Strong Feller Property for SDEs Driven by Multiplicative Cylindrical Stable Noise

Tadeusz Kulczycki¹ · Michał Ryznar¹ · Paweł Sztonyk¹

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Abstract

We consider the stochastic differential equation $dX_t = A(X_{t-})dZ_t$, $X_0 = x$, driven by cylindrical α -stable process Z_t in $\mathbb{R}^d$, where $\alpha \in (0, 1)$ and $d \geq 2$. We assume that the determinant of $A(x) = (a_{ij}(x))$ is bounded away from zero, and $a_{ij}(x)$ are bounded and Lipschitz continuous. We show that for any fixed $\gamma \in (0, \alpha)$ the semigroup P_t of the process X_t satisfies $|P_t f(x) - P_t f(y)| \leq ct^{-\gamma/\alpha} |x - y|^\gamma \|f\|_\infty$ for arbitrary bounded Borel function f . Our approach is based on Levi's method.

Keywords Strong Feller property · Stochastic differential equations · Cylindrical stable processes · Semigroups of operators

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1 Introduction

Let $Z_t = (Z_t^{(1)}, \dots, Z_t^{(d)})^T$ be a cylindrical α -stable process, that is $Z_t^{(1)}, \dots, Z_t^{(d)}$ are independent one-dimensional symmetric standard α -stable processes of index $\alpha \in (0, 1)$, $d \in \mathbb{N}$, $d \geq 2$. We consider the stochastic differential equations

$$dX_t = A(X_{t-})dZ_t, \quad X_0 = x \in \mathbb{R}^d, \quad (1)$$

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✉ Michał Ryznar
Michal.Ryznar@pwr.edu.pl

Tadeusz Kulczycki
Tadeusz.Kulczycki@pwr.edu.pl

Paweł Sztonyk
Pawel.Sztonyk@pwr.edu.pl

¹ Faculty of Pure and Applied Mathematics, Wrocław University of Science and Technology, Wyb. Wyspiańskiego 27, 50-370, Wrocław, Poland

where $A(x) = (a_{ij}(x))$ is a $d \times d$ matrix and there are constants $\eta_1, \eta_2, \eta_3 > 0$, such that for any $x, y \in \mathbb{R}^d$, $i, j \in \{1, \dots, d\}$,

$$|a_{ij}(x)| \leq \eta_1, \quad (2)$$

$$\det(A(x)) \geq \eta_2, \quad (3)$$

$$|a_{ij}(x) - a_{ij}(y)| \leq \eta_3 |x - y|. \quad (4)$$

It is well known that the SDE (1) has a unique strong solution X_t , see e.g. [30, Theorem 34.7 and Corollary 35.3]. By [33, Corollary 3.3], X_t is a Feller process.

Let $\mathbb{E}^x$ denote the expected value of the process X starting from x and $\mathcal{B}_b(\mathbb{R}^d)$ denote the set of all Borel bounded functions $f: \mathbb{R}^d \rightarrow \mathbb{R}$. For any $t \geq 0$, $x \in \mathbb{R}^d$ and $f \in \mathcal{B}_b(\mathbb{R}^d)$ we put

$$P_t f(x) = \mathbb{E}^x f(X_t). \quad (5)$$

The main result of this paper is the following theorem.

Theorem 1.1 *For any $\gamma \in (0, \alpha)$, $\tau > 0$, $t \in (0, \tau]$, $x, y \in \mathbb{R}^d$ and $f \in \mathcal{B}_b(\mathbb{R}^d)$ we have*

$$|P_t f(x) - P_t f(y)| \leq ct^{-\gamma/\alpha} |x - y|^\gamma \|f\|_\infty, \quad (6)$$

where c depends on $\tau, \alpha, d, \eta_1, \eta_2, \eta_3, \gamma$.

Recently, estimates of the type $|P_t f(x) - P_t f(y)| \leq c_t |x - y|^\gamma \|f\|_\infty$ or $|\nabla_x P_t f(x)| \leq c_t \|f\|_\infty$ have been intensively studied for semigroups of solutions of SDEs $dX_t = A(X_t) dZ_t + b(X_t) dt$ driven by Lévy processes Z with jumps. In [37] such results were proved for the Orstein-Uhlenbeck jump process X . In [38] the above estimates were shown in the case when the driving process Z is a subordinated Brownian motion. There are also known results when the Lévy measure ν of Z satisfies $\nu(dz) \geq c 1_{|z| \leq r} |z|^{-d-\alpha} dz$ for some $\alpha \in (0, 2)$ and $c, r > 0$, see [27]. Another well explored case is when Z has a non-degenerate diffusion part, as done in [40]. The above estimates were also studied in [32] for solutions of SDEs in infinite dimensional Hilbert space driven by an additive cylindrical stable noise (see [32, Theorem 5.7]).

Note that Theorem 1.1 implies the strong Feller property of the semigroup P_t . This property also follows from [10], see Remark 4.23. The strong Feller property for SDEs driven by additive cylindrical Lévy processes have been intensively studied recently (see e.g. [11, 32, 39]). It is worth mentioning that the strong Feller property and gradient estimates for the semigroups associated to SDEs driven by Lévy processes in $\mathbb{R}^d$ with jumps, with absolutely continuous Lévy measures, have been examined for many years (see e.g. [25, 28, 34, 36, 38, 40, 42]).

The SDE (1) (with multiplicative noise) was studied by Bass and Chen in [1]. They proved existence and uniqueness of weak solutions of the SDE (1) under very mild assumptions on matrices $A(x)$ (i.e. they assumed that $A(x)$ are continuous and bounded in x and nondegenerate for each x). In [24] the SDE (1) was considered for diagonal matrices $A(x)$ whose diagonal coefficients are bounded away from zero, from infinity and Hölder continuous. Under these assumptions the corresponding transition density $p^A(t, x, y)$ was constructed and Hölder estimates $x \rightarrow p^A(t, x, y)$ were obtained. These estimates imply the strong Feller property of the corresponding semigroup.

The case of non-diagonal matrices $A(x)$, treated in this paper, is much more difficult. We use Levi's method to construct the semigroup P_t and to prove Theorem 1.1. However, there are many problems in applying this method to the case when multiplicative coefficient $A(x)$ in the SDE (1) may perform a non-trivial rotation. Therefore we had to introduce some new ideas. Below we briefly describe the main steps in our approach.

Let us put

$$\mathcal{K}f(x) = \mathcal{A}_\alpha \sum_{i=1}^d \lim_{\zeta \rightarrow 0^+} \int_{|w|>\zeta} [f(x + a_i(x)w) - f(x)] \frac{dw}{|w|^{1+\alpha}},$$

where $a_i(x) = (a_{1i}(x), \dots, a_{di}(x))$ and $\mathcal{A}_\alpha = 2^\alpha \Gamma((1+\alpha)/2) / (\pi^{1/2} |\Gamma(-\alpha/2)|)$. It is well known (see e.g. ([1, Proposition 4.1]) that $\mathcal{K}$ is the generator of the process X .

We divide $\mathcal{K}$ into two parts

$$\mathcal{K}f(x) = \mathcal{L}f(x) + \mathcal{R}f(x), \quad (7)$$

where

$$\mathcal{L}f(x) = \sum_{i=1}^d \lim_{\zeta \rightarrow 0^+} \int_{|w|>\zeta} [f(x + a_i(x)w) - f(x)] \mu(w) dw,$$

and $\mu(w) dw$ is the appropriate truncation of the Lévy measure $|w|^{-1-\alpha} dw$ ($\text{supp}(\mu) \subset [-1, 1]$). The definition of the truncated measure is presented at the beginning of Section 2. Roughly speaking, $\mathcal{L}$ corresponds to small jumps of the process X and $\mathcal{R}$ corresponds to big jumps of the process X . Our first aim is to construct the heat kernel $u(t, x, y)$ corresponding to the operator $\mathcal{L}$. This is done by using Levi's method, which is heuristically explained below.

At first we choose a kernel $\tilde{u}(t, x, y)$ which serves as the first order approximation of $u(t, x, y)$. By $r(t, x, y)$ we denote the resulting error, that is $u(t, x, y) = \tilde{u}(t, x, y) + r(t, x, y)$. Put

$$q_0(t, x, y) = \left(\frac{\partial}{\partial t} - \mathcal{L} \right) r(t, \cdot, y)(x) = - \left(\frac{\partial}{\partial t} - \mathcal{L} \right) \tilde{u}(t, \cdot, y)(x). \quad (8)$$

Since $u(t, x, y)$ is the heat kernel for $\mathcal{L}$, at least formally, one has

$$r(t, x, y) = \int_0^t \int_{\mathbb{R}^d} u(t-s, x, z) q_0(s, z, y) dz ds,$$

so

$$u(t, x, y) = \tilde{u}(t, x, y) + \int_0^t \int_{\mathbb{R}^d} u(t-s, x, z) q_0(s, z, y) dz ds.$$

The formal solution of this equation is given by the formula

$$u(t, x, y) = \tilde{u}(t, x, y) + \sum_{n=0}^{\infty} \int_0^t \int_{\mathbb{R}^d} \tilde{u}(t-s, x, z) q_n(s, z, y) dz ds, \quad (9)$$

where q_n is defined inductively as

$$q_n(t, x, y) = \int_0^t \int_{\mathbb{R}^d} q_0(t-s, x, z) q_{n-1}(s, z, y) dz ds. \quad (10)$$

As the approximation $\tilde{u}(t, x, y)$ we choose the heat kernel of the operator $\mathcal{L}$ frozen at $y \in \mathbb{R}^d$. More precisely, we put $\tilde{u}(t, x, y) = p_y(t, x - y)$, where $p_y(t, \cdot)$ is the heat kernel of $\mathcal{L}^y$ (i.e. $\frac{\partial}{\partial t} p_y(t, z) = \mathcal{L}^y p_y(t, \cdot)(z)$, $t > 0$, $z \in \mathbb{R}^d$), where

$$\mathcal{L}^y f(x) = \mathcal{A}_\alpha \sum_{i=1}^d \lim_{\zeta \rightarrow 0^+} \int_{|w|>\zeta} [f(x + a_i(y)w) - f(x)] \mu(w) dw. \quad (11)$$

Typically, in many papers using Levi's method, the first step was to obtain precise bounds for $q_0(t, x, y)$ which allow to estimate $q_n(t, x, y)$ inductively point-wise. In our case it seems impossible to obtain such precise bounds, hence we prove (see Proposition 3.9) some

crude estimates for $q_0(t, x, y)$ and its integrals with respect to dx or dy , which are sufficient for our purposes. The main tools to prove Proposition 3.9 are Lemma 3.6 and the estimate (18). This key estimate (18) is proven using the techniques and results from [23], [22] and [35]. After constructing the transition density $u(t, x, y)$ we use the technique developed by Knopova and Kulik [19] to show that $U_t f(x) := \int_{\mathbb{R}^d} u(t, x, y) f(y) dy$ satisfies the appropriate heat equation in the so-called approximate setting. In the next step we construct the semigroup T_t for the solution of the SDE (1) (driven by the not truncated process). Roughly speaking, this construction is based on adding big jumps to the truncated process (Meyer-type construction). Next we show that $T_t f(x)$ satisfies the appropriate heat equation in the approximate setting (see Lemma 4.18), which allows to prove that the constructed semigroup T_t is in fact the semigroup P_t .

Let us introduce the operator $\mathcal{K}$ frozen at a point $y \in \mathbb{R}^d$,

$$\mathcal{K}^y f(x) = \mathcal{A}_\alpha \sum_{i=1}^d \lim_{\zeta \rightarrow 0^+} \int_{|w| > \zeta} [f(x + a_i(y)w) - f(x)] \frac{dw}{|w|^{1+\alpha}}.$$

Let $p_y^\mathcal{K}(t, \cdot)$ be the heat kernel of $\mathcal{K}^y$, that is $\frac{\partial}{\partial t} p_y^\mathcal{K}(t, z) = \mathcal{K}^y p_y^\mathcal{K}(t, \cdot)(z)$ for any $t > 0$, $z \in \mathbb{R}^d$. One may ask why the Lévy method is applied to the operator $\mathcal{L}$ (corresponding to the truncated Lévy measure) and not to the operator $\mathcal{K}$. The reason is that (in general) the heat kernel $p_y^\mathcal{K}(t, \cdot)$ of the operator $\mathcal{K}^y$ does not have good integrability properties, i.e. $\int_{\mathbb{R}^d} p_y^\mathcal{K}(t, x - y) dy = \infty$ for some choices of matrices $A(x)$ and some $t > 0$, $x \in \mathbb{R}^d$. Therefore it cannot be used as the first order approximation of the heat kernel of $\mathcal{K}$.

Our current technique is restricted to the case $\alpha \in (0, 1)$. The main difficulty for $\alpha \in [1, 2)$ is that in such case one has to effectively estimate the expression

$$p_y(t, x + a_i(x)w) + p_y(t, x - a_i(x)w) - 2p_y(t, x) \quad (12)$$

instead of

$$p_y(t, x + a_i(x)w) - p_y(t, x), \quad (13)$$

where $p_y(t, x)$ is the frozen density for the truncated process (see Section 3 for the precise definition of $p_y(t, x)$). Our crucial estimate (18) allows a suitable estimate of (13) but fails to bound (12) in a way sufficient for our purpose.

We point out that the existence of transition densities $p(t, x, y)$ of the process X is already well known, see [10]. In our paper we obtain a representation of this densities, see Remark 4.26. One may ask about the boundedness of $p(t, x, y)$. It turns out that for some choices of matrices $A(x)$ (satisfying (2), (3), (4)) and for some $t > 0$, $x \in \mathbb{R}^d$ we have $p(t, x, \cdot) \notin L^\infty(\mathbb{R}^d)$, see Remarks 4.24 and 4.25. Nevertheless we have the following regularity result.

Theorem 1.2 *For any $\gamma \in (0, \alpha/d)$, $\tau > 0$, $t \in (0, \tau]$, $x \in \mathbb{R}^d$ and $f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ we have*

$$|P_t f(x)| \leq ct^{-\gamma d/\alpha} \|f\|_\infty^{1-\gamma} \|f\|_1^\gamma,$$

where c depends on $\tau, \alpha, d, \eta_1, \eta_2, \eta_3, \gamma$.

The paper is organized as follows. In Section 2 we study properties of the transition density of a suitably truncated one-dimensional stable process. These properties are crucial in the sequel. In Section 3 we construct the transition density $u(t, x, y)$ of the solution of (1) driven by the truncated process. We also show that it satisfies the appropriate equation in the approximate setting. In Section 4 we construct the transition semigroup of the solution of (1). We also prove Theorems 1.1 and 1.2.

2 Preliminaries

All constants appearing in this paper are positive and finite. In the whole paper we fix $\tau > 0$, $\alpha \in (0, 1)$, $d \in \mathbb{N}$, $d \geq 2$, η_1, η_2, η_3 , where η_1, η_2, η_3 appear in (2), (3) and (4). We adopt the convention that constants denoted by c (or $c_1, c_2, \dots$) may change their value from one use to the next. In the whole paper, unless is explicitly stated otherwise, we understand that constants denoted by c (or $c_1, c_2, \dots$) depend on $\tau, \alpha, d, \eta_1, \eta_2, \eta_3$. We also understand that they may depend on the choice of the constants ε and γ . We write $f(x) \approx g(x)$ for $x \in A$ if $f, g \geq 0$ on A and there is a constant $c \geq 1$ such that $c^{-1}f(x) \leq g(x) \leq cf(x)$ for $x \in A$. The standard inner product for $x, y \in \mathbb{R}^d$ we denote by $x \cdot y$.

For any $t > 0$, $x \in \mathbb{R}^d$ we define the measure $\sigma_t(x, \cdot)$ by

$$\sigma_t(x, A) = \mathbb{P}^x(X_t \in A), \quad (14)$$

for any Borel set $A \subset \mathbb{R}^d$. $\mathbb{P}^x$ denotes the distribution of the process X starting from $x \in \mathbb{R}^d$. For any $t > 0$, $x \in \mathbb{R}^d$ we have

$$P_t f(x) = \int_{\mathbb{R}^d} f(y) \sigma_t(x, dy), \quad f \in \mathcal{B}_b(\mathbb{R}^d). \quad (15)$$

It is well known that the density of the Lévy measure of the one-dimensional symmetric standard α -stable process is given by $\mathcal{A}_\alpha |x|^{-1-\alpha}$. In the sequel we will need to truncate this density. The truncated density will be denoted by $\mu^{(\delta)}(x)$. For $\delta \in (0, 1]$ we would like to construct a function $\mu^{(\delta)} : \mathbb{R} \setminus \{0\} \rightarrow [0, \infty)$ such that for $x \in (0, \delta]$ we have $\mu^{(\delta)}(x) = \mathcal{A}_\alpha |x|^{-1-\alpha}$, for $x \in (\delta, 2\delta)$ we have $\mu^{(\delta)}(x) \in (0, \mathcal{A}_\alpha |x|^{-1-\alpha})$ and $\mu^{(\delta)}(x) = 0$ for $x \geq 2\delta$. Moreover, $\mu^{(\delta)}$ is defined so that it is nonincreasing, convex and C^1 on $(0, \infty)$ and satisfies $\mu^{(\delta)}(-x) = \mu^{(\delta)}(x)$ for $x \in (0, \infty)$.

At first we find $\mu^{(\delta)}$ for $\delta = 1$. Let $v(x) = \mathcal{A}_\alpha |x|^{-1-\alpha}$. Observe that the tangent line to the graph of $v(x)$ at $(1, v(1))$ is $l : y - \mathcal{A}_\alpha = -\mathcal{A}_\alpha(1 + \alpha)(x - 1)$ and it crosses the horizontal axes at $x_0 = 1 + \frac{1}{1+\alpha} < 2$. Let $(x - 2)^2 + (y - r)^2 = r^2$, $r > 0$, be a circle tangent to the horizontal axes at the point $(2, 0)$ and tangent to the line l . Let $(\tilde{x}, \tilde{y})$ be the point at which the circle touches the line l . It is clear that $1 < \tilde{x} < x_0$. We put

$$\mu^{(1)}(x) = \begin{cases} v(x) & \text{for } 0 < |x| \leq 1, \\ \mathcal{A}_\alpha - \mathcal{A}_\alpha(1 + \alpha)(|x| - 1) & \text{for } 1 < |x| \leq \tilde{x}, \\ r - \sqrt{r^2 - (|x| - 2)^2} & \text{for } \tilde{x} < |x| \leq 2, \\ 0 & \text{for } |x| > 2. \end{cases}$$

It is obvious that $\mu^{(1)}$ is convex, decreasing, continuously differentiable on $(0, \infty)$ and satisfies all the other requirements if $\delta = 1$. For arbitrary $\delta > 0$ we define

$$\mu^{(\delta)}(x) = \delta^{-\alpha-1} \mu^{(1)}(x/\delta).$$

Since $v(x) = \delta^{-\alpha-1} v(x/\delta)$ we see that $\mu^{(\delta)}$ possess all desired properties since $\mu^{(1)}$ does.

We also define

$$\mathcal{G}^{(\delta)} f(x) = \lim_{\zeta \rightarrow 0^+} \int_{|w| > \zeta} (f(x+w) - f(x)) \mu^{(\delta)}(w) dw.$$

By $g_t^{(\delta)}$ we denote the heat kernel corresponding to $\mathcal{G}^{(\delta)}$ that is

$$\frac{\partial}{\partial t} g_t^{(\delta)}(x) = \mathcal{G}^{(\delta)} g_t^{(\delta)}(x), \quad t > 0, x \in \mathbb{R},$$

$$\int_{\mathbb{R}} g_t^{(\delta)}(x) dx = 1, \quad t > 0.$$

It is well known that $g_t^{(\delta)}$ belongs to $C^1((0, \infty))$ as a function of t and belongs to $C^2(\mathbb{R})$ as a function of z . We also note that as a function of x it is symmetric and nonincreasing on $[0, \infty)$. This is due to the fact that $\mu^{(\delta)}$ also has that property.

For any $\varepsilon \in (0, 1]$, $\tau > 0$, $t \in (0, \infty)$ and $x \in \mathbb{R}$ we define

$$h_t^{(\varepsilon)}(x) = \begin{cases} \frac{t}{(|x|+t^{1/\alpha})^{1+\alpha}} & \text{for } |x| < \varepsilon, \\ c_\varepsilon t^{1+(d-1)/\alpha} e^{-|x|} & \text{for } |x| \geq \varepsilon, \end{cases} \quad (16)$$

where $c_\varepsilon = \frac{e^\varepsilon}{(1+\tau^{1/\alpha})^{1+\alpha} \tau^{(d-1)/\alpha}}$. The constant c_ε is chosen so that for any $t \in (0, \tau]$ the function $x \rightarrow h_t^{(\varepsilon)}(x)$ is nonincreasing on $[0, \infty)$.

Lemma 2.1 *For any $\varepsilon \in (0, 1]$, there exists c such that for $\delta = \varepsilon \min\{\frac{\alpha}{8(\alpha+d+2)}, \frac{1}{4d(\eta_1 \vee 1)^2}\}$, and any $t \in (0, \tau]$, $x, y \in \mathbb{R}$, we have*

$$g_t^{(\delta)}(x) \leq c h_t^{(\varepsilon)}(|x|), \quad (17)$$

$$|g_t^{(\delta)}(x) - g_t^{(\delta)}(y)| \leq c|x - y| \left(\frac{h_t^{(\varepsilon)}(|x|)}{t^{1/\alpha} + |x|} + \frac{h_t^{(\varepsilon)}(|y|)}{t^{1/\alpha} + |y|} \right). \quad (18)$$

Proof Our proof, specially of (18), will rely on the relationship between transition densities of isotropic and unimodal Lévy processes living in spaces of different dimensions. Namely, under certain conditions, the gradient of the transition density of one process living in $\mathbb{R}^n$ can be expressed in terms of the transition density of the other process living in $\mathbb{R}^{n+2}$, see Theorem 1.5 in [23].

Hence, at first we consider a general case of heat kernels $g^{(\delta, n)}$ on $\mathbb{R}^n$ for $\delta \in (0, 1]$, $n \in \{1, 2, \dots\}$ such that

$$g_t^{(\delta, n)}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-ix \cdot u} e^{-t\Phi_\delta^{(n)}(u)} du,$$

where

$$\Phi_\delta^{(n)}(u) = \int_{\mathbb{R}^n} (1 - \cos(u \cdot y)) \mu^{(\delta, n)}(y) dy, \quad u \in \mathbb{R}^n,$$

and $\mu^{(\delta, n)}(y) = \mu^{(\delta, n)}(|y|)$ is isotropic unimodal Lévy density such that $\mu^{(\delta, n)}(y) \approx |y|^{n-\alpha}$ for $|y| \leq \delta$, and $\mu^{(\delta, n)}(y) = 0$ for $|y| \geq 2\delta$. In the proof of this lemma we assume that constants c may additionally depend on n . It follows from Lemma 1 in [3] that

$$\frac{2}{n\pi^2} H(|u|) \leq \Phi_\delta^{(n)}(u) \leq 2H(|u|), \quad u \in \mathbb{R}^n,$$

where

$$H(r) = \int_{\mathbb{R}^n} (1 \wedge (r|y|)^2) \mu^{(\delta, n)}(y) dy, \quad r \geq 0,$$

hence we easily obtain

$$\Phi_\delta^{(n)}(u) \geq c|u|^2 \int_0^{\frac{1}{|u|} \wedge \delta} r^{1-\alpha} dr = c \frac{1}{\delta^\alpha} \left((\delta|u|)^2 \wedge (\delta|u|)^\alpha \right), \quad u \in \mathbb{R}^n.$$

Similarly,

$$\Phi_\delta^{(n)}(u) \leq c \frac{1}{\delta^\alpha} \left((\delta|u|)^2 \wedge (\delta|u|)^\alpha \right), \quad u \in \mathbb{R}^n.$$

In particular the symbol $\Phi_\delta^{(n)}$ has global weak lower scaling property with index α (see [3]). This yields, by Theorem 21 of [3],

$$g_t^{(\delta,n)}(x) \leq c \min\{(H^{-1}(1/t))^n, tH(1/|x|)|x|^{-n}\}, \quad t > 0, x \in \mathbb{R}^n.$$

Observing that $H(r) \leq cr^\alpha$ for $r \geq 0$ and $H^{-1}(1/t) \leq c\frac{1}{t^{1/\alpha}}$, for $t \leq \tau$, we have

$$g_t^{(\delta,n)}(x) \leq c \frac{t}{(|x| + t^{1/\alpha})^{n+\alpha}}, \quad x \in \mathbb{R}^n, t \in (0, \tau]. \quad (19)$$

Let $t \leq 1 \wedge \tau$. Using Lemma 4.2 from [35] we get

$$\begin{aligned} g_t^{(\delta,n)}(x) &\leq e^{\frac{-|x|}{8\delta} \log\left(\frac{\delta|x|}{tm_0}\right)} g_t^{(\delta,n)}(0) = \left(\frac{tm_0}{\delta|x|}\right)^{\frac{|x|}{8\delta}} g_t^{(\delta,n)}(0) \\ &\leq ct^{\frac{|x|}{8\delta} - \frac{n}{\alpha}} e^{\frac{-|x|}{8\delta} \log\left(\frac{\delta|x|}{m_0}\right)}, \quad |x| \geq \frac{em_0}{\delta} t, \end{aligned}$$

where $m_0 = m_0(\delta, n) = \int_{\mathbb{R}^n} |y|^2 \mu^{(\delta,n)}(y) dy < \infty$. This yields

$$g_t^{(\delta,n)}(x) \leq ct^{1+\frac{d-1}{\alpha}} e^{\frac{-|x|}{8\delta} \log\left(\frac{\delta|x|}{m_0}\right)},$$

provided $|x| \geq \max\{8\delta(1 + \frac{n+d-1}{\alpha}), \frac{em_0}{\delta} t\}$. We observe that there exists $c_1 = c_1(\delta, \alpha, d, n)$ such that

$$e^{\frac{-|x|}{8\delta} \log\left(\frac{\delta|x|}{m_0}\right)} \leq c_1 e^{-(\alpha+d)|x|/\alpha}, \quad x \in \mathbb{R}^n,$$

so we obtain

$$g_t^{(\delta,n)}(x) \leq c_2 t^{1+\frac{d-1}{\alpha}} e^{-(\alpha+d)|x|/\alpha}, \quad |x| \geq \max\left\{8\delta\left(1 + \frac{n+d-1}{\alpha}\right), \frac{em_0}{\delta} t\right\}, \quad (20)$$

with $c_2 = c_2(\delta, \alpha, d, n)$.

