A Comparison Theorem for Nonsmooth Nonlinear Operators

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Abstract

We prove a comparison theorem for super- and sub-solutions with non-vanishing gradients to semilinear PDEs provided a nonlinearity f is L^p function with $p > 1$. The proof is based on a strong maximum principle for solutions of divergence type elliptic equations with VMO leading coefficients and with lower order coefficients from a Kato class. An application to estimation of periodic water waves profiles is given.

Keywords Semi-linear elliptic equation · Non-smooth nonlinearity · Comparison principal · VMO coefficients · Kato classes · Strong maximum principle

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1 Introduction

Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$. We will consider super- and sub-solutions of the equation

$$\Delta u + f(u) = 0 \quad \text{in } \Omega, \quad (1)$$

where f is a real valued function from $L^p_{\text{loc}}(\mathbb{R})$ with some $p > 1$. To make the term $f(u)$ well-defined (measurable and belonging to $L^p_{\text{loc}}(\Omega)$) we will assume that $u \in C^1(\Omega)$ and $\nabla u \neq 0$ in Ω . Usually, f is supposed to be continuous in almost all papers dealing with Eq. 1 and its generalisations (see, for example, [6] and [11] and numerous papers citing these notes).

It was shown in [4, Remark 2.3] that the strong maximum principle may fail if the function f is only Hölder continuous with an exponent less than 1. Optimal conditions on smoothness of f for validity of the strong maximum principle can be found in [12]. The main difference in our approach is that we compare functions in a neighborhood of a point

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where the gradients are not vanishing. This allows to remove any smoothness assumptions on f .

One of the main results of this paper is the following assertion:

Theorem 1.1 *Let $f \in L^p_{\text{loc}}(\Omega)$, $p > 1$. Also let $u_1, u_2 \in C^1(\Omega)$ have non-vanishing gradients in Ω and satisfy the inequalities*

$$\Delta u_1 + f(u_1) \geq 0 \quad \text{and} \quad \Delta u_2 + f(u_2) \leq 0 \quad (2)$$

in the weak sense. If $u_1 \leq u_2$ and $u_1(x_0) = u_2(x_0)$ for some $x_0 \in \Omega$ then $u_1 = u_2$ in the whole Ω .

We note that the theorem is not true without assumptions that the gradients do not vanish, which follows from [4] (see [7]).

In the case $p > n$ this theorem was proved in [7]. The proof there was based on a weak Harnack inequality for non-negative solutions to the second order equation in divergence form

$$\mathcal{L}u := D_j(a_{ji}(x)D_i u) + b_j(x)D_j u = 0 \quad \text{in } \Omega \quad (3)$$

and closely connected with L^p properties of the coefficients b_j . Therefore one of our main concerns is a strong maximum principle for solutions to Eq. 3. We always assume that the matrix (a_{ij}) is symmetric and uniformly elliptic:

$$\nu|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \nu^{-1}|\xi|^2, \quad x \in \Omega, \quad \xi \in \mathbb{R}^n.$$

It was proved in [14] that if $|b| \in L^p_{\text{loc}}(\Omega)$ (here $b = (b_1, \dots, b_n)$) with $p > n$ then a non-negative weak solution to Eq. 3 satisfies (here $B_\rho(x)$ stands for the ball of radius ρ centered at x)

$$\rho^{-n/\gamma} \|u\|_{L^\gamma(B_{2\rho}(x^0))} \leq C \inf_{x \in B_\rho(x^0)} u(x), \quad (4)$$

where $B_{3\rho}(x^0) \subset \Omega$ and $\gamma \in (1, n/(n-2))$. So the restriction $p > n$ in this assertion inherits in the theorem in [7].¹

For our purpose another type of assumptions on the coefficients b_j are more appropriate. It is called the Kato condition, see [3] and [13].

Definition 1 We say that $f \in K_{n,\alpha}$ if

$$\sup_x \int_{|x-y| \leq r} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy \rightarrow 0 \quad \text{as } r \rightarrow 0. \quad (5)$$

It was proved in [8] that inequality (4) still holds if $|b|^2 \in K_{n,2}$. For Hölder continuous coefficients a_{ji} (4) was proved in [15] under the assumption $|b| \in K_{n,1}$. We note that from the last assertion it follows (4) when $|b| \in L^p_{\text{loc}}$, $p > 1$, depends only on one variable and the leading coefficients are Hölder continuous.

