quad (2.13)$$

This completes the proof of Lemma 2.8. $\square$

2.3 The Poisson Kernel Revisited

Setting 2.9 Assume Setting 2.3, let $P: \mathbb{N}_0 \times \mathbb{Z} \times \mathbb{Z}^{d-2} \rightarrow \mathbb{R}$ be the function which satisfies for all $(z, x, y) \in \mathbb{N}_0 \times \mathbb{Z} \times \mathbb{Z}^{d-2}$ that $P_z(x, y) = \mathbb{P}[S_T = (x, y, 0) | S_0 = (0, 0, z)]$, and let $M \in [0, \infty]$ be the real extended number given by

$$M = \sup_{z \in \mathbb{N}} \left[z \sum_{(x,y) \in \mathbb{Z} \times \mathbb{Z}^{d-2}} |P_z(x, y) - P_z(x-1, y)| \right]. \quad (2.14)$$

Lemma 2.10 Assume Setting 2.9. Then $M < \infty$.

Proof of Lemma 2.10 Throughout this proof let $\omega_d \in (0, \infty)$ be the surface area of the $(d-1)$ -dimensional unit sphere and denote by $|\cdot|: \cup_{n \in \mathbb{N}} \mathbb{R}^n \rightarrow [0, \infty)$ the Euclidean norm. The definition of P and Theorem 8.1.2 in Lawler and Limic [18] (applied with $z = (x, y) \leftarrow (-x, -y, z)$ for $(z, x, y) \in \mathbb{N}_0 \times \mathbb{Z} \times \mathbb{Z}^{d-2}$ and combined with the definition of the Poisson kernel at the beginning of Section 8.1.1 in [18]) shows that there exist $s_1, s_2: \mathbb{Z} \times \mathbb{Z}^{d-2} \times \mathbb{N}_0 \rightarrow \mathbb{R}$, $c_1, c_2 \in (0, \infty)$ which satisfy for all $(x, y, z) \in \mathbb{Z} \times \mathbb{Z}^{d-2} \times \mathbb{N}$ that

$$\begin{aligned} P_z(x, y) &= \mathbb{P}[S_T = (x, y, 0) | S_0 = (0, 0, z)] = \mathbb{P}[S_T = (0, 0, 0) | S_0 = (-x, -y, z)] \\ &= \frac{2z(1 + s_1(x, y, z))}{\omega_d (|x|^2 + |y|^2 + |z|^2)^{d/2}} + s_2(x, y, z), \end{aligned} \quad (2.15)$$

$$\begin{aligned} |s_1(x, y, z)| &\leq \frac{c_1 z}{(|x|^2 + |y|^2 + |z|^2)} \quad \text{and} \\ |s_2(x, y, z)| &\leq \frac{c_2}{(|x|^2 + |y|^2 + |z|^2)^{(d+1)/2}}. \end{aligned} \quad (2.16)$$

The triangle inequality then implies for all $(x, y, z) \in \mathbb{Z} \times \mathbb{Z}^{d-2} \times \mathbb{N}$ that

$$|P_z(x, y) - P_z(x-1, y)| \leq \frac{2z}{\omega_d} \left| \frac{1}{(|x|^2 + |y|^2 + |z|^2)^{d/2}} - \frac{1}{(|x-1|^2 + |y|^2 + |z|^2)^{d/2}} \right| \\ + \sum_{\xi \in [x, x-1]} \left[\frac{2z|s_1(\xi, y, z)|}{\omega_d (|\xi|^2 + |y|^2 + |z|^2)^{d/2}} + |s_2(\xi, y, z)| \right]. \quad (2.17)$$

Next, Eq. 2.16 implies that for all $(\xi, y, z) \in \mathbb{Z} \times \mathbb{Z}^{d-2} \times \mathbb{N}$ it holds that

$$\frac{2z|s_1(\xi, y, z)|}{\omega_d (|\xi|^2 + |y|^2 + |z|^2)^{d/2}} + |s_2(\xi, y, z)| \\ \leq \frac{2c_1 z^2}{\omega_d (|\xi|^2 + |y|^2 + |z|^2)^{\frac{d}{2}+1}} + \frac{c_2}{(|\xi|^2 + |y|^2 + |z|^2)^{(d+1)/2}} \\ \leq \frac{2c_1 + \omega_d c_2}{\omega_d (|\xi|^2 + |y|^2 + |z|^2)^{d/2}}. \quad (2.18)$$

Furthermore, the mean value theorem and the fact that for all $a \in (0, \infty)$ the function $\mathbb{R} \ni \xi \mapsto 1/(\xi^2 + a) \in \mathbb{R}$ never attains local maxima on $\mathbb{R} \setminus \mathbb{Z}$ imply for all $(x, y, z) \in \mathbb{Z} \times \mathbb{Z}^{d-2} \times \mathbb{N}$ that

$$2z \left| \frac{1}{(x^2 + |y|^2 + |z|^2)^{d/2}} - \frac{1}{((x-1)^2 + |y|^2 + |z|^2)^{d/2}} \right| \\ \leq 2z \sup_{\xi \in [x-1, x]} \left| \frac{d}{d\xi} \left[(\xi^2 + |y|^2 + |z|^2)^{-d/2} \right] \right| \\ \leq 2z \sup_{\xi \in [x-1, x]} \left| -\frac{d}{2} (\xi^2 + |y|^2 + |z|^2)^{-\frac{d}{2}-1} \cdot 2\xi \right| \\ \leq \sup_{\xi \in [x-1, x]} \frac{2d}{(\xi^2 + |y|^2 + |z|^2)^{\frac{d}{2}}} \leq \sum_{\xi \in [x-1, x]} \frac{2d}{(\xi^2 + |y|^2 + |z|^2)^{\frac{d}{2}}}. \quad (2.19)$$

Combining this, Eqs. 2.17 and 2.18 we obtain

$$\sum_{(x, y) \in \mathbb{Z} \times \mathbb{Z}^{d-2}} |P_z(x, y) - P_z(x-1, y)| \leq \sum_{(x, y) \in \mathbb{Z} \times \mathbb{Z}^{d-2}} \sum_{\xi \in [x-1, x]} \frac{2d + 2c_1 + \omega_d c_2}{\omega_d (|\xi|^2 + |y|^2 + |z|^2)^{\frac{d}{2}}} < \infty. \quad (2.20)$$

The fact that $\sup_{z \in \mathbb{N}} (z \sum_{x \in \mathbb{Z}^{d-1}} (|x|^2 + z^2)^{-d/2}) < \infty$ and Eq. 2.14 then show that $M < \infty$ and complete the proof of Lemma 2.10. $\square$

Lemma 2.11 (weak L^1 -estimate) *Assume Settings 2.1 and 2.9 and let $L \in \mathbb{N}$, $u \in \mathbb{H}_{d, L, \geq 0}$, $p \in [1, \infty)$. Then it holds for all $t \in (0, \infty)$ that*

$$\left| \left\{ z \in \mathbb{N} : \sum_{y \in \mathbb{I}_L^{d-2}} \left| u(0, y, z) - \frac{1}{|\mathbb{I}_L|} \sum_{x \in \mathbb{I}_L} u(x, y, z) \right|^p > t^p \right\} \right| \leq \frac{M}{t} \sum_{x \in \mathbb{I}_L} \left[\sum_{y \in \mathbb{I}_L^{d-2}} |u(x, y, 0)|^p \right]^{1/p}. \quad (2.21)$$

Proof of Lemma 2.11 Throughout the proof let

$$A \in \mathbb{R}, \quad f: \mathbb{N} \rightarrow \mathbb{R}, \quad D_x^+ u: \mathbb{Z} \times \mathbb{Z}^{d-2} \times \mathbb{N}_0 \rightarrow \mathbb{K}, \quad D_x^- P_z: \mathbb{Z} \times \mathbb{Z}^{d-2} \rightarrow \mathbb{R} \quad (z \in \mathbb{N}_0) \quad (2.22)$$

which satisfy for all $x \in \mathbb{Z}$, $y \in \mathbb{Z}^{d-2}$, $z \in \mathbb{N}_0$ that

$$A = \sum_{x \in \mathbb{I}_L} \left[\sum_{y \in \mathbb{I}_L^{d-2}} |u(x, y, 0)|^p \right]^{1/p},$$

$$f(z) = \left[\sum_{y \in \mathbb{I}_L^{d-2}} \left| u(0, y, z) - \frac{1}{|\mathbb{I}_L|} \sum_{x \in \mathbb{I}_L} u(x, y, z) \right|^p \right]^{1/p}, \quad (2.23)$$

$$(\mathbf{D}_x^+ u)(x, y, z) = u(x+1, y, z) - u(x, y, z) \quad (2.24)$$

and

$$(\mathbf{D}_x^- P_z)(x, y) = P_z(x-1, y) - P_z(x, y). \quad (2.25)$$

The triangle inequality and a telescope sum argument then show for all $y \in \mathbb{I}_L^{d-2}$, $z \in \mathbb{N}$ that

$$\begin{aligned} \left| u(0, y, z) - \frac{1}{|\mathbb{I}_L|} \sum_{x \in \mathbb{I}_L} u(x, y, z) \right| &\leq \frac{1}{|\mathbb{I}_L|} \sum_{x \in \mathbb{I}_L} |u(0, y, z) - u(x, y, z)| \\ &\leq \frac{1}{|\mathbb{I}_L|} \sum_{x \in \mathbb{I}_L} \sum_{x' \in \mathbb{I}_L} |u(x'+1, y, z) - u(x', y, z)| = \sum_{x \in \mathbb{I}_L} |(\mathbf{D}_x^+ u)(x, y, z)|. \end{aligned} \quad (2.26)$$

This, Eq. 2.23, and the triangle inequality imply for all $z \in \mathbb{N}$ that

$$f(z) \leq \left[\sum_{y \in \mathbb{I}_L^{d-2}} \left[\sum_{x \in \mathbb{I}_L} |(\mathbf{D}_x^+ u)(x, y, z)|^p \right]^{1/p} \right]^{1/p} \leq \sum_{x \in \mathbb{I}_L} \left[\sum_{y \in \mathbb{I}_L^{d-2}} |(\mathbf{D}_x^+ u)(x, y, z)|^p \right]^{1/p}. \quad (2.27)$$

Furthermore, Lemma 2.4 shows for all $(x, y, z) \in \mathbb{Z} \times \mathbb{Z}^{d-2} \times ([1, N] \cap \mathbb{Z})$ that

$$\begin{aligned} u(x, y, z) &= \mathbb{E} \left[u(S_T + (x, y, 0)) \middle| S_0 = (0, z) \right] \\ &= \sum_{(\tilde{x}, \tilde{y}) \in \mathbb{Z} \times \mathbb{Z}^{d-2}} \mathbb{P} \left[S_T = (\tilde{x}, \tilde{y}, 0) \middle| S_0 = (0, z) \right] u(x + \tilde{x}, y + \tilde{y}, 0) \\ &= \sum_{(\tilde{x}, \tilde{y}) \in \mathbb{Z} \times \mathbb{Z}^{d-2}} P_z(\tilde{x}, \tilde{y}) u(x + \tilde{x}, y + \tilde{y}, 0). \end{aligned} \quad (2.28)$$

The substitution $\mathbb{Z} \ni \tilde{x} \mapsto \tilde{x} - 1 \in \mathbb{Z}$ then proves for all $(x, y, z) \in \mathbb{Z} \times \mathbb{Z}^{d-2} \times ([1, N] \cap \mathbb{Z})$ that

$$\begin{aligned} u(x+1, y, z) &= \sum_{(\tilde{x}, \tilde{y}) \in \mathbb{Z} \times \mathbb{Z}^{d-2}} P_z(\tilde{x}, \tilde{y}) u(x+1 + \tilde{x}, y + \tilde{y}, 0) \\ &= \sum_{(\tilde{x}, \tilde{y}) \in \mathbb{Z} \times \mathbb{Z}^{d-2}} P_z(\tilde{x}-1, \tilde{y}) u(x + \tilde{x}, y + \tilde{y}, 0). \end{aligned} \quad (2.29)$$

This, Eqs. 2.28, 2.24 and 2.25 yield for all $(x, y, z) \in \mathbb{Z} \times \mathbb{Z}^{d-2} \times ([1, N] \cap \mathbb{Z})$ that

$$(\mathbf{D}_x^+ u)(x, y, z) = \sum_{(\tilde{x}, \tilde{y}) \in \mathbb{Z}^{d-1}} (\mathbf{D}_x^- P_z)(\tilde{x}, \tilde{y}) u(\tilde{x} + x, \tilde{y} + y, 0). \quad (2.30)$$

This, the triangle inequality, and the fact that

$$\forall (\tilde{x}, \tilde{y}) \in \mathbb{Z} \times \mathbb{I}_L^{d-2}: \quad \sum_{y \in \mathbb{I}_L^{d-2}} |u(x, y, 0)|^p = \sum_{y \in \mathbb{I}_L^{d-2}} |u(x + \tilde{x}, y + \tilde{y}, 0)|^p, \quad (2.31)$$

which is a consequence of the periodicity, Eqs. 2.25, 2.24 and 2.14 imply for all $z \in \mathbb{N}$ that

$$\begin{aligned} f(z) &\leq \sum_{x \in \mathbb{I}_L} \left(\sum_{y \in \mathbb{I}_L^{d-2}} |(\mathbb{D}_x^- u)(x, y, z)|^p \right)^{1/p} = \sum_{x \in \mathbb{I}_L} \left(\sum_{y \in \mathbb{I}_L^{d-2}} \left| \sum_{(\tilde{x}, \tilde{y}) \in \mathbb{Z} \times \mathbb{Z}^{d-2}} (\mathbb{D}_x^- P)(z, \tilde{x}, \tilde{y}) u(x + \tilde{x}, y + \tilde{y}, 0) \right|^p \right)^{1/p} \\ &\leq \sum_{x \in \mathbb{I}_L} \sum_{\tilde{x} \in \mathbb{Z}} \sum_{\tilde{y} \in \mathbb{Z}^{d-2}} \left(\sum_{y \in \mathbb{I}_L^{d-2}} |(\mathbb{D}_x^- P)(z, \tilde{x}, \tilde{y}) u(x + \tilde{x}, y + \tilde{y}, 0)|^p \right)^{1/p} \\ &= \sum_{\tilde{x} \in \mathbb{Z}} \left[\sum_{\tilde{y} \in \mathbb{Z}^{d-2}} \left(|(\mathbb{D}_x^- P)(z, \tilde{x}, \tilde{y})| \sum_{x \in \mathbb{I}_L} \left[\sum_{y \in \mathbb{I}_L^{d-2}} |u(x + \tilde{x}, y + \tilde{y}, 0)|^p \right]^{1/p} \right) \right] \\ &= \left[\sum_{(\tilde{x}, \tilde{y}) \in \mathbb{Z} \times \mathbb{Z}^{d-2}} |P(z, \tilde{x}, \tilde{y}) - P(z, \tilde{x} - 1, \tilde{y})| \right] \sum_{x \in \mathbb{I}_L} \left[\sum_{y \in \mathbb{I}_L^{d-2}} |u(x, y, 0)|^p \right]^{1/p} \leq \frac{MA}{z}. \end{aligned} \quad (2.32)$$

This shows for all $t \in (0, \infty)$ that

$$t |\{z \in \mathbb{N}: |f(z)| > t\}| \leq t \left| \left\{ z \in \mathbb{N}: \left| \frac{MA}{z} \right| > t \right\} \right| = t |\{z \in \mathbb{N}: \left| \frac{MA}{t} \right| > z\}| \leq MA. \quad (2.33)$$

This and Eq. 2.23 complete the proof of Lemma 2.11. $\square$

Corollary 2.12 Assume Settings 2.1 and 2.9, let $L \in \mathbb{N}$, $u \in \mathbb{H}_{d,L,\geq 0}$, $\bar{r} \in (0, \infty)$, $p \in (1, \infty)$, $N \in \mathbb{N}$, and assume that $N/L \leq \bar{r}$. Then it holds that

$$\begin{aligned} &\left[\sum_{z \in \mathbb{N}} \sum_{y \in \mathbb{I}_L^{d-2}} \left| u(0, y, z) - \frac{1}{|\mathbb{I}_L|} \sum_{x \in \mathbb{I}_L} u(x, y, z) \right|^p \right]^{1/p} \\ &\leq 2 \left(\frac{Mp}{p-1} \right)^{1/p} 2^{1-\frac{1}{p}} \left[\sum_{x \in \mathbb{I}_L} \sum_{y \in \mathbb{I}_L^{d-2}} |u(x, y, 0)|^p \right]^{1/p} \end{aligned} \quad (2.34)$$

and

$$\begin{aligned} &\left[\sum_{z \in \mathbb{N}} \sum_{y \in \mathbb{I}_L^{d-2}} |u(0, y, z)|^p \right]^{1/p} \\ &\leq \left(2 \left(\frac{Mp}{p-1} \right)^{1/p} 2^{1-\frac{1}{p}} + (\bar{r}/2)^{1/p} \right) \left[\sum_{x \in \mathbb{I}_L} \sum_{y \in \mathbb{I}_L^{d-2}} |u(x, y, 0)|^p \right]^{1/p}. \end{aligned} \quad (2.35)$$

Proof of Corollary 2.12 An interpolation argument (see, e.g., Lemma A.3)) and Lemmas 2.7 and 2.11 imply (2.34). Next, the triangle inequality, (2.34), and Lemma 2.8 prove (2.35). This completes the proof of Corollary 2.12. $\square$

3 Fourier Analysis for Harmonic Functions on the Haft Space

3.1 Main Result

In this section we continue considering harmonic functions on the discrete haft space with periodic boundary conditions, however, from the viewpoint of Fourier analysis. The main results are summarized in Corollary 3.1 below, whose main part is illustrated by Fig. 2. As Bella, Fehrman, and Otto [3] we call the first inequality in Eq. 3.1 the Dirichlet case and the second inequality in Eq. 3.1 the Neumann case. In order to show Corollary 3.1 we combine Corollary 3.14 and, in particular, Corollary 3.14 (the Neumann case) and Corollary 3.29 (the Dirichlet case).

As in the proof in the continuum case [3] our proof is based on Marcinkiewicz-type multiplier theorems and the observation that the tangential derivatives and the normal derivatives of harmonic functions on the haft space are related by mean of Fourier multipliers. After having finished his dissertation [4], the author realized that for the argument with telescope sequences (see the paragraph below inequality (88) in [3]) it suffices to consider haft spaces instead of strips. The calculations here are therefore much simpler than that in [4]. However, we still have to overcome some tedious calculations with the discreteness when estimating the higher derivatives of the multipliers. Another issue is to adapt carefully the paragraph between (83) and (84) in [3] into the discrete case for which we have to work with the dyadic sets, see Section 3.4.

Corollary 3.1 *For every $L \in \mathbb{N}$, $d \in [2, \infty) \cap \mathbb{N}$ let $\mathbb{I}_L$ be the set given by $\mathbb{I}_L = [-L + 1, L] \cap \mathbb{Z}$ and let $\mathbb{H}_{d,L,\geq 0}$ be the set of all bounded functions $u: \mathbb{Z}^{d-1} \times \mathbb{N}_0 \rightarrow \mathbb{R}$ with the properties that*

- a) *it holds for all $x \in \mathbb{Z}^{d-1} \times \mathbb{N}_0$, $i \in [1, d-1] \cap \mathbb{Z}$ that $u(x) = u(x + 2Le_i^d)$ and*

Fig. 2 Estimate (3.1) in Corollary 3.1 bounds by means of L^p -norms, $p \in (1, \infty)$, the derivatives with respect to the vertical (blue) edges by that with respect to the horizontal (red) edges and vice versa

b) it holds for all $x \in \mathbb{Z}^{d-1} \times \mathbb{N}$ that $(\Delta u)(x) = 0$.

For every $L \in \mathbb{N}$, $d \in [2, \infty) \cap \mathbb{N}$, $u \in \mathbb{H}_{d,L,\geq 0}$ let $\mathbf{D}_i^+ u: \mathbb{Z}^{d-1} \times \mathbb{N}_0 \rightarrow \mathbb{R}$, $i \in [1, d] \cap \mathbb{Z}$, be the functions which satisfy for all $i \in [1, d] \cap \mathbb{Z}$, $x \in \mathbb{Z}^{d-1} \times \mathbb{N}_0$ that $(\mathbf{D}_i^+ u)(x) = u(x + \mathbf{e}_i^d) - u(x)$. Then there exist functions $C_1: ([2, \infty) \cap \mathbb{N}) \times (1, \infty) \times (0, \infty) \rightarrow (0, 1)$, $C_2: ([2, \infty) \cap \mathbb{N}) \times (1, \infty) \rightarrow (0, \infty)$ such that

- i) it holds for all $d \in [2, \infty) \cap \mathbb{N}$, $p \in (1, \infty)$, $N, L \in \mathbb{N}$, $u \in \mathbb{H}_{d,L,\geq 0}$ with $N/L \geq r$ and $\sum_{x \in \mathbb{I}_L^{d-1}} u(x) = 0$ that $\|u\|_{L^p(\mathbb{I}_L^{d-1} \times \{N\})} \leq C_1(d, p, r) \|u\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})}$ and
- ii) it holds for all $d \in [2, \infty) \cap \mathbb{N}$, $p \in (1, \infty)$, $L \in \mathbb{N}$, $u \in \mathbb{H}_{d,L,\geq 0}$ that

$$\begin{aligned} \frac{1}{C_2(d, p)} \|\mathbf{D}_d^+ u\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})} &\leq \left[\sum_{i=1}^{d-1} \|\mathbf{D}_i^+ u\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})} \right] \\ &\leq C_2(d, p) \|\mathbf{D}_d^+ u\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})}. \end{aligned} \quad (3.1)$$

3.2 Notation and Settings

Instead of $\mathbb{Z}^d$, $d \in \mathbb{N}$, we will work with $h\mathbb{Z}^d$, $h \in \pi/\mathbb{N}$, $d \in \mathbb{N}$. In fact, our notation in Setting 3.2 is inspired by Jovanović and Süli [19, Section 2.5] so that we can easily use the Marcinkiewicz multiplier therein. To make the notation consistent we introduce Setting 3.3.

Setting 3.2 (Periodic functions and L_h^p -norms) For every $N \in \mathbb{N}$ let $\mathbb{I}_N$ be the set given by $\mathbb{I}_N = [-N + 1, N] \cap \mathbb{Z}$. For every $h \in \pi/\mathbb{N}$ let ω_h be the set given by

$$\omega_h = h\mathbb{I}_{\pi/h} = h \left(\left[-\left(\frac{\pi}{h} - 1\right), \frac{\pi}{h} \right] \cap \mathbb{Z} \right) = \left\{ -h\left(\frac{\pi}{h} - 1\right), -h\left(\frac{\pi}{h} - 2\right), \dots, h\left(\frac{\pi}{h} - 1\right), h\frac{\pi}{h} \right\}. \quad (3.2)$$

For every $N \in \mathbb{N}$, $d \in \mathbb{N}$ let $\mathcal{P}_{2N}(\mathbb{Z}^d, \mathbb{C})$ be the set of all $(2N)$ -periodic functions on $\mathbb{Z}^d$, i.e.,

$$\mathcal{P}_{2N}(\mathbb{Z}^d, \mathbb{C}) = \left\{ a: \mathbb{Z}^d \rightarrow \mathbb{C}: \forall k \in \mathbb{Z}^d, i \in [1, d] \cap \mathbb{Z}: a(k) = a(k + 2N\mathbf{e}_i^d) \right\}. \quad (3.3)$$

For every $d \in \mathbb{N}$, $h \in \pi/\mathbb{N}$ let $h\mathbb{Z}^d$ be the lattice given by $h\mathbb{Z}^d = \{hx: x \in \mathbb{Z}^d\}$ and let $\mathcal{P}_{2\pi}(h\mathbb{Z}^d, \mathbb{C})$ be the set of all 2π -periodic functions defined on $h\mathbb{Z}^d$, i.e.,

$$\mathcal{P}_{2\pi}(h\mathbb{Z}^d, \mathbb{C}) = \left\{ v: h\mathbb{Z}^d \rightarrow \mathbb{C}: \forall x \in h\mathbb{Z}^d, i \in [1, d] \cap \mathbb{Z}: v(x) = v(x + 2\pi\mathbf{e}_i^d) \right\}. \quad (3.4)$$

For every $d \in \mathbb{N}$, $p \in [1, \infty)$, $h \in \pi/\mathbb{N}$, $f \in \mathcal{P}_{2\pi}(h\mathbb{Z}^d, \mathbb{C})$ let $\|f\|_{L_h^p(\omega_h^d)} \in \mathbb{R}$ be the real number which satisfies that

$$\|f\|_{L_h^p(\omega_h^d)} = \left[h^d \sum_{x \in \omega_h^d} |f(x)|^p \right]^{1/p}, \quad (3.5)$$

which is distinguished from $\|\cdot\|_{L^p(A)}$ in Eq. 1.4 by a normalized factor h^d . Denote by $\mathcal{F}$ the so-called *discrete Fourier transform*, i.e.,

$$\mathcal{F}: \left[\bigcup_{d \in \mathbb{N}, h \in \pi/\mathbb{N}} \mathcal{P}_{2\pi}(h\mathbb{Z}^d, \mathbb{C}) \right] \rightarrow \left[\bigcup_{d, N \in \mathbb{N}} \mathcal{P}_{2N}(\mathbb{Z}^d, \mathbb{C}) \right] \quad (3.6)$$

is the operator which satisfies for every $d \in \mathbb{N}$, $h \in \pi/\mathbb{N}$, $v \in \mathcal{P}_{2\pi}(h\mathbb{Z}^d, \mathbb{C})$, $k \in \mathbb{Z}^d$ that

$$\mathcal{F}(v) \in \mathcal{P}_{2\pi/h}(\mathbb{Z}^d, \mathbb{C}) \quad \text{and} \quad (\mathcal{F}(v))(k) = h^d \sum_{x \in \omega_h^d} v(x) e^{-ik \cdot x}. \quad (3.7)$$

Setting 3.3 (Discrete Laplacian, finite differences, and harmonic functions) For every $d \in [2, \infty) \cap \mathbb{N}$, $h \in \pi/\mathbb{N}$, $u: (h\mathbb{Z}^{d-1}) \times (h\mathbb{N}_0) \rightarrow \mathbb{K}$ let $\Delta^h: (h\mathbb{Z}^{d-1}) \times (h\mathbb{N}) \rightarrow \mathbb{K}$ (recall: $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$) be the *discrete Laplacian* with mesh h , i.e., the function which satisfies for all $(x, y) \in (h\mathbb{Z}^{d-1}) \times (h\mathbb{N})$ that

$$\begin{aligned} &(\Delta^h u)(x, y) \\ &= \left[\sum_{i=1}^{d-1} (u(x + h\mathbf{e}_i^{d-1}, y) + u(x - h\mathbf{e}_i^{d-1}, y)) \right] + u(x, y+h) + u(x, y-h) - 2du(x, y), \end{aligned} \quad (3.8)$$

let $D_y^h u, D_{x,i}^h u: (h\mathbb{Z}^{d-1}) \times (h\mathbb{N}_0) \rightarrow \mathbb{K}$, $i \in [1, d-1] \cap \mathbb{Z}$, be the functions which satisfy for all $(x, y) \in (h\mathbb{Z}^{d-1}) \times (h\mathbb{N}_0)$, $i \in [1, d-1] \cap \mathbb{Z}$ that

$$\begin{aligned} (D_{x,i}^h u)(x, y) &= \frac{u(x + h\mathbf{e}_i^{d-1}, y) - u(x, y)}{h} \quad \text{and} \\ (D_y^h u)(x, y) &= \frac{u(x, y+1) - u(x, y)}{h}, \end{aligned} \quad (3.9)$$

and we write $D_x^h u = (D_{x,1}^h u, \dots, D_{x,d-1}^h u): (h\mathbb{Z}^{d-1}) \times (h\mathbb{N}_0) \rightarrow \mathbb{K}^{d-1}$. For every $h \in \pi/\mathbb{N}$, $d \in [2, \infty) \cap \mathbb{N}$ let $\mathbb{H}_{d,h,\geq 0}$ be the set of all bounded functions $u: (h\mathbb{Z}^{d-1}) \times (h\mathbb{N}_0) \rightarrow \mathbb{K}$ which satisfy that

- i) it holds for all $(x, y) \in (h\mathbb{Z}^{d-1}) \times (h\mathbb{N}_0)$, $i \in [1, d-1] \cap \mathbb{Z}$ that $u(x, y) = u(x + 2\pi\mathbf{e}_i^{d-1}, y)$ and
- ii) it holds for all $(x, y) \in (h\mathbb{Z}^{d-1}) \times (h\mathbb{N})$ that $(\Delta^h u)(x, y) = 0$.

In order to obtain Corollary 3.14 using a Riesz-Thorin interpolation argument we choose $\mathbb{K} = \mathbb{C}$ in Setting 3.3 above. For other results we only need $\mathbb{K} = \mathbb{R}$.

3.3 Some Simple Calculations

The main results of this subsection, Corollaries 3.11 and 3.12, prove that the discrete normal and tangential derivatives are related by means of Fourier multipliers. We start with Setting 3.4 below that defines the functions which are used to represent the Fourier transform of harmonic functions and their discrete derivatives. It is useful to consider Q and f in Eq. 3.10 as functions of a complex variable. The names Q and λ are inspired by Guadie [20] who considers harmonic functions on infinite strips with $L^2(\mathbb{Z}^{d-1})$ boundary conditions.

