Backward Stochastic Differential Equations and Dirichlet Problems of Semilinear Elliptic Operators with Singular Coefficients

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Abstract In this paper, we prove that there exists a unique solution to the Dirichlet boundary value problem for a general class of semilinear second order elliptic differential operators which do not necessarily have the maximum principle and are non-symmetric in general. Our method is probabilistic. It turns out that we need to solve a class of backward stochastic differential equations with singular coefficients, which is of independent interest itself. The theory of Dirichlet forms also plays an important role.

Keywords Dirichlet boundary value problem · Semilinear second order elliptic differential equations · Dirichlet forms · Backward stochastic differential equations

Mathematics Subject Classifications (2010) Primary 60H60 · Secondary 60H30, 31J25

1 Introduction

In this paper, we will use probabilistic methods to solve the Dirichlet boundary value problem for the semilinear second order elliptic partial differential equations (PDEs) of the following form:

$$\begin{cases} \mathcal{A}u(x) = -f(x, u(x)), & \forall x \in D, \\ u(x)|_{\partial D} = \varphi, & \forall x \in \partial D, \end{cases} \quad (1.1)$$

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where D is a bounded domain in $\mathbb{R}^d$, $f(\cdot, \cdot)$ is a nonlinear function and $\varphi \in C(\partial D)$. The operator $\mathcal{A}$ is given by

$$\mathcal{A}u = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial u}{\partial x_j}) + \sum_{i=1}^d b_i(x) \frac{\partial u}{\partial x_i} - \text{"div}(\hat{b}u)\text{"} + q(x)u,$$

where $a(x) = (a_{ij}(x))_{1 \leq i, j \leq d}$ is a Borel measurable, (not necessarily symmetric) matrix-valued function on $\mathbb{R}^d$ satisfying

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^d a_{ij}(x) \xi_i \xi_j \quad \forall \xi \in \mathbb{R}^d, x \in \mathbb{R}^d \quad (1.2)$$

and

$$a_{ij}(x) \leq \frac{1}{\lambda} \quad \forall x \in \mathbb{R}^d, 1 \leq i, j \leq d \quad (1.3)$$

for some constant $0 < \lambda \leq 1$; $b = (b_1, \dots, b_d)^*$, $\hat{b} = (\hat{b}_1, \dots, \hat{b}_d)^*$ and q are Borel measurable functions and satisfy $|b|^2 I_D \in L^p(D; dx)$, $|\hat{b}|^2 I_D \in L^p(D; dx)$ and $q I_D \in L^p(D; dx)$ for some constant $p > \frac{d}{2}$. Here $*$ stands for the transpose of a vector or matrix.

There is a big literature regarding the probabilistic approaches of solving boundary value problems. The pioneering work is traced back to Kunitani [12] who used Brownian motion to represent the solution of the classical Dirichlet problem for Laplacian operators. In the linear case, i.e. $f(\cdot, \cdot) = 0$, if $\hat{b} = 0$, $q \leq 0$, the problem (1.1) is solved by the Feynman-Kac formula

$$u(x) = E_x[e^{\int_0^{\tau_D} q(X(s))ds} \varphi(X_{\tau_D})],$$

where $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, P_x, x \in \mathbb{R}^d)$ is the diffusion process associated with the generator L_1 given by

$$L_1 u = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial u}{\partial x_j}) + \sum_{i=1}^d b_i(x) \frac{\partial u}{\partial x_i} \quad (1.4)$$

and τ_D is the first exit time of X from D . We refer the readers to Chen and Zhao [8] for details. The first result on probabilistic interpretation of viscosity solutions of semilinear parabolic PDEs was obtained by Peng in [17] and [16], through the backward stochastic differential equations, which is different from the weak solution considered in this paper.

Since $\hat{b}$ is merely measurable, some of the coefficients of the operator $\mathcal{A}$ are very singular (just distributions). When $\hat{b} \neq 0$, $f(\cdot, \cdot) \neq 0$ and the matrix $a(x)$ is symmetric, the boundary value problem (1.1) was considered by Zhang [18]. The time reversal of symmetric Markov processes and the backward stochastic differential equations (BSDEs) with random terminal time play an important role in [18]. When a is non-symmetric and $f(\cdot, \cdot) = 0$ (i.e. the linear case), Chen, Sun and Zhang [5] obtained a probabilistic representation of the solution to problem (1.1) under the condition that

$$q - \text{div} \hat{b} \leq g \quad (1.5)$$

in the sense of distribution for a sufficient small non-negative function $g \in L^p(D)$.

In this paper, we consider the weak/Sobolev solution (see the definition in the next section) of the semilinear problem (1.1). We will show that there exists a unique, continuous weak solution to the Dirichlet value problem (1.1) under appropriate conditions. We will use the h-transform method to tackle the singular term “div($\hat{b} \cdot$)”. To solve the semilinear boundary

value problem (1.1), we first produce a candidate to the solution by appealing to the theory of BSDEs. More precisely, we will solve a class of BSDEs with very singular coefficients and random terminal time. The BSDEs are driven by the martingale part of a diffusion process. The study of this class of BSDEs is of independent interest. It turns out that the classical L^2 setting of BSDEs is not suitable here. We have to work in the framework of L^1 and deal with class D stochastic processes. We refer the readers to the nice article [4] for related results.

The rest of the paper is organized as follows. In Section 2, we set up the precise framework. Section 3 is devoted to establishing the existence and uniqueness of solutions of a backward stochastic differential equation with singular coefficients. The boundary value problem is solved in Section 4.

2 Framework

We assume that $d \geq 3$. Hereafter we fix $p > \frac{d}{2}$. We will use $\langle \cdot, \cdot \rangle$ and $(\cdot, \cdot)$ to denote respectively the inner product of the Euclidean space $\mathbb{R}^d$ and the inner product of the $L^2(D)$ -space. $\|\cdot\|_\infty$ denotes the L^∞ normal. Recall that $W^{1,2}(D)$ is the Sobolev space on D with the norm $\|u\|_{W^{1,2}(D)} = (\int_D \langle \nabla u, \nabla u \rangle dx + (u, u))^{1/2}$ and that $W_0^{1,2}(D)$ denotes the completion of $C_0^\infty(D)$ under the same norm $\|\cdot\|_{W^{1,2}(D)}$. The operator $\mathcal{A}$ introduced in Section 1 is rigorously determined by the quadratic form $(\mathcal{Q}, D(\mathcal{Q}))$ with $D(\mathcal{Q}) = W^{1,2}(D)$ and for $u, v \in W^{1,2}(D)$

$$\begin{aligned} \mathcal{Q}(u, v) &= (-\mathcal{A}u, v)_{L^2(\mathbb{R}^d)} \\ &= \frac{1}{2} \sum_{i,j=1}^d \int_{\mathbb{R}^d} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx - \sum_{i=1}^d \int_{\mathbb{R}^d} b_i(x) \frac{\partial u}{\partial x_i} v(x) dx \\ &\quad - \sum_{i=1}^d \int_{\mathbb{R}^d} \hat{b}_i(x) \frac{\partial v}{\partial x_i} u(x) dx - \int_D q(x) u(x) v(x) dx. \end{aligned}$$

Definition 2.1 $u \in W^{1,2}(D) \cap C(\bar{D})$ is called a weak solution of problem (1.1), if u satisfies

$$\mathcal{Q}(u, \phi) = \int_D f(x, u(x)) \phi(x) dx \quad \text{for any } \phi \in C_0^\infty(D),$$

and $u(x) = \varphi(x)$ when $x \in \partial D$.