For our applications we need the leading coefficients to be only continuous. So all above mentioned results are not enough for our purpose. Here we prove a theorem which deals with slightly discontinuous leading coefficients and allows $b^\alpha \in K_{n,\alpha}$ with α close to 1 for lower order coefficients. In order to formulate this result we need some definitions.

¹It was pointed out in [10, Theorem 2.5'] that Eq. 4 holds if $|b| \log^{1/2}(1 + |b|) \in L^2_{\text{loc}}(\Omega)$ for $n = 2$ and if $|b| \in L^n_{\text{loc}}(\Omega)$ for $n \geq 3$.

Definition 2 Let $f(x)$ be a measurable and locally integrable function. Define a quantity

$$f^{\#r}(x) := \frac{1}{|B_r|} \int_{B_r(x)} \left| f(y) - \frac{1}{|B_r|} \int_{B_r(x)} f(z) dz \right| dy; \quad \omega_f(\rho) = \sup_x \sup_{r \leq \rho} f^{\#r}(x).$$

We say that $f \in VMO(\mathbb{R}^n)$ if $\omega_f(\rho)$ is bounded and $\omega_f(\rho) \rightarrow 0$ as $\rho \rightarrow 0$. In this case the function $\omega_f(\rho)$ is called VMO-modulus of f .

For a bounded Lipschitz domain Ω the space $f \in VMO(\Omega)$ is introduced in the same way but the integrals in the definition of $f^{\#r}(x)$ are taken over $B_r(x) \cap \Omega$.

Definition 3 We say that a function $\sigma : [0, 1] \rightarrow \mathbb{R}_+$ belongs to the Dini class $\mathcal{D}$ if σ is increasing, $\sigma(t)/t$ is summable and decreasing.

It should be noted that assumption about the decay of $\sigma(t)/t$ is not restrictive (see Remark 1.2 in [1] for more details). We use the notation $\varkappa(\rho) = \int_0^\rho \frac{\sigma(t)}{t} dt$.

Theorem 1.2 Let $n \geq 3$. Assume that the leading coefficients $a_{ij} \in VMO(\Omega)$. Suppose that $|b|^\alpha \in K_{n,\alpha}$ and

$$\sup_x \int_{r/2 \leq |x-y| \leq r} \frac{|b(y)|^\alpha}{|x-y|^{n-\alpha}} dy \leq \sigma^\alpha(r), \quad (6)$$

for some $\alpha > 1$ and $\sigma \in \mathcal{D}$.

If a function $u \in W^{1,p}(\Omega)$, $p > n$, satisfies $u \geq 0$ and $\mathcal{L}u \geq 0$ in Ω then either $u > 0$ in Ω or $u \equiv 0$ in Ω .

Remark 1 Notice that the assumption $|b|^{\alpha_2} \in K_{n,\alpha_2}$ does not imply $|b|^{\alpha_1} \in K_{n,\alpha_1}$ for any $1 < \alpha_1 < \alpha_2$. However, if the condition (6) holds with $\alpha = \alpha_2$ then the Hölder inequality ensures $|b|^{\alpha_1} \in K_{n,\alpha_1}$, and Eq. 6 holds with $\alpha = \alpha_1$ (and another function $\sigma \in \mathcal{D}$).

For $n = 2$ we need a stronger assumption.

Theorem 1.3 Let $n = 2$. Assume that the leading coefficients $a_{ij} \in VMO(\Omega)$. Suppose that

$$\sup_x \int_{r/2 \leq |x-y| \leq r} \frac{|b(y)|^\alpha \log^\alpha \frac{r}{|x-y|}}{|x-y|^{2-\alpha}} dy \leq \sigma^\alpha(r), \quad (7)$$

for some $\alpha > 1$ and $\sigma \in \mathcal{D}$.

If a function $u \in W^{1,p}(\Omega)$, $p > 2$, satisfies $u \geq 0$ and $\mathcal{L}u \geq 0$ in Ω then either $u > 0$ in Ω or $u \equiv 0$ in Ω .

