Correction to: The Mixed Problem for the Laplacian in Lipschitz Domains

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In an earlier article, we use a technical estimate [4, Equation (7.4)] which is not correct. In this note, we give an example to show that the estimate is not correct and provide a substitute argument which allows us to repair the proof of the result of our earlier work. The argument in [4] has been applied in several subsequent works [3, 5] and this correction should be applied to these subsequent works as well. We thank L. Croyle for pointing out this error. Her dissertation [2] includes a different approach to correcting this error. A recent manuscript with Croyle [1] incorporates the argument below to give a proof of the main results in Croyle’s dissertation.

We introduce some notation that we will need to study the incorrect claim in our previous work. We recall the notation $\Delta_r(x)$, $\Psi_r(x)$, $\Gamma(x)$ and u^* introduced in [4, pp. 1336–7]. Below, (x, y) with x a number are references to results in our earlier work, [4]. References to this note will be of the form (A, y) . We will use the Hardy-Littlewood maximal function

$$M(f)(x) = \sup_{s>0} \int_{\Delta_s(x)} |f| d\sigma.$$

We fix a scale $r > 0$ and define the truncated Hardy-Littlewood maximal functions by

$$M_0(f)(x) = \sup_{0<s<r} \int_{\Delta_s(x)} |f| d\sigma, \quad M_\infty(f)(x) = \sup_{r<s} \int_{\Delta_s(x)} |f| d\sigma.$$

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In our applications, r will be comparable to the sidelength of a cube Q which appears in Shen's argument as outlined in (7.1-7.3). For u a function on Ω , we define truncated non-tangential maximal functions by

$$u^\nabla(x) = \sup_{y \in \Gamma(x), 0 < |x-y| < cr} |u(y)|, \quad u^\Delta(x) = \sup_{y \in \Gamma(x), cr \leq |x-y|} |u(y)|.$$

Here, r is as in the definition of the truncated Hardy-Littlewood maximal functions and c is a small constant to be determined.

Using standard arguments to compare averages on surface cubes to averages on surface balls, the problematic claim (7.4) would follow from the assertion

$$u^\Delta(x) \leq C \int_{\Delta_r(x)} u^*(y) d\sigma \quad (\text{A.1})$$

applied to $u = \nabla w$. While it is easy to see that the estimate (A.1) holds when Ω is a half-space, the example below shows that it may fail in Lipschitz domains. In our correction, Lemma A.5 will substitute for the false assertion (A.1).

Example We will find a domain Ω and a family of functions u_ϵ for ϵ in an interval $(0, \epsilon_0)$ for which the estimate (A.1) cannot hold for any finite constant C . We give examples for α large and $c > 0$, but assume these parameters are fixed in the construction below. Recall that α is the parameter used to define the non-tangential approach regions $\Gamma(x)$.

Consider the domain Ω in $\mathbf{R}^2$ which lies above the graph of the Lipschitz function $\phi(x_1) = \max(-2|x_1|, 2|x_1| - 4)$. Let $\epsilon > 0$ be small and find $y \in \Gamma(0)$ with $|y|$ large and $\epsilon = (1 + \alpha) \text{dist}(y, \partial\Omega) - |y|$. We will need α large in order to guarantee that such y exists. If $x \in \Delta_{1/2}(0)$, then we may find a constant $a > 0$ so that $|y| + a|x| \leq |x - y|$. This holds because the line segments which form the boundary of $\partial\Omega$ near 0 lie outside of and are not tangent to the circle $\{z : |z - y| = |y|\}$. Then we will have $\{x : y \in \Gamma(x), x \in \Delta_{1/2}(0)\} \subset \Delta_{\epsilon/a}(0)$. To see this containment, observe that for $y \in \Gamma(x)$, we have $|y| + a|x| \leq |x - y| < (1 + \alpha) \text{dist}(y, \partial\Omega)$ which implies $a|x| < \epsilon$. If we let $u_\epsilon(z) = \chi_{\{y\}}(z)$, then we have

$$u_\epsilon^\Delta(0) = 1, \quad \text{and} \quad \int_{\Delta_{1/2}(0)} u_\epsilon^* d\sigma \leq C\epsilon.$$

Thus the estimate (A.1) with $r = 1/2$ cannot hold uniformly over the family u_ϵ .