Now we need to recall the notion of VMO functions in order to apply the $W^{1,p}$ estimates for the divergence operators. A locally integrable function g is said to be in the VMO space, if

$$\limsup_{\varrho \rightarrow 0} \sup_{r \leq \varrho} \sup_{B_r \subset D} \frac{1}{|B_r|} \int_{B_r} |g(x) - g_{B_r}| dx = 0,$$

where B_r denotes a ball of radius r and

$$g_{B_r} = \frac{1}{|B_r|} \int_{B_r} g(x) dx.$$

Assumption I

(I.1) D is a bounded, connected C^1 -domain.

(I.2) Each a_{ij} , $1 \leq i, j \leq d$, is continuous and belongs to the VMO space.

Remark 2.1 Under the Assumption I, it follows from Theorem 1 in [3] that there exists a unique element $v \in W_0^{1,2p}(D)$ such that

$$\int_D \langle a(x) \nabla v(x), \nabla \phi(x) \rangle dx = -2 \int_D \langle \hat{b}(x), \nabla \phi(x) \rangle dx, \quad \forall \phi \in W_0^{1,2}(D).$$

Set $L_0 u = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial u}{\partial x_j})$. Let $(X_t, P_x^0, x \in \mathbb{R}^d)$ be the diffusion process generated by the infinitesimal operator L_0 . It is well-known that (X_t, P_x^0) is a conservative Feller process on $\mathbb{R}^d$ that has continuous transition density function which admits a two-sided Aronson's heat kernel estimate (see Aronson [1, 2]). Moreover, by Theorem IV.2.5 in [14], for quasi-everywhere (*q.e.*) $x \in \mathbb{R}^d$ we have the Fukushima's decomposition

$$X(t) - X(0) = M_t^0 + N_t^0, \quad P_x^0\text{-a.s.}, \quad (2.1)$$

where M_t^0 is a martingale additive functional (MAF) and N_t^0 is a continuous additive functional (CAF) of locally zero energy. However the decomposition (2.1) can be strengthened to hold for every $x \in \mathbb{R}^d$. This follows from the existence of the heat kernel, Theorem 3.5.4 in [15] and Theorem 2 in [11]. Let v be the function stated in Remark 2.1. By Sobolev embedding theorem, $v \in C(\mathbb{R}^d)$ if we extend $v = 0$ on D^c . Moreover, for any $x \in \mathbb{R}^d$ the Fukushima's decomposition holds

$$v(X(t)) - v(X(0)) = M_t^v + N_t^v, \quad P_x^0\text{-a.e.}, \quad (2.2)$$

where M_t^v is a MAF and N_t^v is a CAF of zero energy.

Assume that $f(x, y)$ is a measurable function, which is continuous w.r.t. y and satisfies

$$\begin{cases} (y_1 - y_2)(f(x, y_1) - f(x, y_2)) \leq -J(x)|y_1 - y_2|^2, \\ |f(x, y)| \leq C(1 + J_1(x)|y|), \end{cases}$$

for some constant C and $J, J_1 \in L^p(D)$. Finally set

$$\tilde{a} = \{\tilde{a}_{ij}(x)\}_{1 \leq i, j \leq d} = \left\{ \frac{a_{ij}(x) + a_{ji}(x)}{2} \right\}_{1 \leq i, j \leq d}.$$

3 BSDEs with Singular Coefficients

In this section, we will obtain the existence and uniqueness of solutions of a class of BSDEs with singular coefficients and random terminal time. The results is of independent interest on its own right and will be used in the subsequent sections. Recall that $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, P_x, x \in \mathbb{R}^d)$ denotes the diffusion process associated with the generator L_1 given by

$$L_1 u = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial u}{\partial x_j}) + \sum_{i=1}^d b_i(x) \frac{\partial u}{\partial x_i}, \quad (3.1)$$

where $\mathcal{F}_t$ is the completed minimal admissible filtration generated by X_t . Similar to (2.1) we have the following Fukushima's decomposition: for any $x \in \mathbb{R}^d$

$$X(t) - X(0) = M_t + N_t, \quad P_x\text{-a.e.}, \quad (3.2)$$

where $M_t = (M_t^1, M_t^2, \dots, M_t^d)$ is a MAF of finite energy and N_t is CAF of locally zero quadratic variation. Moreover,

$$\langle M^i, M^j \rangle_t = \int_0^t \tilde{a}_{ij}(X(s)) ds.$$

Given a $h \in L^p(D)$. Let F be a Borel measurable function on D satisfying

$$|F(x)| \leq C(1 + |h(x)|), \quad (3.3)$$

for some constant $C > 0$. Then similar to Theorem 3.18 and Theorem 4.6 in [9], we have the following result.

Lemma 3.1 Assume there exists a $x_0 \in D$ such that

$$E_{x_0}[e^{-\int_0^{\tau_D} h(X(s)) ds}] < \infty, \quad (3.4)$$

then for any F satisfying (3.3) we have

$$\sup_{x \in D} E_x[\int_0^{\tau_D} e^{-\int_0^t h(X(s)) ds} F(X(t)) dt] < \infty. \quad (3.5)$$

Proof Let $G_D(x, y)$ denote the Green function on the domain D associated with the operator L_1 . It is known from (2.14) in [5] that

$$\int_D G_D(x, y) |h(y)| dy = E_x[\int_0^{\tau_D} |h(X(s))| ds] \leq \sigma \|h\|_{L^p(D)}, \quad (3.6)$$

for some constant σ . Since h is L^p integrable, one can choose $\delta > 0$ such that for any $B \subset D$ with $m(B) < \delta$, we have $\|I_B h\|_{L^p(D)} < \frac{1}{\sigma}$. Hence

$$\sup_{B \subset D: m(B) < \delta} \sup_{x \in D} \int_B G_D(x, y) |h(y)| dy < 1.$$

So h belongs to the class of functions K_1 defined in [6]. It thus follows from Theorem 2.2 in [6] and (3.4) that

$$E_x[e^{-\int_0^{\tau_D} h(X(s)) ds}] \leq C_1, \quad \forall x \in D, \quad (3.7)$$

where C_1 is a positive constant. Moreover by (3.6)

$$\inf_{x \in D} E_x[e^{-\int_0^{\tau_D} h(X(s)) ds}] \geq \inf_{x \in D} \exp(-E_x[\int_0^{\tau_D} h(X(s)) ds]) \geq \exp(-\sigma \|h\|_{L^p(D)}). \quad (3.8)$$

On the other hand, it was shown in [1, 2] that the transition density function $p(t, x, y)$ of $(X_t, P_x, x \in \mathbb{R}^d)$ has a Gaussian upper bound estimate:

$$p(t, x, y) \leq \frac{\sigma_1}{t^{d/2}} e^{-\frac{\sigma_2 |x-y|^2}{t}}, \quad \forall (t, x, y) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d \quad (3.9)$$

for some constants $\sigma_1, \sigma_2 > 0$. This bound together with $h \in L^p(D)$ implies that

$\sup_{x \in D} E_x[\int_0^\delta |h(X(s))| ds] < 1$ for some $\delta > 0$. Hence similar to the proof of Theorem 4.6 in [9] we can show that

$$\int_0^\infty \sup_{x \in D} E_x[e^{-\int_0^t h(X(s)) ds}; t < \tau_D] dt < \infty.$$

The remainder of the proof of (3.5) is the same as that of Theorem 3.18 in [9]. we omit the details. $\square$

Fix a probability measure μ on $\mathbb{R}^d$ and let $P = P_\mu$ denote the probability law of the diffusion process X starting with the initial distribution μ , namely, $P(\cdot) = \int_{\mathbb{R}^d} P_x(\cdot) \mu(dx)$. We now consider BSDEs driven by the martingale part M in (3.2) with random terminal time on the probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Denote by E the expectation under P .