For $\gamma \in (0, 1)$ we define the annulus

$$\mathcal{X}_{r,\gamma}(x) = \{y \in B_r(x) : |x-y| > \gamma r\}. \quad (8)$$

If the location of the center is not important we write simply B_r and $\mathcal{X}_{r,\gamma}$.

As usual, for a bounded domain Ω we denote by $W_0^{1,q}(\Omega)$, $q > 1$, the closure in $W^{1,q}(\Omega)$ of the set of smooth compactly supported function, with the norm

$$\|u\|_{W_0^{1,q}(\Omega)} = \left(\int_{\Omega} |\nabla u|^q dx \right)^{1/q}.$$

2 Strong Maximum Principle for Operators with Lower Order Terms

2.1 Coercivity

Let Ω' be a bounded subdomain in Ω . Consider the problem

$$\mathcal{L}_0 u := D_j(a_{ij}(x)D_i u) = f \quad \text{in } \Omega'; \quad u = 0 \quad \text{on } \partial\Omega'. \quad (9)$$

We say that the operator $\mathcal{L}_0$ is q -coercive in Ω' for some $q > 1$, if for each $f \in W^{-1,q}(\Omega')$ the problem (9) has a unique solution $u \in W_0^{1,q}(\Omega')$ and this solution satisfies

$$\|u\|_{W_0^{1,q}(\Omega')} \leq C_q \|f\|_{W^{-1,q}(\Omega')} \quad (10)$$

with C_q independent on f and u .

It is well known that for arbitrary measurable and uniformly elliptic coefficients the operator $\mathcal{L}_0$ is 2-coercive in arbitrary bounded Ω' . Further, if the coefficients $a_{ij} \in VMO(\Omega)$ then the operator $\mathcal{L}_0$ is q -coercive for arbitrary $q > 1$ in arbitrary bounded $\Omega' \subset \Omega$ with $\partial\Omega' \in C^1$, see [2]. The coercivity constant C_q depends on Ω' and VMO-moduli of a_{ij} . Moreover, by dilation we can see that for $\Omega' = B_r$, $r \leq 1$, this constant does not depend on r . For $\Omega' = \mathcal{X}_{r,\gamma}$, $r \leq 1$, C_q depends on γ but not on r .

Let now the operator in Eq. 9 be q -coercive for certain $q > 2$. Put

$$f = \nabla \cdot \mathbf{f} + f_0,$$

where $\mathbf{f} = (f_1, \dots, f_n) \in (L^q(\Omega'))^n$ and $f_0 \in L^{nq/(n+q)}(\Omega')$. Then by the imbedding theorem $f \in W^{-1,q}(\Omega')$ and Eq. 10 takes the form

$$\|u\|_{W_0^{1,q}(\Omega')} \leq C \left(\sum_{j=1}^n \|f_j\|_{L^q(\Omega')} + \|f_0\|_{L^{nq/(n+q)}(\Omega')} \right).$$

We need the following local estimate.

Theorem 2.1 *Let Ω' be a bounded Lipschitz subdomain of Ω and let the operator in Eq. 9 be q -coercive for certain $q > 2$. Let also $u \in W^{1,q}(\Omega')$ be such that*

$$D_j(a_{ji}(x)D_i u) = 0 \quad \text{in } \Omega' \cap B_r; \quad u = 0 \quad \text{on } \partial\Omega' \cap B_r.$$

Then for a fixed $\lambda \in (0, 1)$

$$\|\nabla u\|_{L^q(B_{\lambda r} \cap \Omega')} \leq C r^{-1} \|u\|_{L^q(B_r \cap \Omega')}, \quad (11)$$

where C may depend on the domain Ω' , q , λ and the coercivity constant C_q but it is independent of r .

Proof First, we claim that the problem (9) is s -coercive for any $s \in [2, q]$. Indeed, we have coercivity for $s = 2$ and $s = q$, and the claim follows by interpolation.