Setting 3.4 Let $d \in [2, \infty) \cap \mathbb{N}$ be fixed, let $R \in \mathbb{C}(\mathbb{C} \setminus (\infty, 0), \mathbb{C})$ be the complex square root, i.e., the function that is holomorphic on $\mathbb{C} \setminus (-\infty, 0] \rightarrow \mathbb{C}$ and satisfies for all $z \in \mathbb{C} \setminus (-\infty, 0)$ that $R(z)^2 = z$ (cf. Lemma A.1), let $Q, f: \mathbb{C} \setminus (-\infty, 1) \rightarrow \mathbb{C}$ be the functions which satisfy for all $z \in \mathbb{C} \setminus (-\infty, 1)$ that

$$Q(z) = z + R(z+1)R(z-1), \quad Q(z) \neq 0, \quad \text{and} \quad f(z) = \frac{1}{Q(z)} - 1, \quad (3.10)$$

let $\lambda: [-\pi, \pi]^{d-1} \rightarrow \mathbb{R}$, $\mathcal{D}_i, \mathcal{N}_i: [-\pi, \pi]^{d-1} \rightarrow \mathbb{C}$, $i \in [1, d-1] \cap \mathbb{Z}$, be the functions which satisfy for all $t \in [-\pi, \pi]^{d-1}$, $i \in [1, d-1] \cap \mathbb{Z}$ that

$$\lambda(t) = d - \sum_{i=1}^{d-1} \cos(t_i), \quad \mathcal{D}_i(t) = \begin{cases} \frac{f(\lambda(t))}{e^{-it_i} - 1} & : t_i \neq 0 \\ 0 & : t_i = 0, \end{cases} \quad \text{and} \\ \mathcal{N}_i(t) = \begin{cases} \frac{e^{-it_i} - 1}{f(\lambda(t))} & : t \neq 0 \\ 0 & : t = 0, \end{cases} \quad (3.11)$$

let $\mathcal{D}_i^h, \mathcal{N}_i^h: [-\pi/h, \pi/h]^{d-1} \rightarrow \mathbb{C}$, $i \in [1, d-1] \cap \mathbb{Z}$, $h \in (0, \infty)$, be the functions which satisfy for all $i \in [1, d-1] \cap \mathbb{Z}$, $h \in (0, \infty)$, $\xi \in [-\pi/h, \pi/h]^{d-1}$ that

$$\mathcal{D}_i^h(\xi) = \mathcal{D}_i(h\xi) \quad \text{and} \quad \mathcal{N}_i^h(\xi) = \mathcal{N}_i(h\xi), \quad (3.12)$$

and let $c \in (0, \infty)$ be the real number (cf. Lemma A.2) which satisfies that $c = \inf_{s \in [-\pi, \pi] \setminus \{0\}} \frac{1 - \cos(s)}{s^2}$.

Lemma 3.5 Assume Setting 3.4. Then it holds for all $t \in [-\pi, \pi]^{d-1}$ that

$$c|t|_\infty^2 \leq \sum_{i=1}^{d-1} c|t_i|^2 \leq |\lambda(t) - 1| \leq \frac{1}{2} \sum_{i=1}^{d-1} |t_i|^2 \leq \frac{1}{2}(d-1)|t|_\infty^2. \quad (3.13)$$

Proof of Lemma 3.5 The fact that $\forall s \in [-\pi, \pi]: cs^2 \leq 1 - \cos(s) \leq s^2/2$ and the definition of λ in Eq. 3.11 complete the proof of Lemma 3.5. $\square$

Lemma 3.6 Assume Setting 3.4 and let $z \in \mathbb{C} \setminus (-\infty, 1)$. Then it holds that

$$Q(z) + \frac{1}{Q(z)} = 2z \quad \text{and} \quad f(z)^2 = \frac{2(z-1)}{Q(z)}. \quad (3.14)$$

Proof of Lemma 3.6 First, Eq. 3.10, the assumption that $\forall \zeta \in \mathbb{C} \setminus (-\infty, 0): R(\zeta)^2 = \zeta$, and the assumption that $z \in \mathbb{C} \setminus (-\infty, 1)$ prove that

$$\begin{aligned} Q(z) \left(z - R(z+1)R(z-1) \right) &= \left(z + R(z+1)R(z-1) \right) \left(z - R(z+1)R(z-1) \right) \\ &= z^2 - R(z+1)^2 R(z-1)^2 \\ &= z^2 - (z+1)(z-1) = 1. \end{aligned} \quad (3.15)$$

Therefore, $1/Q(z) = z - R(z+1)R(z-1)$. This and Eq. 3.10 show that $Q(z) + \frac{1}{Q(z)} = 2z$. Multiplying with $Q(z)$ yields that $Q(z)^2 - 2zQ(z) + 1 = 0$. This and Eq. 3.10 show that

$$\begin{aligned} f(z)^2 &= \left(\frac{1 - Q(z)}{Q(z)} \right)^2 = \frac{(Q(z)^2 - 2zQ(z) + 1) + 2Q(z)(z-1)}{Q(z)^2} \\ &= \frac{2Q(z)(z-1)}{Q(z)^2} = \frac{2(z-1)}{Q(z)}. \end{aligned} \quad (3.16)$$

The proof of Lemma 3.6 is thus completed. $\square$

Lemma 3.7 below is a classical result and is included for convenience of the reader.

Lemma 3.7 (Plancherel's identity) *Assume Setting 3.2. Let $d \in \mathbb{N}$, $h \in \pi/\mathbb{N}$. Then*

$$\sum_{k \in \mathbb{I}_{\pi/h}^d} |(\mathcal{F}(v))(k)|^2 = (2\pi h)^d \sum_{x, y \in \omega_h^d} |v(x)|^2. \quad (3.17)$$

Proof of Lemma 3.7 The fact that $\forall x, y \in \omega_h^d: \sum_{k \in \mathbb{I}_{\pi/h}^d} e^{-ik \cdot (x-y)} = \delta_{xy} (2\pi/h)^d$ implies that

$$\begin{aligned} \sum_{k \in \mathbb{I}_{\pi/h}^d} |(\mathcal{F}(v))(k)|^2 &= \sum_{k \in \mathbb{I}_{\pi/h}^d} (\mathcal{F}(v))(k) \overline{(\mathcal{F}(v))(k)} \\ &= \sum_{k \in \mathbb{I}_{\pi/h}^d} \left(h^d \sum_{x \in \omega_h^d} v(x) e^{-ik \cdot x} \right) \left(h^d \sum_{y \in \omega_h^d} \overline{v(y)} e^{ik \cdot y} \right) = h^{2d} \left[\sum_{x, y \in \omega_h^d} v(x) \overline{v(y)} \sum_{k \in \mathbb{I}_{\pi/h}^d} e^{-ik \cdot (x-y)} \right] \\ &= h^{2d} \left[\sum_{x, y \in \omega_h^d} v(x) \overline{v(y)} (2\pi/h)^d \delta_{xy} \right] = (2\pi)^d \left[h^d \sum_{x, y \in \omega_h^d} |v(x)|^2 \right]. \end{aligned} \quad (3.18)$$

This completes the proof of Lemma 3.7. $\square$

Lemma 3.8 is straightforward and its proof is therefore omitted.

Lemma 3.8 *Assume Setting 3.2. Let $d \in \mathbb{N}$, $h \in \pi/\mathbb{N}$, $f \in \mathcal{P}_{2\pi}(h\mathbb{Z}^d, \mathbb{C})$, $k \in \mathbb{I}_{\pi/h}^d$, $i \in [1, d] \cap \mathbb{Z}$. Then*

- i) *it holds that $[\mathcal{F}(f(\cdot + h\mathbf{e}_i^d))](k) = [\mathcal{F}(f)](k) e^{-ihk_i}$,*
- ii) *it holds that $[\mathcal{F}(f(\cdot - h\mathbf{e}_i^d))](k) = [\mathcal{F}(f)](k) e^{+ihk_i}$, and*
- iii) *it holds that $[\mathcal{F}(f(\cdot + h\mathbf{e}_i^d)) + (f(\cdot - h\mathbf{e}_i^d))](k) = 2[\mathcal{F}(f)](k) \cos(hk_i)$.*

Lemma 3.9 (Fourier transform of the solution) *Assume Settings 3.2 and 3.4. Let $h \in \pi/\mathbb{N}$, $u \in \mathbb{H}_{d,h,\geq 0}$. Then it holds for all $k \in \mathbb{Z}^{d-1}$, $n \in \mathbb{N}_0$ that*

$$[\mathcal{F}(u(\cdot, nh))](k) = ((Q \circ \lambda)(hk))^{-n} [\mathcal{F}(u(\cdot, 0))](k). \quad (3.19)$$

Proof of Lemma 3.9 Throughout the proof let $v : (h\mathbb{Z}^{d-1}) \times (h\mathbb{N}_0) \rightarrow \mathbb{C}$ be the function which satisfies for all $k \in \mathbb{Z}^{d-1}$, $(x, y) \in (h\mathbb{Z}^{d-1}) \times (h\mathbb{N}_0)$, $i \in [1, d-1] \cap \mathbb{Z}$, $n \in \mathbb{N}$ that

$$\begin{aligned} v(x, y) &= v(x + 2\pi \mathbf{e}_i^{d-1}, y) \quad \text{and} \quad [\mathcal{F}(v(\cdot, nh))](k) \\ &= ((Q \circ \lambda)(hk))^{-n} [\mathcal{F}(u(\cdot, 0))](k), \end{aligned} \quad (3.20)$$

defined through its Fourier transform, and let $\tilde{v}, \tilde{u} : \mathbb{Z}^{d-1} \times \mathbb{N}_0 \rightarrow \mathbb{C}$ be the function given by

$$\forall (x, y) \in \mathbb{Z}^{d-1} \times \mathbb{N}_0: \quad \tilde{v}(x, y) = v(hx, hy) \quad \text{and} \quad \tilde{u}(x, y) = u(hx, hy). \quad (3.21)$$

Lemma 3.6 then proves for all $k \in \mathbb{I}_{2\pi/h}^{d-1}$, $n \in \mathbb{N}$ that

$$\begin{aligned} &[\mathcal{F}(v(\cdot, (n+1)h))](k) + [\mathcal{F}(v(\cdot, (n-1)h))](k) \\ &= ((Q \circ \lambda)(hk))^{-(n+1)} [\mathcal{F}(u(\cdot, 0))](k) + ((Q \circ \lambda)(hk))^{-(n-1)} [\mathcal{F}(u(\cdot, 0))](k) \\ &= 2\lambda(hk) ((Q \circ \lambda)(hk))^{-n} \mathcal{F}(u(\cdot, 0)) = 2\lambda(hk) [\mathcal{F}(v(\cdot, nh))](k). \end{aligned} \quad (3.22)$$

Lemma 3.8 (with $d \leftarrow d - 1$, $f \leftarrow v(\cdot, nh)$) for $n \in \mathbb{N}$, $k \in \mathbb{I}_{\pi/h}^{d-1}$, $i \in [1, d - 1] \cap \mathbb{Z}$) then shows for all $k \in \mathbb{I}_{\pi/h}^{d-1}$, $n \in \mathbb{N}$ that

$$\begin{aligned} & [\mathcal{F}((\Delta^h v)(\cdot, nh))](k) \\ &= \left[\sum_{i=1}^{d-1} 2[\mathcal{F}(v(\cdot, nh))](k) \cos(hk_i) \right] + [\mathcal{F}(v(\cdot, (n+1)h))](k) \\ &\quad + [\mathcal{F}(v(\cdot, (n-1)h))](k) - 2d[\mathcal{F}(v(\cdot, nh))](k) \\ &= [\mathcal{F}(v(\cdot, (n+1)h))](k) + [\mathcal{F}(v(\cdot, (n-1)h))](k) \\ &\quad - 2\lambda(hk)[\mathcal{F}(v(\cdot, nh))](k) = 0 \end{aligned} \quad (3.23)$$

This proves for all $(x, y) \in (h\mathbb{Z}^{d-1}) \times (h\mathbb{N})$ that $(\Delta^h v)(x, y) = 0$. A scaling argument and Eq. 3.21 then yield for all $(x, y) \in \mathbb{Z}^{d-1} \times \mathbb{N}$ that $\Delta \tilde{v}(x, y) = 0$. Furthermore, Lemma 3.7 and the fact that $\forall t \in [-\pi, \pi]^{d-1}$: $Q(\lambda(t)) \geq \lambda(t) \geq 1$ imply for all $N \in \mathbb{N}_0$ that

$$\begin{aligned} & (2\pi h)^{d-1} \sup_{x \in \omega_h^{d-1}} |v(x, Nh)|^2 \leq (2\pi h)^{d-1} \sum_{x \in \omega_h^{d-1}} |v(x, Nh)|^2 \\ &= \sum_{k \in \mathbb{I}_{\pi/h}^{d-1}} |[\mathcal{F}(v(\cdot, Nh))](k)|^2 \\ &= \sum_{k \in \mathbb{I}_{\pi/h}^{d-1}} \left| Q^{-N}(\lambda(kh))[\mathcal{F}(v(\cdot, 0))](k) \right|^2 \leq \sum_{k \in \mathbb{I}_{\pi/h}^{d-1}} |[\mathcal{F}(v(\cdot, 0))](k)|^2 \\ &= (2\pi h)^{d-1} \sum_{x \in \omega_h^{d-1}} |v(x, 0)|^2. \end{aligned} \quad (3.24)$$

Hence, v is a bounded functions. This and Eq. 3.21 imply that $\tilde{v}$ is bounded. Moreover, the fact that $\forall (x, y) \in (h\mathbb{Z}^{d-1}) \times (h\mathbb{N})$: $(\Delta^h u)(x, y) = 0$, Eqs. 3.8 and 3.21, and a scaling argument prove that $\forall (x, y) \in \mathbb{Z}^{d-1} \times \mathbb{N}$: $\Delta \tilde{u}(x, y) = 0$. This, the assumption that u is bounded, the fact that $\forall (x, y) \in \mathbb{Z}^{d-1} \times \mathbb{N}$: $(\Delta \tilde{v})(x, y) = (\Delta \tilde{u})(x, y) = 0$, the fact that $\tilde{v}$ is bounded, Lemma 2.4 (with $u \leftarrow \tilde{v}$ and $u \leftarrow \tilde{u}$), and the fact that $\forall x \in (h\mathbb{Z}^{d-1}) \times \{0\}$: $\tilde{v}(x) = \tilde{u}(x)$ (see Eqs. 3.20 and 3.21) ensure that $\tilde{u} = \tilde{v}$. This and Eq. 3.21 imply that $u = v$. Combining this with Eq. 3.20 we complete the proof. $\square$

Lemma 3.10 Assume Settings 3.2-3.4 and let $k \in \mathbb{I}_{\pi/h}^{d-1}$, $i \in [1, d - 1] \cap \mathbb{Z}$, $h \in \pi/\mathbb{N}$, $u \in \mathbb{H}_{d,h,\geq 0}$. Then

- i) it holds that $[\mathcal{F}(\mathcal{D}_{x,i}^h u)(\cdot, 0)](k) = h^{-1}[\mathcal{F}(u(\cdot, 0))](k)(e^{-ihk_i} - 1)$ and
- ii) it holds that $[\mathcal{F}(\mathcal{D}_y^h u)(\cdot, 0)](k) = h^{-1}(Q(\lambda(hk))^{-1} - 1)[\mathcal{F}(u(\cdot, 0))](k)$.

Proof of Lemma 3.10 Observe that Eq. 3.9, Lemma 3.8 (with $d \leftarrow d - 1$, $f \leftarrow u(\cdot, 0)$), and the fact that $u(\cdot, 0) \in \mathcal{P}_{2\pi}(h\mathbb{Z}^{d-1}, \mathbb{C})$ imply that

$$\begin{aligned} & [(\mathcal{F}(\mathcal{D}_{x,i}^h u)(\cdot, 0))](k) = h^{-1}[\mathcal{F}(u(\cdot + h\mathbf{e}_i^{d-1}, 0)) - \mathcal{F}(u(\cdot, 0))](k) \\ &= h^{-1}[\mathcal{F}(u(\cdot, 0))](k)(e^{-ihk_i} - 1) \end{aligned}$$

and

$$\begin{aligned} [\mathcal{F}((\mathcal{D}_y^h u)(\cdot, 0))](k) &= h^{-1} [\mathcal{F}(u(\cdot, h) - u(\cdot, 0))](k) \\ &= h^{-1} \left(Q(\lambda(hk))^{-1} - 1 \right) [\mathcal{F}(u(\cdot, 0))](k). \end{aligned}$$

The proof of Lemma 3.10 is thus completed. $\square$

Corollary 3.11 (Multipliers in the Neumann case) *Assume Settings 3.2-3.4 and let $h \in \pi/\mathbb{N}$, $u \in \mathbb{H}_{d,h,\geq 0}$, $k \in \mathbb{I}_{\pi/h}^{d-1}$, $i \in [1, d-1] \cap \mathbb{Z}$. Then it holds that $[\mathcal{F}((\mathcal{D}_{x,i}^h u)(\cdot, 0))](k) = \mathcal{N}_i^h(k) [\mathcal{F}((\mathcal{D}_y^h u)(\cdot, 0))](k)$.*

Proof of Corollary 3.11 Lemma 3.10 and Eq. 3.11 prove that in the case $k = 0$ it holds that

$$[\mathcal{F}((\mathcal{D}_{x,i}^h u)(\cdot, 0))](k) = [\mathcal{F}((\mathcal{D}_y^h u)(\cdot, 0))](k) = \mathcal{N}_i(hk) = 0 \quad (3.25)$$

and in the case $k \neq 0$ it holds that $Q(\lambda(hk)) \neq 1$ and

$$\begin{aligned} [\mathcal{F}((\mathcal{D}_{x,i}^h u)(\cdot, 0))](k) &= h^{-1} [\mathcal{F}(u(\cdot, 0))](k) (e^{-ihk_i} - 1) \\ &= \frac{e^{-ihk_i} - 1}{Q(\lambda(hk))^{-1} - 1} h^{-1} \left(Q(\lambda(hk))^{-1} - 1 \right) [\mathcal{F}(u(\cdot, 0))](k) \\ &= \frac{e^{-ihk_i} - 1}{Q(\lambda(hk))^{-1} - 1} [\mathcal{F}((\mathcal{D}_y^h u)(\cdot, 0))](k) \\ &= \mathcal{N}_i^h(k) [\mathcal{F}((\mathcal{D}_{x,i}^h u)(\cdot, 0))](k). \end{aligned} \quad (3.26)$$

This completes the proof of Corollary 3.11. $\square$

In Corollary 3.12 below we see that in the Dirichlet case there are $(d-1)$ multipliers, which are the quotients $\mathcal{F}((\mathcal{D}_y^h u)(\cdot, 0))/\mathcal{F}((\mathcal{D}_{x,i}^h u)(\cdot, 0))$, $i \in [1, d-1] \cap \mathbb{Z}$, and therefore not everywhere defined. Fortunately, we can still show that for each dyadic rectangle there is a multiplier well-defined on it. In Section 3.4 we will develop a Marcinkiewicz-type multiplier theorem to deal with this situation.

Corollary 3.12 (Multipliers in the Dirichlet case) *Assume Settings 3.2-3.4, let $h \in \pi/\mathbb{N}$, $u \in \mathbb{H}_{d,h,\geq 0}$, $k \in \mathbb{Z}^{d-1} \setminus \{0\}$, $i \in [1, d-1] \cap \mathbb{Z}$, $v \in \prod_{j=1}^d (D(k_j) \cap \mathbb{I}_{\pi/h})$, assume that $k_i \neq 0$, and let $D(\ell) \subseteq \mathbb{R}$, $\ell \in \mathbb{Z}$, be the intervals given by*

$$\forall \ell \in \mathbb{N}: D(\ell) = [2^{\ell-1}, 2^\ell], \quad D(0) = (-1, 1), \quad \forall \ell \in (-\mathbb{N}): D(\ell) = (-2^{|\ell|}, -2^{|\ell|-1}]. \quad (3.27)$$

Then

- i) it holds that $[\mathcal{F}((\mathcal{D}_y^h u)(\cdot, 0))](0) = [\mathcal{F}((\mathcal{D}_{x,1}^h u)(\cdot, 0))](0) = \dots = [\mathcal{F}((\mathcal{D}_{x,d-1}^h u)(\cdot, 0))](0) = 0$ and
- ii) it holds that $[\mathcal{F}((\mathcal{D}_y^h u)(\cdot, 0))](v) = \mathcal{D}_i^h(v) [\mathcal{F}((\mathcal{D}_{x,i}^h u)(\cdot, 0))](v)$.

Proof of Corollary 3.12 First, note that Lemma 3.10 implies Item (i). Next, observe that Eq. 3.27, the assumption that $k_i \neq 0$, and the assumption that $v_i \in D(k_i) \cap \mathbb{I}_{\pi/h}$ prove that $e^{-ihv_i} - 1 \neq 0$. Lemma 3.10 and Eq. 3.11 therefore show that

$$\begin{aligned} [\mathcal{F}((\mathcal{D}_y^h u)(\cdot, 0))](v) &= h^{-1} \left(Q(\lambda(hk))^{-1} - 1 \right) [\mathcal{F}(u(\cdot, 0))](k) \\ &= \frac{(Q(\lambda(hk))^{-1} - 1) h^{-1} [\mathcal{F}(u(\cdot, 0))](k) (e^{-ihv_i} - 1)}{(e^{-ihv_i} - 1)} = \frac{(Q(\lambda(hk))^{-1} - 1) [\mathcal{F}((\mathcal{D}_{x,i}^h u)(\cdot, 0))](v)}{(e^{-ihv_i} - 1)} \\ &= \mathcal{D}_i^h(v) [\mathcal{F}((\mathcal{D}_{x,i}^h u)(\cdot, 0))](v). \end{aligned} \quad (3.28)$$

This completes the proof of Corollary 3.12. $\square$

Lemma 3.13 Assume Settings 3.2-3.4, let $h \in \pi/\mathbb{N}$, $u \in \mathbb{H}_{d,h,\geq 0}$, let $\underline{r} \in [0, \infty)$, $N \in \mathbb{N}$ satisfy that $Nh \geq \underline{r}$, and assume that $\sum_{x \in \omega_h^{d-1}} u(x, 0) = 0$. Then

- i) it holds for all $k \in \mathbb{I}_{\pi/h}^{d-1} \setminus \{0\}$ that $Q^{-N}(\lambda(kh)) \leq (1 + \sqrt{c\underline{r}})^{-1}$ and
- ii) it holds that $\|u(\cdot, Nh)\|_{L_h^2(\omega_h^{d-1})} \leq (1 + \sqrt{c\underline{r}})^{-1} \|u(\cdot, 0)\|_{L_h^2(\omega_h^{d-1})}$.

Proof of Lemma 3.13 Observe that Eq. 3.11 and Lemma 3.5 show for all $t \in [-\pi, \pi]^{d-1}$ that

$$Q(\lambda(t)) - 1 = \lambda(t) - 1 + \sqrt{\lambda(t)^2 - 1} \geq \sqrt{\lambda(t) - 1} \geq \sqrt{c}|t|_\infty. \quad (3.29)$$

This, Bernoulli's inequality, and the assumption that $Nh \geq \underline{r}$ show for all $k \in \mathbb{I}_L^{d-1} \setminus \{0\}$ that

$$\frac{1}{Q^N(\lambda(kh))} \leq \frac{1}{(1 + \sqrt{c}|kh|_\infty)^N} \leq \frac{1}{1 + \sqrt{c}N|kh|_\infty} \leq \frac{1}{1 + \sqrt{c}|k|_\infty \underline{r}} \leq \frac{1}{1 + \sqrt{c}\underline{r}}. \quad (3.30)$$

This proves Item (i). Observe that Eq. 3.7 and the assumption that $\sum_{x \in \omega_h^{d-1}} u(x, 0) = 0$ imply that $[\mathcal{F}(u(\cdot, 0))](0) = 0$. The Plancherel identity (for details see Lemma 3.7), Lemma 3.9, and Eq. 3.30 hence demonstrate that

$$\begin{aligned} (2\pi h)^{d-1} \sum_{x \in \omega_h^{d-1}} |u(x, Nh)|^2 &= \sum_{k \in \mathbb{I}_{\pi/h}^{d-1}} |[\mathcal{F}(u(\cdot, Nh))](k)|^2 \\ &= \sum_{k \in \mathbb{I}_{\pi/h}^{d-1}} \left| Q^{-N}(\lambda(kh)) [\mathcal{F}(u(\cdot, 0))](k) \right|^2 \\ &= \sum_{k \in \mathbb{I}_{\pi/h}^{d-1} \setminus \{0\}} \left| Q^{-N}(\lambda(kh)) [\mathcal{F}(u(\cdot, 0))](k) \right|^2 \leq (1 + \sqrt{c}\underline{r})^{-2} \sum_{k \in \mathbb{I}_{\pi/h}^{d-1} \setminus \{0\}} |[\mathcal{F}(u(\cdot, 0))](k)|^2 \\ &= (1 + \sqrt{c}\underline{r})^{-2} \sum_{k \in \mathbb{I}_{\pi/h}^{d-1}} |[\mathcal{F}(u(\cdot, 0))](k)|^2 \\ &= (1 + \sqrt{c}\underline{r})^{-2} (2\pi h)^{d-1} \sum_{x \in \omega_h^{d-1}} |u(x, 0)|^2. \end{aligned} \quad (3.31)$$

This and Eq. 3.5 imply Item (ii). The proof of Lemma 3.13 is thus completed. $\square$

Combining Lemmas 3.13 and 2.5, a scaling argument, and a Riesz-Thorin-type interpolation argument we obtain the following result, Corollary 3.14. For later use we only need the fact that the multiplicative constants do not depend on N .

Corollary 3.14 Assume Settings 3.2 and 3.3. Let $h \in \pi/\mathbb{N}$, $u \in \mathbb{H}_{d,h,\geq 0}$, let $\underline{r} \in [0, \infty)$, $N \in \mathbb{N}$ satisfy that $Nh \geq \underline{r}$, and assume that $\sum_{x \in \omega_h^{d-1}} u(x, 0) = 0$. Then

- i) it holds for all $p \in [1, 2]$ that $\|u(\cdot, N)\|_{L_h^p(\omega_h^{d-1})} \leq (1 + \sqrt{c}\underline{r})^{-(2-\frac{2}{p})} \|u(\cdot, 0)\|_{L_h^p(\omega_h^{d-1})}$ and
- ii) it holds for all $p \in [2, \infty]$ that $\|u(\cdot, N)\|_{L_h^p(\omega_h^{d-1})} \leq (1 + \sqrt{c}\underline{r})^{-\frac{2}{p}} \|u(\cdot, 0)\|_{L_h^p(\omega_h^{d-1})}$.

3.4 A Marcinkiewicz-Type Theorem for More Than One Multipliers

This subsection slightly extends the classical Marcinkiewicz multiplier theorem to the case of more than one multipliers (see Corollary 3.22). It will be used to bound the normal component by $(d-1)$ tangential components. In this case there are $(d-1)$ multipliers, however, each multiplier is *not everywhere well-defined* as seen in Corollary 3.12. In the continuum setting this issue is overcome by considering a partition of unity (see the paragraph between (83) and (84) in [3]).

The argument here also relies on local properties of the multipliers. Roughly speaking, the function locvar in Setting 3.15 below, called the *local variation*, measures the variation of a function on each dyadic rectangle. Corollary 3.22 proves that we still obtain L^p -estimates, $p \in (1, \infty)$, if for each dyadic rectangle there is a nice multiplier defined on it. Moreover, in order to conveniently verify an assumption in Corollary 3.22 we use Lemma 3.18.

The notation in Setting 3.15 below, e.g., Eqs. 3.33–3.35, is again inspired by [19, Section 2.5].

Setting 3.15 Let Setting 3.2 be given. Let $D(\ell) \subseteq \mathbb{R}$, $\ell \in \mathbb{Z}$, be the intervals given by

$$\forall \ell \in \mathbb{N}: D(\ell) = [2^{\ell-1}, 2^\ell], \quad D(0) = (-1, 1), \quad \forall \ell \in (-\mathbb{N}): D(\ell) = (-2^{|\ell|}, -2^{|\ell|-1}]. \quad (3.32)$$

For every $\beta \in [0, 1]$ and every finite set $A \subseteq \mathbb{Z}$ we write

$$\sum_{v \in A}^\beta = \begin{cases} \sum_{v \in A} & : \beta = 1 \\ \sup_{v \in A} & : \beta = 0. \end{cases} \quad (3.33)$$

For every $d \in \mathbb{N}$, $\alpha \in \{0, 1\}$, $f: \mathbb{Z}^d \rightarrow \mathbb{C}$ let $\Delta_i^\alpha f: \mathbb{Z}^d \rightarrow \mathbb{C}$ be the functions given by

$$\forall i \in [1, d] \cap \mathbb{Z}, x \in \mathbb{Z}^d: \quad (\Delta_i^\alpha) f(x) = \begin{cases} f(x + \mathbf{e}_i^d) - f(x) & : \alpha = 1 \\ f(x) & : \alpha = 0. \end{cases} \quad (3.34)$$

Let $\text{var}: \bigcup_{d, L \in \mathbb{N}} \mathcal{P}_{2L}(\mathbb{Z}^d, \mathbb{C}) \rightarrow \mathbb{R}$ be the so-called *total variation*, i.e., the function which satisfies for all $d, L \in \mathbb{N}$, $a \in \mathcal{P}_{2L}(\mathbb{Z}^d, \mathbb{C})$ that

$$\text{var}(a) = \sup_{k \in \mathbb{Z}^d} \max_{\alpha \in \{0, 1\}^d} \left[\sum_{v_1 \in D(k_1) \cap \mathbb{I}_L}^{\alpha_1} \dots \sum_{v_d \in D(k_d) \cap \mathbb{I}_L}^{\alpha_d} |(\Delta_1^{\alpha_1} \dots \Delta_d^{\alpha_d} a)(v)| \right]. \quad (3.35)$$

For every $\beta \in [0, 1]$ and every finite set $A \subseteq \mathbb{Z}$ we write

$$\sum_{v \in A}^{\beta, 1} = \begin{cases} \sum_{v \in A \setminus \{\max A\}} & : \beta = 1 \\ \sup_{v \in A} & : \beta = 0. \end{cases} \quad (3.36)$$

Let $\text{locvar}: \bigcup_{d, L \in \mathbb{N}} (\mathcal{P}_{2L}(\mathbb{Z}^d, \mathbb{C}) \times \mathbb{Z}^d) \rightarrow \mathbb{R}$ be the function, called the *local variation*, which satisfies for all $d, L \in \mathbb{N}$, $a \in \mathcal{P}_{2L}(\mathbb{Z}^d, \mathbb{C})$, $k \in \mathbb{Z}^d$ that

$$\text{locvar}(a, k) = \max_{\alpha \in \{0, 1\}^d} \left[\sum_{v_1 \in D(k_1) \cap \mathbb{I}_L}^{\alpha_1, 1} \dots \sum_{v_d \in D(k_d) \cap \mathbb{I}_L}^{\alpha_d, 1} |(\Delta_1^{\alpha_1} \dots \Delta_d^{\alpha_d} a)(v_1, \dots, v_d)| \right]. \quad (3.37)$$

In Lemma 3.16 below we explain the purpose of introducing (3.36) and (3.37).

Lemma 3.16 Assume Setting 3.15, let $d, L \in \mathbb{N}$, $f, g \in \mathcal{P}_{2L}(\mathbb{Z}^d, \mathbb{C})$, $k \in \mathbb{Z}^d$, and assume for all $v \in \prod_{j=1}^d (D(k_j) \cap \mathbb{I}_L)$ that $f(v) = g(v)$. Then it holds that $\text{locvar}(f, k) = \text{locvar}(g, k)$.