Let $g(t, y, \omega) : [0, \infty] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ be a progressively measurable function. For notational convenience, we omit the random parameter ω . Given a finite stopping time τ and a random variable $\xi \in \mathcal{F}_\tau$. Let us recall the definition of a solution of the BSDEs with random terminal time.

Definition 3.1 A pair (Y, Z) is a solution to the following BSDE

$$Y(t) = \xi + \int_{t \wedge \tau}^{\tau} g(s, Y(s)) ds - \int_{t \wedge \tau}^{\tau} \langle Z(s), dM_s \rangle, \quad (3.10)$$

if Y is a $\mathbb{R}$ -valued progressively measurable process and Z is a $\mathbb{R}^d$ -valued predictable process such that:

$$g(\cdot, Y) \in L^1([0, \tau], ds), \quad Z \in L^2([0, \tau], ds), \quad P\text{-a.s.}$$

and (3.10) is valid P -a.s. for every $t \geq 0$,

To prove the existence and uniqueness of a solution to the BSDE (3.10), we need the following lemma which can be found in [4].

Lemma 3.2 If (Y, Z) is a solution to the BSDE (3.10), then for any $0 \leq t \leq u \leq \tau$,

$$|Y(t)| \leq |Y(u)| + \int_t^u \frac{Y(s)}{|Y(s)|} I_{\{Y(s) \neq 0\}} g(s, Y(s)) ds - \int_t^u \left\langle \frac{Y(s)}{|Y(s)|} I_{\{Y(s) \neq 0\}} Z(s), dM_s \right\rangle.$$

Suppose that g is continuous with respect to the variable y and satisfies

$$\begin{cases} (y_1 - y_2)(g(t, y_1) - g(t, y_2)) \leq -K_1(t)|y_1 - y_2|^2, \\ |g(t, y) - g(t, 0)| \leq K_2(t) + K_3(t)|y|, \end{cases} \quad (3.11)$$

where $K_1(t), K_2(t) \geq 0, K_3(t) \geq 0$ are progressively measurable stochastic processes.

For any $0 < \beta < 1$, denote by φ^β the set of the real-valued, adapted, continuous processes $Y(t)$ such that

$$\|Y\|_{\varphi^\beta} = E[\sup_{t \geq 0} |Y(t)|^\beta] < \infty. \quad (3.12)$$

If we define $\rho(Y_1, Y_2) = \|Y_1 - Y_2\|_{\varphi^\beta}$ as a distance in φ^β , then φ^β is complete under this distance. Denote by M^β the set of $\mathbb{R}^d$ -valued predictable processes $Z(t)$ such that

$$\|Z\|_{M^\beta} = E[(\int_0^\infty |Z(t)|^2 dt)^{\frac{\beta}{2}}] < \infty. \quad (3.13)$$

Then M^β is also complete under the distance $\rho'(Z_1, Z_2) = \|Z_1 - Z_2\|_{M^\beta}$.

Let Γ denote the set of the stopping times. We say that a progressively measurable process Y is in Class (D) if the family $\{Y(T)I_{\{T < \infty\}} : T \in \Gamma\}$ is uniformly integrable. We put

$$\|Y\|_1 = \sup\{E[|Y(T)|I_{\{T < \infty\}}] : T \in \Gamma\}.$$

Then it is known that the space of adapted continuous processes which belong to class (D) is complete under $\|\cdot\|_1$ norm, see Chapter VI 21 in [10]. Here is the main result of this section.

Theorem 3.2 Assume

$$E[e^{-\int_0^\tau K_1(s)ds} \xi] < \infty, \quad (3.14)$$

and

$$E\left[\int_0^\tau (e^{-\int_0^t K_1(s)ds} (K_2(t) + |g(t, 0)|) + |K_1(t)| + K_3(t))dt\right] < \infty. \quad (3.15)$$

Then the BSDE (3.10) has a unique solution (Y, Z) such that $e^{-\int_0^t K_1(s)ds} Y(t)$ belongs to Class (D). Moreover, for any $0 < \beta < 1$, $(e^{-\int_0^t K_1(s)ds} Y(t), e^{-\int_0^t K_1(s)ds} Z(t)) \in \varphi^\beta \times M^\beta$.

Proof We set $g(s, y) = K_1(s) = K_2(s) = K_3(s) = 0$ for $s \geq \tau$ for convenience. Set $\tilde{Y}(t) = e^{-\int_0^t K_1(s)ds} Y(t)$, $\tilde{Z}(t) = e^{-\int_0^t K_1(s)ds} Z(t)$, $\tilde{g}(t, y) = e^{-\int_0^t K_1(s)ds} g(t, e^{\int_0^t K_1(s)ds} y) + K_1(t)y$ and $\tilde{\xi} = e^{-\int_0^\tau K_1(s)ds} \xi$. Then it is easy to see that (Y, Z) is a solution to the BSDE (3.10) if and only if $(\tilde{Y}, \tilde{Z})$ satisfies the following BSDE:

$$\tilde{Y}(t) = \tilde{\xi} + \int_{t \wedge \tau}^\tau \tilde{g}(s, \tilde{Y}(s))ds - \int_{t \wedge \tau}^\tau \langle \tilde{Z}(s), dM_s \rangle. \quad (3.16)$$

Therefore, we only need to establish the existence and uniqueness of the BSDE (3.16). Note that $\tilde{g}(t, y)$ is also continuous with respect to y and

$$\begin{cases} (y_1 - y_2)(\tilde{g}(t, y_1) - \tilde{g}(t, y_2)) \leq 0, \\ E[\int_0^\tau |\tilde{g}(t, 0)|dt] < \infty. \end{cases} \quad (3.17)$$

Moreover for any $n \in \mathbb{N}$,

$$E\left[\int_0^n e^{-\int_0^t K_1(s)ds} K_2(t)dt\right] = E\left[\int_0^{\tau \wedge n} e^{-\int_0^t K_1(s)ds} K_2(t)dt\right] \leq E\left[\int_0^\tau e^{-\int_0^t K_1(s)ds} K_2(t)dt\right] < \infty,$$

where the first inequality is due to the fact $K_1(s) = K_2(s) = 0$ when $s \geq \tau$.