Let $1 \leq t \leq \tau$. Using again Lemma 4.2 from [35] we get

$$\begin{aligned} g_t^{(\delta,n)}(x) &\leq e^{\frac{-|x|}{8\delta} \log\left(\frac{\delta|x|}{tm_0}\right)} g_t^{(\delta,n)}(0) \leq e^{\frac{-|x|}{8\delta}} g_1^{(\delta,n)}(0) \\ &\leq ct^{1+\frac{d-1}{\alpha}} e^{\frac{-|x|}{8\delta}}, \quad |x| \geq \frac{em_0}{\delta} t. \end{aligned} \quad (21)$$

Observe that with $\delta = \varepsilon \min\left\{\frac{\alpha}{8(\alpha+d+2)}, \frac{1}{4d(\eta_1 \vee 1)^2}\right\}$ the estimate (17) follows from Eq. 19, Eq. 20 and Eq. 21 with $n = 1$.

The function $\mu^{(\delta,1)} = \mu^{(\delta)}$ satisfies the assumption of Theorem 1.5 in [23] which yields that there exists a Lévy process $X_t^{(3)}$ in $\mathbb{R}^3$ with the characteristic exponent $\Phi_\delta^{(3)}(u) = \Phi_\delta^{(3)}(|u|)$, $u \in \mathbb{R}^3$ and the radial, radially nonincreasing transition density $g_t^{(\delta,3)}(x) = g_t^{(\delta,3)}(|x|)$ satisfying

$$g_t^{(\delta,3)}(r) = \frac{-1}{2\pi r} \frac{d}{dr} g_t^{(\delta,1)}(r), \quad r > 0.$$

Furthermore it follows from the proof of Theorem 1.5 in [23] that the Lévy measure of the process $X_t^{(3)}$ is given by $\nu^{(\delta,3)}(dx) = \mu^{(\delta,3)}(|x|) dx$, where $\mu^{(\delta,3)}(r) = \frac{-1}{2\pi r} \frac{d\mu^{(\delta,1)}}{dr}(r)$, $r > 0$. In particular, $\mu^{(\delta,3)}(r)$ is nonincreasing on $(0, \infty)$ and we have $\mu^{(\delta,3)}(x) = \frac{\mathcal{A}_\alpha}{2\pi(2+\alpha)} |x|^{-3-\alpha}$ for $|x| \leq \delta$, $\mu^{(\delta,3)}(x) \leq \frac{\mathcal{A}_\alpha}{2\pi(2+\alpha)} \delta^{-3-\alpha}$, for $\delta \leq |x| \leq 2\delta$, and $\mu^{(\delta,3)}(x) = 0$ for $|x| > 2\delta$.

By Eqs. 19, 20 and 21, with $n = 3$, we obtain

$$g_t^{(\delta,3)}(x) \leq c \frac{t}{(|x| + t^{1/\alpha})^{3+\alpha}}, \quad x \in \mathbb{R}^3, t \in (0, \tau]$$

and

$$g_t^{(\delta,3)}(x) \leq ct^{1+\frac{d-1}{\alpha}} e^{-(\alpha+d)|x|/\alpha} \leq c \frac{t^{1+\frac{d-1}{\alpha}}}{(|x|+t^{1/\alpha})^2} e^{-|x|}, \quad |x| > \max\{\varepsilon, c_3 t\}, t \in (0, \tau].$$

The above two inequalities yield

$$g_t^{(\delta,3)}(x) \leq c \frac{h_t^{(\varepsilon)}(|x|)}{(|x|+t^{1/\alpha})^2}, \quad x \in \mathbb{R}^3, t \in (0, \tau],$$

and

$$\left| \frac{d}{dr} g_t^{(\delta,1)}(r) \right| \leq c \frac{r h_t^{(\varepsilon)}(r)}{(r+t^{1/\alpha})^2} \leq c \frac{h_t^{(\varepsilon)}(r)}{r+t^{1/\alpha}}, \quad r > 0, t \in (0, \tau].$$

Since $h_t^{(\varepsilon)}$ is nonincreasing, by the Lagrange theorem, we get

$$|g_t^{(\delta,1)}(x) - g_t^{(\delta,1)}(y)| \leq c|x-y| \left(\frac{h_t^{(\varepsilon)}(|x|)}{|x|+t^{1/\alpha}} + \frac{h_t^{(\varepsilon)}(|y|)}{|y|+t^{1/\alpha}} \right), \quad x, y \in \mathbb{R}.$$

□

Lemma 2.2 Let $\varepsilon \in (0, 1]$. For any $t \in (0, \tau]$, $x, x' \in \mathbb{R}$, if $|x - x'| \leq t^{1/\alpha}$ and $|x - x'| \leq \varepsilon/4$, then

$$h_t^{(\varepsilon)}(x') \leq c h_t^{(\varepsilon)}(x/2).$$

Proof Assume first that $|x| \wedge |x'| \leq \varepsilon/2$. Then by the definition of $h_t^{(\varepsilon)}(x)$ we have

$$h_t^{(\varepsilon)}(x) = \frac{t}{(t^{1/\alpha} + |x|)^{1+\alpha}} \geq \frac{t}{(t^{1/\alpha} + |x - x'| + |x'|)^{1+\alpha}} \geq \frac{t}{(2t^{1/\alpha} + 2|x'|)^{1+\alpha}} \geq c h_t^{(\varepsilon)}(x').$$

Assume now that $|x| \wedge |x'| > \varepsilon/2$. Then we have $|x'| \geq |x| - |x - x'| \geq |x| - |x|/2 = |x|/2$. Hence, $h_t^{(\varepsilon)}(x') \leq h_t^{(\varepsilon)}(x/2)$. □

Lemma 2.3 Let $\varepsilon \in (0, 1]$, $\delta = \varepsilon \min\{\frac{\alpha}{8(\alpha+d+2)}, \frac{1}{4d(\eta_1 \vee 1)^2}\}$. For any $t \in (0, \tau]$, $x, x' \in \mathbb{R}^d$, if $|x - x'| \leq t^{1/\alpha}$ and $|x - x'| \leq \delta$, then

$$\left| \prod_{i=1}^d g_t^{(\delta)}(x_i) - \prod_{i=1}^d g_t^{(\delta)}(x'_i) \right| \leq c \left(\prod_{i=1}^d h_t^{(\varepsilon)}(x_i/2) \right) \left[1 \wedge \sum_{j=1}^d t^{-1/\alpha} |x_j - x'_j| \right]. \quad (22)$$

Proof By Lemma 2.1 we get

$$\begin{aligned} & \left| \prod_{i=1}^d g_t^{(\delta)}(x_i) - \prod_{i=1}^d g_t^{(\delta)}(x'_i) \right| \\ & \leq \sum_{j=1}^d \left[\left| g_t^{(\delta)}(x_j) - g_t^{(\delta)}(x'_j) \right| \prod_{i \neq j, 1 \leq i \leq d} g_t^{(\delta)}(|x_i| \wedge |x'_i|) \right] \\ & \leq c \left(\prod_{i=1}^d h_t^{(\varepsilon)}(|x_i| \wedge |x'_i|) \right) \sum_{j=1}^d \frac{|x_j - x'_j|}{t^{1/\alpha}}. \end{aligned}$$

Clearly, since $g_t^{(\delta)}$ is nonincreasing on $[0, \infty)$, we have

$$\left| \prod_{i=1}^d g_t^{(\delta)}(x_i) - \prod_{i=1}^d g_t^{(\delta)}(x'_i) \right| \leq \prod_{i=1}^d g_t^{(\delta)}(|x_i| \wedge |x'_i|).$$

Now the assertion follows from Lemmas 2.1 and 2.2. $\square$

Lemma 2.4 *Let $\varepsilon \in (0, 1]$, $\delta = \varepsilon \min \left\{ \frac{\alpha}{8(\alpha+d+2)}, \frac{1}{4d(\eta_1 \vee 1)^2} \right\}$ and let $a \in \mathbb{R}$ be such that $|a| \leq \frac{\varepsilon}{4\delta}$. Then there exists c such that for any $t \in (0, \tau]$, $x \in \mathbb{R}$ we have*

$$\int_{\mathbb{R}} |g_t^{(\delta)}(x+aw) - g_t^{(\delta)}(x)| \mu^{(\delta)}(w) dw \leq \frac{c|a|^\alpha h_t^{(\varepsilon)}\left(\frac{x}{2}\right)}{t}. \quad (23)$$

Proof First we note that

$$\int_{\mathbb{R}} |g_t^{(\delta)}(x+aw) - g_t^{(\delta)}(x)| \mu^{(\delta)}(w) dw \leq \int_{|w| < 2\delta} |g_t^{(\delta)}(x+aw) - g_t^{(\delta)}(x)| \mathcal{A}_\alpha |w|^{-1-\alpha} dw,$$

and by the substitution $s = aw$ we have

$$\int_{|w| < 2\delta} |g_t^{(\delta)}(x+aw) - g_t^{(\delta)}(x)| |w|^{-1-\alpha} dw = |a|^\alpha \int_{|s| < 2\delta|a|} |g_t^{(\delta)}(x+s) - g_t^{(\delta)}(x)| |s|^{-1-\alpha} ds.$$

Now we estimate the latter integral. Let

$$\begin{aligned} \int_{|s| < 2\delta|a|} |g_t^{(\delta)}(x+s) - g_t^{(\delta)}(x)| |s|^{-1-\alpha} ds &= \int_{|s| < t^{1/\alpha} \wedge (2\delta|a|)} |g_t^{(\delta)}(x+s) - g_t^{(\delta)}(x)| |s|^{-1-\alpha} ds \\ &\quad + \int_{t^{1/\alpha} \wedge (2\delta|a|) \leq |s| < 2\delta|a|} |g_t^{(\delta)}(x+s) - g_t^{(\delta)}(x)| |s|^{-1-\alpha} ds \\ &=: I_1 + I_2. \end{aligned}$$

Using (18) we get

$$\begin{aligned} I_1 &\leq c \int_{|s| < t^{1/\alpha} \wedge (2\delta|a|)} |s|^{-\alpha} \left(\frac{h_t^{(\varepsilon)}(x+s)}{t^{1/\alpha} + |x+s|} + \frac{h_t^{(\varepsilon)}(x)}{t^{1/\alpha} + |x|} \right) ds \\ &\leq ct^{-1/\alpha} \int_{|s| < t^{1/\alpha} \wedge (2\delta|a|)} |s|^{-\alpha} \left(h_t^{(\varepsilon)}(|x+s|) + h_t^{(\varepsilon)}(|x|) \right) ds. \end{aligned}$$

If $|x| \geq 2t^{1/\alpha}$ then for $|s| < t^{1/\alpha}$ we have $h_t^{(\varepsilon)}(|x+s|) \leq h_t^{(\varepsilon)}\left(\frac{|x|}{2}\right)$, since $h_t^{(\varepsilon)}$ is nonincreasing. If $|x| \leq 2t^{1/\alpha}$ and $|s| < t^{1/\alpha}$ then $h_t^{(\varepsilon)}\left(\frac{|x|}{2}\right) \geq ct^{1/\alpha}$ and $h_t^{(\varepsilon)}(|x+s|) \leq t^{1/\alpha}$, hence we obtain $h_t^{(\varepsilon)}(|x+s|) \leq ch_t^{(\varepsilon)}\left(\frac{|x|}{2}\right)$. This yields

$$I_1 \leq ct^{-1/\alpha} h_t^{(\varepsilon)}\left(\frac{|x|}{2}\right) \int_{|s| < t^{1/\alpha}} |s|^{-\alpha} ds = \frac{c}{t} h_t^{(\varepsilon)}\left(\frac{|x|}{2}\right).$$

Now we estimate I_2 . If $t^{1/\alpha} > 2\delta|a|$ then $I_2 = 0$ so we assume that $t^{1/\alpha} \leq 2\delta|a|$ and using Eq. 17 we obtain

$$I_2 \leq c \int_{t^{1/\alpha} \leq |s| \leq 2\delta|a|} \left(h_t^{(\varepsilon)}(|x+s|) + h_t^{(\varepsilon)}(|x|) \right) |s|^{-1-\alpha} ds.$$

Since we have

$$\int_{|s| \geq t^{1/\alpha}} h_t^{(\varepsilon)}(|x|) |s|^{-1-\alpha} ds = \frac{c}{t} h_t^{(\varepsilon)}(|x|),$$

we only need to estimate

$$J = \int_{t^{1/\alpha} \leq |s| \leq 2\delta|a|} h_t^{(\varepsilon)}(|x+s|)|s|^{-1-\alpha} ds.$$

Let $g_t^{(\infty)}$ denote the transition density of the one-dimensional symmetric standard α -stable process. It follows from [2] that $g_t^{(\infty)}(x) \approx \frac{t}{(t^{1/\alpha} + |x|)^{1+\alpha}}$, hence $h_t^{(\varepsilon)}(y) \leq c g_t^{(\infty)}(y)$, for all $t \in (0, \tau]$, $y \in \mathbb{R}$. Noting also that $|s|^{-1-\alpha} \leq c \frac{g_t^{(\infty)}(s)}{t}$ for $|s| \geq t^{1/\alpha}$, and using the Chapman-Kolmogorov equation for $g_t^{(\infty)}$ we get

$$\begin{aligned} J &\leq \int_{|s| \geq t^{1/\alpha}} h_t^{(\varepsilon)}(x+s)|s|^{-1-\alpha} ds \\ &\leq \frac{c}{t} \int_{\mathbb{R}} g_t^{(\infty)}(x+s) g_t^{(\infty)}(s) ds = \frac{c}{t} g_{2t}^{(\infty)}(x), \end{aligned}$$

which yields $J \leq \frac{c h_t^{(\varepsilon)}(x)}{t}$ for $|x| \leq |\varepsilon|$.

Let now $|x| \geq \varepsilon \geq 4\delta|a| \geq 2t^{1/\alpha}$. Then $|x+s| \geq |x|/2$ for $s \leq 2\delta|a|$, and we obtain

$$\begin{aligned} J &\leq \int_{t^{1/\alpha} \leq |s| \leq 2\delta|a|} h_t^{(\varepsilon)}(|x+s|)|s|^{-1-\alpha} ds \leq h_t^{(\varepsilon)}\left(\frac{|x|}{2}\right) \int_{t^{1/\alpha} \leq |s|} |s|^{-1-\alpha} ds \\ &\leq \frac{c}{t} h_t^{(\varepsilon)}\left(\frac{|x|}{2}\right). \end{aligned}$$

□

Lemma 2.5 For any $\varepsilon \in (0, 1]$, $\delta = \varepsilon \min\{\frac{\alpha}{8(\alpha+d+2)}, \frac{1}{4d(\eta_1 \vee 1)^2}\}$ and $m, n \in \mathbb{N}$, $n \geq 2$ there exists $c = c(m, n, \tau, \alpha, d, \eta_1, \eta_2, \eta_3, \varepsilon, \delta)$ such that for any $t \in (0, \tau]$, $x \in \mathbb{R}$ we have

$$\left| \frac{d^m}{dx^m} g_t^{(\delta)}(x) \right| \leq c t^{-(1+m)/\alpha} (1 + |x|)^{-n}.$$

Proof In the proof we assume that constants c may additionally depend on m and n . We use Theorem 3 of [16]. Let $f(s) = \mathcal{A}_\alpha s^{-1-\alpha}$ for $s \leq \delta$ and $f(s) = \mathcal{A}_\alpha \delta^{n-1-\alpha} s^{-n}$ for $s > \delta$. It is then obvious that the assumptions (1) and (2) of Theorem 3 in [16] hold and it follows that

$$\left| \frac{d^m}{dx^m} g_t^{(\delta)}(x) \right| \leq c t^{-(1+m)/\alpha} \min \left\{ 1, t^{1+1/\alpha} f(|x|) + \left(1 + \frac{|x|}{t^{1/\alpha}} \right)^{-n} \right\}, \quad x \in \mathbb{R}, t \in (0, \tau].$$

Clearly, for $|x| \geq 1$ and $t \in (0, \tau]$ we have $f(|x|) \approx |x|^{-n}$ and $\left(1 + \frac{|x|}{t^{1/\alpha}} \right)^{-n} \leq c |x|^{-n}$. This implies the assertion of the lemma. □

Lemma 2.6 There is a constant $C = C(\alpha)$ such that for $\delta \in (0, 1]$, $a \geq 0$, and any $t > 0$,

$$\int_{\mathbb{R}} \left(\frac{(a + |w|)|w|}{t^{1/\alpha}} \wedge 1 \right) \mu^{(\delta)}(w) dw \leq C \frac{t^{1/2} + a^\alpha}{t}.$$

Proof We have

$$\left(\frac{(a + |w|)|w|}{t^{1/\alpha}} \wedge 1 \right) \leq \left(\frac{a|w|}{t^{1/\alpha}} \wedge 1 \right) + \left(\frac{w^2}{t^{1/\alpha}} \wedge 1 \right).$$

Hence, using $\mu^{(\delta)}(w) \leq \frac{\mathcal{A}_\alpha}{|w|^{1+\alpha}}$, we obtain

$$\int_{\mathbb{R}} \left(\frac{(a + |w|)|w|}{t^{1/\alpha}} \wedge 1 \right) \mu^{(\delta)}(w) dw \leq \mathcal{A}_\alpha \int_{\mathbb{R}} \left(\frac{a|w|}{t^{1/\alpha}} \wedge 1 \right) \frac{dw}{|w|^{1+\alpha}} + \mathcal{A}_\alpha \int_{\mathbb{R}} \left(\frac{w^2}{t^{1/\alpha}} \wedge 1 \right) \frac{dw}{|w|^{1+\alpha}}.$$

Next,

$$\int_{\mathbb{R}} \left(\frac{a|w|}{t^{1/\alpha}} \wedge 1 \right) \frac{dw}{|w|^{1+\alpha}} = \frac{a^\alpha}{t} \int_{\mathbb{R}} (|w| \wedge 1) \frac{dw}{|w|^{1+\alpha}}$$

and

$$\int_{\mathbb{R}} \left(\frac{w^2}{t^{1/\alpha}} \wedge 1 \right) \frac{dw}{|w|^{1+\alpha}} = t^{-1/2} \int_{\mathbb{R}} (w^2 \wedge 1) \frac{dw}{|w|^{1+\alpha}}.$$

□

In the sequel we will need a version of the inverse map theorem for a Lipschitz function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $n \in \mathbb{N}$. The corresponding theorem is the main result in [9], however it is not formulated in a suitable way for our purpose. Below, closely following the arguments from [9], we provide a version we need.

It is well known that y almost surely the Jacobi matrix $\mathcal{J}_f(y)$ of f exists. For any $y_0 \in \mathbb{R}^n$ we define (see Definition 1 in [9]) the generalized Jacobian denoted $\partial f(y_0)$ as the convex hull of the set of matrices which can be obtained as limits of $\mathcal{J}_f(y_n)$, when $y_n \rightarrow y_0$.

We denote by $B(x, r)$ an open ball of the center $x \in \mathbb{R}^n$ and radius $r > 0$. For any matrix M we denote by $\|M\|_\infty$ the maximum of its entries.

Lemma 2.7 *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a Lipschitz map and $x \in \mathbb{R}^n$. Suppose that for any $y \in \mathbb{R}^n$, the generalized Jacobian $\partial f(y)$ consist of the matrices which can be represented as $M(x) + R$, where matrices $M(x)$, R satisfy the following conditions: there are positive β and η such that $\|R\|_\infty \leq \eta|x - y|$ and $|vM(x)^T| \geq 2\beta$ for every $v \in \mathbb{R}^n$, $|v| = 1$. Then f is injective on $B(x, \beta/(n\eta))$ and we have $B(f(x), \beta^2/(2n\eta)) \subset f(B(x, \beta/(n\eta)))$.*

Proof Let v be an arbitrary unit vector in $\mathbb{R}^n$. Let $M \in \partial f(y)$ and let $z = vM(x)^T$. Since $M^T = M(x)^T + R^T$ the scalar product of z and $w = vM^T = z + vR^T$ can be estimated as follows

$$z \cdot w = z \cdot (z + vR^T) = |z|^2 + z \cdot (vR^T) \geq |z|^2 - n\eta|z||x - y|.$$

Next, taking $w^* = z/|z|$ we have for $|x - y| \leq \beta/(n\eta)$,

$$w^* \cdot (vM^T) \geq |z| - n\eta|x - y| \geq \beta.$$

Using this fact we can apply Lemma 3 and Lemma 4 of [9] to claim that for every $y_1, y_2 \in B(x, \beta/(n\eta))$ we have

$$|f(y_1) - f(y_2)| \geq \beta|y_1 - y_2|,$$

which shows that f is injective in a ball $B(x, \beta/(n\eta))$. Next, by similar arguments, we show that

$$|vM^T| \geq |vM(x)^T| - |vR^T| \geq 2\beta - n\eta|x - y| \geq \beta, \quad |y - x| \leq \beta/(n\eta),$$

which proves that all matrices from the set $\partial f(y)$ are of full rank if $|y - x| \leq \beta/(n\eta)$.

Finally, we can apply Lemma 5 of [9] to show that the f image of the ball $B(x, \beta/(n\eta))$ contains the ball $B(f(x), \beta^2/(2n\eta))$. □

3 Construction and Properties of the Transition Density of the Solution of (1) Driven by the Truncated Process

The approach in this section is based on Levi's method (cf. [12, 26, 29]). This method was applied in the framework of pseudodifferential operators by Kochubei [20] to construct a fundamental solution to the related Cauchy problem as well as transition density for the corresponding Markov process. In recent years it was used in several papers to study transition densities of Lévy-type processes see e.g. [5, 7, 8, 13, 15, 17–19, 21]. Levi's method was also used to study gradient and Schrödinger perturbations of fractional Laplacians see e.g. [4, 6, 41].

Let us fix $\varepsilon \in (0, 1]$ (it will be chosen later). Recall that for given ε the constant δ is chosen according to Lemma 2.1. For such fixed ε, δ we abbreviate $\mu(x) = \mu^{(\delta)}(x)$, $\mathcal{G} = \mathcal{G}^{(\delta)}$, $g_t(x) = g_t^{(\delta)}(x)$, $h_t(x) = h_t^{(\varepsilon)}(x)$, $x \in \mathbb{R}$. Note that $h_t^{(\varepsilon)}$ depends also on $\tau > 0$ which we keep fixed.

Let us recall that

$$\mathcal{L}f(x) = \sum_{i=1}^d \lim_{\zeta \rightarrow 0^+} \int_{|w| > \zeta} [f(x + a_i(x)w) - f(x)] \mu(w) dw.$$

Our first aim in this section will be to construct the heat kernel $u(t, x, y)$ corresponding to the operator $\mathcal{L}$. This will be done by using Levi's method.

Let $G_t(x) = g_t(x_1) \dots g_t(x_d)$ and $H_t(x) = h_t(x_1) \dots h_t(x_d)$ for $t > 0$ and $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. We also denote $B(x) = (b_{ij}(x)) = A^{-1}(x)$. Note that the coordinates of $B(x)$ satisfy the conditions (2) and (4) with possibly different constants η_1^* and η_3^* , but taking maximums we can assume that $\eta_1^* = \eta_1$ and $\eta_3^* = \eta_3$.

For any $y \in \mathbb{R}^d$, $i = 1, \dots, d$ we put

$$b_i(y) = (b_{i1}(y), \dots, b_{id}(y)).$$

We also denote $\|B\|_\infty = \max\{|b_{ij}| : i, j \in \{1, \dots, d\}\}$. For any $t > 0$, $x, y \in \mathbb{R}^d$ we define

$$\begin{aligned} p_y(t, x) &= \det(B(y)) G_t(x(B(y))^T) \\ &= \det(B(y)) g_t(b_1(y) \cdot x) \dots g_t(b_d(y) \cdot x). \end{aligned}$$

Recall that $\mathcal{L}^y$ is the "freezing" operator given by (11). It may be easily checked that for each fixed $y \in \mathbb{R}^d$ the function $p_y(t, \cdot)$ is the heat kernel of $\mathcal{L}^y$ that is

$$\frac{\partial}{\partial t} p_y(t, x) = \mathcal{L}^y p_y(t, \cdot)(x), \quad t > 0, x \in \mathbb{R}^d, \quad (24)$$

$$\int_{\mathbb{R}^d} p_y(t, x) dx = 1, \quad t > 0.$$

For any $t > 0$, $x, y \in \mathbb{R}^d$ we also define

$$\begin{aligned} r_y(t, x) &= H_t(x(B(y))^T) \\ &= h_t(b_1(y) \cdot x) \dots h_t(b_d(y) \cdot x). \end{aligned}$$

Now we use the approach which was explained in details in the Introduction. The approximation $\tilde{u}(t, x, y)$ of $u(t, x, y)$ is defined by $\tilde{u}(t, x, y) = p_y(t, x - y)$. For $x, y \in \mathbb{R}^d$, $t > 0$, the kernel $q_0(t, x, y)$ is given by (8). We have

$$\mathcal{L}\tilde{u}(t, \cdot, y)(x) = \mathcal{L}^x p_y(t, \cdot)(x - y),$$

hence by (24) we get

$$q_0(t, x, y) = \mathcal{L}^x p_y(t, \cdot)(x - y) - \mathcal{L}^y p_y(t, \cdot)(x - y).$$

For $x, y \in \mathbb{R}^d$, $t > 0$ and $n \in \mathbb{N}$ kernels $q_n(t, x, y)$ are given by (10). For $x, y \in \mathbb{R}^d$, $t > 0$ we define

$$q(t, x, y) = \sum_{n=0}^{\infty} q_n(t, x, y).$$

The kernel $u(t, x, y)$ is given by (9). We have

$$u(t, x, y) = p_y(t, x - y) + \int_0^t \int_{\mathbb{R}^d} p_z(t - s, x - z) q(s, z, y) dz ds. \quad (25)$$

In this section we will show that $q_n(t, x, y)$, $q(t, x, y)$, $u(t, x, y)$ are well defined and we will obtain estimates of these functions. First, we will get some simple properties of $p_y(t, x)$ and $r_y(t, x)$.