Proof of Lemma 3.16 Let us show that for all $\alpha \in \{0, 1\}^d$ it holds that

$$\begin{aligned} & \sum_{v_1 \in D(k_1) \cap \mathbb{I}_L}^{\alpha_1, 1} \cdots \sum_{v_d \in D(k_d) \cap \mathbb{I}_L}^{\alpha_d, 1} (\Delta_1^{\alpha_1} \cdots \Delta_d^{\alpha_d} f)(v) \\ &= \sum_{v_1 \in D(k_1) \cap \mathbb{I}_L}^{\alpha_1, 1} \cdots \sum_{v_d \in D(k_d) \cap \mathbb{I}_L}^{\alpha_d, 1} (\Delta_1^{\alpha_1} \cdots \Delta_d^{\alpha_d} g)(v). \end{aligned} \quad (3.38)$$

First, we consider the case $d = 1$. If $\alpha = 0$, then Eq. 3.38 directly follows from Eq. 3.36. If $\alpha = 1$, observe that for all $v \in (D(k) \cap \mathbb{I}_L) \setminus \max\{D(k) \cap \mathbb{I}_L\}$ it holds that $f(v+1) = g(v+1)$, $f(v) = g(v)$, and hence $(\Delta f)(v) = (\Delta g)(v)$ and Eq. 3.38 then follows from Eq. 3.36. Applying the result for $d = 1$ successively we obtain (3.38) in the case $d \geq 2$. Using (3.37) then completes the proof of Lemma 3.16. $\square$

Lemma 3.17 Assume Setting 3.17 and let $d, L \in \mathbb{N}$, $a \in \mathcal{P}_{2L}(\mathbb{Z}^d, \mathbb{C})$. Then $\text{locvar}(a, 0) = |a(0)|$.

Proof of Lemma 3.17 Observe that Eq. 3.36 and the fact that $D(0) \cap \mathbb{I}_L = \{0\}$ show for all $f \in \mathcal{P}_{2L}(\mathbb{Z}, \mathbb{C})$ that $\sum_{v \in D(0) \cap \mathbb{I}_L}^{1,1} f(v)$ is an empty sum and $\sum_{v \in D(0) \cap \mathbb{I}_L}^{0,1} f(v) = \sup_{v \in D(0) \cap \mathbb{I}_L} f(v) = f(0)$. This, applied successively to each variable, and Eq. 3.37 prove that $\text{locvar}(a, 0) = |a(0)|$. $\square$

For the proof of Lemma 3.18 below we use the mean value theorem. This is a routine idea (cf. the proof of Item (b) in Theorem 2.49 in [19]). The proof is included only for convenience of the reader.

Lemma 3.18 (A sufficient condition to bound local variations) Assume Setting 3.15, let $d, L \in \mathbb{N}$, $M \in (0, \infty)$, $k \in \mathbb{Z}^d$, let J be the set given by $J = \prod_{j=1}^d (D(k_j) \cap [-L+1, L])$, assume that $J \neq \emptyset$, let $A \in C(J, \mathbb{C})$, $a \in \mathcal{P}_{2L}(\mathbb{Z}^d, \mathbb{C})$ satisfy for all $v \in J \cap \mathbb{Z}^d$ that $a(k) = A(k)$, and assume for all $\alpha \in \{0, 1\}^d$, $\xi \in J \setminus \mathbb{I}_L^d$ that $\partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} A \in C(J \setminus \mathbb{I}_L^d, \mathbb{C})$ and $|\xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d} (\partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} A)(\xi)| \leq M$. Then it holds that $\text{locvar}(a, k) \leq M$.

Proof of Lemma 3.18 First, Eqs. 3.32 and 3.36 imply that for all $k \in \mathbb{Z}$, $\xi \in D(k)$, $\alpha \in \{0, 1\}$ it holds that $\sum_{v \in D(k) \cap \mathbb{I}_L}^{\alpha, 1} 1 \leq |\xi|^\alpha$ where the sum is an empty sum for $\alpha = 1$, $k = 0$. This, the assumption that $\forall v \in J \cap \mathbb{Z}^d: a(k) = A(k)$, the mean value theorem (applied to all x_j with $\alpha_j \neq 0$), and the assumption that $\forall \alpha \in \{0, 1\}^d$, $\xi \in J \setminus \mathbb{I}_L^d: |\xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d} (\partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} A)(\xi)| \leq M$ show for all $\alpha \in \{0, 1\}^d$ that

$$\sum_{v_1 \in D(k_1) \cap \mathbb{I}_L}^{\alpha_1, 1} \cdots \sum_{v_d \in D(k_d) \cap \mathbb{I}_L}^{\alpha_d, 1} |(\Delta_1^{\alpha_1} \cdots \Delta_d^{\alpha_d} a)(v)|$$

$$\begin{aligned}
&\leq \sum_{v_1 \in D(k_1) \cap \mathbb{I}_L}^{\alpha_{1,1}} \cdots \sum_{v_d \in D(k_d) \cap \mathbb{I}_L}^{\alpha_{d,1}} \left[\sup_{\xi \in J \setminus \mathbb{I}_L^d} |(\partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} A)(\xi)| \right] \\
&= \left[\sum_{v_1 \in D(k_1) \cap \mathbb{I}_L}^{\alpha_{1,1}} 1 \right] \cdots \left[\sum_{v_d \in D(k_d) \cap \mathbb{I}_L}^{\alpha_{d,1}} 1 \right] \left[\sup_{\xi \in J \setminus \mathbb{I}_L^d} |(\partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} A)(\xi)| \right] \\
&\leq \left[\sup_{\xi \in J \setminus \mathbb{I}_L^d} |\xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d} (\partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} A)(\xi)| \right] \leq M.
\end{aligned} \tag{3.39}$$

Combining this with Eq. 3.37 completes the proof of Lemma 3.18. $\square$

Lemma 3.19 (From local variations to total variations) *Assume Setting 3.15 and let $d, L \in \mathbb{N}$, $a \in \mathcal{P}_{2L}(\mathbb{Z}^d, \mathbb{C})$, $M \in (0, \infty)$ satisfy for all $k \in \mathbb{Z}^d$ that $\text{locvar}(a, k) \leq M$. Then $\text{var}(a) \leq 4^d M$.*

Proof of Lemma 3.19 Throughout this proof for every $\alpha \in \{0, 1\}$, $f \in \mathcal{P}_{2L}(\mathbb{Z}, \mathbb{C})$, $A \subseteq \mathbb{I}_L$ let ∂A be the set given by $\partial A = \{\max A, (\max A + 1) \bmod (2L)\}$ and write

$$\sum_{v \in A}^{\alpha, 0} f(v) = \sum_{v \in \partial A} f(v). \tag{3.40}$$

Then Eq. 3.37 and the assumption that $\forall k \in \mathbb{Z}^d$: $\text{locvar}(a, k) \leq M$ prove that for all $\ell \in [0, d] \cap \mathbb{Z}$, $s, \alpha \in \{0, 1\}^d$ with $s_1 = \dots = s_\ell = 0$ and $s_{\ell+1} = s_d = 1$ it holds that

$$\begin{aligned}
&\sum_{v_1 \in D(k_1) \cap \mathbb{I}_L}^{\alpha_{1,s_1}} \cdots \sum_{v_d \in D(k_d) \cap \mathbb{I}_L}^{\alpha_{d,s_d}} |(\Delta_1^{\alpha_{1,s_1}} \cdots \Delta_d^{\alpha_{d,s_d}} a)(v)| \\
&= \sum_{v_1 \in \partial(D(k_1) \cap \mathbb{I}_L)} \cdots \sum_{v_\ell \in \partial(D(k_\ell) \cap \mathbb{I}_L)} \left(\sum_{v_{\ell+1} \in D(k_{\ell+1}) \cap \mathbb{I}_L}^{\alpha_{\ell+1,1}} \cdots \sum_{v_d \in D(k_d) \cap \mathbb{I}_L}^{\alpha_{d,1}} |(\Delta_{\ell+1}^{\alpha_{\ell+1}} \cdots \Delta_d^{\alpha_d} a)(v)| \right) \\
&\leq \sum_{v_1 \in \partial(D(k_1) \cap \mathbb{I}_L)} \cdots \sum_{v_\ell \in \partial(D(k_\ell) \cap \mathbb{I}_L)} M \leq 2^d M.
\end{aligned} \tag{3.41}$$

A permutation of the coordinates hence shows for all $\ell \in [0, d] \cap \mathbb{Z}$, $s, \alpha \in \{0, 1\}^d$ that

$$\sum_{v_1 \in J(k_1)}^{\alpha_{1,s_1}} \cdots \sum_{v_d \in J(k_d)}^{\alpha_{d,s_d}} |(\Delta_1^{\alpha_{1,s_1}} \cdots \Delta_d^{\alpha_{d,s_d}} a)(v)| \leq 2^d M. \tag{3.42}$$

Next, the notation given in Eqs. 3.33, 3.34 and 3.40 and the fact that $\forall a, b \in \mathbb{R}$: $|a - b| \leq |a| + |b|$ demonstrate, in the one-dimensional case, that for all $f \in \mathcal{P}_{2L}(\mathbb{Z}, \mathbb{C})$, $k \in \mathbb{Z}$, $\alpha \in \{0, 1\}$ it holds that

$$\begin{aligned}
\sum_{v \in D(k) \cap \mathbb{I}_L}^{\alpha} |(\Delta^\alpha f)(v)| &\leq \left[\sum_{v \in D(k) \cap \mathbb{I}_L}^{\alpha, 1} |\Delta^\alpha f(v)| \right] + \left[\sum_{v \in D(k) \cap \mathbb{I}_L}^{\alpha, 0} |f(v)| \right] \\
&= \sum_{s=0}^1 \sum_{v \in D(k) \cap \mathbb{I}_L}^{\alpha, s} |\Delta^{\alpha s} f(v)|.
\end{aligned} \tag{3.43}$$

This (applied to each variable) and Eq. 3.42 imply for all $k \in \mathbb{Z}^d$, $\alpha \in \{0, 1\}^d$ that

$$\begin{aligned} & \sum_{v_1 \in D(k_1) \cap \mathbb{I}_L}^{\alpha_1} \cdots \sum_{v_d \in D(k_d) \cap \mathbb{I}_L}^{\alpha_d} |(\Delta_1^{\alpha_1} \cdots \Delta_d^{\alpha_d} a)(v)| \\ & \leq \sum_{s \in \{0, 1\}^d} \left[\sum_{v_1 \in D(k_1) \cap \mathbb{I}_L}^{\alpha_1, s_1} \cdots \sum_{v_d \in D(k_d) \cap \mathbb{I}_L}^{\alpha_d, s_d} |(\Delta_1^{\alpha_1 s_1} \cdots \Delta_d^{\alpha_d s_d} a)(v)| \right] \\ & \leq \sum_{s \in \{0, 1\}^d} (2^d M) = 4^d M. \end{aligned} \quad (3.44)$$

This and Eq. 3.35 complete the proof of Lemma 3.19. $\square$

Setting 3.20 Let Setting 3.15 be given. Let $m : (1, \infty) \rightarrow [0, \infty]$ be the function which satisfies for all $p \in (1, \infty)$ that $m(p) \in [0, \infty]$ is the smallest real extended number with the property that for all $d \in \mathbb{N}$, $h \in \pi/\mathbb{N}$, $U, u \in \mathcal{P}_{2\pi}(h\mathbb{Z}^d, \mathbb{C})$, $a \in \mathcal{P}_{2\pi/h}(\mathbb{Z}^d, \mathbb{C})$, $M \in (0, \infty)$ with $\mathcal{F}(U) = a\mathcal{F}(u)$ and $\text{var}(a) \leq M$ it holds that

$$\|U\|_{L_h^p(\omega_h^d)} \leq m(p)M \|u\|_{L_h^p(\omega_h^d)}. \quad (3.45)$$

Lemma 3.21 below recalls the discrete Marcinkiewicz multiplier theorem (cf. [19, Theorem 2.49]):

Lemma 3.21 (Marcinkiewicz' theorem) *Assume Setting 3.20 and let $p \in (1, \infty)$. Then $m(p) < \infty$.*

Corollary 3.22 *Assume Setting 3.20, let $d, m \in \mathbb{N}$, $h \in \pi/\mathbb{N}$, $U, u_1, \dots, u_m \in \mathcal{P}_{2\pi}(h\mathbb{Z}^d, \mathbb{C})$, $p \in (1, \infty)$, $L \in \mathbb{N}$, assume that $h = \pi/L$, let $(A_k)_{k \in \mathbb{Z}^d} \subseteq \mathcal{P}_{2L}(\mathbb{Z}^d, \mathbb{C})$, $J : \mathbb{Z}^d \rightarrow [1, m] \cap \mathbb{Z}$ satisfy for all $k \in \mathbb{Z}^d$, $v \in \prod_{j=1}^d (D(k_j) \cap \mathbb{I}_L)$ that*

$$\text{locvar}(A_k, k) \leq M \quad \text{and} \quad (\mathcal{F}(U))(v) = A_k(v)(\mathcal{F}(u_{J(k)}))(v). \quad (3.46)$$

Then it holds that $m(p) < \infty$ and

$$\|U\|_{L_h^p(\omega_h^d)} \leq m(p)4^d M \sum_{i=1}^m \|u_i\|_{L_h^p(\omega_h^d)}. \quad (3.47)$$

Proof of Corollary 3.22 The discrete Marcinkiewicz multiplier theorem (cf. [19, Theorem 2.49]) proves that $m(p) < \infty$. For the rest of this proof let $K : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$ be the mapping which is $2L$ -periodic, i.e., $\forall v \in \mathbb{Z}^d$, $i \in [1, d] \cap \mathbb{Z}$: $K(v) = K(v + 2Le_i^d)$ and satisfies for all $v \in \mathbb{I}_L^d$ that $\prod_{j=1}^d D(K_j(v))$ is the unique dyadic rectangle in the family $\left\{ \prod_{j=1}^d D(k_j) : k \in \mathbb{Z}^d \right\}$ that contains v , let $a, b \in \mathcal{P}_{2L}(\mathbb{Z}^d, \mathbb{C})$ be the functions which satisfy for all $v \in \mathbb{Z}^d$ that $a(v) = A_{K(v)}(v)$ and let $w : \omega_h^d \rightarrow \mathbb{C}$ be the function which satisfies for all $x \in \omega_h^d$ that

$$w(x) = (2\pi)^{-d} \sum_{v \in \mathbb{I}_L^d} [\mathcal{F}(u_{J(K(v))})](v) e^{i v x} \quad (3.48)$$

Then it holds for all $k \in \mathbb{Z}^d$, $v \in \prod_{j=1}^d (D(k_j) \cap \mathbb{I}_L)$ that $K(v) = k$, and $a(v) = A_k(v)$. Lemma 3.16 then implies for all $k \in \mathbb{Z}^d$ with $\prod_{j=1}^d (D(k_j) \cap \mathbb{I}_L) \neq \emptyset$ that $\text{locvar}(a, k) = \text{locvar}(A_k, k)$. This, Eq. 3.46, and Lemma 3.19 ensure that $\text{var}(a) \leq 4^d M$. Moreover, Eq. 3.48 and the Fourier inverse formula (combined with the assumption that $L = \pi/h$) show for all $v \in \mathbb{I}_L^d$ that

$$\begin{aligned} [\mathcal{F}(w)](v) &= [\mathcal{F}(u_{J(K(v))})](v) \quad \text{and} \quad |w(x)| \leq \sum_{i=1}^m \left| (2\pi)^{-d} \sum_{v \in \mathbb{I}_L^d} (\mathcal{F}(u_i))(v) e^{ivx} \right| \\ &= \sum_{i=1}^m |u_i(x)|. \end{aligned} \quad (3.49)$$

Hence, Eq. 3.46 (with $k \leftarrow K(v)$), and the fact that $\forall v \in \mathbb{Z}^d$: $a(v) = A_{K(v)}(v)$ ensure for all $v \in \mathbb{I}_L^d$ that $(\mathcal{F}(U))(v) = A_{K(v)}(v)(\mathcal{F}(u_{J(K(v))}))(v) = a(v)(\mathcal{F}(w))(v)$. Hence, $\mathcal{F}(U) = a\mathcal{F}(w)$. This, Eq. 3.45 (with $M \leftarrow 4^d M$, $u \leftarrow w$), the fact that $\text{var}(a) \leq 4^d M$, and the triangle inequality demonstrate that

$$\|U\|_{L_h^p(\omega_h^d)} \leq \mathfrak{m}(p) 4^d M \|w\|_{L_h^p(\omega_h^d)} \leq \mathfrak{m}(p) 4^d M \sum_{i=1}^m \|u_i\|_{L_h^p(\omega_h^d)}. \quad (3.50)$$

This completes the proof of Corollary 3.22. $\square$

3.5 Cauchy's Integral Formula Revisited

Lemma 3.18 requires estimates on the higher derivatives of the multipliers. Returning to a classical result we can avoid many tedious calculations.

Setting 3.23 Assume Setting 3.4, let $C, m, M \in [0, \infty]$ be the real extended number given by

$$\begin{aligned} m &= \left\lceil \inf_{\xi \in [1, 3d] \times [-2d, 2d]} |Q(\xi)| \right\rceil, \quad M = \left\lceil \sup_{\xi \in [1, 3d] \times [-2d, 2d]} |Q(\xi)| \right\rceil, \\ C^2 &= \max \left\{ \frac{3(d-1)}{2m}, \frac{M}{c(d-1)} \right\}, \end{aligned} \quad (3.51)$$

and let $\mathcal{C}(z) \subseteq \mathbb{C}$, $z \in (1, \infty) \times \{0\}$, be the sets (see Fig. 3) which satisfy for all $z \in (1, \infty) \times \{0\}$ that

$$\mathcal{C}(z) = \left\{ \zeta \in \mathbb{C} : |\zeta - z| = \frac{1}{2}|z - 1| \right\}. \quad (3.52)$$

Lemma 3.24 Assume Setting 3.23. Then it holds for all $t \in [-\pi, \pi]^{d-1} \setminus \{0\}$ that

$$C, m, M \in (0, \infty), \quad \left[\sup_{\zeta \in \mathcal{C}(\lambda(t))} |f(\zeta)| \right] \leq C|t|_\infty, \quad \text{and} \quad \left[\sup_{\zeta \in \mathcal{C}(\lambda(t))} \frac{1}{|f(\zeta)|} \right] \leq \frac{C}{|t|_\infty}. \quad (3.53)$$

Proof of Lemma 3.24 The extreme value theorem and the fact that $\forall z \in \mathbb{C} \setminus (-\infty, 0)$: $Q(z) \neq 0$ ensure that $m, M \in (0, \infty)$ and hence $C \in (0, \infty)$. Furthermore, the triangle inequality and Eq. 3.52 imply that for all $z \in [1, \infty)$, $\zeta \in \mathcal{C}(z)$ it holds that

$$|\zeta - 1| = |(z - 1) + (\zeta - z)| \geq |z - 1| - |\zeta - z| = |z - 1| - \frac{1}{2}|z - 1| = \frac{1}{2}|z - 1| \quad (3.54)$$

and

$$\begin{aligned} |\zeta - 1| &= \frac{1}{2} |(\zeta - z) + (\zeta - 1) + (z - 1)| \leq \frac{1}{2} |\zeta - z| + \frac{1}{2} |\zeta - 1| + \frac{1}{2} |z - 1| \\ &= \frac{1}{4} |z - 1| + \frac{1}{2} |\zeta - 1| + \frac{1}{2} |z - 1| = \frac{3}{4} |z - 1| + \frac{1}{2} |\zeta - 1|. \end{aligned} \quad (3.55)$$

This shows for all $z \in [1, \infty)$, $\zeta \in \mathcal{C}(z)$ that $\frac{1}{2} |z - 1| \leq |\zeta - 1| \leq \frac{3}{2} |z - 1|$. This (with $z \leftarrow \lambda(t)$ for $t \in [-\pi, \pi]^{d-1}$) demonstrate that for all $t \in [-\pi, \pi]^{d-1}$, $\zeta \in \mathcal{C}(\lambda(t))$ it holds that

$$\frac{c}{2} (d-1) |t|_\infty^2 \leq \frac{1}{2} |\lambda(t) - 1| \leq |\zeta - 1| \leq \frac{3}{2} |\lambda(t) - 1| \leq \frac{3}{4} (d-1) |t|_\infty^2. \quad (3.56)$$

Furthermore, Eq. 3.11 implies for all $t \in [-\pi, \pi]^{d-1}$ that $1 \leq \lambda(t) \leq 2d$. Hence, Eq. 3.52 shows that for all $t \in [-\pi, \pi]^{d-1}$ it holds that $\mathcal{C}(\lambda(t)) \subseteq [1, 3d] \times [-2d, 2d] \subseteq \mathbb{C} \setminus (-\infty, 1)$. This, Lemma 3.6, and Eq. 3.56 imply for all $t \in [-\pi, \pi]^{d-1} \setminus \{0\}$, $\zeta \in \mathcal{C}(\lambda(t))$ that

$$|f(\zeta)|^2 = \frac{2|\zeta - 1|}{|Q(\zeta)|} \leq \frac{\frac{3}{2}(d-1)|t|_\infty^2}{m} \leq C^2 |t|_\infty^2 \quad (3.57)$$

and

$$\frac{1}{|f(\zeta)|^2} = \frac{|Q(\zeta)|}{2|\zeta - 1|} \leq \frac{M}{c(d-1)|t|_\infty^2} \leq \frac{C^2}{|t|_\infty^2}. \quad (3.58)$$

This completes the proof of Lemma 3.24 (Fig. 3). $\square$

Lemma 3.25 (Higher t -derivatives) *Assume Setting 3.23 and let $h: \{\zeta \in \mathbb{C}: \Re(\zeta) > 1\} \rightarrow \mathbb{C}$ be a holomorphic function. Then it holds for all $t \in [-\pi, \pi]^{d-1} \setminus \{0\}$, $\alpha \in \{0, 1\}^{d-1}$ that*

$$\left| \left[\left(\frac{\partial}{\partial t_1} \right)^{\alpha_1} \left(\frac{\partial}{\partial t_2} \right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial t_{d-1}} \right)^{\alpha_{d-1}} (h \circ \lambda) \right] (t) \right| \leq \frac{(2/c)^{|\alpha|} (|\alpha|)!}{|t|_\infty^{|\alpha|}} \left[\sup_{\zeta \in \mathcal{C}(\lambda(t))} |h(\zeta)| \right]. \quad (3.59)$$

Proof of Lemma 3.25 Cauchy's integral formula together with the assumption that h is holomorphic and Eq. 3.52 proves for all $z \in (1, \infty) \times \{0\}$, $n \in \mathbb{N}_0$ that

$$\begin{aligned} |h^{(n)}(z)| &= \left| \frac{n!}{2\pi i} \int_{\mathcal{C}(z)} \frac{h(\zeta) d\zeta}{(\zeta - z)^{n+1}} \right| \leq \frac{n!}{2\pi \left(\frac{1}{2} |z - 1| \right)^{n+1}} \left[\sup_{\zeta \in \mathcal{C}(z)} |h(\zeta)| \right] \left[\int_{\zeta \in \mathcal{C}(z)} d|\zeta| \right] \\ &= \frac{n!}{2\pi \left(\frac{1}{2} |z - 1| \right)^{n+1}} \left[\sup_{\zeta \in \mathcal{C}(z)} |h(\zeta)| \right] \left[2\pi \frac{1}{2} |z - 1| \right] = \frac{2^n n!}{|z - 1|^n} \left[\sup_{\zeta \in \mathcal{C}(z)} |h(\zeta)| \right]. \end{aligned} \quad (3.60)$$

Fig. 3 The circle $\mathcal{C}(z) = \{\zeta \in \mathbb{C}: |\zeta - z| = \frac{1}{2} |z - 1|\}$ for some $z \in (1, \infty) \times \{0\}$

Next, Eq. 3.11 and some elementary facts imply for all $t \in [-\pi, \pi]^{d-1} \setminus \{0\}$, $i \in [1, d-1] \cap \mathbb{Z}$ that

$$\lambda(t) \in (1, \infty), \quad \frac{\partial}{\partial t_i} \lambda(t) = \sin(t_i), \quad \text{and} \quad |\sin(t_i)| \leq |t_i|. \quad (3.61)$$

This, Eq. 3.52 (with $z \leftarrow \lambda(t)$), and Lemma 3.5, show for all $t \in [-\pi, \pi]^{d-1} \setminus \{0\}$, $\alpha \in \{0, 1\}^{d-1}$ that

$$\begin{aligned} & \left| \left[\left(\frac{\partial}{\partial t_1} \right)^{\alpha_1} \left(\frac{\partial}{\partial t_2} \right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial t_{d-1}} \right)^{\alpha_{d-1}} (h \circ \lambda) \right] (t) \right| = \left| h^{(|\alpha|)}(\lambda(t)) \prod_{i=1}^{d-1} (\sin t_i)^{\alpha_i} \right| \\ & \leq \left[\left(\frac{2}{|\lambda(t) - 1|} \right)^{|\alpha|} (|\alpha|)! \right] \left[\sup_{\zeta \in \mathcal{C}(\lambda(t))} |h(\zeta)| \right] |t|_\infty^{|\alpha|} \\ & \leq \left[\left(\frac{2}{c|t|_\infty^2} \right)^{|\alpha|} (|\alpha|)! \right] \left[\sup_{\zeta \in \mathcal{C}(\lambda(t))} |h(\zeta)| \right] |t|_\infty^{|\alpha|}. \end{aligned} \quad (3.62)$$

This shows (3.59) and completes the proof of Lemma 3.25. $\square$

3.6 Total Variations of the Multipliers

Lemma 3.26 Assume Settings 3.15 and 3.23, let $a, \hat{C} \in [0, \infty)$ be the real numbers (cf. Lemmas 3.24 and A.2) given by

$$a = \sup_{s \in [-\pi, \pi] \setminus \{0\}} \max \left\{ \left| \frac{d}{ds} (e^{-is} - 1) \right|, \left| \frac{e^{-is} - 1}{s} \right| \right\} \quad \text{and} \quad \hat{C} = 2aC((2/c)^d \vee 1)d!, \quad (3.63)$$

and let $h \in \pi/\mathbb{N}$. Then it holds for all $i \in [1, d-1] \cap \mathbb{Z}$ that $\text{var}(\mathcal{N}_i^h) \leq 4^d \hat{C}$.

Proof of Lemma 3.26 We first do some simple calculations on the derivatives. First, Eq. 3.11 shows for all $i \in [1, d-1] \cap \mathbb{Z}$, $\alpha \in \{0, 1\}^{d-1}$ with $\alpha_i = 0$ that

$$\begin{aligned} (\partial_1^{\alpha_1} \cdots \partial_{d-1}^{\alpha_{d-1}} \mathcal{N}_i)(t) &= \left(\frac{\partial}{\partial t_1} \right)^{\alpha_1} \left(\frac{\partial}{\partial t_2} \right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial t_{d-1}} \right)^{\alpha_{d-1}} \left(\frac{e^{-it_i} - 1}{f(\lambda(t))} \right) \\ &= \left[\left(\frac{\partial}{\partial t_1} \right)^{\alpha_1} \left(\frac{\partial}{\partial t_2} \right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial t_{d-1}} \right)^{\alpha_{d-1}} \left(\frac{1}{f(\lambda(t))} \right) \right] (e^{-it_i} - 1). \end{aligned} \quad (3.64)$$

Next, note that for all $i \in [1, d-1] \cap \mathbb{Z}$ it holds that

$$\frac{\partial}{\partial t_i} \left(\frac{e^{-it_i} - 1}{f(\lambda(t))} \right) = \frac{1}{f(\lambda(t))} \left[\frac{\partial}{\partial t_i} (e^{-it_i} - 1) \right] + \left[\frac{\partial}{\partial t_i} \left(\frac{1}{f(\lambda(t))} \right) \right] (e^{-it_i} - 1). \quad (3.65)$$

This and Eq. 3.11 ensure for all $i \in [1, d-1] \cap \mathbb{Z}$, $\alpha \in \{0, 1\}^{d-1}$ with $\alpha_i = 1$ that

$$\begin{aligned} (\partial_1^{\alpha_1} \cdots \partial_{d-1}^{\alpha_{d-1}} \mathcal{N}_i)(t) &= \left(\frac{\partial}{\partial t_1} \right)^{\alpha_1} \left(\frac{\partial}{\partial t_2} \right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial t_{d-1}} \right)^{\alpha_{d-1}} \left(\frac{e^{-it_i} - 1}{f(\lambda(t))} \right) \\ &= \left[\left(\frac{\partial}{\partial t_1} \right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial t_{i-1}} \right)^{\alpha_{i-1}} \left(\frac{\partial}{\partial t_{i+1}} \right)^{\alpha_{i+1}} \cdots \left(\frac{\partial}{\partial t_{d-1}} \right)^{\alpha_{d-1}} \left(\frac{1}{f(\lambda(t))} \right) \right] \left[\frac{\partial}{\partial t_i} (e^{-it_i} - 1) \right] \\ &\quad + \left[\left(\frac{\partial}{\partial t_1} \right)^{\alpha_1} \left(\frac{\partial}{\partial t_2} \right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial t_{d-1}} \right)^{\alpha_{d-1}} \left(\frac{1}{f(\lambda(t))} \right) \right] (e^{-it_i} - 1). \end{aligned} \quad (3.66)$$

Moreover, Lemma 3.25 (applied with $h \leftarrow 1/f$), Lemma 3.24, and the fact that $\forall \alpha \in \{0, 1\}^{d-1}$: $|\alpha| \leq d$ ensure that for all $\alpha \in \{0, 1\}^{d-1}$ it holds that

$$\left| \left(\frac{\partial}{\partial t_1} \right)^{\alpha_1} \left(\frac{\partial}{\partial t_2} \right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial t_{d-1}} \right)^{\alpha_{d-1}} \left(\frac{1}{f(\lambda(t))} \right) \right| \leq \frac{C((2/c)^d \vee 1)d!}{|t|_\infty^{|\alpha|+1}} \quad (3.67)$$

This (with $\alpha \leftarrow (\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_d)$ and with $\alpha \leftarrow \alpha$ for $\alpha \in \{0, 1\}^{d-1}$, $i \in [1, d-1] \cap \mathbb{Z}$, Eqs. 3.64 and 3.66, the triangle inequality, and Eq. 3.63 prove for all $i \in [1, d-1] \cap \mathbb{Z}$, $\alpha \in \{0, 1\}^{d-1}$ that

$$\begin{aligned} |t|^{|\alpha|} |(\partial_1^{\alpha_1} \dots \partial_{d-1}^{\alpha_{d-1}} \mathcal{N}_i)(t)| &\leq |t|^{|\alpha|} C((2/c)^d \vee 1) d! \left[\frac{a}{|t|^{|\alpha|}} + \frac{a|t|^\infty}{|t|^{|\alpha|+1}} \right] \\ &= 2aC((2/c)^d \vee 1) d! = \hat{C}. \end{aligned} \quad (3.68)$$

This, Eq. 3.11, and the substitution $\xi \leftarrow t/h$ show for all $i \in [1, d-1] \cap \mathbb{Z}$, $\alpha \in \{0, 1\}^{d-1}$, $\xi \in [-\frac{\pi}{h}, \frac{\pi}{h}]^{d-1} \setminus \{0\}$ that $|\xi|^{|\alpha|} |(\partial_1^{\alpha_1} \dots \partial_{d-1}^{\alpha_{d-1}} \mathcal{N}_i^h)(\xi)| \leq \hat{C}$. Then Lemma 3.18 shows for all $i \in [1, d-1] \cap \mathbb{Z}$, $k \in \mathbb{Z}^d \setminus \{0\}$ that $\text{locvar}(\mathcal{N}_i^h, k) \leq \hat{C}$. This and the fact that $\text{locvar}(\mathcal{N}_i^h, 0) = |\mathcal{N}_i^h(0)| = 0$ (see Eqs. 3.11 and 3.12, and Lemma 3.17) prove for all $i \in [1, d-1] \cap \mathbb{Z}$, $k \in \mathbb{Z}^d$ that $\text{locvar}(\mathcal{N}_i^h, k) \leq \hat{C}$. Hence, Lemma 3.19 ensures for all $i \in [1, d-1] \cap \mathbb{Z}$ that $\text{var}(\mathcal{N}_i^h) \leq 4^d \hat{C}$. This completes the proof of Lemma 3.26. $\square$

Combining Lemmas 3.26 and 3.21, and Corollary 3.11 we obtain Corollary 3.27 below.