Second, the estimate (11) for $q = 2$ follows by the Caccioppoli inequality. For $q > 2$ we choose a cut-off function ζ such that $\zeta = 1$ on $B_{\lambda r}$ and $\zeta = 0$ outside $B_{\lambda_1 r}$, where $\lambda < \lambda_1 < 1$, and $\nabla \zeta \leq cr^{-1}$. Then

$$D_j(a_{ji}(x)D_i(\zeta u)) = D_j(a_{ji}(x)uD_i\zeta) + (D_j\zeta)a_{ji}(x)D_iu \quad \text{in } \Omega'; \quad \zeta u = 0 \quad \text{on } \partial\Omega'.$$

Then by the s -coercivity of the operator we have

$$\|\nabla u\|_{L^s(B_{\lambda r})} \leq Cr^{-1}(\|u\|_{L^s(B_{\lambda_1 r})} + \|\nabla u\|_{L^{ns/(n+s)}(B_{\lambda_1 r})})$$

We choose $s = \min(q, 2n/(n-2))$. Then the last term in the right-hand side is estimated by $Cr^{n/s-n/2}\|\nabla u\|_{L^2(B_{\lambda_1 r})}$, and hence by the proved estimate for $q = 2$, we obtain (11) for $q = s$. Repeating this argument (but using now the estimate (11) for $q = s$) we arrive finally at Eq. 11. $\square$

2.2 Estimates of the Green Functions

Let $\mathcal{L}$ be an operator of the form Eq. 3, and the assumptions of Theorem 1.2 are fulfilled. We establish the existence and some estimates of the Green function $G = G(x, y)$ for the problem

$$\mathcal{L}u = f \quad \text{in } B_r; \quad u = 0 \quad \text{on } \partial B_r, \quad (12)$$

in sufficiently small ball $B_r \subset \Omega$.

Lemma 2.1 *Let $n \geq 3$. There exists a positive constant R depending on n , the ellipticity constant ν , VMO-moduli of coefficients a_{ij} , the exponent α and the function σ from Eq. 6 such that for any $r \leq R$ there is the Green function $G(x, y)$ of the problem (12) in a ball $B_r \subset \Omega$. Moreover, it is continuous w.r.t. x for $x \neq y$ and satisfies the estimates*

$$|G(x, y)| \leq \frac{C_1}{|x - y|^{n-2}};$$

$$\text{if } |x - y| \leq \text{dist}(x, \partial B_r)/2 \quad \text{then} \quad G(x, y) \geq \frac{C_2}{|x - y|^{n-2}}, \quad (13)$$

where the constants C_1 and C_2 depend on the same quantities as R .

Proof We use the idea from [15]. Denote by $G_0(x, y)$ the Green function of the problem (9) in the ball B_r . The estimates (13) for G_0 were proved in [9] (see also [5]):

$$0 < G_0(x, y) \leq \frac{C_{10}}{|x - y|^{n-2}};$$

$$\text{if } |x - y| \leq \text{dist}(x, \partial B_r)/2 \quad \text{then} \quad G_0(x, y) \geq \frac{C_{20}}{|x - y|^{n-2}}, \quad (14)$$

where the constants C_{10} and C_{20} depend only on n and ν .

By Remark 1, we can assume without loss of generality $\alpha < n/(n-1)$. Put $q = \alpha' > n$ and denote by C_q the coercivity constant for $\mathcal{L}_0$ in the ball. We begin with the estimate for any $B_\rho(y) \subset \Omega$

$$\begin{aligned} \int_{B_\rho(y)} |b(x)| |D_x G_0(x, y)| dx &= \sum_{j=0}^{\infty} \int_{\mathcal{X}_{\frac{\rho}{2^j}, \frac{1}{2}}(y)} |b(x)| |D_x G_0(x, y)| dx \\ &\leq \sum_{j=0}^{\infty} \left(\int_{\mathcal{X}_{\frac{\rho}{2^j}, \frac{1}{2}}(y)} |D_x G_0(x, y)|^q dx \right)^{\frac{1}{q}} \left(\int_{\mathcal{X}_{\frac{\rho}{2^j}, \frac{1}{2}}(y)} |b(x)|^{q'} dx \right)^{\frac{1}{q'}} =: \sum_{j=0}^{\infty} A_{j1} A_{j2}. \end{aligned}$$