Lemma 3.1 Choose $\gamma \in (0, 1]$. For any $t \in (0, \tau]$, $x, x', y \in \mathbb{R}^d$ we have

$$|p_y(t, x) - p_y(t, x')| \leq c(1 \wedge (t^{-\gamma/\alpha} |x - x'|^\gamma))(r_y(t, x/2) + r_y(t, x'/2)).$$

Proof Of course, we may assume that $\gamma = 1$. We have

$$p_y(t, x) - p_y(t, x') = \det(B(y)) \left(\prod_{i=1}^d g_t(b_i(y) \cdot x) - \prod_{i=1}^d g_t(b_i(y) \cdot x') \right)$$

If $|x - x'| \geq t^{1/\alpha} \|B\|_\infty$ or $|x - x'| \geq \delta \|B\|_\infty$ then the assertion clearly holds. So, we may assume that $|x - x'| \leq t^{1/\alpha} \|B\|_\infty$ and $|x - x'| \leq \delta \|B\|_\infty$. Then the assertion follows easily from Lemma 2.3. $\square$

For $x, y \in \mathbb{R}^d$ we have

$$|B(y)x^T|^2 = |xB(y)^T|^2 = (b_1(y) \cdot x)^2 + \dots + (b_1(y) \cdot x)^2.$$

Lemma 3.2 For any $x, y \in \mathbb{R}^d$ and $i \in \{1, \dots, d\}$ we have

$$\max_{1 \leq i \leq d} (b_i(y) \cdot x)^2 \geq \frac{1}{\eta_1^2 d^3} |x|^2.$$

Proof Indeed, for any $u, x \in \mathbb{R}^d$ we have $|uA(y)^T| \leq \eta_1 d |u|$. Setting $u = xB(y)^T$ we obtain that

$$|xB(y)^T| \geq \frac{1}{\eta_1 d} |x|.$$

Since

$$|xB(y)^T|^2 = |b_1(y) \cdot x|^2 + |b_2(y) \cdot x|^2 + \dots + |b_1(y) \cdot x|^2,$$

it follows that there is $1 \leq k \leq d$ such that $|b_k(y) \cdot x| \geq \frac{1}{\eta_1 d \sqrt{d}} |x|$. $\square$

Corollary 3.3 Assume that $\varepsilon \leq \frac{1}{\eta_1 d \sqrt{d}}$. For any $t \in (0, \tau + 1]$, $x, y \in \mathbb{R}^d$, we have

$$r_y(t, x - y) \leq c_1 t^{-d/\alpha} e^{-c|x-y|}. \quad (26)$$

For any $t \in (0, \tau + 1]$, $x, y \in \mathbb{R}^d$, $|x - y| \geq \varepsilon \eta_1 d^{3/2}$, we have

$$r_y(t, x - y) \leq c_1 t e^{-c|x-y|}. \quad (27)$$

Proof For any $t \in (0, \tau + 1]$, $z \in \mathbb{R}$, by definition of h_t , we have

$$h_t(z) \leq ct^{-1/\alpha} e^{-|z|}. \quad (28)$$

Fix $x, y \in \mathbb{R}^d$, $t \in (0, \tau + 1]$. By Lemma 3.2, there exists $i \in \{1, \dots, d\}$ such that $|b_i(y) \cdot (x - y)| \geq \frac{1}{\eta_1 d \sqrt{d}} |x - y|$. Using this and (28) we get (26). For any $t \in (0, \tau + 1]$, $z \in \mathbb{R}$, $|z| \geq \varepsilon$, by definition of h_t , we have

$$h_t(z) \leq ct^{1+(d-1)/\alpha} e^{-|z|}.$$

If $|x - y| \geq \varepsilon \eta_1 d^{3/2}$ then $\frac{1}{\eta_1 d \sqrt{d}} |x - y| \geq \varepsilon$ hence, by the same arguments as above, we get (27). $\square$

Using the definition of $p_y(t, x)$ and properties of $g_t(x)$ we obtain the following regularity properties of $p_y(t, x)$.

Lemma 3.4 *The function $(t, x, y) \rightarrow p_y(t, x)$ is continuous on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$. The function $t \rightarrow p_y(t, x)$ is in $C^1((0, \infty))$ for each fixed $x, y \in \mathbb{R}^d$. The function $x \rightarrow p_y(t, x)$ is in $C^2(\mathbb{R}^d)$ for each fixed $t > 0$, $y \in \mathbb{R}^d$.*

Lemma 3.5 *For any $y \in \mathbb{R}^d$ we have*

$$\left| \frac{\partial}{\partial x_i} p_y(t, x - y) \right| \leq \frac{c}{t^{(d+1)/\alpha} (1 + |x - y|)^{d+1}}, \quad i \in \{1, \dots, d\}, t \in (0, \tau], x \in \mathbb{R}^d,$$

$$\left| \frac{\partial^2}{\partial x_i \partial x_j} p_y(t, x - y) \right| \leq \frac{c}{t^{(d+2)/\alpha} (1 + |x - y|)^{d+1}}, \quad i, j \in \{1, \dots, d\}, t \in (0, \tau], x \in \mathbb{R}^d.$$

Proof The estimates follow from Lemma 2.5 and the same arguments as in the proof of (26). $\square$

Lemma 3.6 *Let $b_i^*(x, y)$, $x, y \in \mathbb{R}^d$; $i = 1, \dots, d$, be real functions such that there are positive η_4, η_5 and*

$$|b_i^*(x, y)| \leq \eta_4, \quad x, y \in \mathbb{R}^d, \quad (29)$$

$$|b_i^*(x, y) - b_i^*(\bar{x}, \bar{y})| \leq \eta_5(|x - \bar{x}| + |y - \bar{y}|), \quad x, y, \bar{x}, \bar{y} \in \mathbb{R}^d. \quad (30)$$

Let, for $i = 1, \dots, d$,

$$\xi_i = \xi_i(w, x, y) = b_i(y) \cdot (x - y) + b_i^*(x, y)w, \quad w \in \mathbb{R}, x, y \in \mathbb{R}^d.$$

We put

$$\Psi(w, x, y) = (w, \xi_1, \dots, \xi_d) \in \mathbb{R}^{d+1}, \quad w \in \mathbb{R}, x, y \in \mathbb{R}^d$$

and for fixed $x \in \mathbb{R}^d$ we define a map $\Psi_x : \mathbb{R}^{d+1} \mapsto \mathbb{R}^{d+1}$ by $\Psi_x = \Psi(\cdot, x, \cdot)$, while for fixed $y \in \mathbb{R}^d$ we define a map $\Phi_y : \mathbb{R}^{d+1} \mapsto \mathbb{R}^{d+1}$ by $\Phi_y = \Psi(\cdot, \cdot, y)$.

There is a positive $\varepsilon_0 = \varepsilon_0(\eta_1, \eta_3, \eta_4, \eta_5, d) \leq \frac{1}{2\eta_5}$ such that the map Ψ_x and its Jacobian determinant denoted by J_{Ψ_x} has the property

$$|\Psi_x(w, y)| \leq 1,$$

$$(1/2)|\det B(y)| \leq |J_{\Psi_x}(w, y)| \leq 2|\det B(y)|,$$

(w, y) almost surely on the set $Q_x = \{(w, y) \in \mathbb{R}^{d+1} : |y - x| \leq \varepsilon_0, |w| \leq \varepsilon_0\}$. The map Ψ_x is injective on Q_x .

Moreover, the map Φ_y and its Jacobian determinant denoted by J_{Φ_y} has the property

$$\begin{aligned} |\Phi_y(w, x)| &\leq 1, \\ (1/2)|\det B(x)| &\leq |J_{\Phi_y}(w, x)| \leq 2|\det B(x)|, \end{aligned}$$

(w, x) almost surely on the set $Q_y = \{(w, x) \in \mathbb{R}^{d+1} : |y - x| \leq \varepsilon_0, |w| \leq \varepsilon_0\}$. The map Φ_y is injective on Q_y .

Proof In the proof we assume that constants c may additionally depend on η_4, η_5 . We prove the statement for the map Ψ_x , only. Since $|\Psi_x(w, y)| \leq \sqrt{d}(1 + \eta_1 + \eta_4)(|w| + |x - y|)$ we have

$$|\Psi_x(w, y)| \leq 1, \text{ if } |w| + |x - y| \leq \frac{1}{\sqrt{d}(1 + \eta_1 + \eta_4)}. \quad (31)$$

Next, we observe that (w, y) almost surely

$$\frac{\partial \xi_k}{\partial y_l} = -b_{kl}(y) + (x - y) \cdot \frac{\partial b_k}{\partial y_l} + w \frac{\partial b_k^*}{\partial y_l}, \quad 1 \leq l, k \leq d. \quad (32)$$

Since $|(y - x) \cdot \frac{\partial b_k}{\partial y_l} + w \frac{\partial b_k^*}{\partial y_l}| \leq (\eta_3 + \eta_5)(|y - x| + |w|)$, it follows that

$$J_{\Psi_x}(w, y) = (-1)^d \det B(y) + R(x, y, w), \quad |R(x, y, w)| \leq c(|y - x| + |w|),$$

with $c = c(d, \eta_1, \eta_3, \eta_5)$. Since $|\det B(y)| = \frac{1}{|\det A(y)|} \geq \frac{1}{d\eta_1^d}$, we have for sufficiently small $\varepsilon_1 = \varepsilon_1(\eta_1, \eta_3, \eta_4, \eta_5, d)$, (w, y) almost surely

$$J_{\Psi_x}(w, y) = (-1)^d \kappa(x, y, w) \det B(y), \quad |y - x| \leq \varepsilon_1, \quad |w| \leq \varepsilon_1, \quad (33)$$

where $\frac{1}{2} \leq \kappa \leq 2$.

Let $\mathcal{J}_{\Psi_x}(w, y)$ be the Jacobi matrix for the map Ψ_x which is defined (w, y) almost surely. From (32) it follows that

$$\mathcal{J}_{\Psi_x}(w, y) = \mathcal{B}(x) + \mathcal{R},$$

where the coordinates $\mathcal{B}_{kl}(x)$, $0 \leq k, l \leq d$ of the matrix $\mathcal{B}(x)$ are

$$\begin{aligned} \mathcal{B}_{kl}(x) &= -b_{kl}(x), \quad k, l \geq 1, \\ \mathcal{B}_{0l}(x) &= 1; \quad \mathcal{B}_{0l}(x) = 0; \quad \mathcal{B}_{l0}(x) = b_l^*(x, x), \quad 1 \leq l \leq d, \end{aligned}$$

while all the entries of $\mathcal{R}$ satisfy $|\mathcal{R}_{kl}| \leq c\sqrt{|w|^2 + |x - y|^2}$ with $c = c(\eta_3, \eta_5)$.

Let $\partial \Psi_x(w, y)$ denote the generalized Jacobian of Ψ_x at the point (w, y) . Then from the form of $\mathcal{J}_{\Psi_x}$ it is clear that every matrix $\mathcal{M} \in \partial \Psi_x(w, y)$ can be written as

$$\mathcal{M} = \mathcal{B}(x) + \mathcal{R},$$

where all the entries of $\mathcal{R}$ satisfy $|\mathcal{R}_{kl}| \leq c\sqrt{|w|^2 + |x - y|^2}$ with $c = c(\eta_3, \eta_5)$.

Now, for every (u, z) , $u \in \mathbb{R}$, $z \in \mathbb{R}^d : |u|^2 + |z|^2 = 1$ we have

$$|(u, z)\mathcal{B}(x)^T| \geq 2\beta > 0,$$

with $\beta = \beta(d, \eta_1, \eta_4)$. Since $\|\mathcal{R}\|_\infty \leq c\sqrt{|w|^2 + |x - y|^2}$ we can apply Lemma 2.7 with $n = d + 1$ to show on the set $\{(w, y); \sqrt{|w|^2 + |x - y|^2} \leq \beta/(c(d + 1))\}$ the map Ψ_x is injective. This fact, combined with (31) and (33), completes the proof if we choose $\varepsilon_0 = \varepsilon_1 \wedge \frac{1}{2\sqrt{d}(1 + \eta_1 + \eta_4)} \wedge \frac{\beta}{2(d + 1)c}$. $\square$

Remark 3.7 Let for $x \in \mathbb{R}^d$, $\tilde{\Psi}_x$ be the map $\mathbb{R}^d \mapsto \mathbb{R}^d$ given by

$$\tilde{\Psi}_x(y) = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d, \quad y \in \mathbb{R}^d,$$

where $\xi_i = b_i(y) \cdot (x - y)$. Then using the same arguments as in the above proof we can find ε_0 such that all the assertions of Lemma 3.6 are true and additionally

$$(1/2)|\det B(y)| \leq |J_{\tilde{\Psi}_x}(y)| \leq 2|\det B(y)|,$$

for $|x - y| \leq \varepsilon_0$, y almost surely. Moreover, the map $\tilde{\Psi}_x$ is injective on $B(x, \varepsilon_0)$. We can also find $\delta_1 = \delta_1(\eta_1, \eta_3, \eta_4, \eta_5, d) > 0$ and $\delta_2 = \delta_2(\eta_1, \eta_3, \eta_4, \eta_5, d) > 0$ such that the $\tilde{\Psi}_x$ image of the ball $B(x, \delta_1)$ contains $B(0, \delta_2)$. To this end we apply the last assertion of Lemma 2.7.

Let $b_i^*(x, y)$ be the functions introduced in Lemma 3.6. We will use the following abbreviations

$$z_i = B_i(x, y) = b_i(y) \cdot (x - y) = b_{i1}(y)(x_1 - y_1) + \dots + b_{id}(y)(x_d - y_d),$$

$$b_i^* = b_i^*(x, y),$$

$$b_{i0}^* = b_i^*(x, x).$$

Let for $l = 1, \dots, d$,

$$A_l = A_l(x, y) = \int_{\mathbb{R}} \prod_{i \neq l} g_t(z_i + b_i^* w) |g_t(z_l + b_l^* w) - g_t(z_l + b_{l0}^* w)| \mu(w) dw.$$

Corollary 3.8 Assume that $2\delta < \varepsilon_0$, where ε_0 is from Remark 3.7. With the assumptions of Lemma 3.6 we have for $t \leq \tau$,

$$\int_{|y-x| \leq \varepsilon_0} A_l dy \leq ct^{-1/2}, \quad x \in \mathbb{R}^d,$$

and

$$\int_{|y-x| \leq \varepsilon_0} A_l dx \leq ct^{-1/2}, \quad y \in \mathbb{R}^d,$$

where $c = c(\tau, \alpha, d, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \varepsilon)$.

Proof In the proof we assume that constants c may additionally depend on η_4, η_5 . For $x, y \in \mathbb{R}^d$ we get $|b_l^* - b_{l0}^*| \leq \eta_5|x - y|$. Hence, from (18), we have for $w \in \mathbb{R}$, $1 \leq l \leq d$,

$$\begin{aligned} |g_t(z_l + b_l^* w) - g_t(z_l + b_{l0}^* w)| &\leq c \left(\frac{|b_l^* - b_{l0}^*||w|}{t^{1/\alpha}} \wedge 1 \right) (h_t(z_l + b_l^* w) + h_t(z_l + b_{l0}^* w)) \\ &\leq c \left(\frac{|x - y||w|}{t^{1/\alpha}} \wedge 1 \right) (g_t^{(\infty)}(z_l + b_l^* w) + g_t^{(\infty)}(z_l + b_{l0}^* w)). \end{aligned}$$

This implies that

$$A_l \leq c(A_l^1 + A_l^2),$$

where

$$A_l^1 = \int_{\mathbb{R}} \left(\prod_{i=1}^d g_t^{(\infty)}(z_i + b_i^* w) \right) \left(\frac{|x - y||w|}{t^{1/\alpha}} \wedge 1 \right) \mu(w) dw$$

and

$$A_l^2 = \int_{\mathbb{R}} \left(\prod_{i=1}^d g_t^{(\infty)}(z_i + \hat{b}_i w) \right) \left(\frac{|x-y||w|}{t^{1/\alpha}} \wedge 1 \right) \mu(w) dw,$$

with $\hat{b}_i = b_i^*$, $i \neq l$ and $\hat{b}_l = b_{l0}^*$. Note that the functions $\hat{b}_i = \hat{b}_i(x, y)$ have the same properties (29, 30) as b_i^* . To evaluate the integral $\int_{|x-y| \leq \varepsilon_0} A_l^1 dy$, which in fact is an integral with respect to $dw dy$, we introduce new variables (w, ξ) in $\mathbb{R}^{d+1}$, given by $(w, \xi) = \Psi_x(w, y)$, where $\xi_i = z_i + b_i^* w$, $i = 1, \dots, d$ (or $\xi_i = z_i + \hat{b}_i w$ if A_l^2 is treated). We recall that Ψ_x was defined in Lemma 3.6. Note that the vector $\xi = (\xi_1, \dots, \xi_d)$ can be written as

$$\xi = (x - y)B(y)^T + wb^*,$$

where $b^* = (b_1^*, \dots, b_d^*)$, hence

$$(\xi - wb^*)A(y)^T = x - y.$$

From this we infer that

$$|w||x - y| \leq c(|\xi| + |w|)|w|.$$

Let $Q_x = \{(w, y) : |y - x| \leq \varepsilon_0, |w| \leq \varepsilon_0\}$. Due to Lemma 3.6, almost surely on Q_x , the absolute value of the Jacobian determinant of the map Ψ_x is bounded from below and above by two positive constants and Ψ_x is an injective transformation. Let $V_x = \Psi_x(Q_x)$. Observing that the support of the density μ is contained in $[-\varepsilon_0, \varepsilon_0]$ and then applying the above change of variables, we have

$$\begin{aligned} \int_{|y-x| \leq \varepsilon_0} A_l^1 dy &\leq c \int_{|y-x| \leq \varepsilon_0} \int_{\mathbb{R}} \left(\prod_{i=1}^d g_t^{(\infty)}(\xi_i) \right) \left(\frac{(|\xi| + |w|)|w|}{t^{1/\alpha}} \wedge 1 \right) \mu(w) dw dy \\ &\leq c \int_{|y-x| \leq \varepsilon_0} \int_{\mathbb{R}} \left(\prod_{i=1}^d g_t^{(\infty)}(\xi_i) \right) \left(\frac{(|\xi| + |w|)|w|}{t^{1/\alpha}} \wedge 1 \right) \\ &\quad \times \mu(w) |J_{\Psi_x}(w, y)| dw dy \\ &= c \int_{V_x} \prod_{i=1}^d g_t^{(\infty)}(\xi_i) \left(\frac{(|\xi| + |w|)|w|}{t^{1/\alpha}} \wedge 1 \right) \mu(w) dw d\xi \end{aligned}$$

where the last equality follows from the general change of variable formula for injective Lipschitz maps (see e.g. [14, Theorem 3]). Since $|\xi| \leq 1$ for $(w, \xi) \in V_x$, we get

$$\int_{|y-x| \leq \varepsilon_0} A_l^1 dy \leq c \int_{|\xi| \leq 1} \prod_{i=1}^d g_t^{(\infty)}(\xi_i) \int_{\mathbb{R}} \left(\frac{(|\xi| + |w|)|w|}{t^{1/\alpha}} \wedge 1 \right) \mu(w) dw d\xi.$$

Applying Lemma 2.6 we have

$$\int_{\mathbb{R}} \left(\frac{(|\xi| + |w|)|w|}{t^{1/\alpha}} \wedge 1 \right) \mu(w) dw \leq c \frac{t^{1/2} + |\xi|^\alpha}{t}.$$

Finally,

$$\int_{|y-x| \leq \varepsilon_0} A_l^1 dy \leq c \int_{|\xi| \leq 1} \prod_{i=1}^d g_t^{(\infty)}(\xi_i) \frac{t^{1/2} + |\xi|^\alpha}{t} d\xi \leq ct^{-1/2}.$$

Similarly we obtain

$$\int_{|y-x| \leq \varepsilon_0} A_l^2 dy \leq ct^{-1/2},$$

which completes the proof of the first bound. To estimate $\int_{|y-x| \leq \varepsilon_0} A_l dx$ we proceed exactly in the same way. $\square$

For fixed $l \in \{1, \dots, d\}$ let us consider a family of functions $b_i^*(x, y) = b_i(y) \cdot a_l(x)$, $i \in \{1, \dots, d\}$. They satisfy the conditions (29) and (30) with $\eta_4 = d\eta_1^2$ and $\eta_5 = d\eta_1\eta_3$. Let $\varepsilon_0 = \varepsilon_0(\eta_1, \eta_3, \eta_4, \eta_5, d)$ be as found in Lemma 3.6 and Remark 3.7. Finally we choose $\varepsilon = \varepsilon(\eta_1, \eta_3, d) = \frac{\varepsilon_0}{4d^{3/2}(\eta_1 \vee 1)}$. From now on we keep $\varepsilon_0, \varepsilon$ fixed as above. Recall that if we fixed ε we fix δ according to Lemma 2.1.

Proposition 3.9 *For any $x, y \in \mathbb{R}^d$, $t \in (0, \tau]$ we have*

$$|q_0(t, x, y)| \leq ct^{-1-(d-1)/\alpha} h_t \left((\varepsilon/\varepsilon_0)|x - y| \mathbf{1}_{[\varepsilon_0, \infty)}(|x - y|) \right). \quad (34)$$

In particular, for $x, y \in \mathbb{R}^d$, $t \in (0, \tau]$, $|y - x| \geq \varepsilon_0$ we have

$$|q_0(t, x, y)| \leq ce^{(-\varepsilon/\varepsilon_0)|x-y|}.$$

For any $t \in (0, \tau]$, $x \in \mathbb{R}^d$ we have

$$\int_{\mathbb{R}^d} |q_0(t, x, y)| dy \leq ct^{-1/2}. \quad (35)$$

For any $t \in (0, \tau]$, $y \in \mathbb{R}^d$ we have

$$\int_{\mathbb{R}^d} |q_0(t, y, x)| dx \leq ct^{-1/2}. \quad (36)$$

Proof We have

$$q_0(t, x, y) = \sum_{i=1}^d \lim_{\zeta \rightarrow 0^+} \int_{|w| > \zeta} [p_y(t, x - y + a_i(x)w) - p_y(t, x - y + a_i(y)w)] \mu(w) dw.$$

For $i = 1, \dots, d$ we put

$$R_i = \lim_{\zeta \rightarrow 0^+} \int_{|w| > \zeta} [p_y(t, x - y + a_i(x)w) - p_y(t, x - y + a_i(y)w)] \mu(w) dw. \quad (37)$$

We have $q_0(t, x, y) = R_1 + \dots + R_d$. It is clear that it is enough to handle R_1 alone. Note that

$$R_1 = \det(B(y)) \lim_{\zeta \rightarrow 0^+} \int_{|w| > \zeta} \left[G_t \left((x - y + we_1(A(x))^T)(B(y))^T \right) - G_t \left((x - y + we_1(A(y))^T)(B(y))^T \right) \right] \mu(w) dw. \quad (38)$$

We will use the following abbreviations

$$\begin{aligned} z_i &= B_i(x, y) = b_i(y) \cdot (x - y) = b_{i1}(y)(x_1 - y_1) + \dots + b_{id}(y)(x_d - y_d), \\ k_i &= \tilde{b}_{i1}(x, y) = b_i(y) \cdot a_1(x), \\ k_{i0} &= \tilde{b}_{i1}(x, x). \end{aligned}$$

Note that $k_{10} = 1$ and $k_{i0} = 0$, $2 \leq i \leq d$.

We can rewrite (38) as

$$R_1 = \det(B(y)) \lim_{\zeta \rightarrow 0^+} \int_{|w| > \zeta} \left[\prod_{i=1}^d g_t(z_i + k_i w) - \prod_{i=1}^d g_t(z_i + k_{i0} w) \right] \mu(w) dw.$$

Hence,

$$R_1 = \det(B(y))(\mathbf{I}_1 + \dots + \mathbf{I}_d) = \det(B(y))(\mathbf{I}'_1 + \dots + \mathbf{I}'_d),$$

where

$$\begin{aligned} \mathbf{I}_l &= \int_{\mathbb{R}} \left(\prod_{i=1}^{l-1} g_t(z_i + k_i w) \right) [g_t(z_l + k_l w) - g_t(z_l + k_{l0} w)] \left(\prod_{i=l+1}^d g_t(z_i) \right) \mu(w) dw, \\ \mathbf{I}'_l &= \int_{\mathbb{R}} \left(\prod_{i=1}^{l-1} g_t(z_i + k_{i0} w) \right) [g_t(z_l + k_l w) - g_t(z_l + k_{l0} w)] \\ &\quad \times \left(\prod_{i=l+1}^d g_t(z_i + k_i w) \right) \mu(w) dw \end{aligned}$$

for $l = 1, \dots, d$ (with convention that $\prod_{i=1}^0 = 1 = \prod_{i=d+1}^d$).