By (3.11) and (3.15), for any $r > 0$, we have

$$\begin{aligned} & \sup_{|y| \leq r} |\tilde{g}(t, y) - \tilde{g}(t, 0)| \\ & \leq \sup_{|y| \leq r} e^{-\int_0^t K_1(s)ds} |g(t, e^{\int_0^t K_1(s)ds} y) - g(t, 0)| + |K_1(t)|r \\ & \leq e^{-\int_0^t K_1(s)ds} K_2(t) + (|K_1(t)| + K_3(t))r \in L^1([0, n] \times \Omega), \quad \forall n \in \mathbb{N}. \end{aligned} \quad (3.18)$$

Set $\tilde{\xi}^N = (-N) \vee \tilde{\xi} \wedge N$. Note that the martingale representation theorem with respect to the martingale part M is valid according to Theorem 2.1 in [18]. Thus, by (3.17)–(3.18) and Proposition 6.4 in [4] for any $n \in \mathbb{N}$, there exists a solution pair $(\tilde{Y}_n^N, \tilde{Z}_n^N)$ to the following BSDE

$$\tilde{Y}_n^N(t) = E[\tilde{\xi}^N | \mathcal{F}_n] + \int_t^n I_{s \leq \tau} \tilde{g}(s, \tilde{Y}_n^N(s))ds - \int_t^n \langle \tilde{Z}_n^N(s), dM_s \rangle, \quad 0 \leq t \leq n, \quad (3.19)$$

and moreover $\tilde{Y}_n^N \in \text{Class (D)}$, $(\tilde{Y}_n^N, \tilde{Z}_n^N) \in \bigcap_{0 < \beta < 1} \varphi^\beta \times M^\beta$, where we set $\tilde{Y}_n^N(t) = E[\tilde{\xi}^N | \mathcal{F}_n]$, $\tilde{Z}_n^N(t) = 0$ for $t > n$.

Next we are going to show that $\{\tilde{Y}_n^N\}_{n \geq 1}$ is a Cauchy sequence under norm $\|\cdot\|_1$.

For $n, i \in \mathbb{N}$, set $\delta \tilde{Y}_{n+i}^{n+i}(s) = \tilde{Y}_{n+i}^{n+i}(s) - \tilde{Y}_n^N(s)$, $\delta \tilde{Z}_{n+i}^{n+i}(s) = \tilde{Z}_{n+i}^{n+i}(s) - \tilde{Z}_n^N(s)$ and $\delta \tilde{g}_n(s, y) = I_{s \leq \tau} [\tilde{g}(s, \tilde{Y}_n^N(s) + y) - \tilde{g}(s, \tilde{Y}_n^N(s)) + \tilde{g}(s, \tilde{Y}_n^N(n))I_{s \geq n}]$. Then $(y_1 - y_2)(\delta \tilde{g}_n(t, y_1) - \delta \tilde{g}_n(t, y_2)) \leq 0$. By (3.19) we have for any $t \geq 0$

$$\begin{aligned} \delta \tilde{Y}_n^{n+i}(t) &= E[\tilde{\xi}^N | \mathcal{F}_{n+i}] - E[\tilde{\xi}^N | \mathcal{F}_n] + \int_{t \wedge (n+i)}^{n+i} \delta \tilde{g}_n(s, \delta \tilde{Y}_n^{n+i}(s))ds \\ &\quad - \int_{t \wedge (n+i)}^{n+i} \langle \delta \tilde{Z}_n^{n+i}(s), dM_s \rangle. \end{aligned} \quad (3.20)$$

Choose a sequence of stopping times $\{\tau_k\}_{k \geq 1}$ that increases to ∞ and such that $\int_0^{t \wedge \tau_k} \langle \frac{\delta \tilde{Y}_n^{n+i}(s)}{|\delta \tilde{Y}_n^{n+i}(s)|} \delta \tilde{Z}_n^{n+i}(s), dM_s \rangle$ is a martingale. Then by (3.20) we have

$$\begin{aligned} \delta \tilde{Y}_n^{n+i}(t \wedge \tau_k) &= \delta \tilde{Y}_n^{n+i}((n+i) \wedge \tau_k) + \int_{t \wedge (n+i) \wedge \tau_k}^{(n+i) \wedge \tau_k} \delta \tilde{g}_n(s, \delta \tilde{Y}_n^{n+i}(s)) ds \\ &\quad - \int_{t \wedge (n+i) \wedge \tau_k}^{(n+i) \wedge \tau_k} \langle \delta \tilde{Z}_n^{n+i}(s), dM_s \rangle. \end{aligned}$$

It follows from Lemma 3.2 that

$$\begin{aligned} |\delta \tilde{Y}_n^{n+i}(t \wedge \tau_k)| &\leq |\delta \tilde{Y}_n^{n+i}((n+i) \wedge \tau_k)| + \int_{t \wedge (n+i) \wedge \tau_k}^{(n+i) \wedge \tau_k} \frac{\delta \tilde{Y}_n^{n+i}(s)}{|\delta \tilde{Y}_n^{n+i}(s)|} \delta \tilde{g}_n(s, \delta \tilde{Y}_n^{n+i}(s)) ds \\ &\quad - \int_{t \wedge (n+i) \wedge \tau_k}^{(n+i) \wedge \tau_k} \langle \frac{\delta \tilde{Y}_n^{n+i}(s)}{|\delta \tilde{Y}_n^{n+i}(s)|} \delta \tilde{Z}_n^{n+i}(s), dM_s \rangle \\ &\leq |\delta \tilde{Y}_n^{n+i}((n+i) \wedge \tau_k)| + \int_{t \wedge (n+i) \wedge \tau_k}^{(n+i) \wedge \tau_k} |\delta \tilde{g}_n(s, 0)| ds \\ &\quad - \int_{t \wedge (n+i) \wedge \tau_k}^{(n+i) \wedge \tau_k} \langle \frac{\delta \tilde{Y}_n^{n+i}(s)}{|\delta \tilde{Y}_n^{n+i}(s)|} \delta \tilde{Z}_n^{n+i}(s), dM_s \rangle. \end{aligned}$$

Take the conditional expectation with respect to $\mathcal{F}_t$ to obtain

$$\begin{aligned} |\delta \tilde{Y}_n^{n+i}(t \wedge \tau_k)| &\leq E[|\delta \tilde{Y}_n^{n+i}((n+i) \wedge \tau_k)| | \mathcal{F}_t] + E[\int_{t \wedge (n+i) \wedge \tau_k}^{(n+i) \wedge \tau_k} |\delta \tilde{g}_n(s, 0)| ds | \mathcal{F}_t] \\ &= E[|\delta \tilde{Y}_n^{n+i}((n+i) \wedge \tau_k)| | \mathcal{F}_t] + E[\int_{t \wedge (n+i) \wedge \tau_k}^{(n+i) \wedge \tau_k} |\delta \tilde{g}_n(s, 0)| ds | \mathcal{F}_t], \end{aligned} \quad (3.21)$$

here we have used the fact that $\int_0^{(n+i) \wedge \tau_k \wedge t} \langle \frac{\delta \tilde{Y}_n^{n+i}(s)}{|\delta \tilde{Y}_n^{n+i}(s)|} \delta \tilde{Z}_n^{n+i}(s), dM_s \rangle, t \geq 0$ is also a martingale with respect to the filtration $\mathcal{F}_t, t \geq 0$.