We now turn to the arguments needed to correct for the failure of (7.4). To simplify the notation, we will use ∇u^* to denote $(\nabla u)^*$, the non-tangential maximal function applied to ∇u and similarly for ∇u^Δ and ∇u^∇ .

As in section 7 of our earlier work [4], we let u be a solution of the mixed problem with Neumann data f in $L^p(N)$ and zero Dirichlet data. The main change in our proof is to apply Shen's argument outlined in (7.1-7.3) to $M((\nabla u^*)^{1/2})^2$ rather than to ∇u^* . At one point in our argument, we will need to assume that α is large and satisfies $\alpha + 1 > \sqrt{1 + M^2}$ where M is the Lipschitz constant for our domain (see [4, p. 1336]). It is easy to see that estimates for ∇u^* for large α imply an estimate for smaller values of α .

We fix a cube Q and let v and w be defined as in section 7 of [4]. We set $F_Q = M((\nabla v^*)^{1/2})^2$ and $R_Q = M((\nabla w^*)^{1/2})^2$. By uniqueness for the L^1 -mixed problem proved

in [4, Theorem 5.1], we have that $u = v + w$ and thus it follows that we have (7.1). By Theorem 4.17, we have the estimate

$$\int_{\partial\Omega} \nabla v^* d\sigma \leq C \|g\|_{H^1(N)} \leq C\sigma(4Q)^{1-1/p} \left(\int_{4Q} |f|^p d\sigma \right)^{1/p}. \quad (\text{A.2})$$

The function g is the Neumann data for v and is defined on p. 1361 of [4]. If Q is not contained in N , then g may not have mean value zero and an additional argument is needed to see that g lies in $H^1(N)$. We may use the condition (2.1) to construct a function whose norm in $L^p(\partial\Omega)$ is controlled by the $L^p(N)$ norm of g , with mean value zero, whose support lies in a surface ball with diameter controlled by the diameter of Q and which restricts to N to give g . This allows us to estimate g in $H^1(N)$. From (A.2) and the Hardy-Littlewood maximal theorem, we obtain the estimate

$$\int_{2Q} M((\nabla v^*)^{\frac{1}{2}})^2 d\sigma \leq C \left(\int_{4Q} |f|^p d\sigma \right)^{1/p} \quad (\text{A.3})$$

which is (7.2).

Before estimating R_Q , we give two technical lemmata.

Lemma A.4 Suppose that x, y are in $\partial\Omega$ and $|x - y| < Ar$, then we have

$$M_\infty(f)(x) \leq C_A M_\infty(f)(y).$$

Proof By the triangle inequality, we have $\Delta_s(x) \subset \Delta_{s+Ar}(y)$. Thus it follows that

$$\int_{\Delta_s(x)} |f| d\sigma \leq \frac{\sigma(\Delta_{s+Ar}(y))}{\sigma(\Delta_s(x))} \int_{\Delta_{s+Ar}(y)} |f| d\sigma.$$

If we require that $s \geq r$, then we have a constant so that $\sigma(\Delta_{s+Ar}(y))/\sigma(\Delta_s(x)) \leq C_A$ and the Lemma follows. $\square$

Lemma A.5 For $p > 0$, we have

$$u^\Delta(x) \leq C M_\infty((u^*)^p)^{1/p}(x).$$

The constant depends on the value of p and the constant c entering into the definition of u^Δ .

The constant in the above lemma depends on α , as well. We continue to follow the convention introduced in [4] to not track the dependence on α .