By Eqs. 11 and 14, we have $A_{j1} \leq C(n, v, C_q)(2^{-j-1}\rho)^{1-\frac{n}{q'}}$, and Eq. 6 gives

$$A_{j1} A_{j2} \leq 2^{\frac{n}{q'}-1} C \left(\int_{\mathcal{X}_{\frac{\rho}{2^j}, \frac{1}{2}}(y)} \frac{|b(x)|^{q'}}{|x-y|^{n-q'}} dx \right)^{\frac{1}{q'}} \leq 2^{\frac{n}{q'}-1} C \sigma(2^{-j}\rho).$$

Therefore,

$$\int_{B_\rho(y)} |b(x)| |D_x G_0(x, y)| dx \leq 2^{\frac{n}{q'}-1} C \sum_{j=0}^{\infty} \sigma(2^{-j}\rho) \leq A(n, v, q, C_q) \int_0^\rho \frac{\sigma(t)}{t} dt \equiv A\chi(\rho). \quad (15)$$

Next, we write down the equation for G

$$\mathcal{L}_0(G - G_0) = -b \cdot DG \iff (I + G_0 * (b \cdot D))(G - G_0) = -b \cdot DG_0 \quad (16)$$

and obtain

$$G = G_0 - G_0 * (b \cdot DG_0) + G_0 * (b \cdot DG_0) * (b \cdot DG_0) - \dots =: \sum_{k=0}^{\infty} J_k$$

provided this series converges.

We claim that

$$|J_k(x, y)| \leq \frac{C_{10} C^k \chi^k(r)}{|x-y|^{n-2}} \implies |J_{k+1}(x, y)| \leq \frac{C_{10} C^{k+1} \chi^{k+1}(r)}{|x-y|^{n-2}}, \quad (17)$$

for a proper constant C . Indeed,

$$J_{k+1}(x, y) = \int_{B_r} J_k(x, z) (b(z) \cdot D_z G_0(z, y)) dz.$$

Denote $2\rho = |x - y|$. Then

$$|J_{k+1}(x, y)| \leq \left(\int_{B_r \setminus B_\rho(x)} + \int_{B_r \cap B_\rho(x)} \right) |J_k(x, z)| |b(z)| |D_z G_0(z, y)| dz =: I_1 + I_2.$$

We have by Eq. 15

$$\begin{aligned} I_1 &\leq \int_{B_r \setminus B_\rho(x)} \frac{C_{10} C^k \varkappa^k(r)}{|x-z|^{n-2}} |b(z)| |D_z G_0(z, y)| dz \\ &\leq \frac{2^{n-2} C_{10} C^k \varkappa^k(r)}{|x-y|^{n-2}} \int_{B_r \setminus B_\rho(x)} |b(z)| |D_z G_0(z, y)| dz \leq \frac{2^{n-1} A C_{10} C^k \varkappa^{k+1}(r)}{|x-y|^{n-2}} \end{aligned}$$

(here we used an evident inequality $\varkappa(2r) \leq 2\varkappa(r)$). Further,

$$I_2 \leq \int_{B_r \cap B_\rho(x)} \frac{A C^k \varkappa^k(r)}{|x-z|^{n-2}} |b(z)| |D_z G_0(z, y)| dz.$$

By Eqs. 11 and 14,

$$\left(\int_{B_r \cap B_\rho(x)} |D G_0(z, y)|^q dy \right)^{\frac{1}{q}} \leq A \rho^{1-\frac{n}{q}}.$$

Using the assumption $q > n$ we get

$$I_1 \leq \frac{A^2 C^k \varkappa^k(r)}{\rho^{\frac{n}{q'}-1}} \left(\int_{B_r \cap B_\rho(x)} \frac{|b(z)|^{q'}}{|x-z|^{(n-2)q'}} dz \right)^{\frac{1}{q'}} \leq \frac{A^2 C^k \varkappa^k(r)}{\rho^{\frac{n}{q'}-1}} \rho^{1-\frac{n}{q}} \varkappa(\rho) \leq \frac{2^{n-2} A^2 C^k \varkappa^{k+1}(r)}{|x-y|^{n-2}},$$

and the claim follows if we put $C = 2^{n-2}(A + 2C_{10})$.