Corollary 3.27 (The Neumann case) *Assume Settings 3.2 and 3.3. Then there exists a function $C: ([2, \infty) \cap \mathbb{N}) \times (1, \infty) \rightarrow (0, \infty)$ such that for all $h \in \pi/\mathbb{N}$, $p \in (1, \infty)$, $d \in [2, \infty) \cap \mathbb{N}$, $u \in \mathbb{H}_{d,h,\geq 0}$ it holds that $\|\mathcal{D}_{x,h}^h u\|_{L_h^p(\omega_h^{d-1})} \leq C(d, p) \|\mathcal{D}_{y,h}^h u\|_{L_h^p(\omega_h^{d-1})}$.*

Lemma 3.28 *Assume Lemmas 3.3, 3.15 and 3.23 let $h \in \pi/\mathbb{N}$, $u \in \mathbb{H}_{d,h,\geq 0}$, let $a, \hat{C} \in [0, \infty)$ be the real numbers (cf. Lemmas 3.24 and A.2) given by*

$$a = \sup_{s \in [-\pi, \pi] \setminus \{0\}} \max \left\{ \left| \frac{s}{e^{-is} - 1} \right|, s^2 \left| \frac{\partial}{\partial s} \left(\frac{1}{e^{-is} - 1} \right) \right| \right\} \quad \text{and} \quad \hat{C} = 6aC((2/c)^d \vee 1) d!, \quad (3.69)$$

let $J: \mathbb{Z}^{d-1} \rightarrow ([1, d-1] \cap \mathbb{Z})$ be the function which satisfies for all $k \in \mathbb{Z}^{d-1}$ that

$$J(k) = \min \left\{ j \in [1, d-1] \cap \mathbb{Z}: |k_j| = \max_{i \in [1, d-1] \cap \mathbb{Z}} |k_i| \right\}. \quad (3.70)$$

Then it holds for all $k \in \mathbb{Z}^{d-1}$, $v \in \prod_{j=1}^{d-1} D(k_j)$ that

$$[\mathcal{F}((\mathcal{D}_{y,h}^h u)(\cdot, 0))](v) = \mathcal{D}_{J(k),h}^h(v) [\mathcal{F}((\mathcal{D}_{x,J(k),h}^h u)(\cdot, 0))](v) \quad \text{and} \quad \text{var}(\mathcal{D}_{J(\cdot),h}^h(\cdot)) \leq 4^d \hat{C}. \quad (3.71)$$

Proof of Lemma 3.28 First, Eq. 3.70 shows for all $k \in \mathbb{Z}^{d-1} \setminus \{0\}$ that $k_{J(k)} \neq 0$. This, Item (ii) in Corollary 3.12 (with $i \leftarrow J(k)$ for $k \in \mathbb{Z}^{d-1} \setminus \{0\}$), and Item (i) in Corollary 3.12 prove for all $k \in \mathbb{Z}^{d-1}$, $v \in \prod_{j=1}^{d-1} D(k_j)$ that

$$[\mathcal{F}((\mathcal{D}_{y,h}^h u)(\cdot, 0))](v) = \mathcal{D}_{J(k),h}^h(v) [\mathcal{F}((\mathcal{D}_{x,J(k),h}^h u)(\cdot, 0))](v). \quad (3.72)$$

For the rest of this proof let μ be the Lebesgue measure on the real line. Note that Eq. 3.32 and a simple scaling argument show that it holds for all $\ell \in \mathbb{Z} \setminus \{0\}$, $\xi \in (hD(\ell))$ that $|\xi| \geq h\mu(D(\ell))$ and it holds for all $\ell \in \mathbb{Z}$, $\xi \in (hD(\ell))$ that $|\xi| \leq 2h\mu(D(\ell))$. Therefore, the fact that $\forall k \in \mathbb{Z}^{d-1} \setminus \{0\}: k_{J(k)} \neq 0$ and Eq. 3.70 prove for all $k \in \mathbb{Z}^{d-1} \setminus \{0\}$, $t \in \prod_{j=1}^{d-1} (hD(k_j))$ that

$$|t_{J(k)}| \geq h\mu(D(k_{J(k)})) = \left[\max_{i \in [1, d-1] \cap \mathbb{Z}} h\mu(D(k_i)) \right] \geq \frac{1}{2} \left[\max_{i \in [1, d-1] \cap \mathbb{Z}} |t_i| \right] = \frac{1}{2} |t|_\infty. \quad (3.73)$$

Next, Eq. 3.11 shows for all $t \in [-\pi, \pi]^{d-1}$, $i \in [1, d-1] \cap \mathbb{Z}$, $\alpha \in \{0, 1\}^{d-1}$ with $\alpha_i = 0$, $t_i \neq 0$ that

$$\begin{aligned} (\partial_1^{\alpha_1} \dots \partial_{d-1}^{\alpha_{d-1}} \mathcal{D}_i)(t) &= \left(\frac{\partial}{\partial t_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial t_2}\right)^{\alpha_2} \dots \left(\frac{\partial}{\partial t_{d-1}}\right)^{\alpha_{d-1}} \left(\frac{f(\lambda(t))}{e^{-it_i} - 1}\right) \\ &= \left[\left(\frac{\partial}{\partial t_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial t_2}\right)^{\alpha_2} \dots \left(\frac{\partial}{\partial t_{d-1}}\right)^{\alpha_{d-1}} (f(\lambda(t)))\right] \left(\frac{1}{e^{-it_i} - 1}\right). \end{aligned} \quad (3.74)$$

Moreover, note that for all $i \in [1, d-1] \cap \mathbb{Z}$, $t \in [-\pi, \pi]^{d-1}$ with $t_i \neq 0$ it holds that

$$\frac{\partial}{\partial t_i} \left(\frac{f(\lambda(t))}{e^{-it_i} - 1}\right) = f(\lambda(t)) \left[\frac{\partial}{\partial t_i} \left(\frac{1}{e^{-it_i} - 1}\right)\right] + \left[\frac{\partial}{\partial t_i} (f(\lambda(t)))\right] \frac{1}{e^{-it_i} - 1}. \quad (3.75)$$

Then Eq. 3.11 ensures for all $t \in [-\pi, \pi]^{d-1}$, $i \in [1, d-1] \cap \mathbb{Z}$, $\alpha \in \{0, 1\}^{d-1}$ with $t_i \neq 0$, $\alpha_i = 1$ that

$$\begin{aligned} (\partial_1^{\alpha_1} \dots \partial_{d-1}^{\alpha_{d-1}} \mathcal{D}_i)(t) &= \left(\frac{\partial}{\partial t_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial t_2}\right)^{\alpha_2} \dots \left(\frac{\partial}{\partial t_{d-1}}\right)^{\alpha_{d-1}} \left(\frac{f(\lambda(t))}{e^{-it_i} - 1}\right) \\ &= \left[\left(\frac{\partial}{\partial t_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial t_{i-1}}\right)^{\alpha_{i-1}} \left(\frac{\partial}{\partial t_{i+1}}\right)^{\alpha_{i+1}} \dots \left(\frac{\partial}{\partial t_{d-1}}\right)^{\alpha_{d-1}} (f(\lambda(t)))\right] \left[\frac{\partial}{\partial t_i} \left(\frac{1}{e^{-it_i} - 1}\right)\right] \\ &\quad + \left[\left(\frac{\partial}{\partial t_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial t_2}\right)^{\alpha_2} \dots \left(\frac{\partial}{\partial t_{d-1}}\right)^{\alpha_{d-1}} (f(\lambda(t)))\right] \left(\frac{1}{e^{-it_i} - 1}\right). \end{aligned} \quad (3.76)$$

Furthermore, Lemma 3.25 (with $h \leftarrow f$) and Lemma 3.24 ensure that for all $t \in [-\pi, \pi]^{d-1} \setminus \{0\}$, $\alpha \in \{0, 1\}^{d-1}$ it holds that

$$\left|\left(\frac{\partial}{\partial t_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial t_2}\right)^{\alpha_2} \dots \left(\frac{\partial}{\partial t_{d-1}}\right)^{\alpha_{d-1}} (f(\lambda(t)))\right| \leq \frac{C((2/c)^d \vee 1)d!}{|t|_{\infty}^{|\alpha|-1}}. \quad (3.77)$$

Combining (3.74), (3.76), and the triangle inequality then shows that for all $t \in [-\pi, \pi]^{d-1} \setminus \{0\}$, $\alpha \in \{0, 1\}^{d-1}$, $i \in [1, d-1] \cap \mathbb{Z}$ it holds that

$$|\partial_1^{\alpha_1} \dots \partial_{d-1}^{\alpha_{d-1}} \mathcal{D}_i(t)| \leq C((2/c)^d \vee 1)d! \left[\frac{1}{|t|_{\infty}^{|\alpha|-2}} \frac{a}{|t_i|^2} + \frac{1}{|t|_{\infty}^{|\alpha|-1}} \frac{a}{|t_i|} \right] \quad (3.78)$$

This (with $i \leftarrow J(k)$) and Eq. 3.73 prove for all $k \in \mathbb{Z}^{d-1} \setminus \{0\}$, $t \in \prod_{j=1}^{d-1} (hD(k_j))$, $\alpha \in \{0, 1\}^{d-1}$ that

$$|t|_{\infty}^{|\alpha|} |(\partial_1^{\alpha_1} \dots \partial_{d-1}^{\alpha_{d-1}} \mathcal{D}_{J(k)})(t)| \leq C((2/c)^d \vee 1)d! [2^2 a + 2a] \leq 6aC((2/c)^d \vee 1)d! = \hat{C}. \quad (3.79)$$

This, Eq. 3.12, and the chain rule prove for all $k \in \mathbb{Z}^{d-1} \setminus \{0\}$, $\xi \in \prod_{j=1}^{d-1} D(k_j)$, $\alpha \in \{0, 1\}^{d-1}$ that $|\xi|_{\infty}^{|\alpha|} |(\partial_1^{\alpha_1} \dots \partial_{d-1}^{\alpha_{d-1}} \mathcal{D}_{J(k)}^h)(\xi)| \leq \hat{C}$. Lemma 3.18 hence ensures for all $k \in \mathbb{Z}^{d-1} \setminus \{0\}$ that $\text{locvar}(\mathcal{D}_{J(k)}^h, k) \leq \hat{C}$. This and the fact that $\text{locvar}(\mathcal{D}_{J(0)}^h, 0) = |\mathcal{D}_1^h(0)| = 0$ (see Eqs. 3.70, 3.11, 3.12, and Lemma 3.17) prove for all $k \in \mathbb{Z}^d$ that $\text{locvar}(\mathcal{D}_{J(k)}^h, k) \leq \hat{C}$. Hence, Lemma 3.19 ensures that $\text{var}(\mathcal{D}_{J(\cdot)}^h(\cdot)) \leq 4^d \hat{C}$. This and Eq. 3.71 complete the proof of Lemma 3.28. $\square$

Combining Lemma 3.28 and Corollary 3.22 we obtain Corollary 3.29 below.

Corollary 3.29 (The Dirichlet case) *Assume Settings 3.2 and 3.3. Then there exists a function $C: ([2, \infty) \cap \mathbb{N}) \times (1, \infty) \rightarrow (0, \infty)$ such that for all $h \in \pi/\mathbb{N}$, $p \in (1, \infty)$, $d \in [2, \infty) \cap \mathbb{N}$, $u \in \mathbb{H}_{d,h,\geq 0}$ it holds that*

$$\|\mathcal{D}_y^h u\|_{L_h^p(\omega_h^{d-1})} \leq C(d, p) \left[\sum_{i=1}^{d-1} \|\mathcal{D}_x^h u\|_{L_h^p(\omega_h^{d-1})} \right]. \quad (3.80)$$

4 Proof of the Main Theorem

This section combines the results in the previous sections to prove Theorem 1.4. Recall that we use the same terminology as in the continuum case [3]: estimate (1.9) is called *the Dirichlet case* and estimate (1.10) is called *the Neumann case*.

Section 4.1 adapts inequality (78) in [3] into Corollary 4.10, which is proven by the same idea as in [3], i.e., by constructing a telescope series of harmonic functions on haft spaces by means of Dirichlet conditions, see Settings 4.2 and 4.5 below.

In Section 4.2, Corollary 4.15 proves the main result in the Dirichlet case. Corollary 4.15 is essentially proved by Lemma 4.14 where we use the idea of odd reflections as in Step 2 in the proof of Lemma 4 in [3] and Lemma 4.12 illustrates this idea in the discrete case. For convenience we first give a heuristic proof of Lemma 4.14 in the case $d = 2$.

Section 4.3 adapts inequality (79) in [3] into Corollary 4.24. Here, we also construct a telescope series of harmonic functions on haft spaces, however, now by means of Neumann conditions. The reader will see that there are quite a lot of similarities between the Dirichlet and the Neumann case. However, the two cases are not identical and it is necessary to adapt rigorously every step of the proof due to the discreteness.

In Section 4.4 we prove carefully the main theorem in the Neumann case, see Theorem 4.27, although the argument is quite straightforward in the continuum case, as said in the last sentence in the proof of Lemma 4 in [3]. The idea of even reflections is explained in Lemma 4.26 where some minor arguments are used to deal with the discreteness. For convenience we give a heuristic proof of Theorem 4.27 in the case $d = 2$.

Throughout this section we always use the notation given by Setting 4.1 below.

Setting 4.1 For every $d \in \mathbb{N}$, $A \subset \mathbb{Z}^d$, $u: A \rightarrow \mathbb{R}$ let $D_i^+ u: \{x \in A: x + \mathbf{e}_i^d \in A\} \rightarrow \mathbb{R}$, $i \in [1, d] \cap \mathbb{Z}$, be the functions which satisfy for all $i \in [1, d] \cap \mathbb{Z}$, $x \in A$ with $x + \mathbf{e}_i^d \in A$ that $(D_i^+ u)(x) = u(x + \mathbf{e}_i^d) - u(x)$, and $D_i^- u: \{x \in A: x - \mathbf{e}_i^d \in A\} \rightarrow \mathbb{R}$, $i \in [1, d] \cap \mathbb{Z}$, be the functions which satisfy for all $i \in [1, d] \cap \mathbb{Z}$, $x \in A$ with $x - \mathbf{e}_i^d \in A$ that $(D_i^- u)(x) = u(x - \mathbf{e}_i^d) - u(x)$. For every finite set A and every function $u: A \rightarrow \mathbb{R}$ let $\langle u \rangle_A \in \mathbb{R}$ be given by $\langle u \rangle_A = \frac{1}{|A|} \sum_{x \in A} u(x)$.

4.1 Construction of Dirichlet Extensions

Setting 4.2 (Harmonic functions and boundary conditions) Let Setting 4.1 be given. For $L \in \mathbb{N}$ let $\mathbb{I}_L$ be the set given by $\mathbb{I}_L = [-L + 1, L] \cap \mathbb{Z}$. For every $N, L \in \mathbb{N}$, $d \in [2, \infty) \cap \mathbb{N}$ let $\mathbb{S}_{d,L,N}$ be the set of all functions $u: \mathbb{Z}^{d-1} \times ([0, N] \cap \mathbb{Z}) \rightarrow \mathbb{R}$ with the properties that

- (i) it holds for all $x \in \mathbb{Z}^{d-1} \times ([0, N] \cap \mathbb{Z})$, $i \in [1, d-1] \cap \mathbb{Z}$ that $u(x) = u(x + 2L\mathbf{e}_i^d)$ and
- (ii) it holds for all $x \in \mathbb{Z}^{d-1} \times ([1, N-1] \cap \mathbb{Z})$ that $(\Delta u)(x) = 0$,

let $\mathbb{B}_{d,L,N}$ be the set of all boundary conditions $u: \mathbb{Z}^{d-1} \times \{0, N\} \rightarrow \mathbb{R}$ which satisfy for all $x \in \mathbb{Z}^{d-1} \times \{0, N\}$, $i \in [1, d-1] \cap \mathbb{Z}$ that $u(x) = u(x + 2L\mathbf{e}_i^d)$, let $V_{d,L,N}^\tau$ be the set of vertices given by

$$V_{d,L,N}^\tau = \left([0, L]^{d-1} \times [0, N]\right) \setminus \left([1, L-1]^{d-1} \times [1, N-1]\right), \quad (4.1)$$

let $E_{d,L,N}^\# \subseteq E_d$ be the set of edges given by

$$E_{d,L,N}^\# = \left\{(x, y) \in E_d: \frac{1}{2}(x + y) \in \left([0, L]^{d-1} \times [0, N]\right) \setminus \left([1, L-1]^{d-1} \times [1, N-1]\right)\right\}, \quad (4.2)$$

let $\mathbb{H}_{d,L,\geq 0}$ be the set of all bounded functions $u: \mathbb{Z}^{d-1} \times \mathbb{N}_0 \rightarrow \mathbb{R}$ with the properties that

- (i) it holds for all $x \in \mathbb{Z}^{d-1} \times \mathbb{N}_0$, $i \in [1, d-1] \cap \mathbb{Z}$ that $u(x) = u(x + 2Le_i^d)$ and
- (ii) it holds for all $x \in \mathbb{Z}^{d-1} \times \mathbb{N}$ that $(\Delta u)(x) = 0$,

and let $\mathbb{H}_{d,L,\leq N}$ be the set of all bounded functions $u: \mathbb{Z}^{d-1} \times ((-\infty, N] \cap \mathbb{Z}) \rightarrow \mathbb{R}$ with the properties that

- (i) it holds for all $x \in \mathbb{Z}^{d-1} \times ((-\infty, N] \cap \mathbb{Z})$, $i \in [1, d-1] \cap \mathbb{Z}$ that $u(x) = u(x + 2Le_i^d)$ and
- (ii) it holds for all $x \in \mathbb{Z}^{d-1} \times ((-\infty, N-1] \cap \mathbb{Z})$ that $(\Delta u)(x) = 0$.

Setting 4.3 and Lemma 4.4 below prepare two important inequalities, which follow from the results in the last sections. We will bound the telescope series by a geometric series using the fact that $\forall d \in [2, \infty)$, $p \in (1, \infty)$: $C_1(d, p) < 1$.

Setting 4.3 (Regularity constants) Assume Setting 4.2 and let $C_1, C_2: ([2, \infty) \cap \mathbb{N}) \times (1, \infty) \rightarrow [0, \infty]$ be the functions which satisfy that

- i) it holds for all $d \in [2, \infty) \cap \mathbb{N}$, $p \in (1, \infty)$ that $C_1(d, p)$ is the smallest real extended number such that for all $N, L \in \mathbb{N}$, $u \in \mathbb{H}_{d,L,\geq 0}$ with $1/4 \leq N/L$ and $\langle u \rangle_{\mathbb{I}_L^{d-1} \times \{0\}} = 0$ it holds that

$$\|u\|_{L^p(\mathbb{I}_L^{d-1} \times \{N\})} \leq C_1(d, p) \|u\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})} \quad (4.3)$$

and

- ii) it holds for all $d \in [2, \infty) \cap \mathbb{N}$, $p \in (1, \infty)$ that $C_2(d, p)$ is the smallest real extended number such that for all $N, L \in \mathbb{N}$, $i \in [1, d-1] \cap \mathbb{Z}$, $u \in \mathbb{H}_{d,L,\geq 0}$ with $N/L \leq 4$ it holds that

$$\|\nabla u\|_{L^p(E_{d,L,N}^\#)} \leq C_2(d, p) \left[\sum_{j=1}^{d-1} \left\| D_j^+ u \right\|_{L^p(\mathbb{I}_L^{d-1} \times \{0, N\})} \right]. \quad (4.4)$$

Lemma 4.4 Assume Setting 4.3 and let $d \in [2, \infty) \cap \mathbb{N}$, $p \in (1, \infty)$ be fixed. Then it holds that $C_1(d, p) < 1$ and $C_2(d, p) < \infty$.

Heuristic proof of Lemma 4.4 First, note that the discrete derivatives of a harmonic function are still harmonic. Using Corollary 2.2 (with the function replaced by the derivatives) we bound the differences with respect to the edges with endpoints on the face $\{x_1 = 0\}$ of the box by the tangential differences on the bottom $\mathbb{Z}^{d-1} \times \{0\}$ (see Eq. 4.5). Next, using Item (i) in Corollary 3.1 we bound the normal differences on the top $\mathbb{Z}^{d-1} \times \{N\}$ by the normal differences on the bottom $\mathbb{Z}^{d-1} \times \{0\}$ (see Eq. 4.6). Furthermore, using Item (ii) in Corollary 3.1 we bound the normal differences on the bottom $\mathbb{Z}^{d-1} \times \{0\}$ (and hence also that on the top) by the tangential differences the edges on the bottom $\mathbb{Z}^{d-1} \times \{0\}$ (see Eq. 4.7). Using a permutation of the coordinates we hence bound the differences with respect to all edges with one endpoints on the boundary of the box by the tangential differences on the bottom. $\square$

Rigorous proof of Lemma 4.4 Corollary 3.1 implies that there exists $c_0: (0, \infty) \rightarrow (0, 1)$ such that for all $N, L \in \mathbb{N}$, $u \in \mathbb{H}_{d,L,\geq 0}$ with $N/L \geq \underline{r}$ and $\langle u \rangle_{\mathbb{I}_L^{d-1} \times \{0\}} = 0$ it holds that $\|u\|_{L^p(\mathbb{I}_L^{d-1} \times \{N\})} \leq c_0(\underline{r}) \|u\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})}$. This (with $\underline{r} \leftarrow 1/4$) proves that $C_1(d, p) < 1$. Next, recall that Corollary 2.2 shows that there exists $c_1: (0, \infty) \rightarrow (0, \infty)$ such that for all $\bar{r} \in (0, \infty)$, $N \in (0, L\bar{r}) \cap \mathbb{N}$, $u \in \mathbb{H}_{d,L,\geq 0}$ it holds that $\|u\|_{L^p(\{0\} \times \mathbb{I}_L^{d-2} \times ([1, N] \cap \mathbb{Z}))} \leq$

$c_1(\bar{r})\|u\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})}$. This (with $\bar{r} \leftarrow 4$ and $u \leftarrow D_i^\pm u$ for $i \in [1, d] \cap \mathbb{Z}$, $L, N \in \mathbb{N}$, $u \in \mathbb{H}_{d,L,\geq 0}$ with $N/L \leq 4$) implies that for all $L, N \in \mathbb{N}$ with $N/L \leq 4$, $u \in \mathbb{H}_{d,L,\geq 0}$ it holds that

$$\|D_i^\pm u\|_{L^p(\{0\} \times \mathbb{I}_L^{d-2} \times ([1, N] \cap \mathbb{Z}))} \leq c_1(4) \|D_i^\pm u\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})}. \quad (4.5)$$

Next, Corollary 2.6 (with $u \leftarrow D_d^+ u$ and $N \leftarrow N - 1$) shows for all $L \in \mathbb{N}$, $u \in \mathbb{H}_{d,L,\geq 0}$ that

$$\|D_d^- u\|_{L^p(\mathbb{I}_L^{d-1} \times \{N\})} \leq \|D_d^+ u\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})}. \quad (4.6)$$

Hence, Corollary 3.1 shows that there exists $c_2 \in (0, \infty)$ such that for all $L \in \mathbb{N}$, $u \in \mathbb{H}_{d,L,\geq 0}$ it holds that

$$\|D_d^- u\|_{L^p(\mathbb{I}_L^{d-1} \times \{N\})} \leq \|D_d^+ u\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})} \leq c_2 \left[\sum_{i=1}^{d-1} \|D_i^+ u\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})} \right]. \quad (4.7)$$

Combining (4.5) and (4.7) then yields that there exists $c_3 \in (0, \infty)$ such that for all $N, L \in \mathbb{N}$, $i \in [1, d-1] \cap \mathbb{Z}$, $u \in \mathbb{H}_{d,L,\geq 0}$ with $N/L \leq 4$ it holds that

$$\|\nabla u\|_{L^p(E_{d,L,N}^\#)} \leq c_3 \left[\sum_{j=1}^{d-1} \|D_j^+ u\|_{L^p(\mathbb{I}_L^{d-1} \times \{0, N\})} \right]. \quad (4.8)$$

This shows that $C_2(d, p) < \infty$. The proof of Lemma 4.4 is thus completed. $\square$

Existence and uniqueness of the solutions to the Dirichlet problems on haft spaces (shown, e.g., by means of Fourier transforms in Setting 4.5) ensure that the sequences $(u_k)_{k \in \mathbb{N}}$ in Setting 4.5 below are well-defined by Eqs. 4.11–4.13.

Setting 4.5 (Telescope sequence for the Dirichlet case) Assume Setting 4.3, let $N, L \in \mathbb{N}$, $p \in (1, \infty)$, $d \in [2, \infty) \cap \mathbb{N}$ be fixed and satisfy that $1/4 \leq L/N \leq 4$, let $v \in \mathbb{B}_{d,L,N}$ be a boundary condition which satisfies that

$$\langle v \rangle_{\mathbb{I}_L^{d-1} \times \{0\}} = 0 \quad \text{and} \quad \forall x \in \mathbb{Z}^{d-1} \times \{N\}: \quad v(x) = 0, \quad (4.9)$$

and let

$$(u_{2k+1})_{k \in \mathbb{N}_0} \subseteq \mathbb{H}_{d,L,\geq 0} \quad \text{and} \quad (u_{2k+2})_{k \in \mathbb{N}_0} \subseteq \mathbb{H}_{d,L,\leq N} \quad (4.10)$$

be the sequences given by

$$\forall x \in \mathbb{Z}^{d-1} \times \{0\}: \quad u_1(x) = v(x), \quad (4.11)$$

$$\forall k \in \mathbb{N}_0, x \in \mathbb{Z}^{d-1} \times \{N\}: \quad u_{2k+2}(x) = u_{2k+1}(x), \quad (4.12)$$

$$\forall k \in \mathbb{N}, x \in \mathbb{Z}^{d-1} \times \{0\}: \quad u_{2k+1}(x) = u_{2k}(x). \quad (4.13)$$

Lemma 4.6 (Convergence of the telescope series) Assume Setting 4.5. Then

- i) it holds for all $n \in \mathbb{N}$ that $\max_{y \in \{0, N\}} \|u_n\|_{L^p(\mathbb{I}_L^{d-1} \times \{y\})} = C_1(d, p)^{n-1} \|u_1\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})}$ and
- ii) it holds for all $x \in \mathbb{Z}^{d-1} \times ([0, N] \cap \mathbb{Z})$ that $\sum_{k=1}^{\infty} |u_k(x)| < \infty$.