We start with the proof of the bound of q_0 . We observe that $|k_l|, |k_{l0}| \leq d\eta_1^2 \leq \varepsilon/(4\delta)$, $l \in \{1, \dots, d\}$. By Lemmas 2.1 and 2.4, we obtain for $l \leq d-1$,

$$\begin{aligned} |\mathbf{I}_l| &\leq g_t(z_d) g_t^{d-2}(0) \int_{\mathbb{R}} |g_t(z_l + k_l w) - g_t(z_l + k_{l0} w)| \mu(w) dw \\ &\leq c(|k_l| + |k_{l0}|)^\alpha g_t(z_d) g_t^{d-2}(0) \frac{h_t(0)}{t} \\ &\leq ct^{-1-(d-1)/\alpha} h_t(z_d). \end{aligned}$$

The same argument leads to

$$\begin{aligned} |\mathbf{I}_d| &\leq g_t^{d-1}(0) \int_{\mathbb{R}} |g_t(z_d + k_d w) - g_t(z_d)| \mu(w) dw \\ &\leq c g_t^{d-1}(0) |k_d|^\alpha \frac{h_t(z_d/2)}{t} \\ &\leq ct^{-1-(d-1)/\alpha} h_t(z_d/2). \end{aligned}$$

The above inequalities yield

$$|R_1| \leq ct^{-1-(d-1)/\alpha} h_t(z_d/2).$$

Since R_1 is invariant with respect to permutations of $z_2, \dots, z_d$ we infer that

$$|R_1| \leq ct^{-1-(d-1)/\alpha} \inf_{2 \leq i \leq d} h_t(z_i/2) = ct^{-1-(d-1)/\alpha} h_t \left(\max_{2 \leq i \leq d} |z_i|/2 \right). \quad (39)$$

On the other hand, again by Lemma 2.4,

$$\begin{aligned} |\mathbf{I}'_1| &\leq g_t^{d-1}(0) \int_{\mathbb{R}} |g_t(z_1 + k_1 w) - g_t(z_1 + w)| \mu(w) dw \\ &\leq c(|k_1| + 1)^\alpha g_t^{d-1}(0) \frac{h_t(z_1/2)}{t} \\ &\leq ct^{-1-(d-1)/\alpha} h_t(z_1/2). \end{aligned}$$

For $l \geq 2$, a similar argument leads to

$$\begin{aligned} |I'_l| &\leq \sup_{|w| \leq 2\delta} g_t(z_1 + w) g_t^{d-2}(0) \int_{\mathbb{R}} |g_t(z_l + k_l w) - g_t(z_l)| \mu(w) dw \\ &\leq c \sup_{|w| \leq 2\delta} g_t(z_1 + w) g_t^{d-2}(0) \frac{h_t(0)}{t} \\ &\leq c t^{-1-(d-1)/\alpha} \sup_{|w| \leq 2\delta} g_t(z_1 + w). \end{aligned}$$

Observing that $|z_1 + w| \geq |z_1|/2$ for $|w| \leq 2\delta \leq 4\delta \leq |z_1|$ we conclude that

$$|R_1| \leq c t^{-1-(d-1)/\alpha} h_t \left(\frac{|z_1|}{2} \mathbf{1}_{[4\delta, \infty)}(|z_1|) \right). \quad (40)$$

Combining (39) and (40) we arrive at

$$|R_1| \leq c t^{-1-(d-1)/\alpha} h_t \left(\frac{\max_{1 \leq i \leq d} |z_i|}{2} \mathbf{1}_{[4\delta, \infty)}(\max_{1 \leq i \leq d} |z_i|) \right).$$

By Lemma 3.2, $\max_{1 \leq i \leq d} |z_i| \geq \frac{1}{\eta_1 d^{3/2}} |x - y|$, and by the choice of $\varepsilon, \varepsilon_0, \delta$ we have $\frac{1}{2\eta_1 d^{3/2}} \geq \varepsilon/\varepsilon_0$ and $4\delta\eta_1 d^{3/2} \leq \varepsilon_0$, hence

$$\begin{aligned} |R_1| &\leq c t^{-1-(d-1)/\alpha} h_t \left(\frac{|x - y|}{2\eta_1 d^{3/2}} \mathbf{1}_{[4\delta\eta_1 d^{3/2}, \infty)}(|x - y|) \right) \\ &\leq c t^{-1-(d-1)/\alpha} h_t \left((\varepsilon/\varepsilon_0) |x - y| \mathbf{1}_{[\varepsilon_0, \infty)}(|x - y|) \right). \end{aligned}$$

Finally, for $|y - x| \geq \varepsilon_0$ we have

$$|R_1| \leq c e^{-(\varepsilon/\varepsilon_0)|x-y|}.$$

Next, we prove the bound of the integral (35). Let $x \in \mathbb{R}^d$ be fixed. Applying Corollary 3.8 with $b_i^* = k_i \mathbf{1}_{\{i \leq l\}}$ we have that

$$\int_{|y-x| \leq \varepsilon_0} |I_t| dy \leq c t^{-1/2}.$$

Hence,

$$\int_{|x-y| \leq \varepsilon_0} |q_0(t, x, y)| dy \leq c t^{-1/2}.$$

For $|y - x| \geq \varepsilon_0$ we have $|q_0(t, x, y)| \leq c e^{-(\varepsilon/\varepsilon_0)|x-y|}$ which implies that

$$\int_{|x-y| \geq \varepsilon_0} |q_0(t, x, y)| dy \leq c.$$

This completes the proof of (35). The estimate (36) is proved exactly in the same way. $\square$

Using similar arguments as in the proof of Proposition 3.9 we obtain the following result.

Proposition 3.10 *For any $t \in (0, \tau]$, $x \in \mathbb{R}^d$ we have*

$$\int_{\mathbb{R}^d} p_y(t, x - y) dy \leq c, \quad (41)$$

$$\int_{\mathbb{R}^d} r_y(t, (x - y)/2) dy \leq c. \quad (42)$$

For any $\delta_1 > 0$,

$$\lim_{t \rightarrow 0^+} \sup_{x \in \mathbb{R}^d} \int_{B^c(x, \delta_1)} p_y(t, x - y) dy = 0. \quad (43)$$

We have

$$\lim_{t \rightarrow 0^+} \int_{\mathbb{R}^d} p_y(t, x - y) dy = 1, \quad (44)$$

uniformly with respect to $x \in \mathbb{R}^d$.

Proof For fixed $x \in \mathbb{R}^d$ we introduce new variables $u = \tilde{\Psi}_x(y)$ given by

$$u = (x - y)B(y)^T.$$

Note that

$$\frac{1}{d\eta_1} |x - y| \leq |u| = |(x - y)B(y)^T| \leq d\eta_1 |x - y|. \quad (45)$$

For $r > 0$, let $V_x(r)$ be the $\tilde{\Psi}_x$ image of the ball $B(x, r)$. By Remark 3.7 we have almost surely

$$|J_{\tilde{\Psi}_x}(y)| \geq (1/2) |\det B(y)| \geq c, \quad |y - x| \leq \varepsilon_0,$$

and $\tilde{\Psi}_x$ is an injective map on $B(x, \varepsilon_0)$. Hence, for $0 < \delta_1 < \varepsilon_0$, by the change of variables formula (see e.g. [14, Theorem 3]), and then by (45) we obtain

$$\begin{aligned} \int_{\delta_1 \leq |x-y| \leq \varepsilon_0} r_y(t, (x-y)/2) dy &\leq c \int_{\delta_1 \leq |x-y| \leq \varepsilon_0} H_t(u/2) |J_{\tilde{\Psi}_x}(y)| dy \\ &= c \int_{V_x(\varepsilon_0) \setminus V_x(\delta_1)} H_t(u/2) du \\ &\leq c \int_{|u| \geq \frac{\delta_1}{2d\eta_1}} H_t(u/2) du = I(t, \delta_1). \end{aligned}$$

It is clear that

$$\lim_{\delta_1 \rightarrow 0^+} I(t, \delta_1) \leq c, \quad t \leq \tau.$$

If $|x - y| \geq \varepsilon_0$ then $|x - y|/2 \geq \varepsilon\eta_1 d^{3/2}$, hence, by (27), we obtain

$$\int_{|x-y| \geq \varepsilon_0} r_y(t, (x-y)/2) dy \leq c_1 t \int_{|x-y| \geq \varepsilon_0} e^{-c|x-y|} dy = c_2 t.$$

The last two inequalities prove that

$$\sup_{x \in \mathbb{R}^d} \sup_{t \leq \tau} \int_{\mathbb{R}^d} r_y(t, (x-y)/2) dy < \infty.$$

Noting that $\lim_{t \rightarrow 0^+} I(t, \delta_1) = 0$ we obtain

$$\lim_{t \rightarrow 0^+} \sup_{x \in \mathbb{R}^d} \int_{|x-y| \geq \delta_1} r_y(t, (x-y)/2) dy = 0.$$

Since $p_y(t, x - y) \leq cr_y(t, (x - y)/2)$ for $t \leq \tau$, $x, y \in \mathbb{R}^d$, the proof of (41, 42, 43) is completed.

Note that the coordinates of the matrix $B(y)$ have partial derivatives y almost surely, bounded uniformly. We can calculate the absolute value of the Jacobian determinant $J_{\tilde{\Psi}_x}(y)$, y almost surely, as

$$|J_{\tilde{\Psi}_x}(y)| = |\det B(y) + R(x, y)|, \quad |R(x, y)| \leq c|y - x|. \quad (46)$$

Next,

$$\begin{aligned} \int_{|x-y| \leq \delta_1} p_y(t, x-y) dy &= \int_{|x-y| \leq \delta_1} G_t(u) \det B(y) dy \\ &= \int_{|x-y| \leq \delta_1} G_t(u) \left| J_{\tilde{\Psi}_x}(y) \right| dy - \int_{|x-y| \leq \delta_1} G_t(u) R(x, y) dy \\ &= I_1 + I_2. \end{aligned}$$

Applying (46), (45) and the change of variable formula we obtain

$$\begin{aligned} |I_2| &\leq c \int_{|x-y| \leq \delta_1} |x-y| G_t(u) dy \\ &\leq c \int_{|x-y| \leq \delta_1} |u| G_t(u) \left| J_{\tilde{\Psi}_x}(y) \right| dy \\ &= c \int_{V_x(\delta_1)} |u| G_t(u) du \\ &\leq c \int_{|u| \leq d\eta_1 \delta_1} |u| G_t(u) du \rightarrow 0, \text{ if } t \rightarrow 0^+. \end{aligned}$$

Now we can pick, independently of x , positive δ_1 and δ_2 such that $B(0, \delta_2) \subset V_x(\delta_1)$ (see Remark 3.7). Applying again the change of variable formula we obtain

$$I_1 = \int_{V_x(\delta_1)} G_t(u) du \geq \int_{|u| \leq \delta_2} G_t(u) du \rightarrow 1, \text{ if } t \rightarrow 0^+.$$

This completes the proof that uniformly with respect to x ,

$$\lim_{t \rightarrow 0^+} \int_{|x-y| \leq \delta_1} p_y(t, x-y) dy = 1,$$

which combined with (43) proves (44). $\square$

In the sequel we will use the following standard estimate. For any $\gamma \in (0, 1]$, $\theta_0 > 0$ there exists $c = c(\gamma, \theta_0)$ such that for any $\theta \geq \theta_0$, $t > 0$ we have

$$\int_0^t (t-s)^{\gamma-1} s^{\theta-1} ds \leq \frac{c}{\theta^\gamma} t^{(\gamma-1)+(\theta-1)+1}. \quad (47)$$

Lemma 3.11 *For any $t > 0$, $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$ the kernel $q_n(t, x, y)$ is well defined. For any $t \in (0, \tau]$, $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$ we have*

$$\int_{\mathbb{R}^d} |q_n(t, x, y)| dy \leq \frac{c_1^{n+1} t^{(n+1)/2-1}}{(n!)^{1/2}}, \quad (48)$$

$$\int_{\mathbb{R}^d} |q_n(t, y, x)| dy \leq \frac{c_1^{n+1} t^{(n+1)/2-1}}{(n!)^{1/2}}. \quad (49)$$

For any $t \in (0, \tau]$, $x, y \in \mathbb{R}^d$ and $n \in \mathbb{N}$ we have

$$|q_n(t, x, y)| \leq c_1 \frac{c_2^n t^{n/2-1}}{(n!)^{1/2} t^{d/\alpha}}. \quad (50)$$

For any $t \in (0, \tau]$, $x, y \in \mathbb{R}^d$ and $n \in \mathbb{N}$, $|x - y| \geq n + 1$ we have

$$|q_n(t, x, y)| \leq c_1 \frac{c_2^n t^{n/2}}{(n!)^{1/2}} e^{-\frac{\lambda|x-y|}{n+1}}, \quad (51)$$

where $\lambda = \varepsilon/\varepsilon_0$.

Proof By Proposition 3.9, there is a constant c^* such that for any $x, y \in \mathbb{R}^d$, $t \in (0, \tau]$ we have

$$|q_0(t, x, y)| \leq c^* \frac{1}{t^{1+d/\alpha}}, \quad (52)$$

$$|q_0(t, x, y)| \leq c^* e^{-\lambda|x-y|}, \quad |x - y| \geq 1. \quad (53)$$

$$\int_{\mathbb{R}^d} |q_0(t, x, u)| du \leq c^* t^{-1/2}, \quad (54)$$

$$\int_{\mathbb{R}^d} |q_0(t, u, x)| du \leq c^* t^{-1/2}. \quad (55)$$

It follows from (47) there is $p \geq 1$ such that for $n \in \mathbb{N}$,

$$\begin{aligned} \int_0^t (t-s)^{-1/2} s^{n/2-1/2} ds &\leq \frac{P}{(n+1)^{1/2}} t^{n/2}, \\ \int_{t/2}^t (t-s)^{-1/2} s^{n/2-1} ds &\leq \frac{P}{(n+1)^{1/2}} t^{(n+1)/2-1}, \\ \int_0^t (t-s)^{-1/2} s^{n/2} ds &\leq \frac{P}{(n+1)^{1/2}} t^{(n+1)/2}. \end{aligned}$$

We define $c_1 = pc^* \geq c^*$ and $c_2 = 2^{d/\alpha+1} c_1(2+p) > c_1$.

We will prove (48), (49), (50) simultaneously by induction. They are true for $n = 0$ by (52, 54, 55) and the choice of c_1 . Assume that (48), (49), (50) are true for $n \in \mathbb{N}$, we will show them for $n + 1$. By the definition of $q_n(t, x, y)$ and the induction hypothesis we obtain

$$\begin{aligned} |q_{n+1}(t, x, y)| &\leq c_1 \frac{2^{d/\alpha+1}}{t^{d/\alpha+1}} \int_0^{t/2} \int_{\mathbb{R}^d} |q_n(s, z, y)| dz ds \\ &\quad + c_1 \frac{c_2^n 2^{d/\alpha+1}}{(n!)^{1/2} t^{d/\alpha}} \int_{t/2}^t \int_{\mathbb{R}^d} |q_0(t-s, x, z)| dz s^{n/2-1} ds \\ &\leq c_1 \frac{c_1^{n+1} 2^{d/\alpha+1}}{(n!)^{1/2} t^{d/\alpha+1}} \int_0^{t/2} s^{(n+1)/2-1} ds \\ &\quad + c_1 \frac{c_2^n 2^{d/\alpha+1} c_1}{(n!)^{1/2} t^{d/\alpha}} \int_{t/2}^t (t-s)^{-1/2} s^{n/2-1} ds \\ &\leq c_1 \frac{c_2^n t^{(n+1)/2}}{((n+1)!)^{1/2} t^{d/\alpha+1}} \left(2c_1 2^{d/\alpha+1} + c_1 2^{d/\alpha+1} p \right) \\ &= c_1 \frac{c_2^{n+1} t^{(n+1)/2}}{((n+1)!)^{1/2} t^{d/\alpha+1}}. \end{aligned}$$

Hence we get (50) for $n + 1$. In particular this gives that the kernel $q_{n+1}(t, x, y)$ is well defined.

By the definition of $q_n(t, x, y)$, (54) and the induction hypothesis we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} |q_{n+1}(t, x, y)| dy &\leq \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |q_0(t-s, x, z)| |q_n(s, z, y)| dz dy ds \\ &\leq c^* \frac{c_1^{n+1}}{(n!)^{1/2}} \int_0^t (t-s)^{-1/2} s^{(n+1)/2-1} ds \\ &\leq c^* \frac{c_1^{n+1}}{(n!)^{1/2}} \frac{p}{(n+1)^{1/2}} t^{n/2} \\ &= \frac{c_1^{n+2}}{((n+1)!)^{1/2}} t^{n/2}, \end{aligned}$$

which proves (48) for $n+1$. Similarly we get (49).

Now we will show (51). For $n=0$ this follows from (53). Assume that (51) is true for $n \in \mathbb{N}$, we will show it for $n+1$.

Using our induction hypothesis, (48) and (49) we get for $|x-y| \geq n+2$,

$$\begin{aligned} |q_{n+1}(t, x, y)| &= \left| \int_0^t \int_{|x-z| \geq \frac{|x-y|}{n+2}} q_0(t-s, x, z) q_n(s, z, y) dz ds \right. \\ &\quad \left. + \int_0^t \int_{|x-z| \leq \frac{|x-y|}{n+2}} q_0(t-s, x, z) q_n(s, z, y) dz ds \right| \\ &\leq c_1 e^{-\frac{\lambda|x-y|}{n+2}} \int_0^t \int_{\mathbb{R}^d} |q_n(s, z, y)| dz ds \\ &\quad + c_1 \frac{c_2^n}{(n!)^{1/2}} e^{-\frac{\lambda|x-y|}{n+2}} \int_0^t \int_{\mathbb{R}^d} |q_0(t-s, x, z)| dz s^{n/2} ds \\ &\leq c_1 \frac{2c_1^{n+1} t^{(n+1)/2}}{((n+1)!)^{1/2}} e^{-\frac{\lambda|x-y|}{n+2}} + c_1 \frac{c_2^n c^* t^{(n+1)/2} p}{((n+1)!)^{1/2}} e^{-\frac{\lambda|x-y|}{n+2}} \\ &= (2c_1 c_1^{n+1} + c_2^n c_1^2) \frac{t^{(n+1)/2}}{((n+1)!)^{1/2}} e^{-\frac{\lambda|x-y|}{n+2}}, \end{aligned}$$

which proves (51) for $n+1$ since by the choice of constants $2c_1 c_1^{n+1} + c_2^n c_1^2 \leq c_1 c_2^{n+1}$. $\square$

By standard estimates, one easily gets

$$\sum_{n=k}^{\infty} \frac{C^n}{(n!)^{1/2}} \leq \frac{C^k}{(k!)^{1/2}} \sum_{n=k}^{\infty} \frac{C^{n-k}}{((n-k)!)^{1/2}} \leq C_1 e^{-k}, \quad k \in \mathbb{N}, \quad (56)$$

where C_1 depends on C .

Proposition 3.12 For any $t \in (0, \infty)$, $x, y \in \mathbb{R}^d$ the kernel $q(t, x, y)$ is well defined. For any $t \in (0, \tau]$, $x, y \in \mathbb{R}^d$ we have

$$|q(t, x, y)| \leq \frac{c}{t^{d/\alpha+1}} e^{-c_3 \sqrt{|x-y|}} \leq \frac{c}{t^{d/\alpha+1} (1 + |x-y|)^{d+1}}.$$

There exists $a > 0$ (a depends on $\tau, \alpha, d, \eta_1, \eta_2, \eta_3$) such that for any $t \in (0, \tau]$, $x, y \in \mathbb{R}^d$, $|x-y| \geq a$ we have

$$|q(t, x, y)| \leq c e^{-c_3 \sqrt{|x-y|}}.$$

For any $t \in (0, \tau]$ and $x \in \mathbb{R}^d$ we have

$$\int_{\mathbb{R}^d} |q(t, x, y)| dy \leq ct^{-1/2}, \quad (57)$$

$$\int_{\mathbb{R}^d} |q(t, y, x)| dy \leq ct^{-1/2}. \quad (58)$$

Proof By (50), we clearly get $\sum_{n=0}^{\infty} |q_n(t, x, y)| \leq ct^{-d/\alpha-1}$. This gives that $q(t, x, y)$ is well defined and we have $|q(t, x, y)| \leq ct^{-d/\alpha-1}$.

For $|x - y| \geq 1$ by (50), (51) and (56) we get

$$\begin{aligned} |q(t, x, y)| &= \left| \sum_{n=0}^{[\sqrt{|x-y|}-1]} q_n(t, x, y) + \sum_{n=[\sqrt{|x-y|}]}^{\infty} q_n(t, x, y) \right| \\ &\leq c_1 \sum_{n=0}^{[\sqrt{|x-y|}-1]} \frac{c_2^n \tau^{n/2}}{(n!)^{1/2}} e^{-\lambda \sqrt{|x-y|}} + c_1 \sum_{n=[\sqrt{|x-y|}]}^{\infty} \frac{c_2^n \tau^{n/2}}{(n!)^{1/2} t^{d/\alpha+1}} \\ &\leq \frac{c}{t^{d/\alpha+1}} e^{-c_3 \sqrt{|x-y|}}, \end{aligned}$$

where $[z]$ denotes the integer part of $[z]$. Take the smallest $n_0 \in \mathbb{N}$ such that $n_0/2 - 1 \geq d/\alpha$. For $\sqrt{|x - y|} \geq n_0$ we get

$$\begin{aligned} |q(t, x, y)| &\leq c_1 \sum_{n=0}^{[\sqrt{|x-y|}-1]} \frac{c_2^n \tau^{n/2}}{(n!)^{1/2}} e^{-\lambda \sqrt{|x-y|}} + c_1 \sum_{n=[\sqrt{|x-y|}]}^{\infty} \frac{c_2^n \tau^{n/2}}{(n!)^{1/2} t^{d/\alpha+1}} \\ &\leq ce^{-c_3 \sqrt{|x-y|}}. \end{aligned}$$

The estimates (57) and (58) follow easily from (48) and (49). $\square$

By (25), Corollary 3.3, Proposition 3.10 and Proposition 3.12, we immediately obtain the following result.

Corollary 3.13 For any $t \in (0, \infty)$, $x, y \in \mathbb{R}^d$ the kernel $u(t, x, y)$ is well defined. For any $t \in (0, \tau]$, $x, y \in \mathbb{R}^d$ we have

$$|u(t, x, y)| \leq \frac{c}{t^{d/\alpha}} e^{-c_1 \sqrt{|x-y|}} \leq \frac{c}{t^{d/\alpha} (1 + |x - y|)^{d+1}}.$$

There exists $a > \varepsilon > 0$ (a depends on $\tau, \alpha, d, \eta_1, \eta_2, \eta_3$) such that for any $t \in (0, \tau]$, $x, y \in \mathbb{R}^d$, $|x - y| \geq a$ we have

$$|u(t, x, y)| \leq ce^{-c_2 \sqrt{|x-y|}}.$$

For any $t \in (0, \tau]$ and $x \in \mathbb{R}^d$ we have

$$\int_{\mathbb{R}^d} |u(t, x, y)| dy \leq c, \quad (59)$$

$$\int_{\mathbb{R}^d} |u(t, y, x)| dy \leq c. \quad (60)$$

Proof By Corollary 3.3, Proposition 3.10, (25), we only need to prove the corresponding bounds for

$$I(t, x, y) = \int_0^t \int_{\mathbb{R}^d} p_z(t-s, x-z) |q(s, z, y)| dz ds.$$

For $0 < s < t/2$ we have

$$p_z(t-s, x-z) \leq \frac{c}{t^{d/\alpha}},$$

and, by Proposition 3.12, for $t/2 < s < t$,

$$|q(s, z, y)| \leq \frac{c}{t^{d/\alpha+1}}.$$

Hence,

$$\begin{aligned} I(t, x, y) &= \int_0^{t/2} \int_{\mathbb{R}^d} p_z(t-s, x-z) |q(s, z, y)| dz ds \\ &\quad + \int_{t/2}^t \int_{\mathbb{R}^d} p_z(t-s, x-z) |q(s, z, y)| dz ds \\ &\leq \frac{c}{t^{d/\alpha}} \int_0^{t/2} \int_{\mathbb{R}^d} |q(s, z, y)| dz ds + \frac{c}{t^{d/\alpha+1}} \int_{t/2}^t \int_{\mathbb{R}^d} p_z(t-s, x-z) dz ds \\ &\leq \frac{c}{t^{d/\alpha}}, \end{aligned} \quad (61)$$

where (58) and Proposition 3.10 were applied to estimate the integrals with respect to the space variable.

Let a the constant found in Proposition 3.12. Assume that $|x-y| \geq 1+a$. By Corollary 3.3, for $0 < s < t$, we have

$$p_z(t-s, x-z) \leq ce^{-c_1|x-y|}, \quad |x-z| > |x-y| > 1.$$

Proposition 3.12 implies that for $0 < s < t$,

$$|q(s, z, y)| \leq ce^{-c_1\sqrt{|x-y|}}, \quad |y-z| > |x-y| > a$$

Hence,

$$\begin{aligned} I(t, x, y) &\leq \int_0^t \int_{|x-z| > |x-y|} \dots dz ds + \int_0^t \int_{|y-z| > |x-y|} \dots dz ds \\ &\leq ce^{-c_1|x-y|} \int_0^t \int_{\mathbb{R}^d} |q(s, z, y)| dz ds \\ &\quad + ce^{-c_1\sqrt{|x-y|}} \int_0^t \int_{\mathbb{R}^d} p_z(t-s, x-z) dz ds \\ &\leq ct^{1/2}e^{-c_1|x-y|} + cte^{-c_1\sqrt{|x-y|}} \\ &\leq ce^{-c_1\sqrt{|x-y|}}. \end{aligned} \quad (62)$$

Combining (61) and (62) we obtain the desired pointwise estimates of $u(t, x, y)$.