Since $\delta \tilde{Y}_n^{n+i} = \tilde{Y}_{n+i}^N - \tilde{Y}_n^N$ is also in Class (D) and $\delta \tilde{Y}_n^{n+i}(t)$ is continuous, we get

$$\lim_{k \rightarrow \infty} \delta \tilde{Y}_n^{n+i}((n+i) \wedge \tau_k) = \delta \tilde{Y}_n^{n+i}(n+i) \text{ in } L^1(P; \Omega). \quad (3.22)$$

Since

$$\begin{aligned} &\int_{t \wedge (n+i) \wedge \tau_k}^{(n+i) \wedge \tau_k} |\delta \tilde{g}_n(s, 0)| ds \\ &\leq \int_n^{(n+i) \wedge \tau} (|\tilde{g}(s, \tilde{Y}_n^N(n)) - \tilde{g}(s, 0)| + |\tilde{g}(s, 0)|) ds \\ &\leq \int_n^{(n+i) \wedge \tau} [K_2(s) e^{-\int_0^s K_1(r) dr} + (|K_1(s)| + K_3(s)) |\tilde{Y}_n^N(n)|] ds + \int_n^{(n+i) \wedge \tau} (|\tilde{g}(s, 0)|) ds \\ &\leq \int_n^{(n+i) \wedge \tau} K_2(s) e^{-\int_0^s K_1(r) dr} ds + N \int_n^{(n+i) \wedge \tau} (|K_1(s)| + K_3(s)) ds \\ &\quad + \int_n^{(n+i) \wedge \tau} (|\tilde{g}(s, 0)|) ds \in L^1(P; \Omega), \end{aligned} \quad (3.23)$$

by the dominated convergence theorem we have

$$\lim_{k \rightarrow \infty} \int_{t \wedge \tau_k}^{(n+i) \wedge \tau_k} |\delta \tilde{g}_n(s, 0)| ds = \int_{t \wedge \tau}^{(n+i) \wedge \tau} |\delta \tilde{g}_n(s, 0)| ds \text{ in } L^1(P; \Omega). \quad (3.24)$$

Letting $k \rightarrow \infty$ in (3.21) and noticing $\delta \tilde{g}_n(s, 0) = 0$ for $s > \tau$, it follows from (3.22) and (3.24) that

$$|\delta \tilde{Y}_n^{n+i}(t)| \leq E[|\delta \tilde{Y}_n^{n+i}(n+i)| | \mathcal{F}_t] + E[\int_{n \wedge \tau}^{(n+i) \wedge \tau} |\delta \tilde{g}_n(s, 0)| ds | \mathcal{F}_t]. \quad (3.25)$$

Hence,

$$\begin{aligned} \|\delta \tilde{Y}_n^{n+i}\|_1 &= \sup_{T: T \in \Gamma} E[|\delta \tilde{Y}_n^{n+i}(T)| I_{\{T < \infty\}}] \\ &\leq \sup_{T: T \in \Gamma} E[E[|\delta \tilde{Y}_n^{n+i}(n+i)| | \mathcal{F}_T]] + E[E[\int_{n \wedge \tau}^{(n+i) \wedge \tau} |\delta \tilde{g}_n(s, 0)| ds | \mathcal{F}_T]] \\ &= E[|\delta \tilde{Y}_n^{n+i}(n+i)|] + E[\int_{n \wedge \tau}^{(n+i) \wedge \tau} |\tilde{g}(s, \tilde{Y}_n^N(n))| ds]. \end{aligned} \quad (3.26)$$

Since $\tilde{\xi}^N$ is bounded, it holds that

$$\delta \tilde{Y}_n^{n+i}(n+i) = E[\tilde{\xi}^N | \mathcal{F}_{n+i}] - E[\tilde{\xi}^N | \mathcal{F}_n] \rightarrow 0 \text{ in } L^1(\Omega) \quad (3.27)$$

as $n \rightarrow \infty$. By (3.11), we have

$$\begin{aligned}
 & |\tilde{g}(s, \tilde{Y}_n^N(n))| \\
 & \leq e^{-\int_0^s K_1(r)dr} (|g(s, e^{\int_0^s K_1(r)dr} \tilde{Y}_n^N(n)) - g(s, 0)| \\
 & \quad + e^{-\int_0^s K_1(r)dr} |g(s, 0)| + N|K_1(s)| \\
 & \leq e^{-\int_0^s K_1(r)dr} K_2(s) + NK_3(s) \\
 & \quad + e^{-\int_0^s K_1(r)dr} |g(s, 0)| + N|K_1(s)|.
 \end{aligned} \tag{3.28}$$

Thus the condition (3.15) implies that

$$E\left[\int_0^\tau |\tilde{g}(s, \tilde{Y}_n^N(n))|ds\right] < \infty. \tag{3.29}$$

Combining (3.26), (3.27) and (3.29) we see that $\{\tilde{Y}_n^N\}_{n \geq 1}$ is a Cauchy sequence with respect to the $\|\cdot\|_1$ norm. Thus there exists a $\tilde{Y}^N \in \text{Class (D)}$ such that $\lim_{n \rightarrow \infty} \|\tilde{Y}_n^N - \tilde{Y}^N\|_1 = 0$. It follows from [10] that there exists a subsequence of $\{\tilde{Y}_n^N\}_{n \geq 1}$ that converges almost surely to $\tilde{Y}^N$ uniformly on $[0, \infty)$. Hence $\tilde{Y}^N$ is a continuous process.

Next we will show that $\tilde{Y}_n^N$ also converges to $\tilde{Y}^N$ in φ^β for any $0 < \beta < 1$.

Let $\eta_n = |\delta \tilde{Y}_n^{n+i}(n+i)| + \int_{n \wedge \tau}^{(n+i) \wedge \tau} |\delta \tilde{g}_n(s, 0)|ds$, $M_n(t) = E[\eta_n | \mathcal{F}_t]$. Then by Lemma 6.1 in [4] we have $E[\sup_{t \geq 0} |M_n(t)|^\beta] \leq \frac{1}{1-\beta} E[|\eta_n|]^\beta$. Together with (3.25) we get

$$E[\sup_{t \geq 0} |\delta \tilde{Y}_n^{n+i}(t)|^\beta] \leq E[\sup_{t \geq 0} |M_n(t)|^\beta] \leq \frac{1}{1-\beta} E[|\eta_n|]^\beta \rightarrow 0$$

as $n \rightarrow \infty$. Therefore $\tilde{Y}^N \in \varphi^\beta$ and

$$\lim_{n \rightarrow \infty} E[\sup_{t \geq 0} |\tilde{Y}_n^N(t) - \tilde{Y}^N(t)|^\beta] = 0. \tag{3.30}$$

Next we will establish the convergence of $\{\tilde{Z}_n^N\}_{n \geq 1}$.

By (3.20) and Ito's formula,

$$\begin{aligned}
 & |\delta \tilde{Y}_n^{n+i}(t)|^2 + \int_t^{n+i} \langle \tilde{a}(X(s)) \delta \tilde{Z}_n^{n+i}(s), \delta \tilde{Z}_n^{n+i}(s) \rangle ds \\
 & = |\delta \tilde{Y}_n^{n+i}(n+i)|^2 + 2 \int_t^{(n+i)} \delta \tilde{Y}_n^{n+i}(s) \delta \tilde{g}_n(s, \delta \tilde{Y}_n^{n+i}(s)) ds \\
 & \quad - 2 \int_t^n \langle \delta \tilde{Y}_n^{n+i}(s) \delta \tilde{Z}_n^{n+i}(s), dM_s \rangle.
 \end{aligned}$$

Using (1.2) and (3.17), we deduce that

$$\begin{aligned}
 \lambda \int_0^{n+i} |\delta \tilde{Z}_n^{n+i}(s)|^2 ds & \leq |\delta \tilde{Y}_n^{n+i}(n+i)|^2 + 2 \int_0^{(n+i)} \delta \tilde{Y}_n^{n+i}(s) \delta \tilde{g}_n(s, 0) ds \\
 & \quad - 2 \int_0^{n+i} \langle \delta \tilde{Y}_n^{n+i}(s) \delta \tilde{Z}_n^{n+i}(s), dM_s \rangle.
 \end{aligned}$$