Proof of Lemma 4.6 Observe that Eqs. 4.11 and 4.9 imply that $\langle u_1 \rangle_{\mathbb{I}_L^{d-1} \times \{0\}} = 0$. This, Eqs. 4.12 and 4.13, and Lemma 2.4 ensure for all $n \in \mathbb{N}$ that $\langle u_n \rangle_{\mathbb{I}_L^{d-1} \times \{0\}} = \langle u_n \rangle_{\mathbb{I}_L^{d-1} \times \{N\}} = 0$. This, Eqs. 4.12 and 4.13, Lemma 2.5, and Eq. 4.3 show for all $k \in \mathbb{N}_0$, $y \in [0, N] \cap \mathbb{Z}$ that

$$\begin{aligned} \|u_{2k+2}\|_{L^p(\mathbb{I}_L^{d-1} \times \{y\})} &\leq \|u_{2k+2}\|_{L^p(\mathbb{I}_L^{d-1} \times \{N\})} \\ &= \|u_{2k+1}\|_{L^p(\mathbb{I}_L^{d-1} \times \{N\})} \leq C_1(d, p) \|u_{2k+1}\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})} \end{aligned} \quad (4.14)$$

and

$$\begin{aligned} \|u_{2k+3}\|_{L^p(\mathbb{I}_L^{d-1} \times \{y\})} &\leq \|u_{2k+3}\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})} \\ &= \|u_{2k+2}\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})} \leq C_1(d, p) \|u_{2k+2}\|_{L^p(\mathbb{I}_L^{d-1} \times \{N\})}. \end{aligned} \quad (4.15)$$

This and an induction argument prove for all $n \in \mathbb{N}$ that

$$\begin{aligned} \max_{y \in [0, N] \cap \mathbb{Z}} \|u_{n+1}\|_{L^p(\mathbb{I}_L^{d-1} \times \{y\})} &\leq C_1(d, p) \left[\max_{y \in [0, N]} \|u_n\|_{L^p(\mathbb{I}_L^{d-1} \times \{y\})} \right] \\ &\leq C_1(d, p)^n \left[\max_{y \in [0, N]} \|u_1\|_{L^p(\mathbb{I}_L^{d-1} \times \{y\})} \right] = C_1(d, p)^n \|u_1\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})}. \end{aligned} \quad (4.16)$$

This shows Item (i). Next, Eq. 4.16, the fact that $C_1 \in (0, 1)$, and the convergence of the geometric series assure for all $y \in [0, N] \cap \mathbb{Z}$ that $\sum_{n=1}^{\infty} \|u_n\|_{L^p(\mathbb{I}_L^{d-1} \times \{y\})} < \infty$. This implies Item (ii). The proof of Lemma 4.6 is thus completed. $\square$

Lemma 4.7 (Upper bound for the telescope series) *Assume Setting 4.5 and let $w: \mathbb{Z}^{d-1} \times ([0, N] \cap \mathbb{Z}) \rightarrow \mathbb{R}$ be the function given by*

$$\forall x \in \mathbb{Z}^{d-1} \times ([0, N] \cap \mathbb{Z}): \quad w(x) = \left[\sum_{k=1}^{\infty} (-1)^{k+1} u_k(x) \right]. \quad (4.17)$$

Then it holds for all $x \in \mathbb{Z}^{d-1} \times \{0, N\}$ that $w(x) = v(x)$, $w \in \mathbb{S}_{d,L,N}$, and

$$\|\nabla w\|_{L^p(E_{d,L,N}^\#)} \leq \frac{C_2(d, p)}{1 - C_1(d, p)} \left[\sum_{i=1}^{d-1} \|D_i^+ v\|_{L^p(\mathbb{I}_L^{d-1} \times \{0, N\})} \right] \quad (4.18)$$

and

$$\|w\|_{L^p(V_{d,L,N}^\dagger)} \leq \frac{\|v\|_{L^p(\mathbb{I}_L^{d-1} \times \{0, N\})}}{1 - C_1(d, p)}. \quad (4.19)$$

Proof of Lemma 4.7 Note that Eqs. 4.17, 4.11–4.13, and a telescope sum argument demonstrate that for all $x \in \mathbb{I}_L^{d-1} \times \{0\}$ it holds that $w(x) = \sum_{k=1}^{\infty} (-1)^{k+1} u_k(x) = u_1(x) = v(x)$ and for all $x \in \mathbb{I}_L^{d-1} \times \{N\}$ it holds that $w(x) = \sum_{k=1}^{\infty} (-1)^{k+1} u_k(x) = 0 = v(x)$. Next, Eq. 4.10 proves that

$$\begin{aligned} \forall n \in \mathbb{N}_0, x \in \mathbb{I}_L^{d-1} \times ([0, N] \cap \mathbb{Z}), i \in [1, d-1] \cap \mathbb{Z}: \quad u_n(x + 2Le_i^d) &= u_n(x) \quad \text{and} \\ \forall n \in \mathbb{N}_0, x \in \mathbb{I}_L^{d-1} \times ([1, N-1] \cap \mathbb{Z}): \quad (\Delta u_n)(x) &= 0. \end{aligned} \quad (4.20)$$

This and Eq. 4.17 imply that $w \in \mathbb{S}_{d,L,N}$. Next, observe that Eq. 4.10 and a simple calculation imply for all $k \in \mathbb{N}_0$, $i \in [1, d-1] \cap \mathbb{Z}$ that $D_i^+ u_{2k+1} \in \mathbb{H}_{d,L,\geq 0}$ and $D_i^+ u_{2k+2} \in \mathbb{H}_{d,L,\leq N}$. Roughly speaking, discrete derivatives of harmonic functions are also harmonic. This, Eq. 4.17, the triangle inequality, Eq. 4.4 (applied with $u \leftarrow u_k$ for $k \in \mathbb{N}$),

Item (i) in Lemma 4.6 (with $v \leftarrow \mathbf{D}_i^+ u_1$ and $(u_k)_{k \in \mathbb{N}} \leftarrow (\mathbf{D}_i^+ u_k)_{k \in \mathbb{N}}$ for $i \in [1, d-1] \cap \mathbb{Z}$), the fact that $\forall x \in (0, 1)$: $\sum_{k=1}^{\infty} x^{k-1} = (1-x)^{-1}$, and Eq. 4.11 ensure that

$$\begin{aligned} & \|\nabla w\|_{L^p(E_{d,L,N}^\#)} \\ & \leq \left\| \sum_{k=1}^{\infty} \nabla u_k \right\|_{L^p(E_{d,L,N}^\#)} \leq \sum_{k=1}^{\infty} \|\nabla u_k\|_{L^p(E_{d,L,N}^\#)} \leq \sum_{k=1}^{\infty} \left[C_2(d, p) \left(\sum_{i=1}^{d-1} \|\mathbf{D}_i^+ u_k\|_{L^p(\mathbb{I}_L^{d-1} \times \{0, N\})} \right) \right] \\ & \leq \sum_{k=1}^{\infty} \left[C_2(d, p) \left(\sum_{i=1}^{d-1} C_1(d, p)^{k-1} \|\mathbf{D}_i^+ u_1\|_{L^p(\mathbb{I}_L^{d-1} \times \{0, N\})} \right) \right] \\ & = \frac{C_2(d, p)}{1 - C_1(d, p)} \left[\sum_{i=1}^{d-1} \|\mathbf{D}_i^+ u_1\|_{L^p(\mathbb{I}_L^{d-1} \times \{0, N\})} \right] \leq \frac{C_2(d, p)}{1 - C_1(d, p)} \left[\sum_{i=1}^{d-1} \|\mathbf{D}_i^+ v\|_{L^p(\mathbb{I}_L^{d-1} \times \{0, N\})} \right]. \quad (4.21) \end{aligned}$$

This shows (4.18). The argument for Eq. 4.19 is similar but simpler since we do not need to compare tangential and normal edges: we use the geometric series and apply Lemma 4.6 with $v \leftarrow u_1$ and $(u_k)_{k \in \mathbb{N}} \leftarrow (u_k)_{k \in \mathbb{N}}$. The proof of Lemma 4.7 is thus completed. $\square$

Lemma 4.8 below considers general boundary conditions (i.e. without the restriction (4.9)).

Lemma 4.8 (Upper bound for harmonic functions on strips) *Assume Setting 4.2, let $N, L \in \mathbb{N}$, $d \in [2, \infty) \cap \mathbb{N}$, $v \in \mathbb{B}_{d,L,N}$, $p \in (1, \infty)$ satisfy that $1/4 \leq L/N \leq 4$. Then there exists $w \in \mathbb{S}_{d,L,N}$ such that for all $x \in \mathbb{Z}^{d-1} \times \{0, N\}$ it holds that $w(x) = v(x)$ and*

$$\|\nabla w\|_{L^p(E_{d,L,N}^\#)} \leq \frac{2C_2(d, p)}{1 - C_1(d, p)} \left[\sum_{i=1}^{d-1} \|\mathbf{D}_i^+ v\|_{L^p(\mathbb{I}_L^{d-1} \times \{0, N\})} \right] + \frac{4}{N} \|v\|_{L^p(\mathbb{I}_L^{d-1} \times \{0, N\})} \quad (4.22)$$

and

$$\|w\|_{L^p(V_{d,L,N}^\#)} \leq \left[\frac{1}{1 - C_1(d, p)} + 2^p d 48^{d-1} \right] \|v\|_{L^p(\mathbb{I}_L^{d-1} \times \{0, N\})}. \quad (4.23)$$

Proof of Lemma 4.8 Throughout this proof let $v_1, v_2, v_3 \in \mathbb{B}_{d,L,N}$, $w_3: \mathbb{Z}^{d-1} \times ([0, N] \cap \mathbb{Z}) \rightarrow \mathbb{R}$ be the functions given by

$$\begin{aligned} \forall x \in \mathbb{Z}^{d-1} \times \{0\}: \quad & v_1(x) = v(x) - \langle v \rangle_{\mathbb{I}_L^{d-1} \times \{0\}}, \quad v_2(x) = 0, \quad v_3(x) = \langle v \rangle_{\mathbb{I}_L^{d-1} \times \{0\}}, \\ \forall x \in \mathbb{Z}^{d-1} \times \{N\}: \quad & v_1(x) = 0, \quad v_2(x) = v(x) - \langle v \rangle_{\mathbb{I}_L^{d-1} \times \{N\}}, \quad v_3(x) = \langle v \rangle_{\mathbb{I}_L^{d-1} \times \{N\}}, \\ \forall n \in [0, N] \cap \mathbb{Z}, x \in \mathbb{Z}^{d-1} \times \{n\}: \quad & w_3(x) = \frac{n}{N} \langle v \rangle_{\mathbb{I}_L^{d-1} \times \{0\}} + \frac{N-n}{N} \langle v \rangle_{\mathbb{I}_L^{d-1} \times \{N\}}. \end{aligned} \quad (4.24)$$

This implies that $w_3 \in \mathbb{S}_{d,L,N}$. Next, Eq. 4.24, the fact that $\forall a, b \in \mathbb{R}$: $|a - b|^p \leq 2^{p-1}(|a|^p + |b|^p)$, the fact that $\|1\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})}^p = \|1\|_{L^p(\mathbb{I}_L^{d-1} \times \{N\})}^p$, Jensen's inequality, and a very rough volume estimate imply that

$$\begin{aligned} \|\nabla w_3\|_{L^p(E_{d,L,N}^\#)}^p &= 2 \|\mathbf{D}_d^+ w_3\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})}^p + 2 \|\mathbf{D}_d^- w_3\|_{L^p(\mathbb{I}_L^{d-1} \times \{N\})}^p \\ &= 2 \left\| \frac{1}{N} \left(\langle v \rangle_{\mathbb{I}_L^{d-1} \times \{0\}} - \langle v \rangle_{\mathbb{I}_L^{d-1} \times \{N\}} \right) \right\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})}^p + 2 \left\| \frac{1}{N} \left(\langle v \rangle_{\mathbb{I}_L^{d-1} \times \{0\}} - \langle v \rangle_{\mathbb{I}_L^{d-1} \times \{N\}} \right) \right\|_{L^p(\mathbb{I}_L^{d-1} \times \{N\})}^p \\ &= \frac{4}{N^p} \left| \langle v \rangle_{\mathbb{I}_L^{d-1} \times \{0\}} - \langle v \rangle_{\mathbb{I}_L^{d-1} \times \{N\}} \right|^p \|1\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})}^p \\ &\leq \frac{4 \cdot 2^{p-1}}{N^p} \left[\langle |v|^p \rangle_{\mathbb{I}_L^{d-1} \times \{0\}} + \langle |v|^p \rangle_{\mathbb{I}_L^{d-1} \times \{N\}} \right] \|1\|_{L^p(\mathbb{I}_L^{d-1} \times \{N\})}^p \leq \frac{4^p}{N^p} \|v\|_{L^p(\mathbb{I}_L^{d-1} \times \{0, N\})}^p. \end{aligned} \quad (4.25)$$

and

$$\begin{aligned} \|w_3\|_{L^p(V_{d,L,N}^\tau)}^p &\leq \|1\|_{L^p(V_{d,L,N}^\tau)}^p \left(\frac{n}{N} \left| \langle v \rangle_{\mathbb{I}_L^{d-1} \times \{0\}} \right| + \frac{N-n}{N} \left| \langle v \rangle_{\mathbb{I}_L^{d-1} \times \{N\}} \right| \right)^p \\ &\leq \frac{2d(12N)^{d-1}}{L^{d-1}} 2^{p-1} \left(\left| \langle v|v| \rangle_{\mathbb{I}_L^{d-1} \times \{0\}} \right| + \left| \langle v|v| \rangle_{\mathbb{I}_L^{d-1} \times \{N\}} \right| \right) \leq 2^p d 48^{d-1} \|v\|_{L^p(\mathbb{I}_L^{d-1} \times \{0,N\})}^p. \end{aligned} \quad (4.26)$$

Furthermore, the fact that

$$\begin{aligned} \forall x \in \mathbb{Z}^{d-1} \times \{0\}: (\mathbf{D}_i^+ v_1)(x) &= (\mathbf{D}_i^+ v)(x), \\ \forall x \in \mathbb{Z}^{d-1} \times \{N\}: (\mathbf{D}_i^+ v_2)(x) &= (\mathbf{D}_i^+ v)(x), \quad \text{and} \quad \langle v_1 \rangle_{\mathbb{I}_L^{d-1} \times \{0\}} = \langle v_2 \rangle_{\mathbb{I}_L^{d-1} \times \{N\}} = 0 \end{aligned} \quad (4.27)$$

and Lemma 4.7 imply that there exist $w_1, w_2 \in \mathbb{S}_{d,L,N}$ such that

$$\forall x \in \mathbb{Z}^{d-1} \times \{0, N\}: \quad w_1(x) = v_1(x) \quad \text{and} \quad w_2(x) = v_2(x) \quad (4.28)$$

and such that for all $j \in \{1, 2\}$ it holds that

$$\begin{aligned} &\|\nabla w_j\|_{L^p(E_{d,L,N}^\#)} \\ &\leq \frac{C_2(d, p)}{1 - C_1(d, p)} \left[\sum_{i=1}^{d-1} \|\mathbf{D}_i^+ v_j\|_{L^p(\mathbb{I}_L^{d-1} \times \{0,N\})} \right] \leq \frac{C_2(d, p)}{1 - C_1(d, p)} \left[\sum_{i=1}^{d-1} \|\mathbf{D}_i^+ v\|_{L^p(\mathbb{I}_L^{d-1} \times \{0,N\})} \right] \end{aligned} \quad (4.29)$$

and

$$\|w_j\|_{L^p(V_{d,L,N}^\tau)} \leq \frac{\|v_j\|_{L^p(\mathbb{I}_L^{d-1} \times \{0,N\})}}{1 - C_1(d, p)} \leq \frac{\|v\|_{L^p(\mathbb{I}_L^{d-1} \times \{0,N\})}}{1 - C_1(d, p)}. \quad (4.30)$$

Now, let $w: \mathbb{Z}^{d-1} \times ([0, N] \cap \mathbb{Z}) \rightarrow \mathbb{R}$ be the function which satisfies that

$$\forall x \in \mathbb{Z}^{d-1} \times ([0, N] \cap \mathbb{Z}): \quad w(x) = w_1(x) + w_2(x) + w_3(x). \quad (4.31)$$

Then Eqs. 4.24 and 4.28 imply for all $x \in \mathbb{Z}^{d-1} \times \{0, N\}$ that $w(x) = v(x)$. Next, Eq. 4.31, the triangle inequality, Eqs. 4.25, 4.29 and 4.30 imply that

$$\begin{aligned} &\|\nabla w\|_{L^p(E_{d,L,N}^\#)} \\ &\leq \sum_{j=1}^3 \|\nabla w_j\|_{L^p(E_{d,L,N}^\#)} \leq \frac{2C_2(d, p)}{1 - C_1(d, p)} \left[\sum_{i=1}^{d-1} \|\mathbf{D}_i^+ v\|_{L^p(\mathbb{I}_L^{d-1} \times \{0,N\})} \right] + \frac{4}{N} \|v\|_{L^p(\mathbb{I}_L^{d-1} \times \{0,N\})}^p \end{aligned} \quad (4.32)$$

and

$$\|w\|_{L^p(V_{d,L,N}^\tau)} \leq \sum_{j=1}^3 \|w_j\|_{L^p(V_{d,L,N}^\tau)} \leq \left[\frac{1}{1 - C_1(d, p)} + 2^p d 48^{d-1} \right] \|v\|_{L^p(\mathbb{I}_L^{d-1} \times \{0,N\})} \quad (4.33)$$

The proof of Lemma 4.8 is thus completed. $\square$

Lemma 4.8 shows the existence of solutions to Dirichlet problems. Combining this with the uniqueness, which easily follows, e.g., from the maximum principle, we obtain Corollary 4.9 below.

Corollary 4.9 *Let $d \in [2, \infty) \cap \mathbb{N}$, $p \in (1, \infty)$, $v: \mathbb{Z}^{d-1} \times \{0, N\} \rightarrow \mathbb{R}$ satisfy for all $x \in \mathbb{Z}^{d-1} \times \{0, N\}$, $i \in [1, d-1] \cap \mathbb{Z}$ that $v(x) = v(x + 2Le_i^d)$. Then there exists uniquely $w: \mathbb{Z}^{d-1} \times ([0, N] \cap \mathbb{Z}) \rightarrow \mathbb{R}$ such that*

- i) *it holds for all $x \in \mathbb{Z}^{d-1} \times \{0, N\}$ that $w(x) = v(x)$ and*
- ii) *it holds for all $x \in \mathbb{Z}^{d-1} \times ([1, N-1] \cap \mathbb{Z})$ that $(\Delta w)(x) = v(x)$.*

The existence and uniqueness, stated in Corollary 4.9, and Lemma 4.8 imply Corollary 4.10 below.

Corollary 4.10 For $L \in \mathbb{N}$ let $\mathbb{I}_L$ be the discrete interval given by $\mathbb{I}_L = [-L + 1, L] \cap \mathbb{Z}$, let $V_{d,L,N}^\tau$ be the set of vertices given by

$$V_{d,L,N}^\tau = \left([0, L]^{d-1} \times [0, N]\right) \setminus \left([1, L-1]^{d-1} \times [1, N-1]\right), \quad (4.34)$$

and let $E_{d,L,N}^\# \subseteq E_d$ be the set of edges given by

$$E_{d,L,N}^\# = \left\{ (x, y) \in E_d : \frac{1}{2}(x + y) \in \left([0, L]^{d-1} \times [0, N]\right) \setminus \left([1, L-1]^{d-1} \times [1, N-1]\right) \right\}. \quad (4.35)$$

Then there exists $C : ([2, \infty) \cap \mathbb{N}) \times (1, \infty) \rightarrow (0, \infty)$ such that for all $d \in [2, \infty) \cap \mathbb{N}$, $p \in (1, \infty)$, $N, L \in \mathbb{N}$, and for all functions $w : \mathbb{Z}^{d-1} \times ([0, N] \cap \mathbb{Z}) \rightarrow \mathbb{R}$ with the property that

$$\begin{aligned} \forall x \in \mathbb{Z}^{d-1} \times ([0, N] \cap \mathbb{Z}), i \in [1, d-1] \cap \mathbb{Z}: \quad w(x) &= w(x + 2L\mathbf{e}_i^d), \\ 1/4 \leq L/N \leq 4, \quad \text{and} \quad \forall x \in \mathbb{Z}^{d-1} \times ([1, N-1] \cap \mathbb{Z}): \quad (\Delta w)(x) &= 0 \end{aligned} \quad (4.36)$$

it holds that

$$\begin{aligned} \|\nabla w\|_{L^p(E_{d,L,N}^\#)} + \frac{1}{N} \|w\|_{L^p(V_{d,L,N}^\tau)} \\ \leq C(d, p) \left[\sum_{i=1}^{d-1} \|\mathbf{D}_i^+ w\|_{L^p(\mathbb{I}_L^{d-1} \times \{0, N\})} + \frac{1}{N} \|w\|_{L^p(\mathbb{I}_L^{d-1} \times \{0, N\})} \right]. \end{aligned} \quad (4.37)$$

4.2 Proof of the Main Result in the Dirichlet Case

Lemma 4.11 Let $N \in \mathbb{N}$, $f : \mathbb{Z} \rightarrow \mathbb{R}$ satisfy that $\forall x \in \mathbb{Z} : f(x) = f(x + 2N)$ and $\forall x \in [-N + 1, N] \cap \mathbb{Z} : f(x) = -f(-x)$. Then $f(0) = f(N) = 0$.

Proof of Lemma 4.11 The fact $f(0) = -f(-0) = -f(0)$ proves that $f(0) = 0$. Next, the fact that $f(N) = f(-N + 2N) = f(-N) = -f(N)$ proves that $f(N) = 0$. This completes the proof of Lemma 4.11. $\square$

Lemma 4.12 (Odd reflections) Let $d \in [2, \infty) \cap \mathbb{N}$, $w : \mathbb{Z}^{d-1} \times ([0, N] \cap \mathbb{Z}) \rightarrow \mathbb{R}$, $j \in [1, d-1] \cap \mathbb{Z}$ satisfy that

$$\forall x \in \mathbb{Z}^{d-1} \times ([0, N] \cap \mathbb{Z}), i \in [1, d-1] \cap \mathbb{Z}: \quad w(x) = w(x + 2L\mathbf{e}_i^d), \quad (4.38)$$

$$\forall x \in \mathbb{Z}^{d-1} \times ([1, N-1] \cap \mathbb{Z}): \quad (\Delta w)(x) = 0, \quad (4.39)$$

$$\forall x \in \mathbb{Z}^{j-1} \times [-N + 1, N] \cap \mathbb{Z} \times \mathbb{Z}^{d-j-1} \times \{0, N\}: \quad w(x) = -w(x - 2x_j\mathbf{e}_j^d). \quad (4.40)$$

Then it holds that

$$\forall x \in \mathbb{Z}^{j-1} \times \{0, N\} \times \mathbb{Z}^{d-j-1} \times ([0, N] \cap \mathbb{Z}): \quad w(x) = 0. \quad (4.41)$$

Proof of Lemma 4.12 Let $\tilde{w} : \mathbb{Z}^{d-1} \times ([0, N] \cap \mathbb{Z}) \rightarrow \mathbb{R}$ be the function which satisfies for all $x \in \mathbb{Z}^{d-1} \times ([0, N] \cap \mathbb{Z})$ that $\tilde{w}(x) = -w(x - 2x_j\mathbf{e}_j^d)$. Then Eq. 4.39 implies that $\forall x \in \mathbb{Z}^{d-1} \times ([1, N-1] \cap \mathbb{Z}): (\Delta \tilde{w})(x) = 0$ and Eq. 4.38 implies that $\forall x \in \mathbb{Z}^{d-1} \times ([0, N] \cap \mathbb{Z}), i \in [1, d-1] \cap \mathbb{Z}: \tilde{w}(x) = \tilde{w}(x + 2L\mathbf{e}_i^d)$. This and Eq. 4.40 yield that $\forall x \in \mathbb{Z}^{d-1} \times \{0, N\}: w(x) = \tilde{w}(x)$. Corollary 4.9 hence shows for all $x \in \mathbb{Z}^{d-1} \times ([0, N] \cap \mathbb{Z})$ that $w(x) = \tilde{w}(x)$, i.e., $w(x) = -w(x - 2x_j\mathbf{e}_j^d)$. This and Lemma 4.11 (with $f \leftarrow (\mathbb{Z} \ni \xi \mapsto w(x_1, \dots, x_{j-1}, \xi, x_{j+1}, \dots, x_d) \in \mathbb{R})$, i.e., applied to the j -th coordinate) complete the proof of Lemma 4.12. $\square$

The sets $E_{d,N}^\#$, $E_{d,N}^\tau$, $V_{d,N}^\tau$ in Setting 4.13 below are illustrated in Fig. 1: $E_{d,N}^\#$ consists of all tangential and normal (red and blue) edges; $E_{d,N}^\tau$ consists of all tangential (red) edges; $V_{d,N}^\tau$ consists of all points on the boundary (red). Furthermore, this setting provides two ingredients that we need for the next step: Corollary 4.10 (see Eq. 4.46) and the Poincaré inequality (see Eq. 4.47).

Setting 4.13 Let Setting 4.1 be given. For every $d, N \in [2, \infty) \cap \mathbb{N}$ let $E_{d,N}^\#, E_{d,N}^\tau \subseteq E_d$, $\mathbb{I}_N \subseteq \mathbb{Z}$, $V_{d,N}^\tau \subseteq \mathbb{Z}^d$ be the sets given by

$$\mathbb{I}_N = [-N + 1, N] \cap \mathbb{Z}, \quad V_{d,N}^\tau = \mathbb{Z}^d \cap ([0, N]^d) \setminus ((0, N)^d), \quad (4.42)$$

$$E_{d,N}^\# = \left\{ (x, y) \in E_d : \frac{1}{2}(x + y) \in ([0, N]^d) \setminus ([1, N - 1]^d) \right\}, \quad (4.43)$$

$$E_{d,N}^\tau = \left\{ (x, y) \in E_d : x, y \in ([0, N]^d) \setminus ((0, N)^d) \right\}. \quad (4.44)$$

Let $c : ([2, \infty) \cap \mathbb{N}) \times (1, \infty) \rightarrow (0, \infty)$ be a function whose existence is ensured by Corollary 4.10 and which satisfies that for all $d, N \in [2, \infty) \cap \mathbb{N}$, $p \in (1, \infty)$, $w : \mathbb{Z}^{d-1} \times ([0, N] \cap \mathbb{Z}) \rightarrow \mathbb{R}$ with

$$\begin{aligned} \forall i \in [1, d - 1] \cap \mathbb{Z}, x \in \mathbb{Z}^{d-1} \times ([0, N] \cap \mathbb{Z}) : \quad w(x + 2N e_i^d) &= w(x) \quad \text{and} \\ \forall i \in [1, d - 1] \cap \mathbb{Z}, x \in \mathbb{Z}^{d-1} \times ([1, N - 1] \cap \mathbb{Z}) : \quad (\Delta w)(x) &= 0 \end{aligned} \quad (4.45)$$

it holds that

$$\|\nabla w\|_{L^p(E_{d,N}^\#)} + \frac{1}{N} \|w\|_{L^p(V_{d,N}^\tau)} \leq c(d, p) \left[\sum_{i=1}^{d-1} \|\mathbf{D}_i^+ w\|_{L^p(\mathbb{I}_N^{d-1} \times \{0, N\})} + \frac{1}{N} \|w\|_{L^p(\mathbb{I}_N^{d-1} \times \{0, N\})} \right]. \quad (4.46)$$

Let $c_{PI} : \mathbb{N} \times (1, \infty) \rightarrow [0, \infty)$ be a function which satisfies the Poincaré inequality, i.e., it holds for all $p \in (1, \infty)$, $d, N \in \mathbb{N} \cap [2, \infty)$, $u \in V_{d,N}^\tau$ that

$$\frac{1}{N} \inf_{a \in \mathbb{R}} \|u - a\|_{L^p(V_{d,N}^\tau)} \leq c_{PI}(d, p) \|\nabla u\|_{L^p(E_{d,N}^\tau)}. \quad (4.47)$$

Let $\mathbb{Q}_{d,N}$ be the set of all functions $u : ([0, N] \cap \mathbb{Z})^d \rightarrow \mathbb{R}$ which satisfy for all $x \in ([1, N - 1] \cap \mathbb{Z})^d$ that $(\Delta u)(x) = 0$.

Lemma 4.14 Assume Setting 4.13 and let $d, N \in [2, \infty) \cap \mathbb{N}$, $p \in (1, \infty)$, $u \in \mathbb{Q}_{d,N}$ be fixed. Then

$$\|\nabla u\|_{L^p(E_{d,N}^\#)} \leq d(3^d c(d, p) + 1)^d \left[\|\nabla u\|_{L^p(E_N^\tau)} + \frac{1}{N} \|u\|_{L^p(V_{d,N}^\tau)} \right]. \quad (4.48)$$

Heuristic proof of Lemma 4.14 for $d = 2$ First, we decompose u into harmonic functions with periodic boundary conditions. The idea is summarized in Fig. 4. Set $w_1 = u$ on $\{0, N\} \times \{0, \dots, N\}$, extend w_1 to $\{0, N\} \times \{0, \dots, 2N\}$ by an even reflection, i.e., require for all $x_1 \in \{0, N\}$, $x_2 \in \{0, \dots, N\}$ that $w_1(x_1, 2N - x_2) = w_1(x_1, x_2)$, and extend w_1 $2N$ -periodically to $\{0, N\} \times \mathbb{Z}$, and to $\{1, \dots, N - 1\} \times \mathbb{Z}$, using Lemma 4.8, so that it is $\Delta w_1 = 0$ on $\{1, \dots, N - 1\} \times \mathbb{Z}$. Next, define $w_2 = u - w_1$ on $\{0, \dots, N\} \times \{0, N\}$. By definition of w_1 it follows that $w_2 = 0$ on $\{0, N\} \times \{0, N\}$. Extend w_2 to $\{0, \dots, 2N\} \times \{0, N\}$ by an odd reflection, i.e., set for all $x_1 \in \{0, \dots, N\}$, $x_2 \in \{0, N\}$ that $w_2(2N - x_1, x_2) := -w_2(x_1, x_2)$, extend w_2 $2N$ -periodically to $\mathbb{Z} \times \{0, N\}$, and extend w_2 to $\mathbb{Z} \times \{1, \dots, N\}$ so that $\Delta w_2 = 0$ on $\mathbb{Z} \times \{1, \dots, N\}$. The odd reflection on the boundary and uniqueness of the extension show that $w_2 = 0$ on $\{0, N\} \times \{0, \dots, N\}$. It is easy to see that $w_1 + w_2 = u$ on the discrete boundary of $\{0, \dots, N\}^2$, i.e.,

$$w_1 + w_2 = u \quad \text{on} \quad (\{0, N\} \times \{0, \dots, N\}) \cup (\{0, \dots, N\} \times \{0, N\}). \quad (4.49)$$

Fig. 4 Dirichlet decomposition,
 $d = 2$

The construction of w_1 , w_2 and Eq. 4.46 imply that

$$\begin{aligned} \|\nabla w_1\|_{L^p(E_{d,N}^\#)} + \frac{1}{N} \|w_1\|_{L^p(V_{d,N}^\tau)} &\lesssim \|D_2^+ w_1\|_{L^p(\{0,N\} \times \{0,\dots,N\})} + \frac{1}{N} \|w_1\|_{L^p(\{0,N\} \times \{0,\dots,N\})} \\ &\lesssim \|\nabla u\|_{L^p(E_{d,N}^\tau)} + \frac{1}{N} \|u\|_{L^p(V_{d,N}^\tau)}, \end{aligned} \quad (4.50)$$

$$\begin{aligned} \|\nabla w_2\|_{L^p(E_{d,N}^\#)} + \frac{1}{N} \|w_2\|_{L^p(V_{d,N}^\tau)} &\lesssim \|D_1^+ w_2\|_{L^p(\{0,\dots,N\} \times \{0,N\})} + \frac{1}{N} \|w_2\|_{L^p(\{0,\dots,N\} \times \{0,N\})} \\ &= \|D_1^+(u - w_1)\|_{L^p(\{0,\dots,N\} \times \{0,N\})} + \frac{1}{N} \|u - w_1\|_{L^p(\{0,\dots,N\} \times \{0,N\})} \\ &\lesssim \|\nabla u\|_{L^p(E_{d,N}^\tau)} + \frac{1}{N} \|u\|_{L^p(V_{d,N}^\tau)} + \|\nabla w_1\|_{L^p(E_{d,N}^\#)} + \frac{1}{N} \|w_1\|_{L^p(V_{d,N}^\tau)} \\ &\lesssim \|\nabla u\|_{L^p(E_{d,N}^\tau)} + \frac{1}{N} \|u\|_{L^p(V_{d,N}^\tau)}, \end{aligned} \quad (4.51)$$

and

$$\|\nabla u\|_{L^p(E_{d,N}^\#)} \leq \|\nabla w_1\|_{L^p(E_{d,N}^\#)} + \|\nabla w_2\|_{L^p(E_{d,N}^\#)} \lesssim \|\nabla u\|_{L^p(E_{d,N}^\tau)} + \frac{1}{N} \|u\|_{L^p(V_{d,N}^\tau)}. \quad (4.52)$$