Thus, the series in Eq. 16 converges if $\varkappa(r)$ is sufficiently small. Moreover, if $2^{n-2}(A + 2C_{10})\varkappa(r) \leq \frac{C_{20}}{C_{20}+2C_{10}}$ then Eqs. 14 implies 13 with $C_1 = C_{10} + \frac{C_{20}}{2}$, $C_2 = \frac{C_{20}}{2}$.

To prove the continuity of G we take $\widehat{x}$ such that $|x - \widehat{x}| \leq \rho/2 = |x - y|/4$. Since $q > n$, the estimate (11) and the Morrey embedding theorem give

$$|G_0(x, y) - G_0(\widehat{x}, y)| \leq C_{30}(n, v, C_q) \frac{|x - \widehat{x}|^{1-\frac{n}{q}}}{|x - y|^{\frac{n}{q}-1}}.$$

We write down the relation

$$G(x, y) - G(\widehat{x}, y) = \sum_{k=0}^{\infty} (J_k(x, y) - J_k(\widehat{x}, y))$$

and deduce, similarly to Eq. 17, that

$$\begin{aligned} |J_k(x, y) - J_k(\widehat{x}, y)| &\leq C_{30} C^k \varkappa^k(r) \frac{|x - \widehat{x}|^{1-\frac{n}{q}}}{|x - y|^{\frac{n}{q}-1}} \\ \implies |J_{k+1}(x, y) - J_{k+1}(\widehat{x}, y)| &\leq C_{30} C^{k+1} \varkappa^{k+1}(r) \frac{|x - \widehat{x}|^{1-\frac{n}{q}}}{|x - y|^{\frac{n}{q}-1}}. \end{aligned}$$

Therefore, if $\varkappa(r)$ is sufficiently small,

$$|G(x, y) - G(\widehat{x}, y)| \leq C_3 \frac{|x - \widehat{x}|^{1-\frac{n}{q}}}{|x - y|^{\frac{n}{q}-1}}.$$

□

Remark 2 In fact, since q can be chosen arbitrarily large, G is locally Hölder continuous w.r.t. x for $x \neq y$ with arbitrary exponent $\beta \in (0, 1)$.

Now let $\mathcal{X}_{r,\gamma} \subset \Omega$. Consider the Dirichlet problem

$$\mathcal{L}u = f \quad \text{in } \mathcal{X}_{r,\gamma}; \quad u = 0 \quad \text{on } \partial\mathcal{X}_{r,\gamma}. \quad (18)$$

Lemma 2.2 *The statement of Lemma 2.1 holds for the problem (18). The constants R , C_1 and C_2 may depend on the same quantities as in Lemma 2.1 and also on γ .*

The proof of Lemma 2.1 runs without changes.

2.3 Approximation Lemma and Weak Maximum Principle

Lemma 2.3 *Under assumptions of Lemma 2.1, let u be a weak solution of the equation*

$$\mathcal{L}u = f \quad \text{in } B_r \subset \Omega,$$

with $r \leq R$ where R is the constant from Lemma 2.1. Let f be a finite signed measure in B_r .

Put $b_i^m := b_i \chi_{\{|b| \leq m\}}$ and define a sequence $f_m \in L^\infty(B_r)$ such that $f_m \rightarrow f$ in the space of measures. Denote by u_m the solution of the problem

$$\mathcal{L}_m u_m := -D_i(a_{ij} D_j u_m) + b_i^m D_i u_m = f^m \quad \text{in } B_r; \quad u^m = u \quad \text{on } \partial B_r.$$

Then

$$\int_{B_r} |u_m(x) - u(x)| dx \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Proof It is easy to see that the difference $v_m = u_m - u$ solves the problem

$$\mathcal{L}_m v_m = (b_i - b_i^m) D_i u + f_m - f \quad \text{in } B_r; \quad v_m = 0 \quad \text{on } \partial B_r.$$

Using the Green function G_m of the operator $\mathcal{L}_m$ in B_r with the Dirichlet boundary conditions we get

$$\int_{B_r} |v_m(x)| dx \leq \left(\int_{B_r} |(b_i - b_i^m) D_i u| + \int_{B_r} |f_m - f| \right) \cdot \sup_y \int_{B_r} |G_m(x, y)| dx.$$