It follows now from the Burkholder's inequality that

$$\begin{aligned}
 & E[(\int_0^\infty |\delta \tilde{Z}_n^{n+i}(s)|^2 ds)^{\frac{\beta}{2}}] = E[(\int_0^{n+i} |\delta \tilde{Z}_n^{n+i}(s)|^2 ds)^{\frac{\beta}{2}}] \\
 & \leq C_\beta E[|\delta \tilde{Y}_n^{n+i}(n+i)|^\beta] + 2E[\sup_{t \geq 0} |\delta \tilde{Y}_n^{n+i}(t)|^{\frac{\beta}{2}} (\int_n^{(n+i)} \delta \tilde{g}_n(s, 0) ds)^{\frac{\beta}{2}}] \\
 & \quad + 2C_\beta E[(\int_0^{n+i} |\delta \tilde{Y}_n^{n+i}(s)|^2 |\delta \tilde{Z}_n^{n+i}(s)|^2 ds)^{\frac{\beta}{4}}] \\
 & \leq C_\beta \{E[|\delta \tilde{Y}_n^{n+i}(n+i)|^\beta] + E[\sup_{t \geq 0} |\delta \tilde{Y}_n^{n+i}(t)|^\beta] + E[(\int_n^{(n+i)} \delta \tilde{g}_n(s, 0) ds)^\beta]\} \\
 & \quad + 2C_\beta E[\sup_{t \geq 0} |\delta \tilde{Y}_n^{n+i}(t)|^{\frac{\beta}{2}} (\int_0^{n+i} |\delta \tilde{Z}_n^{n+i}(s)|^2 ds)^{\frac{\beta}{4}}] \\
 & \leq C_\beta \{2 \sup_{t \geq 0} E[|\delta \tilde{Y}_n^{n+i}(t)|^\beta] + [E(\int_n^{(n+i)} \delta \tilde{g}_n(s, 0) ds)^\beta]\} \\
 & \quad + 2C_\beta^2 E[\sup_{t \geq 0} |\delta \tilde{Y}_n^{n+i}(t)|^\beta] + \frac{1}{2} E[(\int_0^{n+i} |\delta \tilde{Z}_n^{n+i}(s)|^2 ds)^{\frac{\beta}{2}}].
 \end{aligned}$$

Thus

$$E[(\int_0^\infty |\delta \tilde{Z}_n^{n+i}(s)|^2 ds)^\beta] \leq (4C_\beta + 4C_\beta^2) E[\sup_{t \geq 0} |\delta \tilde{Y}_n^{n+i}(t)|^\beta] + 2C_\beta [E(\int_n^{(n+i)} \delta \tilde{g}_n(s, 0) ds)]^\beta \rightarrow 0$$

as $n \rightarrow \infty$, where (3.17) was used. So there exists $\tilde{Z}^N \in M^\beta$ such that $\lim_{n \rightarrow \infty} E[(\int_0^\infty |\tilde{Z}_n^{n+i}(s) - \tilde{Z}^N(s)|^2 ds)^\beta] = 0$.

Now we will show that $(\tilde{Y}^N, \tilde{Z}^N)$ is a solution of the equation

$$\tilde{Y}^N(t) = \tilde{\xi}^N + \int_{t \wedge \tau}^\tau \tilde{g}(s, \tilde{Y}^N(s)) ds - \int_{t \wedge \tau}^\tau \langle \tilde{Z}^N(s), dM_s \rangle. \quad (3.31)$$

By (3.30) we can find a subsequence $\{n_k\}_{k \geq 1}$ such that

$$\lim_{n_k \rightarrow \infty} \sup_{s \geq 0} |\tilde{Y}_{n_k}^N(s) - \tilde{Y}^N(s)| = 0, \quad P\text{-a.e.}$$

By (3.17) and (3.18) for P-a.e. ω ,

$$\begin{aligned} |\tilde{g}(s, \tilde{Y}_{n_k}^N(s))| &\leq |\tilde{g}(s, 0)| + K_2(s) e^{-\int_0^s K_1(r) dr} + |K_1(s) \tilde{Y}_{n_k}^N(s)| + |K_3(s) \tilde{Y}_{n_k}^N(s)| \\ &\leq |\tilde{g}(s, 0)| + K_2(s) e^{-\int_0^s K_1(r) dr} + C(\omega)(|K_1(s)| + |K_3(s)|) \in L^1([0, \tau(\omega)], ds), \end{aligned}$$

where $C(\omega) = \sup_{s \geq 0} \sup_{n_k: k \geq 1} |\tilde{Y}_{n_k}^N(s, \omega)| + |\tilde{Y}^N(s, \omega)|$. Hence using the dominate convergence theorems we get

$$\lim_{n_k \rightarrow \infty} \int_{t \wedge \tau}^\tau \tilde{g}(s, \tilde{Y}_{n_k}^N(s)) ds = \int_{t \wedge \tau}^\tau \tilde{g}(s, \tilde{Y}^N(s)) ds, \quad P\text{-a.s.}$$

On the other hand, since $\tilde{Z}_n^N$ converges to $\tilde{Z}^N$ in M^β , $\int_{t \wedge \tau}^\tau \langle \tilde{Z}_n^N(s), dM_s \rangle$ converges to $\int_{t \wedge \tau}^\tau \langle \tilde{Z}^N(s), dM_s \rangle$ in probability as $n \rightarrow \infty$. Note also that $E[\tilde{\xi}^N | \mathcal{F}_{n_k}]$ converges to $\tilde{\xi}^N$ in probability as $k \rightarrow \infty$. So replacing n by n_k in (3.19) and letting $k \rightarrow \infty$ we see that $(\tilde{Y}^N, \tilde{Z}^N)$ satisfies (3.31).

Finally we show that there exists a solution to BSDE (3.16). For $N, N_1 \in \mathbb{N}$, let $\delta \tilde{Y}^N(t) = \tilde{Y}^{N+N_1}(t) - \tilde{Y}^N(t)$, $\delta \tilde{Z}^N(t) = \tilde{Z}^{N+N_1}(t) - \tilde{Z}^N(t)$ and $\delta \tilde{g}(t, y) = \tilde{g}(t, y + \tilde{Y}^N(t)) - \tilde{g}(t, \tilde{Y}^N(t))$. Then by (3.17)

$$y \cdot \delta \tilde{g}(t, y) = y(\tilde{g}(t, y + \tilde{Y}^N(t)) - \tilde{g}(t, \tilde{Y}^N(t))) \leq 0. \quad (3.32)$$

From the equation (3.31) we have

$$\delta \tilde{Y}^N(t) = \tilde{\xi}^{N+N_1} - \tilde{\xi}^N + \int_{t \wedge \tau}^\tau \delta \tilde{g}(s, \delta \tilde{Y}^N(s)) ds - \int_{t \wedge \tau}^\tau \langle \delta \tilde{Z}^N(s), dM_s \rangle. \quad (3.33)$$

Since $\delta \tilde{Y}^N(t) = \tilde{Y}^{N+N_1}(t) - \tilde{Y}^N(t)$ is also in Class (D), using Lemma 2.2 and the similar arguments as in the proof of (3.25), we obtain

$$|\delta \tilde{Y}^N(t)| \leq E[|\tilde{\xi}^{N+N_1} - \tilde{\xi}^N| | \mathcal{F}_t]. \quad (3.34)$$