Next, (59) and (60) immediately follow from (57), (58) and Proposition 3.10. □

For any $\zeta > 0$ and $x, y \in \mathbb{R}^d$ we put

$$\mathcal{L}_\zeta f(x) = \sum_{i=1}^d \int_{|w|>\zeta} [f(x + a_i(x)w) - f(x)] \mu(w) dw,$$

$$\mathcal{L}_\zeta^y f(x) = \sum_{i=1}^d \int_{|w|>\zeta} [f(x + a_i(y)w) - f(x)] \mu(w) dw.$$

Lemma 3.14 For any $\xi \in (0, 1]$, $\zeta > 0$, $x, y, v \in \mathbb{R}^d$ and $t \in (\xi, \tau + \xi]$ we have

$$\sum_{i=1}^d \int_{\mathbb{R}} |p_y(t, x - y + a_i(v)w) - p_y(t, x - y)| \mu(w) dw \leq c(\xi) e^{-c|x-y|}, \quad (63)$$

$$\sum_{i=1}^d \int_{|w|\leq\zeta} |p_y(t, x - y + a_i(v)w) - p_y(t, x - y)| \mu(w) dw \leq c(\xi) \zeta^{1-\alpha}, \quad (64)$$

where $c(\xi)$ is a constant depending on $\xi, \tau, \alpha, d, \eta_1, \eta_2, \eta_3, \varepsilon$.

Proof We estimate the term for $i = 1$. By Lemma 3.1, for $\gamma = 1$, we get for $w \in \mathbb{R}$

$$\begin{aligned} & |p_y(t, x - y + a_1(v)w) - p_y(t, x - y)| \\ & \leq ct^{-1/\alpha}|w| \left(r_y \left(t, \frac{x-y}{2} \right) + r_y \left(t, \frac{x-y+a_1(v)w}{2} \right) \right). \end{aligned}$$

Recall that, if $|w| \geq 2\delta$, then $\mu(w) = 0$. So we may assume that $|w| \leq 2\delta$. By Corollary 3.3, we get

$$r_y \left(t, \frac{x-y}{2} \right) + r_y \left(t, \frac{x-y+a_1(v)w}{2} \right) \leq c_1 t^{-d/\alpha} e^{-c|x-y|}.$$

Now (63) and (64) follow by the fact that $\mu(w) \leq c_1 t^{-d/\alpha} |w|^{-1-\alpha}$. $\square$

Lemma 3.15 Let $\tau_2 > \tau_1 > 0$. Assume that a function $f_t(x)$ is bounded and uniformly continuous on $[\tau_1, \tau_2] \times \mathbb{R}^d$. Then

$$\sup_{t \in [\tau_1, \tau_2], x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} p_y(\varepsilon_1, x - y) f_t(y) dy - f_t(x) \right| \rightarrow 0 \quad \text{as } \varepsilon_1 \rightarrow 0^+.$$

Proof The lemma follows easily by Proposition 3.10. $\square$

For any $t > 0$, $x, y \in \mathbb{R}^d$ we define

$$\varphi_y(t, x) = \int_0^t \int_{\mathbb{R}^d} p_z(t-s, x-z) q(s, z, y) dz ds.$$

Clearly we have

$$u(t, x, y) = p_y(t, x - y) + \varphi_y(t, x).$$

For any $t > 0$, $x, y \in \mathbb{R}^d$, $f \in \mathcal{B}_b(\mathbb{R}^d)$ we define

$$\begin{aligned}\Phi_t f(x) &= \int_{\mathbb{R}^d} \varphi_y(t, x) f(y) dy, \\ U_t f(x) &= \int_{\mathbb{R}^d} u(t, x, y) f(y) dy, \\ Q_t f(x) &= \int_{\mathbb{R}^d} q(t, x, y) f(y) dy.\end{aligned}$$

Now, following ideas from [19], we will define the so-called approximate solutions.

For any $t \geq 0$, $\xi \in [0, 1]$, $t + \xi > 0$, $x, y \in \mathbb{R}^d$ we define

$$\varphi_y^{(\xi)}(t, x) = \int_0^t \int_{\mathbb{R}^d} p_z(t-s+\xi, x-z) q(s, z, y) dz ds$$

and

$$u^{(\xi)}(t, x, y) = p_y(t+\xi, x-y) + \varphi_y^{(\xi)}(t, x).$$

For any $t \geq 0$, $\xi \in [0, 1]$, $t + \xi > 0$, $x, y \in \mathbb{R}^d$, $f \in \mathcal{B}_b(\mathbb{R}^d)$ we define

$$\begin{aligned}\Phi_t^{(\xi)} f(x) &= \int_{\mathbb{R}^d} \varphi_y^{(\xi)}(t, x) f(y) dy, \\ U_t^{(\xi)} f(x) &= \int_{\mathbb{R}^d} u^{(\xi)}(t, x, y) f(y) dy, \\ \Phi_0 f(x) &= 0, \quad U_0^{(0)} f(x) = U_0 f(x) = f(x).\end{aligned}$$

By the same arguments as in the proof of Corollary 3.13 we obtain the following result.

Corollary 3.16 *For any $t \in [0, \infty)$, $\xi \in [0, 1]$, $t + \xi > 0$, $x, y \in \mathbb{R}^d$ the kernel $u^{(\xi)}(t, x, y)$ is well defined. For any $t \in (0, \tau]$, $\xi \in [0, 1]$, $x, y \in \mathbb{R}^d$ we have*

$$|u^{(\xi)}(t, x, y)| \leq \frac{c}{(t+\xi)^{d/\alpha}(1+|x-y|)^{d+1}}.$$

For any $t \in (0, \tau]$, $\xi \in [0, 1]$ and $x \in \mathbb{R}^d$ we have

$$\begin{aligned}\int_{\mathbb{R}^d} |u^{(\xi)}(t, x, y)| dy &\leq c, \\ \int_{\mathbb{R}^d} |u^{(\xi)}(t, y, x)| dy &\leq c.\end{aligned}$$

Lemma 3.17 *Let $f \in C_0(\mathbb{R}^d)$ and $0 < \tau_1 < \tau_2 < \tau$. Then $Q_t f(x)$ as a function of (t, x) is uniformly continuous on $[\tau_1, \tau_2] \times \mathbb{R}^d$. We have $\lim_{|x| \rightarrow \infty} Q_t f(x) = 0$ uniformly in $t \in [\tau_1, \tau_2]$. For each $t > 0$ we have $Q_t f \in C_0(\mathbb{R}^d)$.*

Proof For any $\zeta > 0$, $y \in \mathbb{R}^d$, by Lemma 3.4, we obtain that

$$(t, x) \rightarrow \mathcal{L}_\zeta^x p_y(t, \cdot)(x-y) - \mathcal{L}_\zeta^y p_y(t, \cdot)(x-y)$$

is continuous on $(0, \infty) \times \mathbb{R}^d$. Using this and (64) we obtain that

$$(t, x) \rightarrow q_0(t, x, y) \quad \text{is continuous on } (0, \infty) \times \mathbb{R}^d. \quad (65)$$

By Proposition 3.9, we have

$$|q_0(t, x, y)| \leq \frac{c}{t^{1+d/\alpha}} e^{-c_1|x-y|}. \quad (66)$$

For any $n \in \mathbb{N}$, $t > 0$, $x \in \mathbb{R}^d$ denote

$$Q_{n,t}f(x) = \int_{\mathbb{R}^d} q_n(t, x, y) f(y) dy.$$

By (65), (66) and the dominated convergence theorem, we obtain that $(t, x) \rightarrow Q_{0,t}f(x)$ is continuous on $(0, \infty) \times \mathbb{R}^d$. By Lemma 3.11, for any $t \in (0, \tau]$, $x \in \mathbb{R}^d$, $n \in \mathbb{N}$ we have

$$|Q_{n,t}f(x)| \leq \frac{c_1^{n+1} t^{(n+1)/2-1}}{(n!)^{1/2}} \|f\|_{\infty}. \quad (67)$$

Note that for any $t > 0$, $x \in \mathbb{R}^d$, $n \in \mathbb{N}$, $n \geq 1$ we have

$$Q_{n,t}f(x) = \int_0^t \int_{\mathbb{R}^d} q_0(t-s, x, z) Q_{n-1,s}f(z) dz ds.$$

For any $\varepsilon_1 \in (0, \tau_1/2)$ using (65), (66) and (67) we obtain that

$$(t, x) \rightarrow \int_0^{t-\varepsilon_1} \int_{\mathbb{R}^d} q_0(t-s, x, z) Q_{n-1,s}f(z) dz ds$$

is continuous on $[\tau_1, \tau_2] \times \mathbb{R}^d$. Note also that for any $\varepsilon_1 \in (0, \tau_1/2)$, $t \in [\tau_1, \tau_2]$, $x \in \mathbb{R}^d$, $n \in \mathbb{N}$, $n \geq 1$ we have by (35)

$$\begin{aligned} \left| \int_{t-\varepsilon_1}^t \int_{\mathbb{R}^d} q_0(t-s, x, z) Q_{n-1,s}f(z) dz ds \right| &\leq c \|f\|_{\infty} \int_{t-\varepsilon_1}^t (t-s)^{-1/2} s^{-1/2} ds \\ &\leq c \tau_1^{-1/2} \varepsilon_1^{1/2} \|f\|_{\infty}. \end{aligned}$$

This implies that $(t, x) \rightarrow Q_{n,t}f(x)$ is continuous on $[\tau_1, \tau_2] \times \mathbb{R}^d$. Using this and (67) we obtain that $(t, x) \rightarrow Q_t f(x) = \sum_{n=0}^{\infty} Q_{n,t}f(x)$ is continuous on $[\tau_1, \tau_2] \times \mathbb{R}^d$. By Proposition 3.12, we obtain that $\lim_{|x| \rightarrow \infty} Q_t f(x) = 0$ uniformly in $t \in [\tau_1, \tau_2]$. This implies the assertion of the lemma. $\square$

Proposition 3.18 Choose $\gamma \in (0, \alpha)$. For any $t \in (0, \tau]$, $x, x' \in \mathbb{R}^d$, $f \in \mathcal{B}_b(\mathbb{R}^d)$ we have

$$|U_t f(x) - U_t f(x')| \leq c t^{-\gamma/\alpha} |x - x'|^\gamma \|f\|_{\infty}.$$

Proof We have

$$\begin{aligned} U_t f(x) - U_t f(x') &= \int_{\mathbb{R}^d} (p_y(t, x - y) - p_y(t, x' - y)) f(y) dy \\ &\quad + \int_0^t \int_{\mathbb{R}^d} (p_z(t-s, x - z) - p_z(t-s, x' - z)) Q_s f(z) dz ds \\ &= \text{I} + \text{II}. \end{aligned}$$

By Lemma 3.1 and Proposition 3.10, we get

$$|\text{II}| \leq c \|f\|_{\infty} |x - x'|^\gamma t^{-\gamma/\alpha} \int_{\mathbb{R}^d} (r_y(t, (x-y)/2) + r_y(t, (x'-y)/2)) dy \leq c \|f\|_{\infty} |x - x'|^\gamma t^{-\gamma/\alpha}.$$

By Lemma 3.1 and Propositions 3.10, 3.12, we obtain

$$\begin{aligned}
 |\text{II}| &\leq c\|f\|_\infty |x - x'|^\gamma \int_0^t \int_{\mathbb{R}^d} (t-s)^{-\gamma/\alpha} (r_z(t-s, (x-z)/2) \\
 &\quad + r_z(t-s, (x'-z)/2)) s^{-1/2} dz ds \\
 &\leq c\|f\|_\infty |x - x'|^\gamma \int_0^t (t-s)^{-\gamma/\alpha} s^{-1/2} ds \\
 &\leq c\|f\|_\infty |x - x'|^\gamma t^{1/2-\gamma/\alpha}.
 \end{aligned}$$

□

Note that, by Lemma 3.14, for any $\xi \in (0, 1]$, $t \in [\xi, \tau + \xi]$, $x, z \in \mathbb{R}^d$ we have

$$\left| \frac{\partial p_z(t, x-z)}{\partial t} \right| = |\mathcal{L}^z p_z(t, \cdot)(x-z)| \leq c(\xi) e^{-c|x-z|}, \quad (68)$$

where $c(\xi)$ is a constant depending on $\xi, \tau, \alpha, d, \eta_1, \eta_2, \eta_3, \varepsilon$.

The next lemma is similar to [19, Lemma 4.1].

Lemma 3.19 (i) For every $f \in C_0(\mathbb{R}^d)$, $\xi \in (0, 1]$ the function $U_t^{(\xi)} f(x)$ belongs to $C^1((0, \infty))$ as a function of t and to $C_0^2(\mathbb{R}^d)$ as a function of x . Moreover we have

$$\left| \frac{\partial}{\partial t} (U_t^{(\xi)} f)(x) \right| \leq c(\xi) t^{-1/2} \|f\|_\infty, \quad (69)$$

for each $f \in C_0(\mathbb{R}^d)$, $t \in (0, \tau]$, $x \in \mathbb{R}^d$, $\xi \in (0, 1]$, where $c(\xi)$ depends on $\xi, \tau, \alpha, d, \eta_1, \eta_2, \eta_3$.

(ii) For every $f \in C_0(\mathbb{R}^d)$ we have

$$\lim_{t, \xi \rightarrow 0^+} \|U_t^{(\xi)} f - f\|_\infty = 0.$$

(iii) For every $f \in C_0(\mathbb{R}^d)$ we have

$$U_t^{(\xi)} f(x) \rightarrow 0, \quad \text{as } |x| \rightarrow \infty,$$

uniformly in $t \in [0, \tau]$, $\xi \in [0, 1]$.

(iv) For every $f \in C_0(\mathbb{R}^d)$ we have

$$\|U_t^{(\xi)} f - U_t f\|_\infty \rightarrow 0, \quad \text{as } \xi \rightarrow 0^+,$$

uniformly in $t \in [0, \tau]$.

Proof (i) Let $f \in C_0(\mathbb{R}^d)$, $t \in (0, \tau]$, $\xi \in (0, 1]$ and $x \in \mathbb{R}^d$. We have

$$\begin{aligned}
 &\lim_{h \rightarrow 0^+} \frac{\Phi_{t+h}^{(\xi)} f(x) - \Phi_t^{(\xi)} f(x)}{h} \\
 &= \lim_{h \rightarrow 0^+} \frac{1}{h} \int_t^{t+h} \int_{\mathbb{R}^d} p_z(t+h-s+\xi, x-z) Q_s f(z) dz ds \\
 &\quad + \lim_{h \rightarrow 0^+} \int_0^t \int_{\mathbb{R}^d} \frac{p_z(t+h-s+\xi, x-z) - p_z(t-s+\xi, x-z)}{h} Q_s f(z) dz ds \\
 &= \text{I} + \text{II}.
 \end{aligned}$$

By Lemmas 3.4, 3.17, Corollary 3.3 and Proposition 3.12, we get

$$I = \int_{\mathbb{R}^d} p_z(\xi, x - z) Q_t f(z) dz.$$

By Lemma 3.4, the dominated convergence theorem, (68) and Proposition 3.12, we get

$$\Pi = \int_0^t \int_{\mathbb{R}^d} \frac{\partial p_z(t - s + \xi, x - z)}{\partial t} Q_s f(z) dz ds.$$

By similar arguments, we get the analogous result for $\lim_{h \rightarrow 0^-} (\Phi_{t+h}^{(\xi)} f(x) - \Phi_t^{(\xi)} f(x)) / h$.

By (68), we obtain

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^d} p_z(t + \xi, x - z) f(z) dz = \int_{\mathbb{R}^d} \frac{\partial}{\partial t} p_z(t + \xi, x - z) f(z) dz.$$

Hence, we have

$$\begin{aligned} \frac{\partial}{\partial t} (U_t^{(\xi)} f)(x) &= \int_{\mathbb{R}^d} \frac{\partial}{\partial t} p_z(t + \xi, x - z) f(z) dz \\ &\quad + \int_{\mathbb{R}^d} p_z(\xi, x - z) Q_t f(z) dz \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \frac{\partial p_z(t - s + \xi, x - z)}{\partial t} Q_s f(z) dz ds. \end{aligned} \quad (70)$$

Using this, (68), Propositions 3.10 and 3.12 we obtain (69). We also obtain that for every $f \in C_0(\mathbb{R}^d)$, $\xi \in (0, 1]$ the function $U_t^{(\xi)} f(x)$ belongs to $C^1((0, \infty))$ as a function of t .

The fact that $U_t^{(\xi)} f \in C_0^2(\mathbb{R}^d)$ for $\xi \in (0, 1]$ follows by Lemmas 3.4, 3.5, Proposition 3.12 and Lemma 3.17.

(ii) Fix $f \in C_0(\mathbb{R}^d)$. For any $\xi \in [0, 1]$, $t \geq 0$, $t + \xi > 0$, $x \in \mathbb{R}^d$ we have

$$U_t^{(\xi)} f(x) = \int_{\mathbb{R}^d} p_y(t + \xi, x - y) f(y) dy + \Phi_t^{(\xi)} f(x).$$

For any $\xi \in [0, 1]$, $t \in [0, \tau]$, $\xi + t > 0$, $x \in \mathbb{R}^d$, by Propositions 3.12 and 3.10, we get

$$\left| \Phi_t^{(\xi)} f(x) \right| \leq c \|f\|_\infty \int_0^t s^{-1/2} ds \leq c \|f\|_\infty t^{1/2}. \quad (71)$$

By (43), Proposition 3.10 and the fact that f is uniformly continuous on $\mathbb{R}^d$, we obtain

$$\lim_{t, \xi \rightarrow 0^+} \int_{\mathbb{R}^d} p_y(t + \xi, x - y) f(y) dy - f(x) = 0$$

uniformly with respect to $x \in \mathbb{R}^d$. This and (71) gives (ii).

(iii) This follows easily from (ii) and Corollary 3.16.

(iv) Fix $f \in C_0(\mathbb{R}^d)$. By Lemma 3.4, Corollary 3.3 and the dominated convergence theorem we obtain that

$$(t, x) \rightarrow \int_{\mathbb{R}^d} p_y(t, x - y) f(y) dy$$

is continuous on $(0, \tau + 1] \times \mathbb{R}^d$. It follows that

$$(\xi, t, x) \rightarrow \int_{\mathbb{R}^d} p_y(t + \xi, x - y) f(y) dy$$

is continuous on $[0, 1] \times (0, \tau] \times \mathbb{R}^d$. Using Lemma 3.4, Corollary 3.3, Proposition 3.12 and the dominated convergence theorem we obtain that for any $s \in (0, \tau)$

$$(\xi, t, x) \rightarrow \int_{\mathbb{R}^d} p_z(t + \xi - s, x - z) Q_s f(z) dz$$

is continuous on $[0, 1] \times (s, \tau] \times \mathbb{R}^d$. Using this, Corollary 3.3, Proposition 3.12 and the dominated convergence theorem we obtain that

$$(\xi, t, x) \rightarrow \Phi_t^{(\xi)} f(x) = \int_0^\tau 1_{(0,t)}(s) \int_{\mathbb{R}^d} p_z(t + \xi - s, x - z) Q_s f(z) dz ds$$

is continuous on $[0, 1] \times (0, \tau] \times \mathbb{R}^d$. Hence $(\xi, t, x) \rightarrow U_t^{(\xi)} f(x)$ is continuous on $[0, 1] \times (0, \tau] \times \mathbb{R}^d$. Using (71) we obtain that

$$(\xi, t, x) \rightarrow U_t^{(\xi)} f(x) \quad \text{is continuous on } [0, 1] \times [0, \tau] \times \mathbb{R}^d. \quad (72)$$

This and (iii) imply (iv). $\square$

By the same arguments as in the proof of Lemma 3.19 (iv) we obtain the following result.

Lemma 3.20 *For any $f \in \mathcal{B}_b(\mathbb{R}^d)$ the function $(t, x) \rightarrow U_t f(x)$ is continuous on $(0, \infty) \times \mathbb{R}^d$. For any $\xi \in (0, 1]$, $f \in \mathcal{B}_b(\mathbb{R}^d)$ the function $(t, x) \rightarrow U_t^{(\xi)} f(x)$ is continuous on $[0, \infty) \times \mathbb{R}^d$.*

Heuristically, now our aim is to show that if ξ is small then $\frac{\partial}{\partial t}(U_t^{(\xi)} f)(x) - \mathcal{L}(U_t^{(\xi)} f)(x)$ is small. For any $t > 0$, $\xi \in (0, 1]$, $x \in \mathbb{R}^d$ we put

$$\Lambda_t^{(\xi)} f(x) = \frac{\partial}{\partial t}(U_t^{(\xi)} f)(x) - \mathcal{L}(U_t^{(\xi)} f)(x).$$

Lemma 3.21 *$\mathcal{L}(U_t^{(\xi)} f)(x)$ is well defined for every $f \in C_0(\mathbb{R}^d)$, $t \in (0, \tau]$, $\xi \in (0, 1]$ and $x \in \mathbb{R}^d$ and we have*

$$\Lambda_t^{(\xi)} f(x) = \int_{\mathbb{R}^d} p_z(\xi, x-z) Q_t f(z) dz - Q_{t+\xi} f(x) + \int_t^{t+\xi} \int_{\mathbb{R}^d} q_0(t-s+\xi, x, z) Q_s f(z) dz ds. \quad (73)$$

Moreover, we have

$$\left| \mathcal{L}_\zeta(U_t^{(\xi)} f)(x) \right| \leq c(\xi) \|f\|_\infty, \quad \zeta > 0, \quad (74)$$

$$\left| \mathcal{L}(U_t^{(\xi)} f)(x) \right| \leq c(\xi) \|f\|_\infty, \quad (75)$$

for each $f \in C_0(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $t \in (0, \tau]$, $\xi \in (0, 1]$, where $c(\xi)$ is a constant depending on $\xi, \tau, \alpha, d, \eta_1, \eta_2, \eta_3, \varepsilon$.

Proof Let $f \in C_0(\mathbb{R}^d)$, $t \in (0, \tau]$, $\xi \in (0, 1]$, $x \in \mathbb{R}^d$ and $\zeta > 0$. We have

$$\begin{aligned} \mathcal{L}_\zeta(U_t^{(\xi)} f)(x) &= \int_{\mathbb{R}^d} \mathcal{L}_\zeta^x p_z(t + \xi, \cdot)(x - z) f(z) dz \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \mathcal{L}_\zeta^x p_z(t - s + \xi, \cdot)(x - z) Q_s f(z) dz ds. \end{aligned} \quad (76)$$

Using this, Lemma 3.14 and Proposition 3.12 we obtain (74). By (76), the dominated convergence theorem, Lemma 3.14 and Proposition 3.12, one gets

$$\begin{aligned}\mathcal{L}(U_t^{(\xi)} f)(x) &= \lim_{\xi \rightarrow 0^+} \mathcal{L}_\xi^x(U_t^{(\xi)} f)(x) \\ &= \int_{\mathbb{R}^d} \mathcal{L}^x p_z(t + \xi, \cdot)(x - z) f(z) dz \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \mathcal{L}^x p_z(t - s + \xi, \cdot)(x - z) Q_s f(z) dz ds.\end{aligned}\quad (77)$$

Using this and again Lemma 3.14 and Proposition 3.12 we obtain (75).

Note that for $s \in [0, t]$, $z \in \mathbb{R}^d$ we have

$$\frac{\partial p_z(t - s + \xi, x - z)}{\partial t} - \mathcal{L}^x p_z(t - s + \xi, \cdot)(x - z) = -q_0(t - s + \xi, x, z).$$

Using this, (70) and (77) we get

$$\begin{aligned}\Lambda_t^{(\xi)} f(x) &= \int_{\mathbb{R}^d} p_z(\xi, x - z) Q_t f(z) dz \\ &\quad - \int_{\mathbb{R}^d} q_0(t + \xi, x, z) f(z) dz \\ &\quad - \int_0^t \int_{\mathbb{R}^d} q_0(t - s + \xi, x, z) Q_s f(z) dz ds.\end{aligned}\quad (78)$$

For $\xi \in (0, 1]$, $t \in (0, \tau]$, $x \in \mathbb{R}^d$, by the definition of $q(t, x, y)$, we obtain

$$\int_{\mathbb{R}^d} q_0(t + \xi, x, z) f(z) dz = Q_{t+\xi} f(x) - \int_0^{t+\xi} \int_{\mathbb{R}^d} q_0(t - s + \xi, x, z) Q_s f(z) dz ds.$$

Using this and (78) we obtain (73). $\square$

The next lemma is similar to [19, Lemma 4.2].

Lemma 3.22 (i) For any $f \in C_0(\mathbb{R}^d)$ we have

$$\Lambda_t^{(\xi)} f(x) \rightarrow 0, \quad \text{as } \xi \rightarrow 0^+,$$

uniformly in $(t, x) \in [\tau_1, \tau_2] \times \mathbb{R}^d$ for every $0 < \tau_1 < \tau_2 \leq \tau$.

(ii) For any $f \in C_0(\mathbb{R}^d)$ we have

$$\int_0^t \Lambda_s^{(\xi)} f(x) ds \rightarrow 0, \quad \text{as } \xi \rightarrow 0^+, \quad (79)$$

uniformly in $(t, x) \in (0, \tau] \times \mathbb{R}^d$.