By Lemma 2.1, $|G_m(x, y)| \leq C|x - y|^{2-n}$ with constant independent of m . Thus, the supremum of the last integral is bounded. The first integral in brackets tends to zero by the Lebesgue Dominated convergence Theorem, and the Lemma follows. $\square$

Corollary 2.1 *If $\mathcal{L}u = f \geq 0$ in B_r , and $u \geq 0$ on ∂B_r , then $u \geq 0$ in B_r .*

This statement follows from standard weak maximum principle and Lemma 2.3.

Lemma 2.4 *Let $\mathcal{X}_{r,\gamma} \subset \Omega$ and let $r \leq R$ where R is the constant from Lemma 2.2. Then the assertion of Lemma 2.3 and the Corollary 2.1 are still true.*

2.4 Strong Maximum Principle

Lemma 2.5 *Let $B_r \subset \Omega$ be a ball and let $r \leq R$ where R satisfies the assumptions of Lemma 2.1 and Lemma 2.2 with $\gamma = 1/4$. Then the Green function of $\mathcal{L}$ in B_r is strictly positive: $G(x, y) > 0$ for $x, y \in B_r$, $x \neq y$.*

Proof First suppose that $G(x^*, y) < 0$ for certain $x^* \in B_r$ and for a positive measure set of y . By continuity of G in x we have $G(x, y) < 0$ for a (maybe smaller) positive measure set of y and an open set of x . Therefore, we can choose a bounded nonnegative function f such that $u(x) = \int_{B_r} G(x, y) f(y) < 0$ on an open set. But this would contradict to the weak maximum principle, see Corollary 2.1. Thus, we can change G on a null measure set and assume it nonnegative.

Next, suppose that for certain y^* the set $\mathcal{S}(y^*) = \{x \in B_r : G(x, y^*) = 0\}$ is nonempty. We choose then $x^0 \in \mathcal{S}(y^*)$ and $y^0 \in B_r \setminus \mathcal{S}(y^*)$ such that $\rho := |x^0 - y^0| = \text{dist}(y^0, \mathcal{S}(y^*))$. Due to the second estimate in Eq. 13 x^0 is separated from y^* , while ρ can be chosen arbitrarily small. So, we can suppose that

$$B_{2\rho}(y^0) \subset B_r \setminus \{y^*\}; \quad B_{\rho/2}(y^0) \cap \mathcal{S}(y^*) = \emptyset.$$

We introduce the Green function $\widehat{G}$ of $\mathcal{L}$ in $B_{2\rho}(y^0)$ and claim that the function $\delta\widehat{G}(\cdot, y^0)$ with sufficiently small $\delta > 0$ is a lower barrier for $G(\cdot, y^*)$ in the annulus $\mathcal{X}_{2\rho, 1/4}(y^0)$. Indeed, the boundary $\partial\mathcal{X}_{2\rho, 1/4}(y^0)$ consists of two spheres. Notice that $G(\cdot, y^*) > 0$ on $\partial B_{\rho/2}(y^0)$ while $\widehat{G}(\cdot, y^0) = 0$ on $\partial B_{2\rho}(y^0)$. Thus, there exists a positive δ such that

$$G(\cdot, y^*) \geq \delta\widehat{G}(\cdot, y^0) \quad \text{on } \partial\mathcal{X}_{2\rho, 1/4},$$

and the claim follows. By Lemma 2.4 $G(\cdot, y^*) \geq \delta\widehat{G}(\cdot, y^0)$ in the whole annulus, and, in particular,

$$G(x^0, y^*) \geq \delta\widehat{G}(x^0, y^0) > 0$$

(the last inequality follows from the second estimate in Eq. 13). The obtained contradiction proves the Lemma. $\square$