This together with (3.14) imply

$$\|\delta \tilde{Y}^N\|_1 = \sup_{T: T \in \Gamma} E[|\delta \tilde{Y}^N(T)| I_{\{T < \infty\}}] \leq E[|\tilde{\xi}^{N+N_1} - \tilde{\xi}^N|] \rightarrow 0$$

as $N \rightarrow \infty$. Hence there exists a continuous process $\tilde{Y} \in \text{Class (D)}$ such that $\lim_{N \rightarrow \infty} \|\tilde{Y}^N - \tilde{Y}\|_1 = 0$. For any $0 < \beta < 1$, using Lemma 6.1 in [4] and (3.34) we get

$$E[\sup_{t \geq 0} |\delta \tilde{Y}^N(t)|^\beta] \leq \frac{1}{1-\beta} (E[|\tilde{\xi}^{N+N_1} - \tilde{\xi}^N|]^\beta). \quad (3.35)$$

This implies that

$$\lim_{N \rightarrow \infty} E[\sup_{t \geq 0} |\tilde{Y}^N(t) - \tilde{Y}(t)|^\beta] = 0. \quad (3.36)$$

Applying Ito's formula to $\tilde{Y}^N$ in (3.33) and using (3.32), we obtain

$$\begin{aligned} & |\delta \tilde{Y}^N(t)|^2 + \int_{t \wedge \tau}^\tau \langle \tilde{a}(X(s)) \delta \tilde{Z}^N(s), \delta \tilde{Z}^N(s) \rangle ds \\ & \leq |\tilde{\xi}^{N+N_1} - \tilde{\xi}^N|^2 + 2 \int_{t \wedge \tau}^\tau \delta \tilde{Y}^N(s) \delta \tilde{g}(s, \delta \tilde{Y}^N(s)) ds \\ & \quad - 2 \int_{t \wedge \tau}^\tau \langle \delta \tilde{Y}^N(s) \delta \tilde{Z}^N(s), dM_s \rangle \\ & \leq |\tilde{\xi}^{N+N_1} - \tilde{\xi}^N|^2 - 2 \int_{t \wedge \tau}^\tau \langle \delta \tilde{Y}^N(s) \delta \tilde{Z}^N(s), dM_s \rangle. \end{aligned} \quad (3.37)$$

By (1.2), (3.37) and Burkholder's inequality, we have

$$\begin{aligned} E[(\int_0^\tau |\delta \tilde{Z}^N(s)|^2 ds)^{\frac{\beta}{2}}] & \leq C_\beta \{E[|\tilde{\xi}|_{\tilde{\xi} \geq N}^\beta] + 2C_\beta E[(\int_0^\tau |\delta \tilde{Y}^N(s) \delta \tilde{Z}^N(s)|^2 ds)^{\frac{\beta}{4}}]\} \\ & \leq C_\beta \{E[|\tilde{\xi}|_{\tilde{\xi} \geq N}^\beta] + 2E[\sup_{s \geq 0} |\delta \tilde{Y}^N(s)|^{\frac{\beta}{2}} (\int_0^\tau |\delta \tilde{Z}^N(s)|^2 ds)^{\frac{\beta}{4}}]\} \\ & \leq C_\beta (E[|\tilde{\xi}|_{\tilde{\xi} \geq N}^\beta] + 2C_\beta^2 E[\sup_{s \geq 0} |\delta \tilde{Y}^N(s)|^\beta] + \frac{1}{2} E[(\int_0^\tau |\delta \tilde{Z}^N(s)|^2 ds)^{\frac{\beta}{2}}]). \end{aligned} \quad (3.38)$$

(3.38), (3.14) and (3.35) imply that

$$E[(\int_0^\tau |\delta \tilde{Z}^N(s)|^2 ds)^{\frac{\beta}{2}}] \leq 2C_\beta (E[|\tilde{\xi}|_{\tilde{\xi} \geq N}^\beta])^\beta + 4C_\beta^2 E[\sup_{s \geq 0} |\delta \tilde{Y}^N(s)|^\beta] \rightarrow 0$$

as $N \rightarrow \infty$. So there exists $\tilde{Z} \in M^\beta$ such that

$$\lim_{N \rightarrow \infty} E[(\int_0^\infty |\tilde{Z}^N(s) - \tilde{Z}(s)|^2 ds)^{\frac{\beta}{2}}] = \lim_{N \rightarrow \infty} E[(\int_0^\tau |\tilde{Z}^N(s) - \tilde{Z}(s)|^2 ds)^{\frac{\beta}{2}}] = 0.$$

Letting $N \rightarrow \infty$ in (3.31), it is easy to see that $(\tilde{Y}, \tilde{Z})$ is a solution to the BSDE (3.16).

Now we proceed to prove the uniqueness of the BSDE (3.16).

Suppose $(\tilde{Y}_1, \tilde{Z}_1)$ is another solution to (3.16) such that $\tilde{Y}_1 \in \text{Class (D)}$. Let $\delta \tilde{Y}(t) = \tilde{Y}(t) - \tilde{Y}_1(t)$, $\delta \tilde{Z}(t) = \tilde{Z}(t) - \tilde{Z}_1(t)$. Choose a sequence of stopping times $\{T_k\}_{k \geq 1}$ that increases to ∞ and such that $\int_0^{t \wedge T_k} \langle \frac{\delta \tilde{Y}(s)}{|\delta \tilde{Y}(s)|} \delta \tilde{Z}(s), dM_s \rangle$ is a uniformly integrable martingale for every k . Then by Lemma 3.2 we have

$$\begin{aligned} |\delta \tilde{Y}(t \wedge T_k)| & \leq |\delta \tilde{Y}(\tau \wedge T_k)| + \int_{t \wedge T_k}^{\tau \wedge T_k} \frac{\delta \tilde{Y}(s)}{|\delta \tilde{Y}(s)|} (\tilde{g}(s, \tilde{Y}_1(s)) - \tilde{g}(s, \tilde{Y}_2(s))) ds \\ & \quad - \int_{t \wedge T_k}^{\tau \wedge T_k} \langle \frac{\delta \tilde{Y}(s)}{|\delta \tilde{Y}(s)|} \delta \tilde{Z}(s), dM_s \rangle \\ & \leq |\delta \tilde{Y}(\tau \wedge T_k)| - \int_{t \wedge T_k}^{\tau \wedge T_k} \langle \frac{\delta \tilde{Y}(s)}{|\delta \tilde{Y}(s)|} \delta \tilde{Z}(s), dM_s \rangle. \end{aligned}$$

Taking the conditional expectation with respect to $\mathcal{F}_t$ on both sides, we obtain

$$|\delta \tilde{Y}(t \wedge T_k)| \leq E[|\delta \tilde{Y}(\tau \wedge T_k)| | \mathcal{F}_t]. \quad (3.39)$$

Since $\delta \tilde{Y}(t) = \tilde{Y}(t) - \tilde{Y}_1(t)$ is in Class (D) and $\lim_{k \rightarrow \infty} \delta \tilde{Y}(\tau \wedge T_k) = 0$ *P-a.e.*, we have that $\lim_{k \rightarrow \infty} \delta \tilde{Y}(\tau \wedge T_k) = 0$ in $L^1(\Omega)$. Letting $k \rightarrow \infty$ in (3.39) we get $|\delta \tilde{Y}(t)| \leq 0$. Hence $\tilde{Y}(t) = \tilde{Y}_1(t)$, $t \geq 0$. From the equation satisfied by the solutions, we further deduce that $\int_0^t \langle \delta \tilde{Z}(s), dM_s \rangle = 0$ and hence $E[\int_0^t \langle \delta \tilde{Z}(s), dM_s \rangle]^2 = 0$. This gives $\tilde{Z} = \tilde{Z}_1$. The proof is complete. $\square$