Proof Fix $x \in \partial\Omega$ and suppose that $y \in \Gamma(x)$. Find $\hat{y} \in \partial\Omega$ so that $|y - \hat{y}| = d(y) = \text{dist}(y, \partial\Omega)$ and observe that if $z \in \Delta_{\alpha d(y)}(\hat{y})$, we have $y \in \Gamma(z)$. This implies $|u(y)| \leq u^*(z)$ for $z \in \Delta_{\alpha d(y)}(\hat{y})$. By the triangle inequality $|x - \hat{y}| \leq |x - y| + |y - \hat{y}| \leq (2 + \alpha)d(y)$. Hence we have that $\Delta_{\alpha d(y)}(\hat{y}) \subset \Delta_{(2+\alpha)d(y)}(x)$. It follows that

$$|u(y)| \leq \frac{\sigma(\Delta_{(2+\alpha)d(y)}(x))}{\sigma(\Delta_{\alpha d(y)}(\hat{y}))} \int_{\Delta_{(2+\alpha)d(y)}(x)} u^* d\sigma.$$

If we assume that $|x - y| > cr$, then we have $d(y) > cr/(1 + \alpha)$ and the estimate

$$u^\Delta(x) \leq C M_\infty(u^*)(x) \quad (\text{A.6})$$

follows immediately if $c \geq 1/2$. For smaller values of c , we note that for $t < s$

$$\int_{\Delta_r(x)} |f| d\sigma \leq \frac{\sigma(\Delta_s(x))}{\sigma(\Delta_t(x))} \int_{\Delta_s(x)} |f| d\sigma$$

and hence (A.6) follows for smaller values of c . This gives our Lemma for $p = 1$. To obtain the Lemma for other values of p , apply the case with $p = 1$ to $|u|^p$. $\square$

Lemma A.7 *There is a small constant c so that if $1 + \alpha > \sqrt{1 + M^2}$, then we have*

$$\int_{\Psi_r(x)} u dy \leq C \int_{\Delta_{r\sqrt{1+M^2}}(x)} u^* d\sigma, \quad 0 < r < cr_0. \quad (\text{A.8})$$

Proof We first observe that if $z \in \partial\Omega$ and the boundary is given by $\{y_n = \phi(y')\}$ near z , then for $M = \|\nabla\phi\|_{L^\infty(\mathbf{R}^{n-1})}$, we will have the truncated cone $\{y : M|y' - z'| < y_n - z_n, y_n - z_n < cr_0\}$ contained in Ω . Thus for small t , we have $\text{dist}(z + te_n, \partial\Omega) \geq t/\sqrt{1 + M^2}$. With our condition on α it follows that $z + te_n \in \Gamma(z)$ for $t \in (0, cr_0)$. We also have $\Psi_r(x) \subset \{y : y = z + te_n, z \in \Delta_{r\sqrt{1+M^2}}(x), 0 < t < 2r\sqrt{1 + M^2}\}$. For $y \in \Delta_{r\sqrt{1+M^2}}(x)$, we have $\int_{\{t: y+te_n \in \Psi_r(x)\}} |u(y + te_n)| dt \leq 2r\sqrt{1 + M^2} u^*(y)$, and we can integrate this inequality over $\Delta_{r\sqrt{1+M^2}}(x)$ to obtain (A.8). $\square$

To estimate R_Q , we will fix $r > c \text{diam}(Q)$ so that for $x \in 2Q$ we have $\Delta_{rC_M}(x) \subset 4Q$ where $C_M = 4\sqrt{1 + M^2}$. This will also be the value of r we use in defining the truncated non-tangential and Hardy-Littlewood maximal functions. Our goal is to show that

$$\begin{aligned} & \left(\int_{\Delta_r(x)} M((\nabla w^*)^{\frac{1}{2}})^{2q} d\sigma \right)^{1/q} \\ & \leq C \left[\int_{\Delta_{rC_M}(x)} M((\nabla u^*)^{\frac{1}{2}})^2 d\sigma + \left(\int_{4Q} |f|^p d\sigma \right)^{1/p} \right]. \end{aligned} \quad (\text{A.9})$$

Covering $2Q$ by a finite collection of such balls we may obtain (7.3) from (A.9).