Proof of Theorem 1.2. We repeat in essential the proof of Lemma 2.5. Denote the set $\mathcal{S} = \{x \in \Omega : u(x) = 0\}$ and suppose that $\mathcal{S} \neq \Omega$. Then we can choose $x^0 \in \mathcal{S}$ and $y^0 \in \Omega \setminus \mathcal{S}$ such that $\rho := |x^0 - y^0| = \text{dist}(y^0, \mathcal{S})$, and ρ can be chosen arbitrarily small. Repeating the proof of Lemma 2.5 we introduce the same Dirichlet Green function $\widehat{G}(\cdot, y^0)$ of $\mathcal{L}$ in $B_{2\rho}(y^0)$ and show that $\delta\widehat{G}(\cdot, y^0)$ with sufficiently small $\delta > 0$ is a lower barrier for u in the annulus $\mathcal{X}_{2\rho, 1/4}(y^0)$. This ends the proof. $\square$

2.5 The Case $n = 2$

The case $n = 2$ is treated basically in the same way as the case $n \geq 3$, but some changes must be done mostly due to the fact that the estimate of the Green function contains logarithm.

Let us explain what changes must be done in the argument in compare with $n \geq 3$. Denote by $G_0(x, y)$ the Green function of the problem (9) in the disc B_r . Then for $x \neq y \in B_r$,

$$0 < G_0(x, y) \leq C'_{10} \log \left(\frac{r}{|x - y|} + 2 \right);$$

$$\text{if } |x - y| \leq \text{dist}(x, \partial B_r)/2 \quad \text{then } G_0(x, y) \geq C'_{20} \log \left(\frac{r}{|x - y|} + 2 \right), \quad (19)$$

where the constants C'_{10} and C'_{20} depend only on the ellipticity constants of the operator $\mathcal{L}_0$. Indeed by [9] these estimates can be reduced to similar estimates for the Laplacian, when they can be verified directly (in this case the Green function can be written explicitly).

Analog of Lemma 2.1 reads as follows.

Lemma 2.6 *Let $n = 2$. There exists a positive constant R depending on the ellipticity constant ν , VMO-moduli of coefficients a_{ij} , the exponent α and the function σ from Eq. 7 such that for any $r \leq R$ there is the Green function $G(x, y)$ of the problem (12) in a disc $B_r \subset \Omega$. Moreover, it is continuous w.r.t. x for $x \neq y$ and satisfies the estimates*

$$|G(x, y)| \leq C'_1 \log \left(\frac{r}{|x - y|} + 2 \right);$$

$$\text{if } |x - y| \leq \text{dist}(x, \partial B_r)/2 \quad \text{then} \quad |G(x, y)| \geq C'_2 \log \left(\frac{r}{|x - y|} + 2 \right), \quad (20)$$

where the constants C'_1 and C'_2 depend on the same quantities as R .

To prove this statement we establish the inequality (15) by using the estimate (19) and the assumption (7) instead of Eq. 6. The rest of the proof runs without changes.

Corresponding analog of Lemma 2.2 is true here also.

The remaining part of the proof of Theorem 1.3 is the same as that of Theorem 1.2.

3 Comparison Theorem for Nonlinear Operators

3.1 Proof of Theorem 1.1

We recall that the statement of Theorem 1.1 with $p > n$ was proved in [7, Theorem 1] by reducing it to the strong maximum principle for the Eq. 3 with continuous leading coefficients and $f(x_1)$ playing the role of a coefficient b_1 . Since the function f depends only on one variable, the assumption $f \in L^p_{\text{loc}}(\mathbb{R})$ with a certain $p > 1$ implies Eqs. 6 and 7 with $\alpha = p$. Thus, we can apply our Theorems 1.2 and 1.3 (for $n \geq 3$ and $n = 2$, respectively) instead of [7, Lemma 1], and the proof of [7, Theorem 1] runs without other changes.

3.2 Application to Water Wave Theory

In the paper [7], two theorems were proved on estimates of the free surface profile of water waves on two-dimensional flows with vorticity in a channel, see Theorems 2 and 3 in [7]. The vorticity function ω in that theorems was assumed to belong to $L^p_{\text{loc}}(\mathbb{R})$ with $p > 2$ and the proof was based on the application of [7, Theorem 1]. Now the application of our Theorem 1.1 allows us to weaken the apriori assumption for the vorticity function to $\omega \in L^p_{\text{loc}}(\mathbb{R})$, $p > 1$.

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