Proof Let $f \in C_0(\mathbb{R}^d)$ and $0 < \tau_1 < \tau_2 \leq \tau$. For any $t > 0$, $x \in \mathbb{R}^d$, $\xi \in (0, 1]$ we put

$$\begin{aligned}\Lambda_t^{(\xi, 1)} f(x) &= \int_{\mathbb{R}^d} p_z(\xi, x - z) Q_t f(z) dz - Q_{t+\xi} f(x). \\ \Lambda_t^{(\xi, 2)} f(x) &= \int_t^{t+\xi} \int_{\mathbb{R}^d} q_0(t - s + \xi, x, z) Q_s f(z) dz ds.\end{aligned}$$

By Lemma 3.17, we get

$$\sup_{t \in [\tau_1, \tau_2], x \in \mathbb{R}^d} |Q_{t+\xi} f(x) - Q_t f(x)| \rightarrow 0 \quad \text{as } \xi \rightarrow 0^+.$$

By Lemmas 3.15 and 3.17, we obtain

$$\sup_{t \in [\tau_1, \tau_2], x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} p_z(\xi, x - z) Q_t f(z) dz - Q_t f(x) \right| \rightarrow 0 \quad \text{as } \xi \rightarrow 0^+.$$

This gives (i) for $\Lambda_t^{(\xi, 1)} f(x)$ instead of $\Lambda_t^{(\xi)} f(x)$.

By Proposition 3.12, for any $t \in (0, \tau]$, $x \in \mathbb{R}^d$, $\xi \in (0, 1]$ we get

$$\left| \Lambda_t^{(\xi, 1)} f(x) \right| \leq c \|f\|_\infty t^{-1/2}. \quad (80)$$

This allows to use the dominated convergence theorem in the integral (79) with $\Lambda_t^{(\xi)} f(x)$ replaced by $\Lambda_t^{(\xi, 1)} f(x)$. So (ii) for $\Lambda_t^{(\xi, 1)} f(x)$ follows from (i) for $\Lambda_t^{(\xi, 1)} f(x)$.

For any $t \in (0, \tau]$, $x \in \mathbb{R}^d$, $\xi \in (0, 1]$, by Propositions 3.9 and 3.12, we get

$$\left| \Lambda_t^{(\xi, 2)} f(x) \right| \leq c \|f\|_\infty \int_t^{t+\xi} ((t-s+\xi)s)^{-1/2} ds. \quad (81)$$

This implies (i) and (ii) for $\Lambda_t^{(\xi, 2)} f(x)$. $\square$

Lemma 3.23 *There exist $\varepsilon_1 \in (0, 1]$ and $t_1 \in (0, 1]$ such that for any $t \in (0, t_1]$, $x, y \in \mathbb{R}^d$, $|x - y| \leq \varepsilon_1 t^{1/\alpha}$ we have*

$$u(t, x, y) \geq c_1 t^{-d/\alpha}.$$

ε_1, t_1 depend on $\alpha, d, \eta_1, \eta_2, \eta_3, \varepsilon$.

Proof By the weak lower scaling property of the symbol $\Phi_\delta^{(1)}$ (see proof of Lemma 2.1) and by [3, formula (23)], we get that $g_t(0) \geq ct^{-1/\alpha}$. Using this and Lemma 2.1 there exist $\varepsilon_2 > 0$, $t_2 > 0$ such that for $|y| \leq \varepsilon_2 t^{1/\alpha}$, $t \leq t_2$ we have $g_t(y) \geq ct^{-1/\alpha}$. It follows that there exist $\varepsilon_3 > 0$, $t_3 > 0$ such that for $x, y \in \mathbb{R}^d$, $|x - y| \leq \varepsilon_3 t^{1/\alpha}$, $t \leq t_3$ we have $p_y(t, x - y) \geq ct^{-d/\alpha}$. By Lemma 3.12 and Proposition 3.10, for $t \in (0, \tau]$, $x, y \in \mathbb{R}^d$ and $a \in [1/2, 1)$, we get

$$\begin{aligned} |\varphi_y(t, x)| &= \left| \int_0^{ta} \int_{\mathbb{R}^d} p_z(t-s, x-z) q(s, z, y) dz ds \right. \\ &\quad \left. + \int_{ta}^t \int_{\mathbb{R}^d} p_z(t-s, x-z) q(s, z, y) dz ds \right| \\ &\leq c(t-ta)^{-d/\alpha} \int_0^{ta} s^{-1/2} ds + ct^{-d/\alpha-1} t(1-a) \\ &\leq ct^{-d/\alpha} (t^{1/2}(1-a)^{-d/\alpha} + 1-a). \end{aligned}$$

By an appropriate choice of a , there exists ε_1 and $t_1 > 0$ such that for any $x, y \in \mathbb{R}^d$, $|x - y| \leq \varepsilon_1 t^{1/\alpha}$, $t \leq t_1$ we have

$$u(t, x, y) = p_y(t, x - y) - \varphi_y(t, x) \geq ct^{-d/\alpha} - c_2 t^{-d/\alpha} (t^{1/2}(1-a)^{-d/\alpha} + 1-a) \geq c_1 t^{-d/\alpha}.$$

$\square$

4 Construction and Properties of the Semigroup of X_t

Let us introduce the following notation

$$\nu(x) = \frac{\mathcal{A}_\alpha}{|x|^{1+\alpha}} - \mu(x), \quad x \in \mathbb{R},$$

$$\lambda = d \int_{\mathbb{R}} \nu(x) dx < \infty.$$

Note that by (7), for any $x \in \mathbb{R}^d$ and $f \in \mathcal{B}_b(\mathbb{R}^d)$, we have

$$\mathcal{R}f(x) = \sum_{i=1}^d \int_{\mathbb{R}} [f(x + a_i(x)w) - f(x)] \nu(w) dw.$$

We denote, for any $x \in \mathbb{R}^d$ and $f \in \mathcal{B}_b(\mathbb{R}^d)$,

$$\mathcal{N}f(x) = \sum_{i=1}^d \int_{\mathbb{R}} [f(x + a_i(x)w)] \nu(w) dw.$$

It is clear that

$$\|\mathcal{N}f\|_\infty \leq \lambda \|f\|_\infty. \quad (82)$$

For any $t \geq 0$, $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$, $n \geq 1$, $f \in \mathcal{B}_b(\mathbb{R}^d)$ we define

$$\Psi_{0,t}f(x) = U_t f(x), \quad (83)$$

$$\Psi_{n,t}f(x) = \int_0^t U_{t-s}(\mathcal{N}(\Psi_{n-1,s}f))(x) ds, \quad n \geq 1. \quad (84)$$

For any $t \geq 0$, $\xi \in [0, 1]$, $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$, $f \in \mathcal{B}_b(\mathbb{R}^d)$ we define

$$\Psi_{0,t}^{(\xi)}f(x) = U_t^{(\xi)}f(x), \quad (85)$$

$$\Psi_{n,t}^{(\xi)}f(x) = \int_0^t U_{t-s}^{(\xi)}(\mathcal{N}(\Psi_{n-1,s}^{(\xi)}f))(x) ds, \quad n \geq 1. \quad (86)$$

We observe that

$$\Psi_{n,t} = \Psi_{n,t}^{(0)}.$$

Finally, we remark that all these functions have nothing in common with the functions Ψ or Ψ_x used in Section 3.

Lemma 4.1 $\Psi_{n,t}f(x)$ and $\Psi_{n,t}^{(\xi)}f(x)$ are well defined for any $t > 0$, $f \in \mathcal{B}_b(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $n \in \mathbb{N}$ and $\xi \in [0, 1]$. For any $f \in \mathcal{B}_b(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $n \in \mathbb{N}$ we have

$$|\Psi_{n,t}f(x)| \leq \frac{c_1^{n+1}t^n}{n!} \|f\|_\infty, \quad t \in (0, \tau], \quad (87)$$

$$|\Psi_{n,t}^{(\xi)}f(x)| \leq \frac{c_1^{n+1}t^n}{n!} \|f\|_\infty, \quad \xi \in (0, 1], \quad t \in [0, \tau]. \quad (88)$$

Proof We will only show the result for $\Psi_{n,t}f(x)$ using the induction. The proof for $\Psi_{n,t}^{(\xi)}f(x)$ is almost the same.

Let c be the constant from (59) and put $c_1 = (\lambda \vee 1)c$. For $n = 0$ (87) follows from (59). Assume that (87) holds for $n \geq 0$, we will show it for $n + 1$. Indeed, applying (59) and (82), we get

$$|\Psi_{n+1,t} f(x)| \leq \int_0^t \int_{\mathbb{R}^d} |u(t-s, x, z)| dz \frac{\lambda c_1^{n+1} s^n}{n!} ds \leq \frac{c_1^{n+2} t^{n+1}}{(n+1)!}.$$

□

For any $x \in \mathbb{R}^d$ we define

$$\begin{aligned} T_t f(x) &= e^{-\lambda t} \sum_{n=0}^{\infty} \Psi_{n,t} f(x), \quad t > 0, \\ T_0 f(x) &= f(x), \\ T_t^{(\xi)} f(x) &= e^{-\lambda t} \sum_{n=0}^{\infty} \Psi_{n,t}^{(\xi)} f(x), \quad t \geq 0, \xi \in [0, 1]. \end{aligned}$$

Our ultimate aim will be to show that for any $t > 0$ we have $T_t = P_t$, where P_t is given by (5).

By Lemma 4.1, we obtain

Corollary 4.2 $T_t f(x)$ and $T_t^{(\xi)} f(x)$ are well defined for any $t \geq 0$, $f \in \mathcal{B}_b(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $\xi \in [0, 1]$ and for $t \in [0, \tau]$ we have $\max\{|T_t f(x)|, |T_t^{(\xi)} f(x)|\} \leq c \|f\|_{\infty}$.

Next, we obtain the following regularity results concerning operators T_t .

Theorem 4.3 For any $\gamma \in (0, \alpha/d)$, $t \in (0, \tau]$, $x \in \mathbb{R}^d$ and $f \in L^1(\mathbb{R}^d) \cap L^{\infty}(\mathbb{R}^d)$ we have

$$|T_t f(x)| \leq c t^{-\gamma d/\alpha} \|f\|_{\infty}^{1-\gamma} \|f\|_1^{\gamma}.$$

Proof For any $t \in (0, \tau]$, $x \in \mathbb{R}^d$ by Corollary 3.13 we get $|U_t f(x)| \leq c \|f\|_{\infty}$, $|U_t f(x)| \leq c t^{-d/\alpha} \|f\|_1$. Fix $\gamma \in (0, \alpha/d)$. It follows that for any $t \in (0, \tau]$, $x \in \mathbb{R}^d$ we have $|U_t f(x)| \leq c t^{-\gamma d/\alpha} \|f\|_{\infty}^{1-\gamma} \|f\|_1^{\gamma}$. Hence $|\Psi_{1,t} f(x)| \leq c \|f\|_{\infty}^{1-\gamma} \|f\|_1^{\gamma}$. Using the same arguments as in Lemma 4.1 for any $t \in (0, \tau]$, $x \in \mathbb{R}^d$, $n \in \mathbb{N}$, $n \geq 1$ one gets $|\Psi_{n,t} f(x)| \leq c^n t^{n-1} \|f\|_{\infty}^{1-\gamma} \|f\|_1^{\gamma} / (n-1)!$, which implies the assertion of the theorem. □

Theorem 4.4 Choose $\gamma \in (0, \alpha)$. For any $t \in (0, \tau]$, $x, x' \in \mathbb{R}^d$, $f \in \mathcal{B}_b(\mathbb{R}^d)$ we have

$$|T_t f(x) - T_t f(x')| \leq c t^{-\gamma/\alpha} |x - x'|^{\gamma} \|f\|_{\infty}.$$

Proof We have

$$T_t f(x) = e^{-\lambda t} U_t f(x) + e^{-\lambda t} \sum_{n=1}^{\infty} \Psi_{n,t} f(x).$$

By Lemma 3.18, it remains to show

$$\left| \sum_{n=1}^{\infty} \Psi_{n,t} f(x) - \sum_{n=1}^{\infty} \Psi_{n,t} f(x') \right| \leq c t^{-\gamma/\alpha} |x - x'|^{\gamma} \|f\|_{\infty}. \quad (89)$$

Let $n \in \mathbb{N}$, $n \geq 1$. By (84) we get

$$\Psi_{n,t}f(x) - \Psi_{n,t}f(x') = \int_0^t (U_{t-s}(\mathcal{N}(\Psi_{n-1,s}f))(x) - U_{t-s}(\mathcal{N}(\Psi_{n-1,s}f))(x')) ds. \quad (90)$$

By Lemma 4.1, we have

$$|\Psi_{n,t}f(x)| \leq \frac{c_1^{n+1}t^n}{n!} \|f\|_\infty.$$

Hence for $s \in (0, \tau]$, by (82), we arrive at

$$|\mathcal{N}(\Psi_{n-1,s}f)(x)| \leq \frac{\lambda c_1^n s^{n-1}}{(n-1)!} \|f\|_\infty.$$

Using this and Lemma 3.18 we get

$$|U_{t-s}(\mathcal{N}(\Psi_{n-1,s}f))(x) - U_{t-s}(\mathcal{N}(\Psi_{n-1,s}f))(x')| \leq c(t-s)^{-\gamma/\alpha} |x-x'|^\gamma \frac{c_1^n s^{n-1}}{(n-1)!} \|f\|_\infty.$$

Combining this with (90) we obtain

$$|\Psi_{n,t}f(x) - \Psi_{n,t}f(x')| \leq c \frac{c_1^n}{(n-1)!} \|f\|_\infty |x-x'|^\gamma t^{n-\gamma/\alpha}.$$

This implies (89), which finishes the proof. $\square$

Clearly, we have, by applying (59) and (82), the following lemma.

Lemma 4.5 *There exists $a \geq 1$ such that for any $f \in \mathcal{B}_b(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $\xi \in [0, 1]$, $t \in (0, \tau]$ we have*

$$\left| \int_0^t U_{t-s}^{(\xi)}(\mathcal{N}(f))(x) ds \right| \leq a \|f\|_\infty,$$

where a depends on $\tau, \alpha, d, \eta_1, \eta_2, \eta_3$.

Lemma 4.6 *Assume that $f \in \mathcal{B}_b(\mathbb{R}^d)$. Put $\beta = 1/(4(d/\alpha + 1))$. For any $t \in (0, \tau]$, $x \in \mathbb{R}^d$, we have*

$$|Q_t f(x)| \leq \frac{c \|f\|_\infty}{t^{3/4}(\text{dist}(x, \text{supp}(f)) + 1)^\beta}.$$

Proof Let $t \in (0, \tau]$ be arbitrary. By Proposition 3.12, we get for $x \in \mathbb{R}^d$,

$$\begin{aligned} |Q_t f(x)| &\leq ct^{-1/2} \|f\|_\infty, \\ |Q_t f(x)| &\leq \frac{c \|f\|_\infty}{t^{d/\alpha+1}(\text{dist}(x, \text{supp}(f)) + 1)^\beta}. \end{aligned}$$

It follows that

$$\begin{aligned} |Q_t f(x)|^{1-\beta} &\leq ct^{(-1/2)(1-\beta)} \|f\|_\infty^{1-\beta} \leq ct^{-1/2} \|f\|_\infty^{1-\beta}, \\ |Q_t f(x)|^\beta &\leq \frac{c \|f\|_\infty^\beta}{t^{1/4}(\text{dist}(x, \text{supp}(f)) + 1)^\beta}. \end{aligned}$$

This implies the assertion of the lemma. $\square$

Lemma 4.7 Assume that $f \in \mathcal{B}_b(\mathbb{R}^d)$. For any $\varepsilon_1 > 0$ there exists $r \geq 1$ (depending on $\varepsilon_1, \tau, \alpha, d, \eta_1, \eta_2, \eta_3$) such that for any $\xi \in [0, 1]$, $t \in [0, \tau]$, $x \in \mathbb{R}^d$, if $\text{dist}(x, \text{supp}(f)) \geq r$, then $\left| U_t^{(\xi)} f(x) \right| \leq \varepsilon_1 \|f\|_\infty$.

Proof Let $\xi \in [0, 1]$, $t \in [0, \tau]$ be arbitrary. Assume that $\text{dist}(x, \text{supp}(f)) \geq 1$ and $t + \xi > 0$. By Lemmas 3.10, 4.6 we get

$$\begin{aligned} |\Phi_t^{(\xi)} f(x)| &= \left| \int_0^t \int_{\mathbb{R}^d} p_z(t-s+\xi, x-z) Q_s f(z) dz ds \right| \\ &\leq \frac{c \|f\|_\infty}{(\text{dist}(x, \text{supp}(f)) + 1)^\beta} \int_0^t \frac{1}{s^{3/4}} ds. \end{aligned}$$

By Corollary 3.3 we get

$$\int_{\mathbb{R}^d} p_z(t+\xi, x-z) f(z) dz \leq c \|f\|_\infty e^{-c_1 \text{dist}(x, \text{supp}(f))}.$$

This gives the assertion of the lemma. $\square$

The proof of the next lemma is standard and it is omitted.

Lemma 4.8 Assume that $f \in \mathcal{B}_b(\mathbb{R}^d)$. For any $\varepsilon_1 > 0$ there exists $r \geq 1$ (depending on $\varepsilon_1, \tau, \alpha, d, \eta_1, \eta_2, \eta_3$) such that, for any $x \in \mathbb{R}^d$, if $\text{dist}(x, \text{supp}(f)) \geq r$, then $|\mathcal{N}f(x)| \leq \varepsilon_1 \|f\|_\infty$.

Lemma 4.9 Assume that $f \in \mathcal{B}_b(\mathbb{R}^d)$. For any $\varepsilon_1 > 0$ there exists $r \geq 1$ (depending on $\varepsilon_1, \tau, \alpha, d, \eta_1, \eta_2, \eta_3$) such that for any $\xi \in [0, 1]$, $t \in [0, \tau]$, $x \in \mathbb{R}^d$, if $\text{dist}(x, \text{supp}(f)) \geq r$, then $\left| \int_0^t U_{t-s}^{(\xi)} (\mathcal{N}(f))(x) ds \right| \leq \varepsilon_1 \|f\|_\infty$.

Proof Let $\xi \in [0, 1]$, $t \in (0, \tau]$ be arbitrary. Choose $\varepsilon_1 > 0$. There exists $a > 0$ such that for any $g \in \mathcal{B}_b(\mathbb{R}^d)$ we have $\|U_t^{(\xi)} g\|_\infty \leq a \|g\|_\infty$, where a depends on $\tau, \alpha, d, \eta_1, \eta_2, \eta_3$. By Lemma 4.8, there exists $r_1 \geq 1$ such that, if $\text{dist}(x, \text{supp}(f)) \geq r_1$, then

$$|\mathcal{N}f(x)| \leq \frac{\varepsilon_1 \|f\|_\infty}{2a\tau},$$

where r_1 depends on $\varepsilon_1, \tau, \alpha, d, \eta_1, \eta_2, \eta_3$. Put $A = \{x \in \mathbb{R}^d : \text{dist}(x, \text{supp}(f)) \geq r_1\}$. We have, applying also (82),

$$|\mathcal{N}f(x)| = |\mathcal{N}f(x) \mathbf{1}_{A^c}(x) + \mathcal{N}f(x) \mathbf{1}_A(x)| \leq \lambda \|f\|_\infty \mathbf{1}_{A^c}(x) + \frac{\varepsilon_1 \|f\|_\infty}{2a\tau} \mathbf{1}_A(x).$$

Put $\tilde{f}(x) = \lambda \|f\|_\infty \mathbf{1}_{A^c}(x)$, $\tilde{\tilde{f}}(x) = \frac{\varepsilon_1 \|f\|_\infty}{2a\tau} \mathbf{1}_A(x)$.

We have

$$\left| \int_0^t U_{t-s}^{(\xi)} (\mathcal{N}(f))(x) ds \right| \leq \int_0^t U_{t-s}^{(\xi)} (\tilde{f})(x) ds + \int_0^t U_{t-s}^{(\xi)} (\tilde{\tilde{f}})(x) ds. \quad (91)$$

For any $x \in \mathbb{R}^d$ we get

$$\int_0^t U_{t-s}^{(\xi)} (\tilde{\tilde{f}})(x) ds \leq \frac{\varepsilon_1 \|f\|_\infty}{2}. \quad (92)$$

By Lemma 4.7, there exists $r_2 \geq r_1$ such that, if $\text{dist}(x, \text{supp}(f)) \geq r_2$ and $s \in (0, t)$, then

$$\left| U_{t-s}^{(\xi)}(\tilde{f})(x) \right| \leq \frac{\varepsilon_1 \|f\|_\infty}{2\tau},$$

where r_2 depends on $\varepsilon_1, r_1, \tau, \alpha, d, \eta_1, \eta_2, \eta_3$. It follows that, if $\text{dist}(x, \text{supp}(f)) \geq r_2$, then

$$\int_0^t U_{t-s}^{(\xi)}(\tilde{f})(x) ds \leq \frac{\varepsilon_1 \|f\|_\infty}{2}. \quad (93)$$

Finally, (91-93) imply the assertion of the lemma. $\square$

Lemma 4.10 Assume that $f \in \mathcal{B}_b(\mathbb{R}^d)$. For any $\varepsilon_1 > 0$ there exists $r \geq 1$ (depending on $\varepsilon_1, \tau, \alpha, d, \eta_1, \eta_2, \eta_3$), such that for any $\xi \in [0, 1], t \in [0, \tau], x \in \mathbb{R}^d$, if $\text{dist}(x, \text{supp}(f)) \geq r$, then $|T_t^{(\xi)} f(x)| \leq \sum_{n=0}^\infty |\Psi_{n,t}^{(\xi)} f(x)| \leq \varepsilon_1 \|f\|_\infty$.

Proof Fix $f \in \mathcal{B}_b(\mathbb{R}^d)$. Put

$$M = \sup_{n \in \mathbb{N}} \frac{c_1^{n+1} \tau^n}{n!},$$

where c_1 is a constant from Lemma 4.1. Let $\xi \in [0, 1], t \in [0, \tau]$ be arbitrary such that $t + \xi > 0$. Choose $\varepsilon_2 > 0$. By Lemma 4.1, there exists n_0 such that for any $x \in \mathbb{R}^d$ we have

$$\sum_{n=n_0}^\infty \left| \Psi_{n,t}^{(\xi)} f(x) \right| \leq \varepsilon_2 \|f\|_\infty. \quad (94)$$

Put $r_{-1} = 1$. Now we will show that for any $n \in \mathbb{N}$ there exists $r_n \geq r_{n-1}$ such that, if $\text{dist}(x, \text{supp}(f)) \geq r_n$, then

$$\left| \Psi_{n,t}^{(\xi)} f(x) \right| \leq 2^n a^n \varepsilon_2 \|f\|_\infty, \quad (95)$$

where $a \geq 1$ is a constant from Lemma 4.5 and r_n depends on $r_{n-1}, \varepsilon_2, \tau, \alpha, d, \eta_1, \eta_2, \eta_3$.

By Lemma 4.7, there exists $r_0 \geq r_{-1}$ such that, if $\text{dist}(x, \text{supp}(f)) \geq r_0$, then

$$\left| \Psi_{0,t}^{(\xi)} f(x) \right| = \left| U_{t-s}^{(\xi)} f(x) \right| \leq \varepsilon_2 \|f\|_\infty, \quad (96)$$

where r_0 depends on $\varepsilon_2, \tau, \alpha, d, \eta_1, \eta_2, \eta_3$.

Assume that (95) holds for $n \in \mathbb{N}$. We will show it for $n+1$. Put $A_n = \{x \in \mathbb{R}^d : \text{dist}(x, \text{supp}(f)) \geq r_n\}$. We have

$$\begin{aligned} \left| \Psi_{n,t}^{(\xi)} f(x) \right| &= \left| \Psi_{n,t}^{(\xi)} f(x) \mathbf{1}_{A_n^c}(x) + \Psi_{n,t}^{(\xi)} f(x) \mathbf{1}_{A_n}(x) \right| \\ &\leq M \|f\|_\infty \mathbf{1}_{A_n^c}(x) + 2^n a^n \varepsilon_2 \|f\|_\infty \mathbf{1}_{A_n}(x). \end{aligned}$$

Put $f_n(x) = M \|f\|_\infty \mathbf{1}_{A_n^c}(x)$, $\tilde{f}_n(x) = 2^n a^n \varepsilon_2 \|f\|_\infty \mathbf{1}_{A_n}(x)$.

We have

$$\Psi_{n+1,t}^{(\xi)} f(x) = \int_0^t U_{t-s}^{(\xi)} (\mathcal{N}(\Psi_{n,s}^{(\xi)} f))(x) ds.$$

Hence,

$$\left| \Psi_{n+1,t}^{(\xi)} f(x) \right| \leq \int_0^t U_{t-s}^{(\xi)} (\mathcal{N}(f_n))(x) ds + \int_0^t U_{t-s}^{(\xi)} (\mathcal{N}(\tilde{f}_n))(x) ds. \quad (97)$$

By Lemma 4.5, for any $x \in \mathbb{R}^d$, we get

$$\int_0^t U_{t-s}^{(\xi)} (\mathcal{N}(\tilde{f}_n))(x) ds \leq a \|\tilde{f}_n\|_\infty \leq 2^n a^{n+1} \varepsilon_2 \|f\|_\infty. \quad (98)$$

By Lemma 4.9, there exists $r_{n+1} \geq r_n$ such that, if $\text{dist}(x, \text{supp}(f)) \geq r_{n+1}$, then

$$\int_0^t U_{t-s}^{(\xi)} (\mathcal{N}(f_n))(x) ds \leq \varepsilon_2 \|f\|_\infty, \quad (99)$$

where r_{n+1} depends on $\varepsilon_2, r_n, \tau, \alpha, d, \eta_1, \eta_2, \eta_3$.