This completes the heuristic of Lemma 4.14 proof for $d = 2$. $\square$

Proof of Lemma 4.14 First, we will successively construct functions $w_i : \mathbb{Z}^{i-1} \times ([0, N] \cap \mathbb{Z}) \times \mathbb{Z}^{d-i} \rightarrow \mathbb{R}$, $i \in [1, d] \cap \mathbb{Z}$, with the property $\mathcal{A}(w_1, \dots, w_d)$ defined as follows: For every $\ell \in [1, d] \cap \mathbb{Z}$, $w_i : \mathbb{Z}^{i-1} \times ([0, N] \cap \mathbb{Z}) \times \mathbb{Z}^{d-i} \rightarrow \mathbb{R}$, $i \in [1, \ell] \cap \mathbb{Z}$ let $\mathcal{A}(w_1, \dots, w_\ell)$ be the statement that

- i) for all $i \in [1, \ell] \cap \mathbb{Z}$, $j \in ([1, d] \cap \mathbb{Z}) \setminus \{i\}$, $x \in \mathbb{Z}^{i-1} \times ([0, N] \cap \mathbb{Z}) \times \mathbb{Z}^{d-i}$ it holds that $w_i(x) = w_i(x + 2N\mathbf{e}_j)$,
- ii) for all $i \in [1, \ell] \cap \mathbb{Z}$, $x \in \mathbb{Z}^{i-1} \times ([1, N-1] \cap \mathbb{Z}) \times \mathbb{Z}^{d-i}$ it holds that $(\Delta w_i)(x) = 0$,
- iii) for all $i \in [1, \ell] \cap \mathbb{Z}$, $x \in ([0, N] \cap \mathbb{Z})^{i-1} \times \{0, N\} \times ([0, N] \cap \mathbb{Z})^{d-i}$ it holds that

$$u(x) = \sum_{v=1}^i w_v(x), \quad (4.53)$$

- iv) for all $i \in [1, \ell] \cap \mathbb{Z}$, $j \in [1, i-1] \cap \mathbb{Z}$, $x \in ([0, N] \cap \mathbb{Z})^{i-1} \times \{0, N\} \times ([0, N] \cap \mathbb{Z})^{d-i}$ it holds that $w_i(x) = 0$, and

v) for all $i \in [1, \ell] \cap \mathbb{Z}$ it holds that

$$\|\nabla w_i\|_{L^p(E_{d,N}^\#)} \leq (3^d c(d, p) + 1)^i \left[\|\nabla u\|_{L^p(E_{d,N}^\tau)} + \frac{1}{N} \|u\|_{L^p(V_{d,N}^\tau)} \right]. \quad (4.54)$$

As a first step, let $v_1: \{0, N\} \times \mathbb{Z}^{d-1} \rightarrow \mathbb{R}$ be the function given by

$$\forall x \in \{0, N\} \times ([0, N] \cap \mathbb{Z})^{d-1}: \quad v_1(x) = u(x), \quad (4.55)$$

$$\forall j \in [2, d] \cap \mathbb{Z}, x \in \{0, N\} \times ([-(N-1), N] \cap \mathbb{Z})^{d-1}:$$

$$v_1(x) = v_1(x - 2x_j \mathbf{e}_j^d), \quad (4.56)$$

$$\forall j \in [2, d] \cap \mathbb{Z}, x \in \{0, N\} \times \mathbb{Z}^{d-1}: \quad v(x) = v(x + 2N \mathbf{e}_j^d) \quad (4.57)$$

and let $w_1: ([0, N] \cap \mathbb{Z}) \times \mathbb{Z}^{d-1} \rightarrow \mathbb{R}$ be the function (cf. Corollary 4.9) which satisfies that

$$\begin{aligned} \forall x \in \{0, N\} \times \mathbb{Z}^{d-1}: \quad w_1(x) &= v_1(x), \\ \forall x \in ([1, N-1] \cap \mathbb{Z}) \times \mathbb{Z}^{d-1}: \quad (\Delta w_1)(x) &= 0. \end{aligned} \quad (4.58)$$

Then Eq. 4.46 (combined with a permutation of coordinates), Eqs. 4.55 and 4.56 imply that

$$\begin{aligned} \|\nabla w_1\|_{L^p(E_N^\#)} &\leq c(d, p) \left[\sum_{i \leq d, i \neq 1} \|\mathbf{D}_i^+ v_1\|_{L^p(\{0, N\} \times \mathbb{Z}^{d-1})} + \frac{1}{N} \|v_1\|_{L^p(\{0, N\} \times \mathbb{Z}^{d-1})} \right] \\ &\leq (3^d c(d, p) + 1) \left[\|\nabla u\|_{L^p(E_{d,N}^\tau)} + \frac{1}{N} \|u\|_{L^p(V_{d,N}^\tau)} \right]. \end{aligned} \quad (4.59)$$

Combining (4.55), (4.57), (4.58), and (4.59) yields that $\mathcal{A}(w_1)$ is true. For the recursive step let $\ell \in [1, d] \cap \mathbb{Z}$ and suppose that we have constructed $w_1, \dots, w_\ell$ so that $\mathcal{A}(w_1, \dots, w_\ell)$ holds. Now, let $v_{\ell+1}: \mathbb{Z}^\ell \times \{0, N\} \times \mathbb{Z}^{d-\ell-1} \rightarrow \mathbb{R}$ be the function which satisfies that

i) for all $x \in ([0, N] \cap \mathbb{Z})^\ell \times \{0, N\} \times ([0, N] \cap \mathbb{Z})^{d-\ell-1}$ it holds that

$$v_{\ell+1}(x) = u(x) - \sum_{v=1}^{\ell} w_v(x), \quad (4.60)$$

ii) for all $j \in [1, \ell] \cap \mathbb{Z}, x \in ([-N+1, N] \cap \mathbb{Z})^\ell \times \{0, N\} \times ([-N+1, N] \cap \mathbb{Z})^{d-\ell-1}$ it holds that

$$v_{\ell+1}(x) = -v_{\ell+1}(x - 2x_j \mathbf{e}_j^d), \quad (4.61)$$

iii) for all $j \in [\ell+2, d] \cap \mathbb{Z}, x \in ([-N+1, N] \cap \mathbb{Z})^\ell \times \{0, N\} \times ([-N+1, N] \cap \mathbb{Z})^{d-\ell-1}$ it holds that

$$v_{\ell+1}(x) = v_{\ell+1}(x - 2x_j \mathbf{e}_j^d), \quad (4.62)$$

and

iv) for all $j \in ([1, d] \cap \mathbb{Z}) \setminus \{\ell+1\}, x \in \mathbb{Z}^\ell \times \{0, N\} \times \mathbb{Z}^{d-\ell-1}$ it holds that

$$v_{\ell+1}(x) = v_{\ell+1}(x + 2N \mathbf{e}_j), \quad (4.63)$$

and let $w_{\ell+1} : \mathbb{Z}^\ell \times ([0, N] \cap \mathbb{Z}) \times \mathbb{Z}^{d-\ell-1} \rightarrow \mathbb{R}$ be the function (cf. Corollary 4.10) given by

$$\forall x \in \mathbb{Z}^\ell \times \{0, N\} \times \mathbb{Z}^{d-\ell-1} : \quad w_{\ell+1}(x) = v_{\ell+1}(x), \quad (4.64)$$

$$\forall x \in \mathbb{Z}^\ell \times ([1, N-1] \cap \mathbb{Z}) \times \mathbb{Z}^{d-\ell-1} : \quad (\Delta w_{\ell+1})(x) = 0, \quad (4.65)$$

$$\begin{aligned} \forall j \in ([1, d] \cap \mathbb{Z}) \setminus \{\ell+1\}, x \in \mathbb{Z}^\ell \times ([0, N] \cap \mathbb{Z}) \times \mathbb{Z}^{d-\ell-1} : \\ w_{\ell+1}(x + 2N\mathbf{e}_j^d) = w_{\ell+1}(x). \end{aligned} \quad (4.66)$$

Then Eq. 4.46 implies that

$$\begin{aligned} \|\nabla w_{\ell+1}\|_{L^p(E_N^\#)} &\leq c(d, p) \\ &\left[\sum_{v \leq d, v \neq \ell+1} \|\mathbf{D}_v^+ v_{\ell+1}\|_{L^p(\mathbb{I}_N^\ell \times \{0, N\} \times \mathbb{I}_N^{d-\ell-1})} + \frac{1}{N} \|v_{\ell+1}\|_{L^p(\mathbb{I}_N^\ell \times \{0, N\} \times \mathbb{I}_N^{d-\ell-1})} \right]. \end{aligned} \quad (4.67)$$

Note that Eqs. 4.60 and 4.64 imply that

$$\forall x \in ([0, N] \cap \mathbb{Z})^\ell \times \{0, N\} \times ([0, N] \cap \mathbb{Z})^{d-\ell-1} : \quad w_{\ell+1}(x) = u(x) - \sum_{\mu=1}^{\ell} w_{\mu}(x). \quad (4.68)$$

To lighten the notation let $U_{d,N}^{\ell+1}, V_{d,N}^{\ell+1} \subseteq \mathbb{Z}^d$ be the sets given by

$$U_{d,N}^{\ell+1} = \mathbb{I}_N^\ell \times \{0, N\} \times \mathbb{I}_N^{d-\ell-1} \quad \text{and} \quad V_{d,N}^{\ell+1} = ([0, N] \cap \mathbb{Z})^\ell \times \{0, N\} \times ([0, N] \cap \mathbb{Z})^{d-\ell-1}. \quad (4.69)$$

Then Eq. 4.67, a rough volume estimate, Eq. 4.68, and the triangle inequality show that

$$\begin{aligned} \|\nabla w_{\ell+1}\|_{L^p(E_N^\#)} &\leq c(d, p) \left[\left(\sum_{v \leq d, v \neq \ell+1} \|\mathbf{D}_v^+ w_{\ell+1}\|_{L^p(U_{d,N}^{\ell+1})} \right) + \frac{1}{N} \|w_{\ell+1}\|_{L^p(U_{d,N}^{\ell+1})} \right] \\ &\leq c(d, p) 3^d \left[\left(\sum_{v \leq d, v \neq \ell+1} \|\mathbf{D}_v^+ w_{\ell+1}\|_{L^p(V_{d,N}^{\ell+1})} \right) + \frac{1}{N} \|w_{\ell+1}\|_{L^p(V_{d,N}^{\ell+1})} \right] \\ &\leq c(d, p) 3^d \left[\sum_{\mu=1}^{\ell} \left(\sum_{v \leq d, v \neq \ell+1} \|\mathbf{D}_v^+ w_{\mu}\|_{L^p(V_{d,N}^{\ell+1})} + \frac{1}{N} \|w_{\mu}\|_{L^p(V_{d,N}^{\ell+1})} \right) \right. \\ &\quad \left. + \sum_{v \leq d, v \neq \ell+1} \|\mathbf{D}_v^+ u\|_{L^p(V_{d,N}^{\ell+1})} + \frac{1}{N} \|u\|_{L^p(V_{d,N}^{\ell+1})} \right] \\ &\leq c(d, p) 3^d \left[\sum_{\mu=1}^{\ell} \left[\|\nabla w_{\mu}\|_{L^p(E_N^\#)} + \frac{1}{N} \|w_{\mu}\|_{L^p(V_{d,N}^\tau)} \right] + \|\nabla u\|_{L^p(E_N^\tau)} + \frac{1}{N} \|u\|_{L^p(V_{d,N}^\tau)} \right] \\ &\leq c(d, p) 3^d \left[\|\nabla u\|_{L^p(E_N^\tau)} + \frac{1}{N} \|u\|_{L^p(V_{d,N}^\tau)} \right] \left[\sum_{\mu=0}^{\ell} (3^d c(d, p) + 1)^\mu \right] \\ &\leq (3^d c(d, p) + 1)^{\ell+1} \left[\|\nabla u\|_{L^p(E_N^\tau)} + \frac{1}{N} \|u\|_{L^p(V_{d,N}^\tau)} \right]. \end{aligned} \quad (4.70)$$

Furthermore, Eqs. 4.61 and 4.65, and uniqueness of the Dirichlet problem show that

$$\forall j \in [1, \ell] \cap \mathbb{Z}, x \in \mathbb{Z}^\ell \times ([0, N] \cap \mathbb{Z}) \times \mathbb{Z}^{d-\ell-1} : \quad w_{\ell+1}(x) = -w_{\ell+1}(x - 2x_j \mathbf{e}_j^d). \quad (4.71)$$

This and Lemma 4.12 (applied to the j -th coordinate for $j \in [1, \ell] \cap \mathbb{Z}$) show that

$$\forall j \in [1, \ell] \cap \mathbb{Z}, x \in ([0, N] \cap \mathbb{Z})^{j-1} \times \{0, N\} \times ([0, N] \cap \mathbb{Z})^{d-j}: \quad w_{\ell+1}(x) = 0. \quad (4.72)$$

Combining this, Eqs. 4.66, 4.65, 4.68 and 4.70, and the fact that $w_1, \dots, w_\ell$ were constructed with the property $\mathcal{A}(w_1, \dots, w_\ell)$ yields that $\mathcal{A}(w_1, \dots, w_{\ell+1})$ is true. Thus we have iteratively constructed $w_1, \dots, w_d$ with $\mathcal{A}(w_1, \dots, w_d)$. This implies for all $x \in V_{d,N}^\tau$ that $u(x) = \sum_{i=1}^d w_i(x)$ and for all $x \in ([1, N] \cap \mathbb{Z})^{d-1}$, $i \in [1, d] \cap \mathbb{Z}$ that $(\Delta w_i)(x) = 0$. The fact that $\forall x \in ([1, N] \cap \mathbb{Z})^{d-1}: (\Delta u)(x) = 0$ hence implies for all $x \in ([1, N] \cap \mathbb{Z})^{d-1}$ that $u(x) = \sum_{i=1}^d w_d(x)$. This, the triangle inequality, and Eq. 4.54 demonstrate that

$$\|\nabla u\|_{L^p(E_{d,N}^\#)} \leq \sum_{i=1}^d \|\nabla w_i\|_{L^p(E_{d,N}^\#)} \leq d(3^d c(d, p) + 1)^d \left[\|\nabla u\|_{L^p(E_{d,N}^\tau)} + \frac{1}{N} \|u\|_{L^p(V_{d,N}^\tau)} \right]. \quad (4.73)$$

This completes the proof of Lemma 4.14. $\square$

Corollary 4.15 Assume Lemma 4.14 and let $d, N \in [2, \infty) \cap \mathbb{N}$, $p \in (1, \infty)$, $u \in \mathbb{Q}_{d,N}$ be fixed. Then it holds that $\|\nabla u\|_{L^p(E_{d,N}^\#)} \leq d(3^d c(d, p) + 1)^d (1 + c_{PI}(d, p)) \|\nabla u\|_{L^p(E_{d,N}^\tau)}$.

Proof of Corollary 4.15 Lemma 4.14 (with $u \leftarrow u - a$ for $a \in \mathbb{R}$) and Eq. 4.47 show that

$$\begin{aligned} \|\nabla u\|_{L^p(E_{d,N}^\#)} &= \inf_{a \in \mathbb{R}} \|\nabla(u - a)\|_{L^p(E_{d,N}^\#)} \\ &\leq d(3^d c(d, p) + 1)^d \left[\inf_{a \in \mathbb{R}} \left(\|\nabla(u - a)\|_{L^p(E_N^\tau)} + \frac{1}{N} \|u - a\|_{L^p(V_{d,N}^\tau)} \right) \right] \\ &= d(3^d c(d, p) + 1)^d \left[\|\nabla u\|_{L^p(E_N^\tau)} + \frac{1}{N} \inf_{a \in \mathbb{R}} \|u - a\|_{L^p(V_{d,N}^\tau)} \right] \\ &\leq d(3^d c(d, p) + 1)^d (1 + c_{PI}(d, p)) \|\nabla u\|_{L^p(E_N^\tau)}. \end{aligned} \quad (4.74)$$

This completes the proof of Corollary 4.15. $\square$

4.3 Construction of the Neumann Extensions

In the Neumann case we also use a telescope sequence. First of all, instead of Settings 4.3 we start with Setting 4.16 below with Neumann conditions on the right hand sides of Eqs. 4.75 and 4.76.

Setting 4.16 Assume Setting 4.2 and let $C_1, C_2: ([2, \infty) \cap \mathbb{N}) \times (1, \infty) \rightarrow [0, \infty]$ be the functions which satisfy for all $d \in [2, \infty) \cap \mathbb{N}$, $p \in (1, \infty)$ that $C_1(d, p), C_2(d, p)$ are the smallest real extended numbers such that for all $N, L \in \mathbb{N}$, $u \in \mathbb{H}_{d,L,\geq 0}$ with $1/4 \leq N/L$ and $N \geq 2$ it holds that

$$\|\mathbf{D}_d^- u\|_{L^p(\mathbb{I}_L^{d-1} \times \{N\})} \leq C_1(d, p) \|\mathbf{D}_d^+ u\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})} \quad (4.75)$$

and

$$\|\nabla u\|_{L^p(E_{d,L,N}^\#)} \leq C_2(d, p) \|\mathbf{D}_d^+ u\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})}. \quad (4.76)$$

Lemma 4.17 Assume Setting 4.16 and let $d \in [2, \infty) \cap \mathbb{N}$, $p \in (1, \infty)$ be fixed. Then it holds that $C_1(d, p) < 1$ and $C_2(d, p) < \infty$.

Heuristic proof of Lemma 4.17 First, note that the discrete derivatives of a harmonic function are still harmonic. Using Item (i) in Corollary 3.1 (with the function replaced by the derivative) we bound the normal differences on the top $\mathbb{Z}^{d-1} \times \{N\}$ by the normal differences on the bottom $\mathbb{Z}^{d-1} \times \{0\}$ (see Eq. 4.78). Next, using Corollary 2.6 (with the function replaced by the derivative) and Item (ii) in Corollary 3.1 we bound the tangential differences on the top and bottom by the normal differences on the bottom (see Eq. 4.81). Furthermore, using Corollary 2.2 we bound the differences with respect to all edges with one endpoint on the face $\{x_1 = 0\}$ by the tangential differences on the bottom (see Eq. 4.79) and hence again by the normal differences on the bottom. A permutation of the coordinates then shows that we can bound the differences with respect to all edges with one endpoints on the boundary by the normal differences on the bottom. $\square$

Rigorous proof of Lemma 4.17 Corollary 3.1 implies that there exists $c_1: (0, \infty) \rightarrow (0, 1)$ such that for all $N, L \in \mathbb{N}, u \in \mathbb{H}_{d,L,\geq 0}$ with $N/L \geq \underline{r}$ and $\langle u \rangle_{\mathbb{I}_L^{d-1} \times \{0\}} = 0$ it holds that

$$\|u\|_{L^p(\mathbb{I}_L^{d-1} \times \{N\})} \leq c_1(\underline{r}) \|u\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})}. \quad (4.77)$$

This (with $N \leftarrow N-1, \underline{r} \leftarrow 1/4$, and $u \leftarrow D_d^+ u$ for $N \in [2, \infty) \cap \mathbb{N}, L \in \mathbb{N}, u \in \mathbb{H}_{d,L,\geq 0}$) and the fact that $\forall N \in \mathbb{N} \cap [2, \infty): (N-1) \geq N/2$ show that for all $N, L \in \mathbb{N}, u \in \mathbb{H}_{d,L,\geq 0}$ with $N/L \geq 1/4$ and $\langle u \rangle_{\mathbb{I}_L^{d-1} \times \{0\}} = 0$ it holds that $(N-1)/L \geq N/(2L) \geq 1/8$ and

$$\|D_d^- u\|_{L^p(\mathbb{I}_L^{d-1} \times \{N\})} = \|D_d^+ u\|_{L^p(\mathbb{I}_L^{d-1} \times \{N-1\})} \leq c_1(1/8) \|D_d^+ u\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})}. \quad (4.78)$$

Furthermore, Corollary 2.2 shows that there exists $c_2: (0, \infty) \rightarrow (0, \infty)$ such that for all $\bar{r} \in (0, \infty), L \in \mathbb{N}, N \in (0, L\bar{r}) \cap \mathbb{N}, u \in \mathbb{H}_{d,L,\geq 0}$ it holds that $\|u\|_{L^p(\{0\} \times \mathbb{I}_L^{d-2} \times ([1, N] \cap \mathbb{Z}))} \leq c_2(\bar{r}) \|u\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})}$. This (with $\bar{r} \leftarrow 4, u \leftarrow D_i^\pm u$, and $u \leftarrow D_d^+ u$ for $i \in [1, d-1] \cap \mathbb{Z}, L, N \in \mathbb{N}$ with $N/L \leq 1/4$) shows for all $L, N \in \mathbb{N}$ with $N/L \leq 4, u \in \mathbb{H}_{d,L,\geq 0}, i \in [1, d-1] \cap \mathbb{Z}$ that

$$\|D_i^\pm u\|_{L^p(\{0\} \times \mathbb{I}_L^{d-2} \times ([1, N] \cap \mathbb{Z}))} \leq c_2(4) \|D_i^\pm u\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})} \quad (4.79)$$

and

$$\|D_d^+ u\|_{L^p(\{0\} \times \mathbb{I}_L^{d-2} \times ([1, N] \cap \mathbb{Z}))} \leq c_2(4) \|D_d^+ u\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})}. \quad (4.80)$$

Corollary 2.6 (with $u \leftarrow D_i^\pm u$, for $i \in [1, d-1] \cap \mathbb{Z}, u \in \mathbb{H}_{d,L,\geq 0}$) and Corollary 3.1 show that there exists $c_3 \in (0, \infty)$ such that for all $u \in \mathbb{H}_{d,L,\geq 0}, i \in [1, d-1] \cap \mathbb{Z}$ it holds that

$$\|D_i^\pm u\|_{L^p(\mathbb{I}_L^{d-1} \times \{N\})} \leq \|D_i^\pm u\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})} \leq c_3 \|D_d^+ u\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})}. \quad (4.81)$$

Combining (4.78–4.81) we obtain that there exists $c_4 \in (0, \infty)$ such that for all $d \in [2, \infty) \cap \mathbb{N}, p \in (1, \infty), L, N \in \mathbb{N}$ with $N/L \leq 4, u \in \mathbb{H}_{d,L,\geq 0}$ it holds that

$$\|\nabla u\|_{L^p(E_{d,L,N}^\#)} \leq c_4 \|D_d^+ u\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})}. \quad (4.82)$$

This shows that $C_2(d, p) < \infty$. The proof of Lemma 4.17 is thus completed. $\square$

Setting 4.18 below introduces a telescope sequence which is similar to that in the Dirichlet case (cf. Setting 4.5). In Eq. 4.85 the means on each layer are set to be zero, since otherwise the Neumann problems on the haft spaces do not determine unique solutions.

Setting 4.18 (Telescope sequence for the Neumann case) Assume Setting 4.2, let $N, L \in \mathbb{N}$, $d \in [2, \infty) \cap \mathbb{N}$, $p \in (1, \infty)$ be fixed and satisfy that $1/4 \leq L/N \leq 4$, let $v \in \mathbb{B}_{d,L,N}$ satisfy that

$$\sum_{x \in \mathbb{I}_L^{d-1} \times \{0\}} v(x) = 0 \quad \text{and} \quad \forall x \in \mathbb{Z}^{d-1} \times \{N\}: \quad v(x) = 0, \quad (4.83)$$

and let

$$(u_{2k+1})_{k \in \mathbb{N}_0} \subseteq \mathbb{H}_{d,L,\geq 0} \quad \text{and} \quad (u_{2k+2})_{k \in \mathbb{N}_0} \subseteq \mathbb{H}_{d,L,\leq N} \quad (4.84)$$

be the sequences which satisfy that

$$\begin{aligned} \forall x \in \mathbb{Z}^{d-1} \times \{0\}: \quad & (\mathbf{D}_i^+ u_1)(x) = v(x), \\ \forall k \in \mathbb{N}_0, x \in \mathbb{Z}^{d-1} \times \{N\}: \quad & (\mathbf{D}_d^- u_{2k+2})(x) = (\mathbf{D}_d^- u_{2k+1})(x), \\ \forall k \in \mathbb{N}, x \in \mathbb{Z}^{d-1} \times \{0\}: \quad & (\mathbf{D}_d^+ u_{2k+1})(x) = (\mathbf{D}_d^+ u_{2k})(x), \quad \text{and} \\ \forall n \in \mathbb{N}, y \in [0, N] \cap \mathbb{Z}: \quad & \langle u_n \rangle_{\mathbb{I}_L^{d-1} \times \{y\}} = 0. \end{aligned} \quad (4.85)$$

Lemma 4.19 Assume Setting 4.18. Then it holds for all $n \in \mathbb{N}$ that

$$\max \left\{ \|\mathbf{D}_d^+ u_n\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})}, \|\mathbf{D}_d^- u_n\|_{L^p(\mathbb{I}_L^{d-1} \times \{N\})} \right\} \leq C_1(d, p) n^{-1} \|\mathbf{D}_d^+ u_1\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})} \quad (4.86)$$

Proof of Lemma 4.6 The assumption that $(u_{2k+1})_{k \in \mathbb{N}_0} \subseteq \mathbb{H}_{d,L,\geq 0}$ in Eq. 4.84 and Corollary 2.6 (with $u \leftarrow \mathbf{D}_d^+ u_{2k+3}$ for $k \in \mathbb{N}_0$) show $k \in \mathbb{N}_0$ that

$$\|\mathbf{D}_d^- u_{2k+1}\|_{L^p(\mathbb{I}_L^{d-1} \times \{N\})} = \|\mathbf{D}_d^+ u_{2k+1}\|_{L^p(\mathbb{I}_L^{d-1} \times \{N-1\})} \leq \|\mathbf{D}_d^+ u_{2k+1}\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})} \quad (4.87)$$

Similarly, the assumption that $(u_{2k+2})_{k \in \mathbb{N}_0} \subseteq \mathbb{H}_{d,L,\leq N}$ in Eq. 4.84 and Corollary 2.6 (together with a simple change of coordinates) show for all $k \in \mathbb{N}_0$ that

$$\|\mathbf{D}_d^+ u_{2k+2}\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})} \leq \|\mathbf{D}_d^- u_{2k+2}\|_{L^p(\mathbb{I}_L^{d-1} \times \{N\})}. \quad (4.88)$$

Next, Eqs. 4.84 and 4.75, and possibly a simple change of coordinates show for all $k \in \mathbb{N}_0$ that

$$\|\mathbf{D}_d^- u_{2k+2}\|_{L^p(\mathbb{I}_L^{d-1} \times \{N\})} = \|\mathbf{D}_d^- u_{2k+1}\|_{L^p(\mathbb{I}_L^{d-1} \times \{N\})} \leq C_1(d, p) \|\mathbf{D}_d^+ u_{2k+1}\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})}. \quad (4.89)$$

and

$$\|\mathbf{D}_d^+ u_{2k+3}\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})} \leq \|\mathbf{D}_d^+ u_{2k+2}\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})} \leq C_1(d, p) \|\mathbf{D}_d^- u_{2k+2}\|_{L^p(\mathbb{I}_L^{d-1} \times \{N\})}. \quad (4.90)$$

Combining (4.87–4.90), an induction argument, and Eq. 4.87 (with $k \leftarrow 0$) proves that for all $n \in \mathbb{N}$ it holds that

$$\begin{aligned} & \max \left\{ \|\mathbf{D}_d^+ u_{n+1}\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})}, \|\mathbf{D}_d^- u_{n+1}\|_{L^p(\mathbb{I}_L^{d-1} \times \{N\})} \right\} \\ & \leq C_1(d, p) \max \left\{ \|\mathbf{D}_d^+ u_n\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})}, \|\mathbf{D}_d^- u_n\|_{L^p(\mathbb{I}_L^{d-1} \times \{N\})} \right\} \\ & \leq C_1(d, p) n^{-1} \max \left\{ \|\mathbf{D}_d^+ u_1\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})}, \|\mathbf{D}_d^- u_1\|_{L^p(\mathbb{I}_L^{d-1} \times \{N\})} \right\} \\ & \leq C_1(d, p) n^{-1} \|\mathbf{D}_d^+ u_1\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})}. \end{aligned} \quad (4.91)$$

The proof of Lemma 4.19 is thus completed. $\square$

Lemma 4.20 Assume Setting 4.18. Then it holds for all $x \in \mathbb{Z}^{d-1} \times ([0, N] \cap \mathbb{Z})$ that $\sum_{n=1}^{\infty} |u_n(x)| < \infty$.