To obtain (A.9), we begin by observing that

$$\begin{aligned} R_Q &= M((\nabla w^*)^{\frac{1}{2}})^2 \\ &\leq C(M_\infty((\nabla w^*)^{\frac{1}{2}})^2 + M_0((\nabla w^\nabla)^{\frac{1}{2}})^2 + M_0((\nabla w^\Delta)^{\frac{1}{2}})^2) \end{aligned} \quad (\text{A.10})$$

and will proceed to estimate the three terms on the right of (A.10).

From Lemma A.4, we have

$$\begin{aligned} \sup_{y \in \Delta_r(x)} M_\infty((\nabla w^*)^{\frac{1}{2}})^2(y) &\leq C \inf_{y \in \Delta_r(x)} M_\infty((\nabla w^*)^{\frac{1}{2}})^2(y) \\ &\leq C \int_{\Delta_r(x)} M_\infty((\nabla w^*)^{\frac{1}{2}})^2 d\sigma. \end{aligned} \quad (\text{A.11})$$

Next we observe that Lemma 6.1 gives that

$$\left(\int_{\Delta_{2r}(x)} (\nabla w^\nabla)^q d\sigma \right)^{1/q} \leq C \left(\int_{\Psi_{4r}(x)} |\nabla w| dy + |\bar{f}_{4Q}| \right).$$

The application of this Lemma fixes the value of c . Using the estimate (A.8) gives the estimate

$$\int_{\Psi_{4r}(x)} |\nabla w| dy \leq C \int_{\Delta_{rC_M}(x)} \nabla w^* d\sigma.$$

If $z \in \Delta_r(x)$, then we have $M_0(f)(z) = M_0(f\chi_{\Delta_{2r}(x)})(z)$. Using the two previous displayed equations, we obtain that

$$\left(\int_{\Delta_r(x)} M_0((\nabla w^\nabla)^{1/2})^{2q} d\sigma \right)^{1/q} \leq C \left(\int_{\Delta_{rC_M}(x)} \nabla w^* d\sigma + |\bar{f}_{4Q}| \right). \quad (\text{A.12})$$

Finally, to estimate $M_0((\nabla w^\Delta)^{\frac{1}{2}})^2$, we may use Lemma A.4 and Lemma A.5 to see that

$$\nabla w^\Delta(y) \leq CM_\infty((\nabla w^*)^{\frac{1}{2}})^2(z), \quad y \in \Delta_{2r}(x), \quad z \in \Delta_{rC_M}(x).$$

Using again our observation about the dependence of the values of $M_0(f)$, we obtain

$$\sup_{y \in \Delta_r(x)} M_0((\nabla w^\Delta)^{\frac{1}{2}})^2(y) \leq C \int_{\Delta_{rC_M}(x)} M((\nabla w^*)^{\frac{1}{2}})^2 d\sigma. \quad (\text{A.13})$$

From (A.10–A.13) we conclude that

$$\left(\int_{\Delta_r(x)} M((\nabla w^*)^{\frac{1}{2}})^{2q} d\sigma \right)^{1/q} \leq C \left[\int_{\Delta_{rC_M}(x)} M((\nabla w^*)^{\frac{1}{2}})^2 d\sigma + \left(\int_{4Q} |f|^p d\sigma \right)^{1/p} \right].$$

To complete the proof of (A.9), we write $w = u - v$ and then use (A.2) to obtain

$$\begin{aligned} & \int_{\Delta_{rC_M}(x)} M((\nabla w^*)^{\frac{1}{2}})^2 d\sigma \\ & \leq C \left(\int_{\Delta_{rC_M}(x)} M((\nabla u^*)^{\frac{1}{2}})^2 d\sigma + \int_{\Delta_{rC_M}(x)} M((\nabla v^*)^{\frac{1}{2}})^2 d\sigma \right) \\ & \leq C \left(\int_{\Delta_{rC_M}(x)} M((\nabla u^*)^{\frac{1}{2}})^2 d\sigma + \left(\int_{4Q} |f|^p d\sigma \right)^{1/p} \right). \end{aligned}$$

Our claim (A.9) follows from the two previous displayed equations.

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Dr. Katharine A. Ott has moved from Kentucky to Bates College in Maine in the years since they first published the article. Thus the new address.

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