By (97–99), we obtain (95) for $n + 1$. By (95), we obtain that, if $\text{dist}(x, \text{supp}(f)) \geq r_{n_0}$, then

$$\sum_{n=0}^{n_0} \left| \Psi_{n,t}^{(\xi)} f(x) \right| \leq \varepsilon_2 \sum_{n=0}^{n_0} 2^n a^n \|f\|_\infty.$$

Using this and (94) we get the assertion of the lemma. $\square$

By Lemma 4.10 and Theorem 4.3, one easily obtains the following result.

Corollary 4.11 *Assume that $f \in \mathcal{B}_b(\mathbb{R}^d)$, for any $n \in \mathbb{N}$, $n \geq 1$ we have $f_n \in \mathcal{B}_b(\mathbb{R}^d)$, $\sup_{n \in \mathbb{N}, n \geq 1} \|f_n\|_\infty < \infty$ and $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ for almost all $x \in \mathbb{R}^d$ with respect to the Lebesgue measure. Then, for any $t > 0$, $x \in \mathbb{R}^d$, we have $\lim_{n \rightarrow \infty} T_t f_n(x) = T_t f(x)$.*

Lemma 4.12 (i) *For every $f \in C_0(\mathbb{R}^d)$ we have*

$$\lim_{t, \xi \rightarrow 0^+} \|T_t^{(\xi)} f - f\|_\infty = 0.$$

(ii) *For every $f \in C_0(\mathbb{R}^d)$ we have*

$$T_t^{(\xi)} f(x) \rightarrow 0, \quad \text{as } |x| \rightarrow \infty,$$

uniformly in $t \in [0, \tau]$, $\xi \in [0, 1]$.

(iii) *For every $f \in C_0(\mathbb{R}^d)$ we have*

$$\|T_t^{(\xi)} f - T_t f\|_\infty \rightarrow 0, \quad \text{as } \xi \rightarrow 0^+,$$

uniformly in $t \in [0, \tau]$.

Proof (i) This follows from Lemma 3.19 (ii) and Lemma 4.1.

(ii) Note that $T_0^{(\xi)} f = U_0^{(\xi)}$ so (ii) for $t = 0$ follows from Lemma 3.19. So we may assume that $t > 0$. Let $t \in (0, \tau]$, $\xi \in [0, 1]$ be arbitrary. By Lemma 4.2, we have

$$\left\| T_t^{(\xi)} f \right\|_\infty \leq c_1 \|f\|_\infty. \quad (100)$$

Choose $\varepsilon_1 > 0$. Since $f \in C_0(\mathbb{R}^2)$, there exists $r_1 > 0$ such that if $|x| \geq r_1$ then $|f(x)| \leq \varepsilon_1/(2c_1)$, where c_1 is a constant from (100). Put $f_1(x) = f(x)\mathbf{1}_{B(0, r_1)}(x)$, $f_2(x) = f(x)\mathbf{1}_{B^c(0, r_1)}(x)$. By Lemma 4.10, there exists $r_2 > r_1$ such that, if $|x| \geq r_2$, then $|T_t^{(\xi)} f_1(x)| \leq \varepsilon_1/2$. Hence for any $|x| \geq r_2$ we have $|T_t^{(\xi)} f(x)| \leq |T_t^{(\xi)} f_1(x) + T_t^{(\xi)} f_2(x)| \leq \varepsilon_1/2 + (\varepsilon_1/(2c_1))c_1 = \varepsilon_1$.

(iii) Let $\xi \in (0, 1]$, $n \in \mathbb{N}$, $n \geq 1$, $x \in \mathbb{R}^d$, $f \in C_0(\mathbb{R}^d)$. Note that for any $t \in (0, \tau]$ we have

$$\begin{aligned} \Psi_{n,t}^{(\xi)} f(x) &= \int_0^t \int_{\mathbb{R}^d} p_y(t-s+\xi, x-y) \mathcal{N}(\Psi_{n-1,s}^{(\xi)} f)(y) dy ds \\ &\quad + \int_0^t \int_0^{t-s} \int_{\mathbb{R}^d} p_z(t-s-r+\xi, x-z) Q_r(\mathcal{N}(\Psi_{n-1,s}^{(\xi)} f))(z) dz dr ds. \end{aligned} \quad (101)$$

By the same arguments as in the proof of Lemma 3.19 (iv), we obtain that $(\xi, t, x) \rightarrow \Psi_{n,t}^{(\xi)} f(x)$ is continuous on $[0, 1] \times [0, \tau] \times \mathbb{R}^d$ for any $n \in \mathbb{N}, n \geq 1$. Using this, (72) and Lemma 4.1 we obtain that $(\xi, t, x) \rightarrow T_t^{(\xi)} f(x)$ is continuous on $[0, 1] \times [0, \tau] \times \mathbb{R}^d$. This and (ii) implies (iii). $\square$

By the same arguments as in the proof of Lemma 3.19 (iv), we obtain the following result.

Lemma 4.13 *For any $f \in \mathcal{B}_b(\mathbb{R}^d)$, $n \in \mathbb{N}$, the function $(t, x) \rightarrow \Psi_{n,t} f(x)$ is continuous on $(0, \infty) \times \mathbb{R}^d$. For any $\xi \in (0, 1]$, $f \in \mathcal{B}_b(\mathbb{R}^d)$, $n \in \mathbb{N}$, the function $(t, x) \rightarrow \Psi_{n,t}^{(\xi)} f(x)$ is continuous on $[0, \infty) \times \mathbb{R}^d$.*

Lemma 4.14 *$\frac{\partial}{\partial t} (\Psi_{n,t}^{(\xi)} f)(x)$, $\mathcal{L} (\Psi_{n,t}^{(\xi)} f)(x)$ are well defined for any $t > 0$, $\xi \in (0, 1]$, $x \in \mathbb{R}^d$, $n \in \mathbb{N}, n \geq 1$ and $f \in C_0(\mathbb{R}^d)$ and we have*

$$\begin{aligned} \frac{\partial}{\partial t} (\Psi_{n,t}^{(\xi)} f)(x) - \mathcal{L} (\Psi_{n,t}^{(\xi)} f)(x) &= \int_{\mathbb{R}^d} p_z(\xi, x - z) \mathcal{N} (\Psi_{n-1,t}^{(\xi)} f)(z) dz \\ &\quad + \int_0^t \Lambda_{t-s}^{(\xi)} (\mathcal{N} (\Psi_{n-1,s}^{(\xi)} f))(x) ds. \end{aligned}$$

Moreover, $\frac{\partial}{\partial t} \Psi_{n,t}^{(\xi)} f(x)$ is continuous as a function of t for $t > 0$.

Proof Let $\xi \in (0, 1]$, $n \in \mathbb{N}, n \geq 1$, $x \in \mathbb{R}^d$, $f \in C_0(\mathbb{R}^d)$. Note that for any $t \in (0, \tau]$, $s \in (0, t)$, we have

$$\begin{aligned} U_{t-s}^{(\xi)} (\mathcal{N} (\Psi_{n-1,s}^{(\xi)} f))(x) &= \int_{\mathbb{R}^d} p_y(t - s + \xi, x - y) \mathcal{N} (\Psi_{n-1,s}^{(\xi)} f)(y) dy \\ &\quad + \int_0^{t-s} \int_{\mathbb{R}^d} p_z(t - s - r + \xi, x - z) \mathcal{Q}_r (\mathcal{N} (\Psi_{n-1,s}^{(\xi)} f))(z) dz dr. \end{aligned}$$

By similar arguments as in the proof of Lemma 3.19 (i), we obtain that $\frac{\partial}{\partial t} U_{t-s}^{(\xi)} (\mathcal{N} (\Psi_{n-1,s}^{(\xi)} f))(x)$ is well defined and continuous as a function of t for $t \in (s, \tau]$. Note that for any $g \in C_0(\mathbb{R}^d)$, and $t \geq 0$ we have $U_t^{(\xi)} g \in C_0(\mathbb{R}^d)$, $\mathcal{N} g \in C_0(\mathbb{R}^d)$. By (86), (69), Lemmas 3.20, 4.13, 3.19 (i) and standard arguments, we get

$$\begin{aligned} \frac{\partial}{\partial t} (\Psi_{n,t}^{(\xi)} f)(x) &= U_0^{(\xi)} (\mathcal{N} (\Psi_{n-1,t}^{(\xi)} f))(x) + \int_0^t \frac{\partial}{\partial t} U_{t-s}^{(\xi)} (\mathcal{N} (\Psi_{n-1,s}^{(\xi)} f))(x) ds \\ &= \int_{\mathbb{R}^d} p_z(\xi, x - z) \mathcal{N} (\Psi_{n-1,t}^{(\xi)} f)(z) dz + \int_0^t \Lambda_{t-s}^{(\xi)} (\mathcal{N} (\Psi_{n-1,s}^{(\xi)} f))(x) ds \\ &\quad + \int_0^t \mathcal{L}^x (U_{t-s}^{(\xi)} (\mathcal{N} (\Psi_{n-1,s}^{(\xi)} f)))(x) ds \end{aligned} \quad (102)$$

This implies that $\frac{\partial}{\partial t} \Psi_{n,t}^{(\xi)} f(x)$ is continuous as a function of t for $t \in (0, \tau]$.

For any $\zeta > 0$ we have

$$\mathcal{L}_\zeta (\Psi_{n,t}^{(\xi)} f)(x) = \int_0^t \mathcal{L}_\zeta^x U_{t-s}^{(\xi)} (\mathcal{N} (\Psi_{n-1,s}^{(\xi)} f))(x) ds.$$

By the dominated convergence theorem and (74), we obtain

$$\mathcal{L}\left(\Psi_{n,t}^{(\xi)} f\right)(x) = \lim_{\zeta \rightarrow 0^+} \mathcal{L}_{\zeta}\left(\Psi_{n,t}^{(\xi)} f\right)(x) = \int_0^t \mathcal{L}^x U_{t-s}^{(\xi)}\left(\mathcal{N}\left(\Psi_{n-1,s}^{(\xi)} f\right)\right)(x) ds.$$

This and (102) gives the assertion of the lemma. $\square$

Lemma 4.15 For any $t > 0$, $\xi \in (0, 1]$, $x \in \mathbb{R}^d$, $i, j \in \{1, \dots, d\}$, $k \in \mathbb{N}$ and $f \in C_0(\mathbb{R}^d)$ we have $\Psi_{k,t}^{(\xi)} f(x) \in C^2(\mathbb{R}^d)$ (as a function of x) and

$$\frac{\partial}{\partial t} \left(\sum_{n=0}^{\infty} \Psi_{n,t}^{(\xi)} f \right)(x) = \sum_{n=0}^{\infty} \frac{\partial}{\partial t} \left(\Psi_{n,t}^{(\xi)} f \right)(x), \quad (103)$$

$$\frac{\partial}{\partial x_i} \left(\sum_{n=0}^{\infty} \Psi_{n,t}^{(\xi)} f \right)(x) = \sum_{n=0}^{\infty} \frac{\partial}{\partial x_i} \left(\Psi_{n,t}^{(\xi)} f \right)(x), \quad (104)$$

$$\frac{\partial^2}{\partial x_i \partial x_j} \left(\sum_{n=0}^{\infty} \Psi_{n,t}^{(\xi)} f \right)(x) = \sum_{n=0}^{\infty} \frac{\partial^2}{\partial x_i \partial x_j} \left(\Psi_{n,t}^{(\xi)} f \right)(x), \quad (105)$$

$$\mathcal{L} \left(\sum_{n=0}^{\infty} \Psi_{n,t}^{(\xi)} f \right)(x) = \sum_{n=0}^{\infty} \mathcal{L} \left(\Psi_{n,t}^{(\xi)} f \right)(x), \quad (106)$$

$$\mathcal{N} \left(\sum_{n=0}^{\infty} \Psi_{n,t}^{(\xi)} f \right)(x) = \sum_{n=0}^{\infty} \mathcal{N} \left(\Psi_{n,t}^{(\xi)} f \right)(x). \quad (107)$$

Proof Fix $f \in C_0(\mathbb{R}^d)$, $\xi \in (0, 1]$. By Lemma 4.14, we know that $t \rightarrow \frac{\partial}{\partial t} (\Psi_{n,t}^{(\xi)} f)(x)$ is continuous on $(0, \tau]$ for each fixed $n \in \mathbb{N}$, $x \in \mathbb{R}^d$. Using this, Lemma 4.1, (102) and (69) we get (103).

Let $t \in (0, \tau]$, $n \in \mathbb{N}$, $i, j \in \{1, \dots, d\}$. The fact that $\frac{\partial}{\partial x_i} (\Psi_{n,t}^{(\xi)} f)(x)$ is well defined and continuous as a function of $x \in \mathbb{R}^d$ follows from (101), Lemmas 3.4, 3.5, Proposition 3.12 and Lemma 3.19. By the above arguments, we also get

$$\left| \frac{\partial}{\partial x_i} (\Psi_{n,t}^{(\xi)} f)(x) \right| \leq c(\xi) \sup_{s \in (0, t]} \left\| \Psi_{n-1,s}^{(\xi)} f \right\|_{\infty}, \quad n \geq 1.$$

Using this, Lemma 3.19 and Lemma 4.1 we arrive at (104). By similar arguments we obtain that $\frac{\partial^2}{\partial x_i \partial x_j} (\Psi_{n,t}^{(\xi)} f)(x)$ is well defined and continuous as a function of $x \in \mathbb{R}^d$ and

$$\left| \frac{\partial^2}{\partial x_i \partial x_j} (\Psi_{n,t}^{(\xi)} f)(x) \right| \leq c(\xi) \sup_{s \in (0, t]} \left\| \Psi_{n-1,s}^{(\xi)} f \right\|_{\infty}, \quad n \geq 1. \quad (108)$$

Using this and Lemma 4.1 we get (105).

For any $\zeta > 0$ we have

$$\mathcal{L}_{\zeta} \left(\sum_{n=0}^{\infty} \Psi_{n,t}^{(\xi)} f \right)(x) = \sum_{n=0}^{\infty} \mathcal{L}_{\zeta} \left(\Psi_{n,t}^{(\xi)} f \right)(x).$$

This, Lemma 4.1, (74) and (75) imply (106). The equality (107) is clear. $\square$

Corollary 4.16 For every $f \in C_0(\mathbb{R}^d)$, $\xi \in (0, 1]$ the function $T_t^{(\xi)} f(x)$ belongs to $C^1((0, \infty))$ as a function of t and to $C_0^2(\mathbb{R}^d)$ as a function of x .

Proof Fix $f \in C_0(\mathbb{R}^d)$, $\xi \in (0, 1]$. The fact that $T_t^{(\xi)} f(x)$ belongs to $C^1((0, \infty))$ as a function of t follows from (103), Lemma 4.14, (102), (69), Lemma 3.16 and Lemma 4.1. From the proof of Lemma 4.15 we know that for each $t > 0$, $n \in \mathbb{N}$, $i, j \in \{1, \dots, d\}$ the function $\frac{\partial^2}{\partial x_i \partial x_j} \left(\Psi_{n,t}^{(\xi)} f \right) (x)$ is continuous as a function of $x \in \mathbb{R}^d$. The fact that $T_t^{(\xi)} f(x)$ belongs to $C_0^2(\mathbb{R}^d)$ as a function of x follows from (104), (105), (108), Lemmas 4.1 and 4.10. $\square$

Heuristically, now our aim is to show that if ξ is small then $\frac{\partial}{\partial t} (T_t^{(\xi)} f)(x) - \mathcal{K}(T_t^{(\xi)} f)(x)$ is small. For any $t > 0$, $\xi \in (0, 1]$, $x \in \mathbb{R}^d$ and $f \in C_0(\mathbb{R}^d)$ let us denote

$$\begin{aligned}\Upsilon_t^{(\xi)} f(x) &= \frac{\partial}{\partial t} \left(T_t^{(\xi)} f \right) (x) - \mathcal{K} \left(T_t^{(\xi)} f \right) (x), \\ \Upsilon_t^{(\xi,1)} f(x) &= e^{-\lambda t} \sum_{n=1}^{\infty} \left[\int_{\mathbb{R}^d} p_z(\xi, x-z) \mathcal{N} \left(\Psi_{n-1,t}^{(\xi)} f \right) (z) dz - \mathcal{N} \left(\Psi_{n-1,t}^{(\xi)} f \right) (x) \right], \\ \Upsilon_t^{(\xi,2)} f(x) &= e^{-\lambda t} \sum_{n=1}^{\infty} \int_0^t \Lambda_{t-s}^{(\xi)} \left(\mathcal{N} \left(\Psi_{n-1,s}^{(\xi)} f \right) \right) (x) ds.\end{aligned}$$

By Lemma 4.1, (80), (81) and the boundedness of $\mathcal{N} : L^\infty(\mathbb{R}^d) \rightarrow L^\infty(\mathbb{R}^d)$, the above series are convergent.

Lemma 4.17 For any $t > 0$, $\xi \in (0, 1]$, $x \in \mathbb{R}^d$ and $f \in C_0(\mathbb{R}^d)$ we have

$$\Upsilon_t^{(\xi)} f(x) = e^{-\lambda t} \Lambda_t^{(\xi)} f(x) + \Upsilon_t^{(\xi,1)} f(x) + \Upsilon_t^{(\xi,2)} f(x).$$

Proof By Lemmas 3.21, 4.14 and 4.15, we get

$$\begin{aligned}\frac{\partial}{\partial t} \left(T_t^{(\xi)} f \right) (x) &= -\lambda e^{-\lambda t} \Psi_{0,t}^{(\xi)} f(x) + e^{-\lambda t} \mathcal{L} \left(\Psi_{0,t}^{(\xi)} f \right) (x) + e^{-\lambda t} \Lambda_t^{(\xi)} f(x) \\ &\quad - \lambda e^{-\lambda t} \sum_{n=1}^{\infty} \Psi_{n,t}^{(\xi)} f(x) + e^{-\lambda t} \sum_{n=1}^{\infty} \mathcal{L} \left(\Psi_{n,t}^{(\xi)} f \right) (x) \\ &\quad + e^{-\lambda t} \sum_{n=1}^{\infty} \mathcal{N} \left(\Psi_{n-1,t}^{(\xi)} f \right) (x) + \Upsilon_t^{(\xi,1)} f(x) + \Upsilon_t^{(\xi,2)} f(x).\end{aligned}$$

Again, by Lemma 4.15, this is equal to

$$\begin{aligned}\mathcal{L} \left(e^{-\lambda t} \sum_{n=0}^{\infty} \Psi_{n,t}^{(\xi)} f \right) (x) - \lambda \left(e^{-\lambda t} \sum_{n=0}^{\infty} \Psi_{n,t}^{(\xi)} f \right) (x) \\ + \mathcal{N} \left(e^{-\lambda t} \sum_{n=0}^{\infty} \Psi_{n,t}^{(\xi)} f \right) (x) + e^{-\lambda t} \Lambda_t^{(\xi)} f(x) + \Upsilon_t^{(\xi,1)} f(x) + \Upsilon_t^{(\xi,2)} f(x).\end{aligned}$$

Using the definition of $T_t^{(\xi)} f$ and noting that $\mathcal{N}g(x) - \lambda g(x) = \mathcal{R}g(x)$ and $\mathcal{L}g(x) + \mathcal{R}g(x) = \mathcal{K}g(x)$ we obtain the assertion of the lemma. $\square$

Lemma 4.18 (i) For any $f \in C_0(\mathbb{R}^d)$ we have

$$\Upsilon_t^{(\xi)} f(x) \rightarrow 0, \quad \text{as } \xi \rightarrow 0^+,$$

uniformly in $(t, x) \in [\tau_1, \tau] \times \mathbb{R}^d$ for every $\tau_1 \in (0, \tau)$.

(ii) For any $f \in C_0(\mathbb{R}^d)$ we have

$$\int_0^t \Upsilon_s^{(\xi)} f(x) ds \rightarrow 0, \quad \text{as } \xi \rightarrow 0^+, \quad (109)$$

uniformly in $(t, x) \in (0, \tau] \times \mathbb{R}^d$.

Proof The lemma follows from Lemma 3.22, Proposition 3.10, Lemmas 4.10, 4.1, (80), (81) and the boundedness of $\mathcal{N} : L^\infty(\mathbb{R}^d) \rightarrow L^\infty(\mathbb{R}^d)$. $\square$

The next result (positive maximum principle) is based on the ideas from [19, Section 4.2]. Its proof is very similar to the proof of [19, Lemma 4.3] and it is omitted.

Lemma 4.19 Let us consider the function $v : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$ and the family of functions $v^{(\xi)} : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$, $\xi \in (0, 1]$. Assume that for each $\xi \in (0, 1]$ $\sup_{t \in (0, \tau], x \in \mathbb{R}^d} |v^{(\xi)}(x, t)| < \infty$, $v^{(\xi)}$ is C^1 in the first variable and C^2 in the second variable. We also assume that (for any $\tau > 0$)

(i)

$$v^{(\xi)}(t, x) \rightarrow v(t, x) \quad \text{as } \xi \rightarrow 0^+,$$

uniformly in $t \in [0, \tau]$, $x \in \mathbb{R}^d$;

(ii)

$$v^{(\xi)}(t, x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty,$$

uniformly in $t \in [0, \tau]$, $\xi \in (0, 1]$;

(iii) for any $0 < \tau_1 < \tau_2 \leq \tau$

$$\frac{\partial}{\partial t} v^{(\xi)}(t, x) - \mathcal{K}v^{(\xi)}(t, x) \rightarrow 0 \quad \text{as } \xi \rightarrow 0^+,$$

uniformly in $t \in [\tau_1, \tau_2]$, $x \in \mathbb{R}^d$;

(iv)

$$v^{(\xi)}(t, x) \rightarrow v(0, x) \quad \text{as } (\xi \rightarrow 0^+ \text{ and } t \rightarrow 0^+),$$

uniformly in $x \in \mathbb{R}^d$;

(v) for any $x \in \mathbb{R}^d$ $v(0, x) \geq 0$.

Then for any $t \geq 0$, $x \in \mathbb{R}^d$ we have $v(t, x) \geq 0$.

Proposition 4.20 $T_t : \mathcal{B}_b(\mathbb{R}^d) \rightarrow \mathcal{B}_b(\mathbb{R}^d)$ is a linear, bounded operator for any $t \in (0, \tau]$. For each $t \in (0, \tau]$, $f \in \mathcal{B}_b(\mathbb{R}^d)$ and $R \geq 1$ there exists a sequence $f_k \in C_0(\mathbb{R}^d)$, $k \in \mathbb{N}$ such that $\lim_{k \rightarrow \infty} f_k(x) = f(x)$ for almost all $x \in B(0, R)$; for any $k \in \mathbb{N}$ we have $\|f_k\|_\infty \leq \|f\|_\infty$ and for any $x \in B(0, R)$ we have $\lim_{k \rightarrow \infty} T_t f_k(x) = T_t f(x)$.

Proof Fix $t \in (0, \tau]$. The fact that $T_t : \mathcal{B}_b(\mathbb{R}^d) \rightarrow \mathcal{B}_b(\mathbb{R}^d)$ is a linear, bounded operator follows by the definition of T_t and Lemma 4.1.

Fix $f \in \mathcal{B}_b(\mathbb{R}^d)$, $R \geq 1$ and $k \in \mathbb{N}$, $k \geq 1$. By Lemma 4.10, there exists $R_k \geq R$ such that for any $x \in B(0, R)$ we have

$$|T_t(f \mathbf{1}_{B^c(0, R_k)})(x)| \leq \frac{1}{k}. \quad (110)$$

Put $g_{1,k}(x) = \mathbf{1}_{B(0,R_k)}(x)f(x)$, $g_{2,k}(x) = \mathbf{1}_{B^c(0,R_k)}(x)f(x)$. By standard methods, there exists $f_k \in C_0(\mathbb{R}^d)$ such that

$$\|f_k - g_{1,k}\|_1 \leq \frac{1}{k}.$$

and $\text{supp}(f_k) \subset B(0, R_k + 1)$, $\|f_k\|_\infty \leq \|f\|_\infty$. By Theorem 4.3, for any $x \in \mathbb{R}^d$, we have

$$|T_t(f_k - g_{1,k})(x)| \leq \frac{c\|f\|_\infty^{1-\alpha/(2d)}}{k^{\alpha/(2d)}t^{1/2}}.$$

This and (110) imply that for any $x \in B(0, R)$ we have $\lim_{k \rightarrow \infty} T_t f_k(x) = T_t f(x)$. We also have $\|f_k \mathbf{1}_{B(0,R)} - f \mathbf{1}_{B(0,R)}\|_1 \leq 1/k$. Hence, there exists a subsequence k_m such that $\lim_{m \rightarrow \infty} f_{k_m}(x) = f(x)$ for almost all $x \in B(0, R)$. $\square$

Proposition 4.21 For any $t \in (0, \infty)$, $x \in \mathbb{R}^d$ and $f \in C_b^2(\mathbb{R}^d)$ we have

$$T_t f(x) = f(x) + \int_0^t T_s(\mathcal{K}f)(x) ds. \quad (111)$$

Proof Step 1. $f \in C_0^2(\mathbb{R}^d)$.

For any $t \geq 0$, $x \in \mathbb{R}^d$, $\xi \in (0, 1]$ put

$$\begin{aligned} v(t, x) &= T_t f(x) - f(x) - \int_0^t T_s(\mathcal{K}f)(x) ds, \\ v^{(\xi)}(t, x) &= T_t^{(\xi)} f(x) - f(x) - \int_0^t T_s^{(\xi)}(\mathcal{K}f)(x) ds. \end{aligned}$$

Note that $\mathcal{K}f \in C_0(\mathbb{R}^d)$. By Lemmas 4.12, 4.18 and Corollary 4.16, we obtain that $v(t, x)$, $v^{(\xi)}(t, x)$ satisfy the assumptions of Lemma 4.19. Note that $v(0, x) = 0$ for all $x \in \mathbb{R}^d$. The assertion of the proposition for $f \in C_0^2(\mathbb{R}^d)$ follows from Lemma 4.19.