Proof of Lemma 4.20 The triangle inequality, a telescope sum argument, Jensen's inequality, and Eq. 4.76 show for all $a \in \{0, N\}$, $x \in \mathbb{I}_L^{d-1} \times \{a\}$, $u \in \mathbb{H}_{d,L,\geq 0} \cup \mathbb{H}_{d,L,\leq N}$ with $\langle u \rangle_{\mathbb{I}_L^{d-1} \times \{a\}} = 0$ that

$$\begin{aligned} |u(x)| &= \frac{1}{|\mathbb{I}_L^{d-1}|} \left| \sum_{y \in \mathbb{I}_L^{d-1} \times \{a\}} u(x) - u(y) \right| \leq \frac{1}{|\mathbb{I}_L^{d-1}|} \sum_{y \in \mathbb{I}_L^{d-1} \times \{a\}} |u(x) - u(y)| \\ &\leq \frac{1}{|\mathbb{I}_L^{d-1}|} \sum_{y \in \mathbb{I}_L^{d-1} \times \{a\}} \sum_{e \in E_{d,L,N}^\#} |\nabla_e u| = \sum_{e \in E_{d,L,N}^\#} |\nabla_e u| \leq \left| E_{d,L,N}^\# \right| \|\nabla u\|_{L^p(E_{d,L,N}^\#)} \\ &\leq \left| E_{d,L,N}^\# \right| C_2(d, p) \max \left\{ \|\mathbf{D}_d^+ u\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})}, \|\mathbf{D}_d^- u\|_{L^p(\mathbb{I}_L^{d-1} \times \{N\})} \right\}, \end{aligned} \quad (4.92)$$

This (with $u \leftarrow u_n$ for $n \in \mathbb{N}$ and combined with Eq. 4.84), and Lemma 4.19 imply for all $x \in \mathbb{I}_L^{d-1} \times ([0, N] \cap \mathbb{Z})$ that

$$\begin{aligned} |u_n(x)| &\leq \left| E_{d,L,N}^\# \right| C_2(d, p) \max \left\{ \|\mathbf{D}_d^+ u_n\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})}, \|\mathbf{D}_d^- u_n\|_{L^p(\mathbb{I}_L^{d-1} \times \{N\})} \right\} \\ &\leq \left| E_{d,L,N}^\# \right| C_2(d, p) C_1(d, p)^{n-1} \|\mathbf{D}_d^+ u_1\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})}. \end{aligned} \quad (4.93)$$

The fact that $C_1(d, p) < 1$ and the fact that $\forall x \in (0, 1): \sum_{k=1}^\infty x^{k-1} = (1-x)^{-1}$ then complete the proof of Lemma 4.20. $\square$

Lemma 4.21 Assume Setting 4.16 and let $w: \mathbb{Z}^{d-1} \times ([0, N] \cap \mathbb{Z}) \rightarrow \mathbb{R}$ satisfy that

$$\forall x \in \mathbb{Z}^{d-1} \times ([0, N] \cap \mathbb{Z}): \quad w(x) = \left[\sum_{k=1}^\infty (-1)^{k+1} u_k(x) \right]. \quad (4.94)$$

Then it holds that

$$w \in \mathbb{S}_{d,L,N}, \quad \forall x \in \mathbb{Z}^{d-1} \times \{0\}: \mathbf{D}_d^+ w(x) = v(x), \quad \forall x \in \mathbb{Z}^{d-1} \times \{N\}: \mathbf{D}_d^- w(x) = v(x), \quad (4.95)$$

and

$$\|\nabla w\|_{L^p(E_{d,L,N}^\#)} \leq \frac{C_2(d, p)}{1 - C_1(d, p)} \|v\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})}. \quad (4.96)$$

Proof of Lemma 4.21 First, Eq. 4.84 proves that for all $n \in \mathbb{N}_0$, $x \in \mathbb{I}_L^{d-1} \times ([1, N-1] \cap \mathbb{Z})$ it holds that $(\Delta u_n)(x) = 0$ and for all $n \in \mathbb{N}_0$, $x \in \mathbb{I}_L^{d-1} \times ([0, N] \cap \mathbb{Z})$, $i \in [1, d-1] \cap \mathbb{Z}$ it holds that $u_n(x + 2Le_i^d) = u_n(x)$. This, Eq. 4.94, and the definition of $\mathbb{S}_{d,L,N}$ imply that $w \in \mathbb{S}_{d,L,N}$. Furthermore, Lemma 4.20, Eqs. 4.94, 4.85 and 4.83 show that

$$\forall x \in \mathbb{Z}^{d-1} \times \{0\} : \quad (\mathbf{D}_d^+ w)(x) = \sum_{k=1}^\infty (-1)^{k+1} (\mathbf{D}_d^+ u_k)(x) = (\mathbf{D}_d^+ u_1)(x) = v(x) \quad \text{and} \quad (4.97)$$

$$\forall x \in \mathbb{Z}^{d-1} \times \{N\} : \quad (\mathbf{D}_d^- w)(x) = \sum_{k=1}^\infty (-1)^{k+1} (\mathbf{D}_d^- u_k)(x) = 0 = v(x). \quad (4.98)$$

This completes (4.95). Next, observe that Eq. 4.94, the triangle inequality, Eq. 4.76 (with $u \leftarrow u_k$ for $k \in \mathbb{N}$), Lemma 4.20, the fact that $\forall x \in (0, 1)$: $\sum_{k=1}^{\infty} x^{k-1} = (1-x)^{-1}$, and Eq. 4.85 ensure that

$$\begin{aligned} \|\nabla w\|_{L^p(E_{d,L,N}^\#)} &\leq \left\| \sum_{k=1}^{\infty} \nabla u_k \right\|_{L^p(E_{d,L,N}^\#)} \leq \sum_{k=1}^{\infty} \|\nabla u_k\|_{L^p(E_{d,L,N}^\#)} \\ &\leq \sum_{k=1}^{\infty} \left[C_2(d, p) \max \left\{ \|\mathbf{D}_d^+ u_k\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})}, \|\mathbf{D}_d^- u_k\|_{L^p(\mathbb{I}_L^{d-1} \times \{N\})} \right\} \right] \\ &\leq \sum_{k=1}^{\infty} \left[C_2(d, p) C_1(d, p)^{k-1} \|\mathbf{D}_d^+ u_1\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})} \right] = \frac{C_2(d, p)}{1 - C_1(d, p)} \left[\sum_{i=1}^{d-1} \|v\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})} \right]. \end{aligned} \quad (4.99)$$

This implies (4.96). The proof of Lemma 4.21 is thus completed. $\square$

Lemma 4.22 Assume Setting 4.16 and let $N, L \in \mathbb{N}$, $d \in [2, \infty) \cap \mathbb{N}$, $p \in (1, \infty)$ $v \in \mathbb{B}_{d,L,N}$ satisfy that $1/4 \leq L/N \leq 4$ and $\langle v \rangle_{\mathbb{I}_L^{d-1} \times \{0, N\}} = 0$. Then there exists $w \in \mathbb{S}_{d,L,N}$ such that

$$\forall x \in \mathbb{I}_L^{d-1} \times \{0\}: \quad \mathbf{D}_d^+ w(x) = v(x), \quad \forall x \in \mathbb{I}_L^{d-1} \times \{N\}: \quad \mathbf{D}_d^- w(x) = v(x), \quad (4.100)$$

and

$$\|\nabla w\|_{L^p(E_{d,L,N}^\#)} \leq \left[\frac{4C_2(d, p)}{1 - C_1(d, p)} + 2d16^d \right] \|v\|_{L^p(\mathbb{I}_L^{d-1} \times \{0, N\})}. \quad (4.101)$$

Proof of Lemma 4.22 Let $v_1, v_2, v_3 \in \mathbb{B}_{d,L,N}$ be the functions which satisfy that

$$\begin{aligned} \forall x \in \mathbb{I}_L^{d-1} \times \{0\} \quad &: \quad v_1(x) = v(x) - \langle v \rangle_{\mathbb{I}_L^{d-1} \times \{0\}}, \quad v_2(x) = 0, \quad v_3(x) = \langle v \rangle_{\mathbb{I}_L^{d-1} \times \{0\}}; \\ \forall x \in \mathbb{I}_L^{d-1} \times \{N\} \quad &: \quad v_1(x) = 0, \quad v_2(x) = v(x) - \langle v \rangle_{\mathbb{I}_L^{d-1} \times \{N\}}, \quad v_3(x) = \langle v \rangle_{\mathbb{I}_L^{d-1} \times \{N\}}, \end{aligned} \quad (4.102)$$

and let $w_3: \mathbb{I}_L^{d-1} \times ([0, N] \cap \mathbb{Z}) \rightarrow \mathbb{R}$ be the function which satisfies that

$$\forall n \in [0, N] \cap \mathbb{Z}, x \in \mathbb{I}_L^{d-1} \times \{n\}: \quad w_3(x) = n \langle v \rangle_{\mathbb{I}_L^{d-1} \times \{0\}}. \quad (4.103)$$

This construction and the fact that $\langle v \rangle_{\mathbb{I}_L^{d-1} \times \{N\}} + \langle v \rangle_{\mathbb{I}_L^{d-1} \times \{0\}} = 0$ show that

$$\begin{aligned} \forall x \in \mathbb{I}_L^{d-1} \times ([1, N-1] \cap \mathbb{Z}): \quad &(\Delta w_3)(x) = 0, \\ \forall x \in \mathbb{I}_L^{d-1} \times \{0\}: \quad &(\mathbf{D}_d^+ w_3)(x) = \langle v \rangle_{\mathbb{I}_L^{d-1} \times \{0\}} = v_3(x), \\ \forall x \in \mathbb{I}_L^{d-1} \times \{N\}: \quad &(\mathbf{D}_d^- w_3)(x) = \langle v \rangle_{\mathbb{I}_L^{d-1} \times \{N\}} = v_3(x). \end{aligned} \quad (4.104)$$

Next, the fact that

$$\{e \in E_{d,L,N}^\#: \nabla_e w_3 \neq 0\} \subseteq \{(x, x \pm \mathbf{e}_d^j): x \in \mathbb{I}_L^{d-1} \times ([0, L]^{d-1} \times [0, N]) \setminus ((0, L)^{d-1} \times (0, N))\}, \quad (4.105)$$

the fact that $[0, L]^{d-1} \times [0, N]$ has $2d$ faces, the fact that $N/L \in [1/4, 4]$, Eqs. 4.104 and 4.102, and Jensen's inequality imply that

$$\|\nabla w_3\|_{L^p(E_{d,L,N}^\#)}^p \leq (2d)16^d \|v\|_{L^p(\mathbb{I}_L^{d-1} \times \{0, N\})}^p, \quad (4.106)$$

which is a very rough estimate, however, gives a constant depending only on d . Furthermore, Lemma 4.21 (together with a simple change of coordinates), shows that there exists $w_1 \in \mathbb{H}_{d,L,\geq 0}$, $w_2 \in \mathbb{H}_{d,L,\leq N}$ such that it holds that

$$\begin{aligned} \forall x \in \mathbb{I}_L^{d-1} \times \{0\}, i \in \{1, 2\}: \quad &\mathbf{D}_d^+ w_i(x) = v_i(x), \\ \forall x \in \mathbb{I}_L^{d-1} \times \{N\}, i \in \{1, 2\}: \quad &\mathbf{D}_d^- w_i(x) = v_i(x), \end{aligned} \quad (4.107)$$

and such that for all $i \in \{1, 2\}$ it holds that

$$\|\nabla w_i\|_{L^p(E_{d,L,N}^\#)} \leq \frac{C_2(d, p)}{1 - C_1(d, p)} \|v_i\|_{L^p(\mathbb{I}_L^{d-1} \times \{0, N\})}. \quad (4.108)$$

Now let $w \in \mathbb{B}_{d,L,N}$ be the function which satisfies for all $x \in \mathbb{I}_L^{d-1} \times \{0, N\}$ that $w(x) = w_1(x) + w_2(x) + w_3(x)$. Then Eqs. 4.107, 4.104 and 4.102 imply (4.100). Furthermore, the triangle inequality, Jensen's inequality, and Eq. 4.102 show that $\forall i \in \{1, 2\}$: $\|v_i\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})} \leq 2 \|v\|_{L^p(\mathbb{I}_L^{d-1} \times \{0, N\})}$. This, the triangle inequality, Eqs. 4.108, 4.102 and 4.106 prove that

$$\|\nabla w\|_{L^p(E_{d,L,N}^\#)} \leq \left[\frac{4C_2(d, p)}{1 - C_1(d, p)} + 2d16^d \right] \|v\|_{L^p(\mathbb{I}_L^{d-1} \times \{0, N\})}. \quad (4.109)$$

This completes the proof of Lemma 4.22. $\square$

Observe that Lemma 4.22 shows the existence of the Neumann problem on strips. Furthermore, the uniqueness is straightforward (e.g. by means of the maximum principle applied to the derivatives $D_d^+ u$ defined on $\mathbb{Z}^{d-1} \times ([0, N-1] \cap \mathbb{Z})$ and harmonic on $\mathbb{Z}^{d-1} \times ([1, N-2] \cap \mathbb{Z})$ and the derivatives $D_d^- u$ defined on $\mathbb{Z}^{d-1} \times ([1, N] \cap \mathbb{Z})$ and harmonic on $\mathbb{Z}^{d-1} \times ([2, N-1] \cap \mathbb{Z})$). We therefore obtain Corollary 4.23 below. However, more important for us is Corollary 4.24 that follows from Lemma 4.22, Lemma 4.17, and the uniqueness in Corollary 4.23.

Corollary 4.23 *Let Setting 4.1 be given. Let $d, N \in [2, \infty) \cap \mathbb{N}$, $p \in (1, \infty)$, $L \in \mathbb{N}$, let $\mathbb{I}_L$ be the set given by $\mathbb{I}_L = [-L+1, L] \cap \mathbb{Z}$, let $v: \mathbb{Z}^{d-1} \times \{0, N\} \rightarrow \mathbb{R}$ satisfy for all $x \in \mathbb{Z}^{d-1} \times \{0, N\}$, $i \in [1, d-1] \cap \mathbb{Z}$ that $v(x) = v(x + 2Le_i^d)$ and $\langle v \rangle_{\mathbb{I}_L^{d-1} \times \{0\}} = \langle v \rangle_{\mathbb{I}_L^{d-1} \times \{N\}} = 0$. Then there exists uniquely $w: \mathbb{Z}^{d-1} \times ([0, N] \cap \mathbb{Z}) \rightarrow \mathbb{R}$ such that*

- i) *it holds that $\langle w \rangle_{\mathbb{I}_L^{d-1} \times \{0\}} = \langle w \rangle_{\mathbb{I}_L^{d-1} \times \{N\}} = 0$,*
- ii) *it holds for all $x \in \mathbb{Z}^{d-1} \times \{0\}$ that $D_d^+ w(x) = v(x)$,*
- iii) *it holds for all $x \in \mathbb{Z}^{d-1} \times \{N\}$ that $D_d^- w(x) = v(x)$, and*
- iv) *it holds for all $x \in \mathbb{Z}^{d-1} \times ([1, N-1] \cap \mathbb{Z})$ that $(\Delta w)(x) = v(x)$.*

Corollary 4.24 *Let Setting 4.1 be given. For $L \in \mathbb{N}$ let $\mathbb{I}_L$ be the set given by $\mathbb{I}_L = [-L+1, L] \cap \mathbb{Z}$ and let $E_{d,L,N}^\# \subseteq E_d$ be the set of edges given by*

$$E_{d,L,N}^\# = \left\{ (x, y) \in E_d : \frac{1}{2}(x+y) \in ([0, L]^{d-1} \times [0, N]) \setminus ([1, L-1]^{d-1} \times [1, N-1]) \right\}, \quad (4.110)$$

Then there exists $C: ([2, \infty) \cap \mathbb{N}) \times (1, \infty) \rightarrow (0, \infty)$ such that for all $d \in [2, \infty) \cap \mathbb{N}$, $p \in (1, \infty)$, $N, L \in \mathbb{N}$, $w: \mathbb{Z}^{d-1} \times ([0, N] \cap \mathbb{Z}) \rightarrow \mathbb{R}$ with $1/4 \leq L/N \leq 4$, $\forall x \in \mathbb{Z}^{d-1} \times ([0, N] \cap \mathbb{Z})$, $i \in [1, d-1] \cap \mathbb{Z}$: $w(x) = w(x + 2Le_i^d)$, and $\forall x \in \mathbb{Z}^{d-1} \times ([1, N-1] \cap \mathbb{Z})$: $(\Delta w)(x) = 0$ it holds that

$$\|\nabla w\|_{L^p(E_{d,L,N}^\#)} \leq C(d, p) \left[\|D_d^+ w\|_{L^p(\mathbb{I}_L^{d-1} \times \{0\})} + \|D_d^- w\|_{L^p(\mathbb{I}_L^{d-1} \times \{N\})} \right]. \quad (4.111)$$

4.4 Proof of the Main Result in the Neumann Case

Observe that if $f \in C^1(\mathbb{R}, \mathbb{R})$ is an even $2N$ -periodic function, then f' is an odd function and in particular $f'(0) = f'(N) = 0$. Lemma 4.25 below adapts this simple observation into the discrete case.

Lemma 4.25 Let $N \in \mathbb{N}$, $c \in \mathbb{R}$, $f: \mathbb{Z} \rightarrow \mathbb{R}$ satisfy that $\forall x \in \mathbb{Z}: f(x) = f(x + 2(N - 1))$ and $\forall x \in [-(N - 2), N - 1] \cap \mathbb{Z}: f(x) = f(1 - x) + c$. Then $f(1) - f(0) = f(N - 1) - f(N) = 0$.

Proof of Lemma 4.25 The fact $f(1) - f(0) = (f(0) + c) - (f(1) + c)$ proves that $f(1) - f(0) = 0$. Next, the fact that $f(N - 1) - f(N) = (f(1 - (N - 1) + 2(N - 1)) + c) - (f(1 - N + 2(N - 1)) + c) = f(N) - f(N - 1)$ proves that $f(N - 1) - f(N) = 0$. This completes the proof of Lemma 4.25. $\square$

Lemma 4.26 below gives the technical details in order to make the Neumann conditions vanish.

Lemma 4.26 (Even reflections) Let $d \in [2, \infty) \cap \mathbb{N}$, $w: \mathbb{Z}^{d-1} \times ([0, N] \cap \mathbb{Z}) \rightarrow \mathbb{R}$, $j \in [1, d - 1] \cap \mathbb{Z}$ satisfy that

$$\begin{aligned} \forall x \in \mathbb{Z}^{d-1} \times ([0, N] \cap \mathbb{Z}), i \in [1, d - 1] \cap \mathbb{Z}: \quad w(x) &= w(x + 2(N - 1)\mathbf{e}_i^d), \\ \forall x \in \mathbb{Z}^{d-1} \times ([1, N - 1] \cap \mathbb{Z}): \quad (\Delta w)(x) &= 0, \end{aligned} \quad (4.112)$$

and

$$\begin{aligned} \forall x \in \mathbb{Z}^{j-1} \times ([-(N - 2), N - 1] \cap \mathbb{Z}) \times \mathbb{Z}^{d-j-1} \times \{0\}: \quad (\mathbf{D}_d^+ w)(x) &= (\mathbf{D}_d^+ w)(x + \mathbf{e}_j^d - 2x_j \mathbf{e}_j^d), \\ \forall x \in \mathbb{Z}^{j-1} \times ([-(N - 2), N - 1] \cap \mathbb{Z}) \times \mathbb{Z}^{d-j-1} \times \{N\}: \quad (\mathbf{D}_d^- w)(x) &= (\mathbf{D}_d^- w)(x + \mathbf{e}_j^d - 2x_j \mathbf{e}_j^d). \end{aligned} \quad (4.113)$$

Then it holds that

$$\forall x \in \mathbb{Z}^{j-1} \times \{0\} \times \mathbb{Z}^{d-j-1} \times ([0, N] \cap \mathbb{Z}): \quad (\mathbf{D}_j^+ w)(x) = 0, \quad (4.114)$$

$$\forall x \in \mathbb{Z}^{j-1} \times \{N\} \times \mathbb{Z}^{d-j-1} \times ([0, N] \cap \mathbb{Z}): \quad (\mathbf{D}_j^- w)(x) = 0. \quad (4.115)$$

Proof of Lemma 4.26 Let $\tilde{w}: \mathbb{Z}^{d-1} \times ([0, N] \cap \mathbb{Z}) \rightarrow \mathbb{R}$ be the function given by

$$\forall x \in \mathbb{Z}^{d-1} \times ([0, N] \cap \mathbb{Z}): \quad \tilde{w}(x) = w(x + \mathbf{e}_j^d - 2x_j \mathbf{e}_j^d). \quad (4.116)$$

Then Eq. 4.112 implies that

$$\begin{aligned} \forall x \in \mathbb{Z}^{d-1} \times ([0, N] \cap \mathbb{Z}), i \in [1, d - 1] \cap \mathbb{Z}: \quad \tilde{w}(x) &= \tilde{w}(x + 2(N - 1)\mathbf{e}_i^d), \\ \forall x \in \mathbb{Z}^{d-1} \times ([1, N - 1] \cap \mathbb{Z}): \quad (\Delta \tilde{w})(x) &= 0. \end{aligned} \quad (4.117)$$

Furthermore, Eqs. 4.116 and 4.113, the fact that $j \neq d$, and the periodicity in Eqs. 4.112 and 4.116 imply that

$$\begin{aligned} \forall x \in \mathbb{Z}^{d-1} \times \{0\}: \quad (\mathbf{D}_d^+ w)(x) &= (\mathbf{D}_d^+ \tilde{w})(x), \\ \forall x \in \mathbb{Z}^{d-1} \times \{N\}: \quad (\mathbf{D}_d^- w)(x) &= (\mathbf{D}_d^- \tilde{w})(x). \end{aligned} \quad (4.118)$$

This and uniqueness (“up to a constant”) of the Neumann problem show that there exists $c \in \mathbb{R}$ such that for all $\mathbb{Z}^{d-1} \times ([0, N] \cap \mathbb{Z})$ it holds that $w(x) = \tilde{w}(x) + c = w(x + \mathbf{e}_j^d - 2x_j \mathbf{e}_j^d) + c$. This and Lemma 4.25 (applied to the j -th variable) complete the proof of Lemma 4.26. $\square$

Theorem 4.27 For every $d, N \in [2, \infty) \cap \mathbb{N}$ let $E_{d,N}^\# \subseteq E_d$ be the sets of edges given by

$$E_{d,N}^\# = \left\{ (x, y) \in E_d: \frac{1}{2}(x + y) \in ([0, L]^{d-1} \times [0, N]) \setminus ([1, L - 1]^{d-1} \times [1, N - 1]) \right\}, \quad (4.119)$$

let $V_{d,N}^v \subseteq \mathbb{Z}^d$ be the set of vertices given by

$$V_{d,N}^v = \left[\bigcup_{i=1}^d ([1, N-1] \cap \mathbb{Z})^{i-1} \times \{0, N\} \times ([1, N-1] \cap \mathbb{Z})^{d-i} \right], \quad (4.120)$$

let $\mathcal{N}_{d,N}$ be the set of all functions $v: V_{d,N}^v \rightarrow \mathbb{R}$ with $\langle v \rangle_{V_{d,N}^v} = 0$, and let $\mathcal{Q}_{d,N}$ be the set of functions $u: ([0, N] \cap \mathbb{Z})^d \rightarrow \mathbb{R}$ with the property that $\forall x \in ([1, N-1] \cap \mathbb{Z})^d: (\Delta u)(x) = 0$. Then there exists a function $C: ([2, \infty) \cap \mathbb{N}) \times (1, \infty) \rightarrow \mathbb{R}$ such that for all $d, N \in [2, \infty) \cap \mathbb{N}$, $p \in (1, \infty)$, $v: \mathcal{N}_{d,N} \rightarrow \mathbb{R}$ there exists a function $u \in \mathcal{Q}_{d,N}$ such that

$$\begin{aligned} \forall i \in [1, d] \cap \mathbb{Z}, x \in ([1, N-1] \cap \mathbb{Z})^{i-1} \times \{0\} \times ([1, N-1] \cap \mathbb{Z})^{d-i}: D_i^+ u &= v, \\ \forall i \in [1, d] \cap \mathbb{Z}, x \in ([1, N-1] \cap \mathbb{Z})^{i-1} \times \{N\} \times ([1, N-1] \cap \mathbb{Z})^{d-i}: D_i^- u &= v, \end{aligned} \quad (4.121)$$

(i.e., v is the Neumann condition of u) and such that $\|\nabla u\|_{L^p(E_{d,N}^\#)} \leq C(d, p) \|v\|_{L^p(V_{d,N}^v)}$.

Figure 1 illustrates the sets $E_{d,N}^\#$ and $V_{d,N}^v$ in Theorem 4.27 above in the case $d = 2$, $N = 10$: the elements of $E_{d,N}^\#$ are all tangential and normal (red and blue) edges and the elements of $V_{d,N}^v$ are all boundary vertices (red) without the ones at four corners.

Heuristic proof of Theorem 4.27 for $d = 2$ The argument for the decomposition in the Neumann case is different from that in the Dirichlet case by the fact that we switch the role of the odd and even reflections, i.e. in contrast to Fig. 4 (first even, then odd) we now have *first odd, then even* (see Fig. 5). This is due to a minor difficulty: in any case (periodic or box) the Neumann condition of a harmonic function must have vanishing mean. First, let $v_1: \{0, N\} \times \{1, \dots, N-1\} \rightarrow \mathbb{R}$ satisfy that $v_1(0, \cdot) = (D_1^+ u)(0, \cdot) = u(1, \cdot) - u(0, \cdot)$ and $v_1(N, \cdot) = (D_1^- u)(N, \cdot) = u(N-1, \cdot) - u(N, \cdot)$. Next, extend v_1 to $\{0, N\} \times \{1, \dots, 2N-2\}$ by an odd reflection, i.e., set for all $x_1 \in \{0, N\}$, $x_2 \in \{N, \dots, 2N-2\}$ that $v_1(x_1, x_2) = -v_1(x_1, 2N-x_2-1)$, see Fig. 5b. Next, extend v_1 $(2N-2)$ -periodically along x_2 -direction to $\{0, N\} \times \mathbb{Z}$. The odd reflection implies that $\sum_{\mathbb{Z}^{d-1} \times \{0\}} v_1 = \sum_{\mathbb{Z}^{d-1} \times \{N\}} v_1 = 0$. This and Corollary 4.23 show that there exists $w_1: \{0, \dots, N\} \times \mathbb{Z} \rightarrow \mathbb{R}$ such that $(\Delta w_1) = 0$ on $\{1, \dots, N-1\} \times \mathbb{Z}$, $(D_1^+ w_1)(0, \cdot) = v(0, \cdot)$, and $(D_1^- w_1)(N, \cdot) = v(N, \cdot)$. Next, let $v_2: \{1, \dots, N-1\} \times \{0, N\} \rightarrow \mathbb{R}$ satisfy that $v_2(\cdot, 0) = (D_2^+ u)(\cdot, 0) - (D_2^+ w_1)(\cdot, 0)$ and $v_2(\cdot, N) = (D_2^- u)(\cdot, N) - (D_2^- w_1)(\cdot, N)$ (see the blue edges in Fig. 5c), extend v_2 to $\{1, \dots, 2N-2\} \times \{0, N\}$ by an even reflection, i.e., for all $x_1 \in \{N, \dots, 2N-2\}$, $x_2 \in \{0, N\}$ we set $v_2(x_1, x_2) = v_2(2N-x_1-1, x_2)$, and extend v_2 $(2N-2)$ -periodically to $\mathbb{Z} \times \{0, N\}$. Note that $\Delta u = \Delta w_1 = 0$ on $\{1, \dots, N-1\}^2$ and therefore the Neumann conditions of u and w_1 on the boundary of $\{0, \dots, N\}^2$ have vanishing means. This and the construction of w_2 show that

$$\begin{aligned} \sum_{x=1}^{N-1} (v_2(x, 0) + v_2(x, N)) &= \sum_{x=1}^{N-1} \left[D_2^+(u - w_1)(x, 0) + D_2^-(u - w_1)(x, N) \right] \\ &= \sum_{x=1}^{N-1} \left[D_1^+(u - w_1)(0, x) + D_1^-(u - w_1)(N, x) \right] = 0 \end{aligned} \quad (4.122)$$

(Roughly speaking, in Fig. 5c: the values of v_2 at horizontal (red) edges have vanishing mean, that at horizontal (red) edges and vertical (blue) edges also have vanishing mean, and therefore that at vertical (blue) edges also have vanishing mean.) This shows that v_2 is a Neumann condition and therefore there exists $w_2: \mathbb{Z} \times \{0, \dots, N\} \rightarrow \mathbb{R}$ such that $\Delta w_2 = 0$ on $\mathbb{Z} \times \{1, \dots, N-1\}$, $D_2^+ w_2(\cdot, 0) = v_2(\cdot, 0)$, and $D_2^- w_2(\cdot, N) = v_2(\cdot, N)$. The even

Fig. 5 Construction of w_1 in the Neumann case, $d = 2$ and $N = 4$

reflection and the fact that $\forall x \in \{1, \dots, N-1\}: w_2(\cdot, x) = w_2(\cdot + 2(N-1), x)$ show for all $x_1, x_2 \in \{1, \dots, N-1\}$ that

$$\begin{aligned} (D_1^+ w_2)(0, x_2) &= w_2(1, x_2) - w_2(0, x_2) = w_2(2N-2, x_2) - w_2(2N-1, x_2) \\ &= -(D_1^+ w_2)(2N-1, x_2) = -(D_1^+ w_2)(0, x_2), \end{aligned}$$

i.e., in Fig. 5c, the gradient of w_2 w.r.t. all horizontal edges (red) edges vanish.

This and the construction of w_2 imply that the Neumann condition of $w_1 + w_2$ is equal to the Neumann condition of u . From now on the argument is similar to the proof in the Dirichlet case. $\square$

Proof of Theorem 4.27 First, let $d, N \in [2, \infty) \cap \mathbb{N}$, $p \in (1, \infty)$, $v \in \mathcal{N}_{d,N}$ be arbitrary but fixed and we will successively construct functions $w_i: \mathbb{Z}^{i-1} \times ([0, N] \cap \mathbb{Z}) \times \mathbb{Z}^{d-i} \rightarrow \mathbb{R}$, $i \in [1, d] \cap \mathbb{Z}$, with the property $\mathcal{A}(w_1, \dots, w_d)$ defined as follows: For every $\ell \in [1, d] \cap \mathbb{Z}$ and every collection of functions $w_i: \mathbb{Z}^{i-1} \times ([0, N] \cap \mathbb{Z}) \times \mathbb{Z}^{d-i} \rightarrow \mathbb{R}$, $i \in [1, \ell] \cap \mathbb{Z}$ let $\mathcal{A}(w_1, \dots, w_\ell)$ be the statement which is true if

- (i) it holds for all $i \in [1, \ell] \cap \mathbb{Z}$, $x \in \mathbb{Z}^{i-1} \times ([1, N-1] \cap \mathbb{Z}) \times \mathbb{Z}^{d-i}$ that $\Delta w_i(x) = 0$,
- (ii) it holds for all $i \in ([1, \ell] \cap \mathbb{Z})$, $j \in ([1, d] \cap \mathbb{Z}) \setminus \{i\}$, $x \in \mathbb{Z}^{i-1} \times ([0, N] \cap \mathbb{Z}) \times \mathbb{Z}^{d-i}$ that

$$w_i(x) = w_i(x + 2(N-1)e_j^d), \quad (4.123)$$

- (iii) it holds for all $i \in [1, \ell] \cap \mathbb{Z}$, $x \in ([1, N-1] \cap \mathbb{Z})^{i-1} \times \{0\} \times ([1, N-1] \cap \mathbb{Z})^{d-i}$ that

$$v(x) = \sum_{v=1}^i (D_i^+ w_v)(x), \quad (4.124)$$

- (iv) it holds for all $i \in [1, \ell] \cap \mathbb{Z}$, $x \in ([1, N-1] \cap \mathbb{Z})^{i-1} \times \{N\} \times ([1, N-1] \cap \mathbb{Z})^{d-i}$ that

$$v(x) = \sum_{v=1}^i (D_i^- w_v)(x), \quad (4.125)$$

- (v) it holds for all $i \in [1, \ell] \cap \mathbb{Z}$, $j \in [1, i-1] \cap \mathbb{Z}$, $x \in ([1, N-1] \cap \mathbb{Z})^{j-1} \times \{0\} \times ([1, N-1] \cap \mathbb{Z})^{d-j}$ that $(D_j^+ w_i)(x) = 0$, and
- (vi) it holds for all $i \in [1, \ell] \cap \mathbb{Z}$, $j \in [1, i-1] \cap \mathbb{Z}$, $x \in ([1, N-1] \cap \mathbb{Z})^{j-1} \times \{N\} \times ([1, N-1] \cap \mathbb{Z})^{d-j}$ that $(D_j^- w_i)(x) = 0$.

As a first step, let $v_1: \{0, N\} \times \mathbb{Z}^{d-1} \rightarrow \mathbb{R}$ be the function which satisfies that

$$\forall x \in \{0, N\} \times ([1, N-1] \cap \mathbb{Z})^{d-1}: v_1(x) = v(x), \quad (4.126)$$

$$\begin{aligned} \forall j \in [2, d] \cap \mathbb{Z}, x \in \{0, N\} \times ([-(N-2), N-1] \cap \mathbb{Z})^{d-1}: v_1(x) \\ = -v_1(x + \mathbf{e}_j^d - 2x_j \mathbf{e}_j^d), \end{aligned} \quad (4.127)$$

$$\forall j \in [2, d] \cap \mathbb{Z}, x \in \{0, N\} \times \mathbb{Z}^{d-1}: v_1(x) = v_1(x + 2(N-1)\mathbf{e}_j^d). \quad (4.128)$$

This implies that $\langle v_1 \rangle_{\{0, N\} \times \mathbb{I}_{N-1}^{d-1}} = 0$. Let $w_1: ([0, N] \cap \mathbb{Z}) \times \mathbb{Z}^{d-1} \rightarrow \mathbb{R}$ be a function whose existence is ensured by Corollary 4.23 and which satisfies that

$$\begin{aligned} \forall j \in [2, d] \cap \mathbb{Z}, x \in \{0, N\} \times \mathbb{Z}^{d-1}: w_1(x) &= w_1(x + 2(N-1)\mathbf{e}_j^d) \\ \forall x \in \{0\} \times \mathbb{Z}^{d-1}: (D_i^+ w_1)(x) &= v_1(x), \\ \forall x \in \{N\} \times \mathbb{Z}^{d-1}: (D_i^- w_1)(x) &= v_1(x), \\ \forall x \in ([1, N-1] \cap \mathbb{Z}) \times \mathbb{Z}^{d-1}: (\Delta w_1)(x) &= 0. \end{aligned} \quad (4.129)$$

Observe that $\mathcal{A}(w_1)$ holds. Next, let $\ell \in [1, d-1] \cap \mathbb{Z}$ and suppose that we have constructed $w_1, \dots, w_\ell$ such that $\mathcal{A}(w_1, \dots, w_\ell)$ holds. Let $v_{\ell+1}: \mathbb{Z}^\ell \times \{0, N\} \times \mathbb{Z}^{d-\ell-1} \rightarrow \mathbb{R}$ be the function which satisfies that

- i) for all $x \in ([1, N-1] \cap \mathbb{Z})^\ell \times \{0\} \times ([1, N-1] \cap \mathbb{Z})^{d-\ell-1}$ it holds that

$$v_{\ell+1}(x) = v(x) - \sum_{v=1}^{\ell} (D_{\ell+1}^+ w_v)(x) \quad (4.130)$$

- ii) for all $x \in ([1, N-1] \cap \mathbb{Z})^\ell \times \{N\} \times ([1, N-1] \cap \mathbb{Z})^{d-\ell-1}$ it holds that

$$v_{\ell+1}(x) = v(x) - \sum_{v=1}^{\ell} (D_{\ell+1}^- w_v)(x), \quad (4.131)$$

- iii) for all $j \in ([1, d] \cap \mathbb{Z}) \setminus \{\ell+1\}$, $x \in \mathbb{Z}^\ell \times \{0, N\} \times \mathbb{Z}^{d-\ell-1}$ it holds that

$$v_{\ell+1}(x) = v_{\ell+1}(x + 2(N-1)\mathbf{e}_j) \quad (4.132)$$

- iv) for all $j \in [1, \ell] \cap \mathbb{Z}$, $x \in ([-(N-2), N-1] \cap \mathbb{Z})^\ell \times \{0, N\} \times ([-(N-2), N-1] \cap \mathbb{Z})^{d-\ell-1}$ it holds that

$$v_{\ell+1}(x) = v_{\ell+1}(x + \mathbf{e}_j^d - 2x_j \mathbf{e}_j^d), \quad (4.133)$$

and

- v) for all $j \in [\ell+2, d] \cap \mathbb{Z}$, $x \in ([-(N-2), N-1] \cap \mathbb{Z})^\ell \times \{0, N\} \times ([-(N-2), N-1] \cap \mathbb{Z})^{d-\ell-1}$ it holds that

$$v_{\ell+1}(x) = -v_{\ell+1}(x + \mathbf{e}_j^d - 2x_j \mathbf{e}_j^d). \quad (4.134)$$

Next, we show that

$$\langle v_{\ell+1} \rangle_{\mathbb{I}_{N-1}^\ell \times \{0, N\} \times \mathbb{I}_{N-1}^{d-\ell-1}} = 0. \quad (4.135)$$

To this end we distinguish two cases: $\ell+1 = d$ and $\ell+1 < d$. First, when $\ell+1 < d$, then the odd reflection in Eq. 4.134 implies (4.135). Next, we consider the case $\ell+1 = d$. Note that in this case $[\ell+2, d] \cap \mathbb{Z} = \emptyset$ and we therefore cannot use Eq. 4.134. In this step, to shorten the notation, for every $i \in [1, d] \cap \mathbb{Z}$, $j \in \{0, N\}$ let F_i^j be the set given by

$$F_i^j = ([1, N-1] \cap \mathbb{Z})^i \times \{j\} \times ([1, N-1] \cap \mathbb{Z})^{d-i-1}. \quad (4.136)$$

The fact that $v \in \mathcal{N}_{d,N}$ implies that $\langle v \rangle_{V_{d,N}^v} = 0$ and hence that

$$\sum_{i=1}^d \left[\sum_{x \in F_i^0} v(x) + \sum_{x \in F_i^N} v(x) \right] = 0. \quad (4.137)$$

The fact that $\forall j \in [1, d-1] \cap \mathbb{Z}$, $x \in ([1, N-1] \cap \mathbb{Z})^d$: $(\Delta w_j)(x) = 0$, following from the induction hypothesis and the case assumption $\ell+1 = d$, shows that the Neumann conditions of w_j , $j \in [1, d-1] \cap \mathbb{Z}$, on $([0, N] \cap \mathbb{Z})^d$ have vanishing means, i.e., it holds for all $j \in [1, d-1] \cap \mathbb{Z}$ that

$$\sum_{i=1}^d \left[\sum_{x \in F_i^0} (\mathbf{D}_i^+ w_j) + \sum_{x \in F_i^N} (\mathbf{D}_i^- w_j)(x) \right] = 0. \quad (4.138)$$

This and Eq. 4.137 prove that

$$\sum_{i=1}^d \left[\sum_{x \in F_i^0} \left(v(x) - \sum_{j=1}^{d-1} (\mathbf{D}_i^+ w_j)(x) \right) + \sum_{x \in F_i^N} \left(v(x) - \sum_{j=1}^{d-1} (\mathbf{D}_i^- w_j)(x) \right) \right] = 0. \quad (4.139)$$

Furthermore, Eqs. 4.124 and 4.125, following from the induction hypothesis, imply that

$$\sum_{i=1}^{d-1} \left[\sum_{x \in F_i^0} \left(v(x) - \sum_{j=1}^{d-1} (\mathbf{D}_i^+ w_j)(x) \right) + \sum_{x \in F_i^N} \left(v(x) - \sum_{j=1}^{d-1} (\mathbf{D}_i^- w_j)(x) \right) \right] = 0. \quad (4.140)$$

This, Eqs. 4.130, 4.131 and 4.139 show that

$$\sum_{x \in F_d^0 \cup F_d^N} v_d(x) = \sum_{x \in F_d^0} \left(v(x) - \sum_{j=1}^{d-1} (\mathbf{D}_d^+ w_j)(x) \right) + \sum_{x \in F_d^N} \left(v(x) - \sum_{j=1}^{d-1} (\mathbf{D}_d^- w_j)(x) \right) = 0. \quad (4.141)$$

This and Eq. 4.133 complete the proof of Eq. 4.135. Now, Eq. 4.135 and Corollary 4.23 imply that there exists $w_{\ell+1}: \mathbb{Z}^\ell \times ([0, N] \cap \mathbb{Z}) \times \mathbb{Z}^{d-\ell-1} \rightarrow \mathbb{R}$ such that

- i) for all $x \in ([1, N-1] \cap \mathbb{Z})^\ell \times \{0\} \times ([1, N-1] \cap \mathbb{Z})^{d-\ell-1}$ it holds that

$$(\mathbf{D}_{\ell+1}^+ w_{\ell+1})(x) = v_{\ell+1}(x) \quad (4.142)$$

- ii) for all $x \in ([1, N-1] \cap \mathbb{Z})^\ell \times \{N\} \times ([1, N-1] \cap \mathbb{Z})^{d-\ell-1}$ it holds that

$$(\mathbf{D}_{\ell+1}^- w_{\ell+1})(x) = v_{\ell+1}(x) \quad (4.143)$$

- iii) for all $j \in ([1, d] \cap \mathbb{Z}) \setminus \{\ell+1\}$, $x \in \mathbb{Z}^\ell \times \{0, N\} \times \mathbb{Z}^{d-\ell-1}$ it holds that

$$w_{\ell+1}(x) = w_{\ell+1}(x + 2(N-1)\mathbf{e}_j^d), \quad (4.144)$$

and

- iv) for all $i \in [1, \ell] \cap \mathbb{Z}$, $x \in \mathbb{Z}^{i-1} \times ([1, N-1] \cap \mathbb{Z}) \times \mathbb{Z}^{d-i}$ that

$$(\Delta w_i)(x) = 0. \quad (4.145)$$

Observe that Eqs. 4.142 and 4.143, the even reflection in Eq. 4.133, and Lemma 4.26 show that

$$\begin{aligned} \forall j \in [1, \ell] \cap \mathbb{Z}, x \in ([1, N-1] \cap \mathbb{Z})^{i-1} \times \{0\} \times ([1, N-1] \cap \mathbb{Z})^{d-j}: \quad & (\mathbf{D}_j^+ w_{\ell+1})(x) = 0, \\ \forall j \in [1, \ell] \cap \mathbb{Z}, x \in ([1, N-1] \cap \mathbb{Z})^{i-1} \times \{N\} \times ([1, N-1] \cap \mathbb{Z})^{d-j}: \quad & (\mathbf{D}_j^- w_{\ell+1})(x) = 0. \end{aligned} \quad (4.146)$$

Combining (4.142), (4.143), (4.130) and (4.131) yields that

$$\begin{aligned} \forall x \in ([1, N-1] \cap \mathbb{Z})^\ell \times \{0\} \times ([1, N-1] \cap \mathbb{Z})^{d-\ell-1}: \quad v(x) &= \sum_{v=1}^{\ell+1} (\mathbf{D}_i^+ w_v)(x), \\ \forall x \in ([1, N-1] \cap \mathbb{Z})^\ell \times \{N\} \times ([1, N-1] \cap \mathbb{Z})^{d-\ell-1}: \quad v(x) &= \sum_{v=1}^{\ell+1} (\mathbf{D}_i^- w_v)(x). \end{aligned} \quad (4.147)$$

This, Eqs. 4.144 and 4.145, and the induction hypothesis $\mathcal{A}(w_1, \dots, w_\ell)$ imply that $\mathcal{A}(w_1, \dots, w_{\ell+1})$ holds. We have thus recursively constructed a sequence $w_1, \dots, w_d$ with $\mathcal{A}(w_1, \dots, w_d)$. Now, let $u: ([0, N] \cap \mathbb{Z})^d \rightarrow \mathbb{R}$ be the function which satisfies for all $x \in ([0, N] \cap \mathbb{Z})^d$ that $u(x) = \sum_{i=1}^d w_i(x)$. The property $\mathcal{A}(w_1, \dots, w_d)$ then implies that $u \in \mathbb{Q}_{d,N}$,

$$\forall i \in [1, d] \cap \mathbb{Z}, x \in ([1, N-1] \cap \mathbb{Z})^{i-1} \times \{0\} \times ([1, N-1] \cap \mathbb{Z})^{d-i}: \quad (\mathbf{D}_i^+ u)(x) = v(x), \quad (4.148)$$

and

$$\forall i \in [1, d] \cap \mathbb{Z}, x \in ([1, N-1] \cap \mathbb{Z})^{i-1} \times \{N\} \times ([1, N-1] \cap \mathbb{Z})^{d-i}: \quad (\mathbf{D}_i^- u)(x) = v(x). \quad (4.149)$$

This, Eqs. 4.124 and 4.125 prove that u and $w_1 + \dots + w_d$ have the same Neumann conditions on the boundary of $\{0, \dots, N\}$. Hence, $\nabla u = \nabla w_1 + \dots + \nabla w_d$ on $E_{d,N}^\#$. The rest of the proof is now clear. We only give a sketch. We write $C(d, p)$ to denote possibly different real numbers that only depend on d, p and write for $i \in [1, d] \cap \mathbb{Z}$ to lighten the notation

$$F_i^j = ([1, N-1] \cap \mathbb{Z})^i \times \{j\} \times ([1, N-1] \cap \mathbb{Z})^{d-i-1}, \quad \hat{F}_i^j = \mathbb{I}_{N-1} \times \{j\} \times \mathbb{I}_{N-1}^{d-i-1}. \quad (4.150)$$

Then the fact that $\nabla u = \nabla w_1 + \dots + \nabla w_d$ on $E_{d,N}^\#$, the triangle inequality, Corollary 4.24, Eq. 4.124, Eq. 4.125, and an induction argument show that

$$\|\nabla u\|_{L^p(E_{d,N}^\#)} \leq \sum_{i=1}^d \|\nabla w_i\|_{L^p(E_{d,N}^\#)} \leq C(d, p) \sum_{i=1}^d \left[\|\mathbf{D}_i^+ w_i\|_{L^p(\hat{F}_i^0)} + \|\mathbf{D}_i^- w_i\|_{L^p(\hat{F}_i^N)} \right] \quad (4.151)$$

$$\leq C(d, p) \sum_{i=1}^d \left[\|v\|_{L^p(F_i^0)} + \|v\|_{L^p(F_i^N)} \right] \leq C(d, p) \|v\|_{L^p(V_{d,N}^v)}. \quad (4.152)$$

The proof of Theorem 4.27 is thus completed. $\square$

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Appendix : A

For convenience we include here some simple results.

A.1 Some Basic Results

Lemma A.1 (Complex square root) *There exists a unique function $R \in C(\mathbb{C} \setminus (-\infty, 0], \mathbb{C})$ such that $R|_{\mathbb{C} \setminus (-\infty, 0]}$ is holomorphic and $\forall z \in \mathbb{C} \setminus (-\infty, 0): R(z)^2 = z$.*

Proof of Lemma A.1 Let $\log: \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$ be the principle branch of the logarithm, i.e., it holds for all $z \in \mathbb{C} \setminus \{0\}$ that $\exp(\log z) = z$ and $-\pi \leq \Im(\log z) \leq \pi$, where $\Im$ denotes the imaginary part (see, e.g., Theorem I.2.11 in Freitag and Busam [21]), and let $R: \mathbb{C} \setminus (-\infty, 0) \rightarrow \mathbb{C}$ be given by

$$\forall z \in \mathbb{C} \setminus (-\infty, 0]: \quad R(z) = \exp\left(\frac{1}{2} \log z\right) \quad \text{and} \quad R(0) = 0. \quad (\text{A.1})$$

An elementary property of the function $\exp$ then shows that

$$\forall z \in \mathbb{C} \setminus (-\infty, 0): \quad R(z)^2 = z. \quad (\text{A.2})$$

Furthermore, the fact that $\exp$ and $\log|_{\mathbb{C} \setminus (-\infty, 0]}$ are holomorphic (cf. Theorem I.5.8 in [21]) and the chain rule then show that $R|_{\mathbb{C} \setminus (-\infty, 0]}$ is holomorphic. Finally, we prove by contradiction that R is continuous at 0. Suppose there exist $(z_n)_{n \in \mathbb{N}} \subseteq \mathbb{C} \setminus (-\infty, 0)$, $\epsilon \in (0, 1)$ such that $\lim_{n \rightarrow \infty} z_n = 0$ and $\forall n \in \mathbb{N}: |R(z_n)| > \epsilon$ and without loss of generality assume for all $n \in \mathbb{N}$ that $|z_n| < 1$. Then Eq. A.2 implies for all $n \in \mathbb{N}$ that $|R(z_n)|^2 = |z_n| \leq 1$. The Bolzano theorem hence proves that there exists a sequence $(n_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ such that $(R(z_{n_k}))_{k \in \mathbb{N}}$ converges. This, Eq. A.2, and the fact that $\lim_{n \rightarrow \infty} z_n = 0$ then demonstrate that $\lim_{k \rightarrow \infty} |R(z_{n_k})|^2 = \lim_{k \rightarrow \infty} |z_{n_k}| = 0$. This contradicts the assumption that $\forall n \in \mathbb{N}: |R(z_n)| > \epsilon$. Thus, R is continuous at 0. The proof of Lemma A.1 is thus completed. $\square$

Lemma A.2 i) *It holds that $0 < \inf_{s \in [-\pi, \pi] \setminus \{0\}} \frac{1 - \cos(s)}{s^2} < \sup_{s \in [-\pi, \pi] \setminus \{0\}} \frac{1 - \cos(s)}{s^2} < \infty$,*
 ii) *it holds that $0 < \inf_{s \in [-\pi, \pi] \setminus \{0\}} \left| \frac{e^{-is} - 1}{s} \right| \leq \sup_{s \in [-\pi, \pi] \setminus \{0\}} \left| \frac{e^{-is} - 1}{s} \right| < \infty$, and*
 iii) *it holds that $\sup_{s \in [-\pi, \pi] \setminus \{0\}} \left| s^2 \frac{d}{ds} \left(\frac{1}{e^{-is} - 1} \right) \right| < \infty$.*

Proof of Lemma A.2 Throughout the proof let $g, h: [-\pi, \pi] \rightarrow \mathbb{R}$ be the functions which satisfy for all $s \in [-\pi, \pi] \setminus \{0\}$ that

$$g(s) = \frac{1 - \cos(s)}{s^2}, \quad h(s) = \left| \frac{e^{-is} - 1}{s} \right|, \quad g(0) = \frac{1}{2}, \quad \text{and} \quad h(0) = 1. \quad (\text{A.3})$$

The fact that $\lim_{s \rightarrow 0} \frac{1 - \cos(s)}{s^2} = \frac{1}{2}$ and the fact that $\left| \frac{e^{-is} - 1}{s} \right| = 1$ show that $g, h \in C([-\pi, \pi], \mathbb{R})$. The extreme value theorem and the fact that $\forall s \in [-\pi, \pi]: \min\{g(s), h(s)\} > 0$ then imply that

$$\begin{aligned} 0 &< \inf_{s \in [-\pi, \pi]} g(s) \leq \inf_{s \in [-\pi, \pi] \setminus \{0\}} \frac{1 - \cos(s)}{s^2} \leq \sup_{s \in [-\pi, \pi] \setminus \{0\}} \frac{1 - \cos(s)}{s^2} \\ &\leq \sup_{s \in [-\pi, \pi]} g(s) < \infty. \end{aligned} \quad (\text{A.4})$$

and

$$\begin{aligned} 0 &< \inf_{s \in [-\pi, \pi]} h(s) \leq \inf_{s \in [-\pi, \pi] \setminus \{0\}} \left| \frac{e^{-is} - 1}{s} \right| \leq \sup_{s \in [-\pi, \pi] \setminus \{0\}} \left| \frac{e^{-is} - 1}{s} \right| \\ &\leq \sup_{s \in [-\pi, \pi]} h(s) < \infty. \end{aligned} \quad (\text{A.5})$$

This proves Items (i) and (ii). Finally, Item (iii) follows from Item(ii). The proof of Lemma A.2 is thus completed. $\square$

A.2 The Simple Random Walk Representation Without Martingale Theory

For convenience of the reader we include an elementary proof without using martingales.

Proof of Item (i) in Lemma 2.4 without martingale theory First, it holds for all $n \in \mathbb{N}$ that $\{S_{n-1} = x, T > n-1\}$ depends only on $X_1, \dots, X_{n-1}$ and is therefore independent of X_n . The fact that $\forall n \in \mathbb{N}: S_n = S_{n-1} + X_n$, the assumption on the distribution of $X_n, n \in \mathbb{N}$, and the assumption that $\forall x \in \mathbb{Z}^{d-1} \times \mathbb{N}: \Delta u(x) = 0$ imply for all $x \in \mathbb{Z}^{d-1} \times \mathbb{N}, n \in \mathbb{N}$ that

$$\begin{aligned} \mathbb{E}[u(S_n) \mathbb{1}_{S_{n-1}=x} \mathbb{1}_{T>n-1}] &= \mathbb{E}[u(S_{n-1} + X_n) \mathbb{1}_{S_{n-1}=x} \mathbb{1}_{T>n-1}] \\ &= \mathbb{E}[u(x + X_n) \mathbb{1}_{S_{n-1}=x} \mathbb{1}_{T>n-1}] = \mathbb{E}[u(x + X_n)] \mathbb{P}(S_{n-1} = x, T > n-1) \\ &= \frac{1}{2d} \left[\sum_{i=1}^d u(x + \mathbf{e}_i^d) + u(x - \mathbf{e}_i^d) \right] \mathbb{P}(S_{n-1} = x, T > n-1) \\ &= u(x) \mathbb{P}(S_{n-1} = x, T > n-1). \end{aligned} \quad (\text{A.6})$$

This and the fact that $\forall x \in \mathbb{Z}^{d-1} \times \{0\}, n \in \mathbb{N}: \mathbb{P}(S_{n-1} = x, T > n-1) = 0$ prove for all $x \in \mathbb{Z}^{d-1} \times \mathbb{N}_0, n \in \mathbb{N}$ that $\mathbb{E}[u(S_n) \mathbb{1}_{S_{n-1}=x} \mathbb{1}_{T>n-1}] = u(x) \mathbb{P}(S_{n-1} = x, T > n-1)$ and

$$\begin{aligned} &\mathbb{E}[u(S_n \wedge T) \mathbb{1}_{S_{(n-1) \wedge T} = x}] \\ &= \mathbb{E}[u(S_n \wedge T) \mathbb{1}_{S_{(n-1) \wedge T} = x} \mathbb{1}_{T>n-1}] + \mathbb{E}[u(S_n \wedge T) \mathbb{1}_{S_{(n-1) \wedge T} = x} \mathbb{1}_{T \leq n-1}] \\ &= \mathbb{E}[u(S_n) \mathbb{1}_{S_{n-1}=x} \mathbb{1}_{T>n-1}] + \mathbb{E}[u(S_T) \mathbb{1}_{S_T=x} \mathbb{1}_{T \leq n-1}] \\ &= \mathbb{E}[u(S_n) \mathbb{1}_{S_{n-1}=x} \mathbb{1}_{T>n-1}] + u(x) \mathbb{P}(S_T = x, T \leq n-1) \\ &= u(x) (\mathbb{P}(S_{n-1} = x, T > n-1) + \mathbb{P}(S_T = x, T \leq n-1)) \\ &= u(x) (\mathbb{P}(S_{(n-1) \wedge T} = x, T > n-1) + \mathbb{P}(S_{(n-1) \wedge T} = x, T \leq n-1)) \\ &= u(x) \mathbb{P}(S_{(n-1) \wedge T} = x). \end{aligned} \quad (\text{A.7})$$

This and the assumption that u is bounded yield that

$$\begin{aligned} \mathbb{E}[u(S_n \wedge T)] &= \mathbb{E} \left[u(S_n \wedge T) \sum_{x \in \mathbb{Z}^{d-1} \times \mathbb{N}_0} \mathbb{1}_{S_{(n-1) \wedge T} = x} \right] = \sum_{x \in \mathbb{Z}^{d-1} \times \mathbb{N}_0} \mathbb{E}[u(S_n \wedge T) \mathbb{1}_{S_{(n-1) \wedge T} = x}] \\ &= \sum_{x \in \mathbb{Z}^{d-1} \times \mathbb{N}_0} u(x) \mathbb{P}(S_{(n-1) \wedge T} = x) = \mathbb{E}[u(S_{(n-1) \wedge T})]. \end{aligned} \quad (\text{A.8})$$

An induction argument shows for all $n \in \mathbb{N}, (x, y) \in \mathbb{Z}^{d-1} \times \mathbb{N}_0$ that $\mathbb{E}[u(S_n \wedge T)] = \mathbb{E}[u(S_0)]$ and $\mathbb{E}[u(S_n \wedge T) | S_0 = (x, y)] = \mathbb{E}[u(S_0) | S_0 = (x, y)] = u(x, y)$. The bounded convergence theorem then ensures with n tending to infinity that for all $x \in \mathbb{Z}^{d-1}, y \in \mathbb{N}_0$ it holds that $\mathbb{E}[u(S_T) | S_0 = (x, y)] = u(x, y)$. This (with $u \leftarrow u(\cdot + (x, 0))$ and $(x, y) \leftarrow$

$(0, y)$ for $x \in \mathbb{Z}^{d-1}$, $y \in \mathbb{N}_0$) establishes that for all $(x, y) \in \mathbb{Z}^{d-1} \times \mathbb{N}_0$ it holds that $u(x, y) = \mathbb{E}[u(S_T + (x, 0)) | S_0 = (0, y)]$. The proof is thus completed. $\square$

A.3 An Interpolation Argument

Lemma A.3 below gives a version of the Marcinkiewicz interpolation theorem in the discrete case. Its formulation is unfortunately not found in the literature, although its proof is quite routine. We follow the proof of Theorem 9.1 in the book by DiBenedetto [22].

Lemma A.3 (L_w^p - L^∞ -interpolation) *Let $N \in \mathbb{N}$, let $E \subseteq \mathbb{Z}^N$ be a finite set, let $p, r \in [1, \infty)$, $N_p, N_\infty \in (0, \infty)$, assume that $1 \leq p < r < \infty$, let $T: E^\mathbb{R} \rightarrow E^\mathbb{R}$ be linear and satisfy for all $f: E \rightarrow \mathbb{R}$ that*

$$|\{y \in E: |(T(f))(y)| > t\}| \leq (N_p/t)^p \quad \text{and} \quad \|T(f)\|_{L^\infty(E)} \leq N_\infty \|f\|_{L^\infty(E)}. \quad (\text{A.9})$$

Then it holds for all $f: E \rightarrow \mathbb{R}$ that

$$\|T(f)\|_{L^r(E)} \leq 2 \left(\frac{r}{r-p} \right)^{1/r} N_p^{\frac{p}{r}} N_\infty^{1-\frac{p}{r}} \|f\|_{L^r(E)}. \quad (\text{A.10})$$

Proof of Lemma A.3 Throughout this proof let $f: E \rightarrow \mathbb{R}$ and let $f_i = (f_i^{t,\lambda}(x))_{t,\lambda \in (0,\infty), x \in E}$, $i \in \{1, 2\}$, be the functions which satisfy that

$$\forall x \in E, t, \lambda \in (0, \infty): \left(f_1^{t,\lambda}(x) = f(x) \mathbb{1}_{f(x) > \lambda t} \right) \wedge \left(f_2^{t,\lambda}(x) = f(x) \mathbb{1}_{f(x) \leq \lambda t} \right) \quad (\text{A.11})$$

First, Markov's inequality and Eq. A.9 show for all $\lambda, t > 0$ that

$$\begin{aligned} \left| \left\{ y \in E: |(T(f_1^{\lambda,t}))(y)| > \frac{t}{2} \right\} \right| &\leq \left(\frac{2}{t} \right)^p \|T(f_1^{\lambda,t})\|_{L^p(E)}^p \\ &\leq \left(\frac{2N_p}{t} \right)^p \sum_{y \in E} |f_1^{\lambda,t}(y)|^p \leq \left(\frac{2N_p}{t} \right)^p \left[\sum_{y \in E} |f(y)|^p \mathbb{1}_{f(y) > \lambda t} \right]. \end{aligned} \quad (\text{A.12})$$

Observe that Eq. A.9 shows for all $\lambda \in [0, (2N_\infty)^{-1}]$, $t \in (0, \infty)$ that

$$\|T(f_2^{\lambda,t})\|_{L^\infty(E)} \leq N_\infty^{-1} \|f_2^{\lambda,t}\|_{L^\infty(E)} \leq N_\infty^{-1} \lambda t \leq \frac{t}{2} \quad (\text{A.13})$$

Hence, it holds for all $\lambda \in [(2N_\infty)^{-1}, \infty)$, $t \in (0, \infty)$ that

$$\left| \left\{ y \in E: |(T(f_2^{\lambda,t}))(y)| > \frac{t}{2} \right\} \right| = 0. \quad (\text{A.14})$$

The fact that T is linear, the fact that $f_1 + f_2 = f$ (see Eq. A.11), the triangle inequality, and Eq. A.12 therefore show for all $\lambda \in [(2N_\infty)^{-1}, \infty)$, $t \in (0, \infty)$ that

$$\begin{aligned} &|\{y \in E: |(T(f))(y)| > t\}| \\ &\leq \left| \left\{ y \in E: |(T(f_1^{\lambda,t}))(y)| > \frac{t}{2} \right\} \right| + \left| \left\{ y \in E: |(T(f_2^{\lambda,t}))(y)| > \frac{t}{2} \right\} \right| \\ &\leq \left(\frac{2N_p}{t} \right)^p \left[\sum_{y \in E} |f(y)|^p \mathbb{1}_{f(y) > \lambda t} \right]. \end{aligned} \quad (\text{A.15})$$

The fact that $\forall x \in [0, \infty): x^r = \int_0^\infty r t^{r-1} \mathbb{1}_{x>t}$ Tonelli's theorem, and a direct calculation hence yield for all $\lambda \in [(2N_\infty)^{-1}, \infty)$, $t \in (0, \infty)$ that

$$\begin{aligned} \sum_{y \in E} |(T(f))(y)|^r &= \sum_{y \in E} \int_0^\infty r t^{r-1} \mathbb{1}_{|(T(f))(y)|>t} dt = \int_0^\infty r t^{r-1} \left[\sum_{y \in E} \mathbb{1}_{|(T(f))(y)|>t} \right] dt \\ &= \int_0^\infty r t^{r-1} |\{y \in E: (T(f))(y) > t\}| dt \leq \int_0^\infty r t^{r-1} \left(\frac{2N_p}{t} \right)^p \left[\sum_{y \in E} |f(y)|^p \mathbb{1}_{f(y)>\lambda t} \right] dt \\ &= r(2N_p)^p \sum_{y \in E} \left[|f(y)|^p \int_0^\infty r t^{r-p-1} \mathbb{1}_{f(y)>\lambda t} dt \right] = \frac{r(2N_p)^p}{(r-p)\lambda^{r-p}} \sum_{y \in E} |f(y)|^r. \end{aligned} \quad (\text{A.16})$$

This (with $\lambda \leftarrow (2N_\infty)^{-1}$) implies that $\|f\|_{L^r(E)}^r \leq \frac{r}{r-p} (2N_p)^p (2N_\infty)^{r-p}$. This and the fact that f was arbitrary complete the proof of Lemma A.3. $\square$

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