Step 2. $f \in C_b^2(\mathbb{R}^d)$.

By standard methods, there exists a sequence $f_n \in C_0^2(\mathbb{R}^d)$, $n = 1, 2, \dots$ such that

$$\sup_{n \in \mathbb{N}} \max_{i,j \in \{1, \dots, d\}} \sup_{x \in \mathbb{R}^d} \left(|f_n(x)| + \left| \frac{\partial f_n}{\partial x_i}(x) \right| + \left| \frac{\partial^2 f_n}{\partial x_i \partial x_j}(x) \right| \right) < \infty.$$

and for any $r \geq 1$ we have $\lim_{n \rightarrow \infty} (\sup_{|x| \leq r} |f_n(x) - f(x)|) = 0$. It follows that $\sup_{n \geq 1, x \in \mathbb{R}^d} |\mathcal{K}f_n(x)| < \infty$ and for each $x \in \mathbb{R}^d$ we have $\mathcal{K}f_n(x) \rightarrow \mathcal{K}f(x)$. By Corollary 4.11, it follows that for each $x \in \mathbb{R}^d$, $t > 0$ and $s \in (0, t]$ we have $T_t f_n(x) \rightarrow T_t f(x)$ and $T_s(\mathcal{K}f_n)(x) \rightarrow T_s(\mathcal{K}f)(x)$. By Step 1 and the dominated convergence theorem, we obtain the assertion of the proposition. $\square$

The following result shows that $\{T_t\}$ is a Feller semigroup.

Theorem 4.22 We have

- (i) $T_t : C_0(\mathbb{R}^d) \rightarrow C_0(\mathbb{R}^d)$ for any $t \in (0, \infty)$,
- (ii) $T_t f(x) \geq 0$ for any $t > 0$, $x \in \mathbb{R}^d$ and $f \in C_0(\mathbb{R}^d)$ such that $f(x) \geq 0$ for all $x \in \mathbb{R}^d$,
- (iii) $T_t \mathbf{1}_{\mathbb{R}^d}(x) = 1$ for any $t > 0$, $x \in \mathbb{R}^d$,
- (iv) $T_{t+s} f(x) = T_t(T_s f)(x)$ for any $s, t > 0$, $x \in \mathbb{R}^d$, $f \in C_0(\mathbb{R}^d)$,
- (v) $\lim_{t \rightarrow 0^+} \|T_t f - f\|_\infty = 0$ for any $f \in C_0(\mathbb{R}^d)$.

(vi) there exists a nonnegative function $p(t, x, y)$ in $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$; for each fixed $t > 0$, $x \in \mathbb{R}^d$ the function $y \rightarrow p(t, x, y)$ is Lebesgue measurable, $\int_{\mathbb{R}^d} p(t, x, y) dy = 1$ and $T_t f(x) = \int_{\mathbb{R}^d} p(t, x, y) f(y) dy$ for $f \in C_0(\mathbb{R}^d)$.

Proof (i) This follows from Theorem 4.4 and Lemma 4.12 (ii).

(ii) Let $f \in C_0(\mathbb{R}^d)$ be such that $f(x) \geq 0$ for all $x \in \mathbb{R}^d$. For $t \geq 0$, $x \in \mathbb{R}^d$, $\xi \in (0, 1]$ put $v(t, x) = T_t f(x)$, $v^{(\xi)}(t, x) = T_t^{(\xi)} f(x)$. By Lemmas 4.12, 4.18 and Corollary 4.16, we obtain that $v(t, x)$, $v^{(\xi)}(t, x)$ satisfy the assumptions of Lemma 4.19. The assertion of Theorem 4.22 (ii) follows from Lemma 4.19.

(iii) The proof is very similar to the proof of [19, Lemma 4.5 b]. Let $f \in C_0^2(\mathbb{R}^2)$ be such that $f \equiv 1$ on $B(0, 1) \subset \mathbb{R}^d$ and let $f_n(x) = f(x/n)$, $x \in \mathbb{R}^d$, $n \in \mathbb{N}$, $n \geq 1$. For any $x \in \mathbb{R}^d$ we have $\lim_{n \rightarrow \infty} f_n(x) = 1$, $\lim_{n \rightarrow \infty} \mathcal{K} f_n(x) = 0$ and $\sup_{n \in \mathbb{N}, n \geq 1} (\|f_n\|_\infty \vee \|\mathcal{K} f_n\|_\infty) < \infty$. By Corollary 4.11, for any $s, t > 0$ and $x \in \mathbb{R}^d$, we get

$$\lim_{n \rightarrow \infty} T_t f_n(x) = T_t \mathbf{1}_{\mathbb{R}^d}(x), \quad \lim_{n \rightarrow \infty} T_s (\mathcal{K} f_n)(x) = 0. \quad (112)$$

Using (111) for f_n and (112) we obtain (iii).

(iv) Let $f \in C_0(\mathbb{R}^d)$. For $s, t \geq 0$, $x \in \mathbb{R}^d$, $\xi \in (0, 1]$ put $v(t, x) = T_{t+s} f(x) - T_t (T_s f)(x)$, $v^{(\xi)}(t, x) = T_{t+s}^{(\xi)} f(x) - T_t^{(\xi)} (T_s f)(x)$. By Lemmas 4.12, 4.18 and Corollary 4.16, we obtain that $v(t, x)$, $v^{(\xi)}(t, x)$ satisfy the assumptions of Lemma 4.19. Note that $v(0, x) = 0$ for all $x \in \mathbb{R}^d$. The assertion of Theorem 4.22 (iv) follows from Lemma 4.19.

(v) Choose $\varepsilon_1 > 0$. Since $f \in C_0(\mathbb{R}^d)$ there exists $\delta_1 > 0$ such that

$$\forall x, y \in \mathbb{R}^d \quad |x - y| < \delta_1 \Rightarrow |f(x) - f(y)| < \varepsilon_1.$$

Fix arbitrary $x \in \mathbb{R}^d$, $t \in (0, \tau]$. Put $f_1(y) = \mathbf{1}_{B(x, \delta_1)}(y)(f(y) - f(x))$, $f_2(y) = \mathbf{1}_{B^c(x, \delta_1)}(y)(f(y) - f(x))$, $y \in \mathbb{R}^d$. By (iii), we have

$$T_t f(x) - f(x) = T_t f_1(x) + T_t f_2(x).$$

We also have

$$|T_t f_1(x)| < c\varepsilon_1,$$

$$|T_t f_2(x)| \leq 2\|f\|_\infty T_t \mathbf{1}_{B^c(x, \delta_1)}(x)$$

and

$$\begin{aligned} T_t \mathbf{1}_{B^c(x, \delta_1)}(x) &= e^{-\lambda t} \int_{B^c(x, \delta_1)} p_y(t, x - y) dy + e^{-\lambda t} \Phi_t \mathbf{1}_{B^c(x, \delta_1)}(x) \\ &\quad + e^{-\lambda t} \sum_{n=1}^{\infty} \Psi_{n,t} \mathbf{1}_{B^c(x, \delta_1)}(x). \end{aligned}$$

By Proposition 3.10, there exists $\tau_1 \in (0, \tau]$ such that

$$\forall t \in (0, \tau_1] \quad \int_{B^c(x, \delta_1)} p_y(t, x - y) dy < \varepsilon_1.$$

By Proposition 3.12, we obtain that

$$\forall t \in (0, \tau_1] \quad |\Phi_t \mathbf{1}_{B^c(x, \delta_1)}(x)| \leq c\tau_1^{1/2}.$$

By Lemma 4.1, we obtain that

$$\forall t \in (0, \tau_1] \quad \left| e^{-\lambda t} \sum_{n=1}^{\infty} \Psi_{n,t} \mathbf{1}_{B^c(x, \delta_1)}(x) \right| \leq ct.$$

This implies (v).

(vi) This follows from (i), (ii), (iii) and Theorem 4.3. $\square$

We are now in a position to provide the proof of Theorems 1.1 and 1.2.

Proof of Theorem 1.1 From Theorem 4.22 we conclude that there is a Feller process $\tilde{X}_t$ with the semigroup T_t on $C_0(\mathbb{R}^d)$. Let $\tilde{\mathbb{P}}^x, \tilde{\mathbb{E}}^x$ be the distribution and expectation of the process $\tilde{X}_t$ starting from $x \in \mathbb{R}^d$.

By Theorem 4.22 (vi), Proposition 4.20 and Lemma 4.10, we get

$$\tilde{\mathbb{E}}^x f(\tilde{X}_t) = T_t f(x) = \int_{\mathbb{R}^d} p(t, x, y) f(y) dy \quad f \in \mathcal{B}_b(\mathbb{R}^d), t > 0, x \in \mathbb{R}^d. \quad (113)$$

By Proposition 4.21, for any function $f \in C_b^2(\mathbb{R}^d)$, the process

$$M_t^{\tilde{X}, f} = f(\tilde{X}_t) - f(\tilde{X}_0) - \int_0^t \mathcal{K} f(\tilde{X}_s) ds$$

is a $(\tilde{\mathbb{P}}^x, \mathcal{F}_t)$ martingale, where $\mathcal{F}_t$ is a natural filtration. That is $\tilde{\mathbb{P}}^x$ solves the martingale problem for $(\mathcal{K}, C_b^2(\mathbb{R}^d))$. On the other hand, according to [1, Theorem 6.3], the unique solution X to the stochastic equation (1) has the law which is the unique solution to the martingale problem for $(\mathcal{K}, C_b^2(\mathbb{R}^d))$. Hence $\tilde{X}$ and X have the same law so for any $t > 0$, $x \in \mathbb{R}^d$ and any Borel bounded set $A \subset \mathbb{R}^d$ we have

$$\sigma_t(x, A) = \int_A p(t, x, y) dy,$$

where $\sigma_t(x, A)$ is defined by (14). Using this, (15) and (113) we obtain

$$P_t f(x) = T_t f(x), \quad t > 0, x \in \mathbb{R}^d, f \in \mathcal{B}_b(\mathbb{R}^d). \quad (114)$$

Now the assertion of Theorem 1.1 follows from Theorem 4.4 and (114). $\square$

Proof of Theorem 1.2 The result follows from Theorem 4.3 and (114). $\square$

Remark 4.23 One of the referees suggested the following elegant approach to the strong Feller property, which combines the Feller property with the results obtained in [10]. By the the main theorem of [10] or rather its proof, the transition density $p(t, x, \cdot)$ of the solution to the equation (1) exists for every x and it belongs to the classical Besov space $B_{1,\infty}^s(\mathbb{R}^d)$ for some $0 < s < 1$, and

$$\sup_{x \in \mathbb{R}^d} \|p(t, x, \cdot)\|_{B_{1,\infty}^s} < \infty,$$

where $\|\cdot\|_{B_{1,\infty}^s}$ is a Besov norm in $B_{1,\infty}^s(\mathbb{R}^d)$. There is a classical but very deep result in the theory of function spaces which states that the Besov space $B_{1,\infty}^s(\mathbb{R}^d)$ is compactly embedded into $L^1(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is an arbitrary nonempty bounded open set. This means that for any sequence $f_n \in B_{1,\infty}^s(\mathbb{R}^d)$ such that $\sup_n \|f_n\|_{B_{1,\infty}^s} < \infty$ the sequence $f_n \mathbf{1}_\Omega$ is relatively compact in $L^1(\Omega)$.

Let $x_n, n \geq 1$, be such that $\lim_{n \rightarrow \infty} x_n = x_0$, f be a bounded Borel function and $\epsilon > 0$. Since $A(\cdot)$ is Lipschitz continuous, the semigroup determined by the solution of to the

equation (1) has the Feller property, see eg. [33]. This implies that the sequence of measures $m_n(dy) = p(t, x_n, y)dy$ is weakly convergent to $m_0(dy) = p(t, x_0, y)dy$. Hence the set of the truncated measures $\mathbf{1}_\Omega m_n(dy)$ is weakly convergent to $\mathbf{1}_\Omega m_0(dy)$ provided that $\partial\Omega$ has the Lebesgue measure 0. By compact embedding any subsequence $\{x_{k_n}\}$ has a subsequence $\{x_{l_n}\}$ such that $p(t, x_{l_n}, \cdot)\mathbf{1}_\Omega$ is convergent to $q(\cdot)$ in $L^1(\Omega)$. Due to the weak convergence of the truncated measures we have that $q(\cdot) = p(t, x_0, \cdot)$ in $L^1(\Omega)$. This implies

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \left| \int f(y) p_t(x_n, y) dy - \int f(y) p_t(x_0, y) dy \right| \\ & \leq \limsup_{n \rightarrow \infty} \|f\|_\infty \left(\|p_t(x_n, \cdot) - p_t(x_0, \cdot)\|_{L^1(\Omega)} + 2 \sup_{n \geq 0} m_n(\Omega^c) \right) \\ & = \limsup_{n \rightarrow \infty} 2\|f\|_\infty \sup_{n \geq 0} m_n(\Omega^c) \\ & \leq \epsilon, \end{aligned}$$

provided Ω is large enough. Indeed, $\sup_{n \geq 0} m_n(\Omega^c)$ can be made arbitrarily small by the tightness of $\{m_n; n \geq 0\}$, which follows from some moment estimates on X_t . This shows that the above limit must be 0. It is worth noticing that this argument works in our setup for all $0 < \alpha < 2$. On the other hand we do not think that from the results of [10] one can obtain our main estimate (6).

Remark 4.24 For any $\alpha \in (0, 1)$, $d \geq 2$ there exist $A(x)$ satisfying (2-4) and $t > 0$ such that $P_t : L^1(\mathbb{R}^d) \rightarrow L^\infty(\mathbb{R}^d)$ is not bounded. For simplicity we will present an example for $d = 2$ but similar examples can be constructed for $d > 2$.

Proof First we define $A(x_1, x_2)$. Let $\kappa(r) : [0, \infty) \rightarrow [0, \infty)$ be defined by $\kappa(r) = 0$ for $r \in [0, 1]$, $\kappa(r) = r - 1$ for $r \in (1, 1 + \pi/4]$, $\kappa(r) = \pi/4$ for $r > 1 + \pi/4$. It is easy to check that $\kappa(r) = ((r - 1) \vee 0) \wedge (\pi/4)$ and that it is a Lipschitz function. Now let us introduce standard polar coordinates (r, φ) , $r \in [0, \infty)$, $\varphi \in [0, 2\pi)$ by $x_1 = r \cos \varphi$, $x_2 = r \sin \varphi$.

We put $A(x_1, x_2) = \tilde{A}(r, \varphi) = \begin{bmatrix} \cos(\tilde{\theta}(r, \varphi)) & -\sin(\tilde{\theta}(r, \varphi)) \\ \sin(\tilde{\theta}(r, \varphi)) & \cos(\tilde{\theta}(r, \varphi)) \end{bmatrix}$, where $\tilde{\theta}(r, \varphi)$ is defined in the following way. $\tilde{\theta}(r, \varphi) = 0$ for $r \in [0, 1]$, $\varphi \in [0, 2\pi)$, $\tilde{\theta}(r, \varphi) = \varphi$ for $r \in (1, 1 + \pi/4]$, $\varphi \in [0, \kappa(r)]$, $\tilde{\theta}(r, \varphi) = 2\kappa(r) - \varphi$ for $r \in (1, 1 + \pi/4]$, $\varphi \in (\kappa(r), 2\kappa(r)]$, $\tilde{\theta}(r, \varphi) = 0$ for $r \in (1, 1 + \pi/4]$, $\varphi \in [2\kappa(r), 2\pi)$, $\tilde{\theta}(r, \varphi) = \varphi$ for $r > 1 + \pi/4$, $\varphi \in [0, \pi/4]$, $\tilde{\theta}(r, \varphi) = \pi/2 - \varphi$ for $r > 1 + \pi/4$, $\varphi \in (\pi/4, \pi/2]$, $\tilde{\theta}(r, \varphi) = 0$ for $r > 1 + \pi/4$, $\varphi \in (\pi/2, 2\pi)$.

One can check that $A(x_1, x_2) = \begin{bmatrix} \cos(\theta(x_1, x_2)) & -\sin(\theta(x_1, x_2)) \\ \sin(\theta(x_1, x_2)) & \cos(\theta(x_1, x_2)) \end{bmatrix}$, where $\theta(0, 0) = 0$ and for $(x_1, x_2) \neq (0, 0)$

$$\theta(x_1, x_2) = \left(\kappa \left(\sqrt{x_1^2 + x_2^2} \right) - \left| \left((\text{Arg}(x_1 + ix_2) \vee 0) \wedge \frac{\pi}{2} \right) - \kappa \left(\sqrt{x_1^2 + x_2^2} \right) \right| \right) \vee 0.$$

It is clear that $A(x)$ satisfies (2-4).

Put $D = B((3, 1), 1)$. Note that for any $x \in \mathbb{R}^2$ such that $x_2 \in [0, x_1]$ and $|x| \geq \pi/4 + 1$ we have $A(x) = A(x_1, x_2) = |x|^{-1} \begin{bmatrix} x_1 & -x_2 \\ x_2 & x_1 \end{bmatrix}$. In particular, this holds for $x \in D$.

For any $f \in \mathcal{B}_b(\mathbb{R}^d)$, $t > 0$, $x \in \mathbb{R}^d$ we have $P_t f(x) = T_t f(x) = e^{-\lambda t} \sum_{n=0}^{\infty} \Psi_{n,t} f(x)$. For our purposes it is enough to study $\Psi_{1,t}$. We have

$$\begin{aligned} \Psi_{1,t} f(x) &= \int_0^t U_{t-s}(\mathcal{N}(U_s f))(x) ds = \\ &= \sum_{i=1}^2 \int_0^t \int_{\mathbb{R}^2} u(t-s, x, z) \int_{\mathbb{R}} \int_{\mathbb{R}^2} u(s, z + a_i(z)w, y) f(y) dy v(w) dw dz ds. \end{aligned} \quad (115)$$

By arguments similar to the proof of Theorem 4.22, one can show that for any $t > 0$, $x \in \mathbb{R}^d$ and almost all $y \in \mathbb{R}^d$ we have $u(t, x, y) \geq 0$ and for any $s, t > 0$, $x \in \mathbb{R}^d$, $f \in \mathcal{B}_b(\mathbb{R}^d)$ we have $U_{t+s} f(x) = U_t(U_s f)(x)$, (we omit the details here). Put

$$u_1(t, x, y) = \sum_{i=1}^2 \int_0^t \int_{\mathbb{R}^2} u(t-s, x, z) \int_{\mathbb{R}} u(s, z + a_i(z)w, y) v(w) dw dz ds. \quad (116)$$

By (115), we have

$$\Psi_{1,t} f(x) = \int_{\mathbb{R}^2} u_1(t, x, y) f(y) dy.$$

By Lemma 3.23 and the semigroup property of U_t , one can easily obtain that there exists $t_2 > 1$ such that for any $t \in [t_2 - 1, t_2]$,

$$\int_D u(t, 0, z) dz \geq c. \quad (117)$$

Now our aim will be to estimate from below $u_1(t_2, 0, y)$ for y which are sufficiently close to 0. Let c_1, ε_1, t_1 be the constants from Lemma 3.23. First, note that for $z \in D$ we have $a_1(z) = (a_{11}(z), a_{21}(z)) = |z|^{-1}(z_1, z_2) = z/|z|$. Hence $|z + a_1(z)w| = |z| + w$ for any $z \in D$, $w \in \mathbb{R}$. It follows that for $z \in D$ we have

$$w \in (-|z| - \varepsilon_1 s^{1/\alpha}/2, -|z| + \varepsilon_1 s^{1/\alpha}/2) \iff |z + a_1(z)w| < \varepsilon_1 s^{1/\alpha}/2.$$

Therefore, for $z \in D$, $|y| \leq \varepsilon_1 s^{1/\alpha}/2$, $w \in (-|z| - \varepsilon_1 s^{1/\alpha}/2, -|z| + \varepsilon_1 s^{1/\alpha}/2)$, we have $|z + a_i(z)w - y| \leq |z + a_i(z)w| + |y| \leq \varepsilon_1 s^{1/\alpha}$. Note also that

$$|y| \leq \varepsilon_1 s^{1/\alpha}/2 \iff (2|y|/\varepsilon_1)^\alpha \leq s.$$

Hence, by Lemma 3.23, for $(2|y|/\varepsilon_1)^\alpha \leq t_1/2$, $s \in [(2|y|/\varepsilon_1)^\alpha, t_1]$, $z \in D$ we have

$$\int_{-|z| - \varepsilon_1 s^{1/\alpha}/2}^{-|z| + \varepsilon_1 s^{1/\alpha}/2} u(s, z + a_1(z)w, y) v(w) dw \geq c \varepsilon_1 s^{1/\alpha} \frac{c_1}{s^{2/\alpha}} = \frac{cc_1 \varepsilon_1}{s^{1/\alpha}}. \quad (118)$$

Recall that $t_1 \in (0, 1]$ and $t_2 > 1 \geq t_1$. By (117), for any $s \in (0, t_1]$, we have

$$\int_D u(t_2 - s, 0, z) dz \geq c. \quad (119)$$

Therefore, by (116), nonnegativity of $u(\cdot, \cdot, \cdot)$, (118), (119) for $(2|y|/\varepsilon_1)^\alpha \leq t_1/2$ we have

$$\begin{aligned} u_1(t_2, 0, y) &\geq \int_{(2|y|/\varepsilon_1)^\alpha}^{t_1} \int_D u(t_2 - s, 0, z) \int_{-|z|-\varepsilon_1 s^{1/\alpha}/2}^{-|z|+\varepsilon_1 s^{1/\alpha}/2} u(s, z + a_1(z)w, y) \\ &\quad \times v(w) dw dz ds \\ &\geq c \int_{(2|y|/\varepsilon_1)^\alpha}^{t_1} s^{-1/\alpha} ds \\ &\geq c|y|^{\alpha-1}. \end{aligned} \quad (120)$$

(One can show that in similar examples for $d > 2$ we have $u_1(t_2, 0, y) \geq c|y|^{\alpha+1-d}$.)

Observe that $(2|y|/\varepsilon_1)^\alpha \leq t_1/2 \iff |y| \leq t_1^{1/\alpha} \varepsilon_1 2^{-1-1/\alpha}$. For $r \in (0, t_1^{1/\alpha} \varepsilon_1 2^{-1-1/\alpha})$ we get by (120)

$$T_{t_2} \mathbf{1}_{B(0,r)}(0) \geq e^{-\lambda t_2} \Psi_{1,t_2} \mathbf{1}_{B(0,r)}(0) = e^{-\lambda t_2} \int_{B(0,r)} u_1(t_2, 0, y) dy \geq cr^{\alpha+1}. \quad (121)$$

By (114) we have $T_t = P_t$. By Theorem 1.1 $x \rightarrow P_t \mathbf{1}_{B(0,r)}(x)$ is continuous so $\|P_t \mathbf{1}_{B(0,r)}\|_\infty \geq P_t \mathbf{1}_{B(0,r)}(0)$. Using this and (121) for $r \in (0, t_1^{1/\alpha} \varepsilon_1 2^{-1-1/\alpha})$ we get

$$\frac{\|P_{t_2} \mathbf{1}_{B(0,r)}\|_\infty}{\|\mathbf{1}_{B(0,r)}\|_1} \geq \frac{P_{t_2} \mathbf{1}_{B(0,r)}(0)}{\|\mathbf{1}_{B(0,r)}\|_1} \geq cr^{\alpha-1},$$

which implies the assertion of the remark. (One can show that in similar examples for $d > 2$ we have $\|P_{t_2} \mathbf{1}_{B(0,r)}\|_\infty / \|\mathbf{1}_{B(0,r)}\|_1 \geq cr^{\alpha+1-d}$.) $\square$

Remark 4.25 In the above example (in $\mathbb{R}^2$) we showed that the transition density $p(t, 0, y)$, for some $t > 0$, is an unbounded function. In fact, the following estimate holds y almost surely

$$p(t, 0, y) \geq c|y|^{\alpha-1}, \quad |y| \leq \varepsilon_1,$$

where c, ε_1 are some positive constants possibly dependent on t .

Hence, we can not expect a general result saying that, with our assumptions, we have the standard estimates for $p(t, x, y)$ of the form

$$p(t, x, y) \leq Ct^{-d/\alpha},$$

as for example in the case of diagonal matrices [24], or matrices satisfying some further regularity assumptions [31]. On the other hand, the assumption $\alpha < 1$ plays an important role (in $\mathbb{R}^2$), since for $\alpha > 1$, by the results of [5], the transition density is bounded.

Remark 4.26 One of the consequences of the results obtained in the paper is the following representation of the transition density

$$p(t, x, y) = p_1(t, x, y) + p_2(t, x, y) + p_3(t, x, y),$$

where $p_1(t, x, y) = \det(B(y))G_t((y-x)(B(y))^T)$ and $p_2(t, x, y) = u(t, x, y) - p_1(t, x, y)$. It is obvious that $\int_{\mathbb{R}^d} p_1(t, x, y) dx = 1$. By Proposition 3.10 and Corollary 3.3 we have $\int_{\mathbb{R}^d} p_1(t, x, y) dy \leq c$, $p_1(t, x, y) \leq ct^{-d/\alpha} e^{-c_1|x-y|}$, $p_1(t, x, \cdot)$ weakly converges to δ_x as $t \rightarrow 0^+$. By Corollary 3.13 we have $|p_2(t, x, y)| \leq ct^{-d/\alpha} e^{-c_1\sqrt{|x-y|}}$. By (25), (57) and (58) we obtain $\int_{\mathbb{R}^d} |p_2(t, x, y)| dy \leq ct^{1/2}$, $\int_{\mathbb{R}^d} |p_2(t, x, y)| dx \leq ct^{1/2}$. By Lemma 4.1 we have $\int_{\mathbb{R}^d} |p_3(t, x, y)| dy \leq ct$.

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