Let $f(x, y) : D \times \mathbb{R} \rightarrow \mathbb{R}$ be a given Borel measurable function. Assume that f is continuous *w.r.t.* the variable y and satisfies

$$\begin{cases} (y_1 - y_2)(f(x, y_1) - f(x, y_2)) \leq -J_1(x)|y_1 - y_2|^2, \\ |f(x, y)| \leq C(1 + |J_1(x)| + J_2(x)|y|), \end{cases}$$

where $J_1(x), J_2(x) \in L^p(D)$ and C is a constant. We have the following important corollary:

Corollary 3.3 Assume there exists a $x_0 \in D$ such that

$$E_{x_0}[e^{-\int_0^{\tau_D} J_1(X(s))ds}] < \infty. \quad (3.40)$$

Then for any $\mathcal{F}_{\tau_D}$ -measurable bounded r.v. ξ and $x \in D$, the following two statements hold:

(i) There exists a unique pair (Y_x, Z_x) satisfying the following BSDE:

$$Y_x(t) = \xi + \int_{t \wedge \tau_D}^{\tau_D} f(X(s), Y_x(s))ds - \int_{t \wedge \tau_D}^{\tau_D} \langle Z_x(s), dM_s \rangle, \quad P_x\text{-a.e.}, \quad (3.41)$$

such that $e^{-\int_0^t J_1(X(s))ds} Y_x(t) \in \text{Class}(D)$. Furthermore,

$$\sup_{x \in D} Y_x(0) < \infty.$$

(ii) There exists at most one solution (Y_x, Z_x) satisfying the BSDE (3.41) such that Y_x is a bounded process.

Remark 3.1 Please be ware of the subtle difference of (i) and (ii). We do not assume $e^{-\int_0^t J_1(X(s))ds} Y_x(t) \in \text{Class}(D)$ in (ii). The statement (ii) is needed in the next section.

Proof (i) Since $J_1, J_2 \in L^p(D)$, we know from (3.6) that $E_x[\int_0^{\tau_D} (|J_1(X(s))| + J_2(X(s)))ds] < \infty$. By Lemma 3.1 and (3.40) we see that all the conditions in Theorem 3.2 are satisfied. Hence for any $x \in D$, there exists a unique solution (Y_x, Z_x) to (3.41) such that $e^{-\int_0^t J_1(X(s))ds} Y_x(t) \in \text{Class}(D)$.

Next we prove the last statement of the Corollary (i).

Set $\tilde{Y}_x(t) = e^{-\int_0^t J_1(X(s))ds} Y_x(t)$, $\tilde{Z}_x(t) = e^{-\int_0^t J_1(X(s))ds} Z_x(t)$ and $\tilde{f}(t, y) = J_1(X(t))y + e^{-\int_0^t J_1(X(s))ds} f(X(t), e^{\int_0^t J_1(X(s))ds} y)$, $\tilde{\xi} = e^{-\int_0^t J_1(X(s))ds} \xi$ as in Theorem 2.1. It follows from (3.41) that

$$\tilde{Y}_x(t) = \tilde{\xi} + \int_{t \wedge \tau_D}^{\tau_D} \tilde{f}(s, \tilde{Y}_x(s))ds - \int_{t \wedge \tau_D}^{\tau_D} \langle \tilde{Z}_x(s), dM_s \rangle. \quad (3.42)$$

Choose a sequence of stopping times $\{\tau_n\}$ that increases to ∞ and such that

$\int_0^{t \wedge \tau_n} \langle \frac{\tilde{Y}_x(s)}{|\tilde{Y}_x(s)|} \tilde{Z}_x(s), dM_s \rangle$ is a uniformly integrable martingale under P_x . By Lemma 3.2, it follows from (3.42) that

$$\begin{aligned} |\tilde{Y}_x(t \wedge \tau_n)| &= |\tilde{Y}_x(\tau_D \wedge \tau_n)| + \int_{t \wedge \tau_D \wedge \tau_n}^{\tau_D \wedge \tau_n} \frac{\tilde{Y}_x(s)}{|\tilde{Y}_x(s)|} \tilde{f}(s, \tilde{Y}_x(s))ds - \int_{t \wedge \tau_D \wedge \tau_n}^{\tau_D \wedge \tau_n} \langle \frac{\tilde{Y}_x(s)}{|\tilde{Y}_x(s)|} \tilde{Z}_x(s), dM_s \rangle \\ &\leq |\tilde{Y}_x(\tau_D \wedge \tau_n)| - \int_{t \wedge \tau_D \wedge \tau_n}^{\tau_D \wedge \tau_n} \langle \frac{\tilde{Y}_x(s)}{|\tilde{Y}_x(s)|} \tilde{Z}_x(s), dM_s \rangle. \end{aligned}$$

Taking the conditional expectation with respect to $\mathcal{F}_t$ on both sides of the above inequality, we have

$$|\tilde{Y}_x(t \wedge \tau_n)| \leq E_x[|\tilde{Y}_x(\tau \wedge \tau_n)| | \mathcal{F}_t]. \quad (3.43)$$

Since $\tilde{Y}_x(t) \in \text{Class}(D)$, letting $n \rightarrow \infty$ in (3.43) we get

$$|\tilde{Y}_x(t)| \leq E_x[|\tilde{\xi}| | \mathcal{F}_t]. \quad (3.44)$$

In particular, we have

$$\begin{aligned} \sup_{x \in D} |Y_x(0)| &= \sup_{x \in D} |\tilde{Y}_x(0)| \leq \sup_{x \in D} E_x[|\tilde{\xi}|] \\ &\leq \|\xi\|_\infty \sup_{x \in D} E_x[e^{-\int_0^{\tau_D} J_1(X(s))ds}] < \infty. \end{aligned}$$

(ii) Suppose that (Y_x, Z_x) and (Y'_x, Z'_x) are solutions to BSDE (3.41) such that Y_x and Y'_x are both bounded processes. Since $J_1 \in L^p(D)$, there exists constant $\alpha > 0, \gamma > 1$ such that $\|\gamma(-J_1 - \alpha)^+\|_{L^p(D)} < \frac{1}{\sigma}$ where σ is defined in (3.6). Then by (3.6) and Kahamiskii's inequality, we have $E_x[e^{\int_0^{\tau_D} \gamma(-J_1 - \alpha)(X(s))ds}] \leq E_x[e^{\int_0^{\tau_D} \gamma(-J_1 - \alpha)^+(X(s))ds}] < \infty$. Moreover by (3.7) and (3.8)

$$\gamma_1 < E_x[e^{\int_0^{\tau_D} \gamma(-J_1 - \alpha)(X(s))ds}] < \gamma_2, \quad \forall x \in \bar{D}, \quad (3.45)$$

for some positive constants $\gamma_1, \gamma_2 > 0$.

Set $\bar{Y}_x(s) = e^{\int_0^{s \wedge \tau_D} (-J_1 - \alpha)(X(r))dr} Y_x(s)$, $\bar{Z}_x(s) = e^{\int_0^{s \wedge \tau_D} (-J_1 - \alpha)(X(r))dr} Z_x(s)$ and $\bar{Y}'_x(s) = e^{\int_0^{s \wedge \tau_D} (-J_1 - \alpha)(X(r))dr} Y'_x(s)$, $\bar{Z}'_x(s) = e^{\int_0^{s \wedge \tau_D} (-J_1 - \alpha)(X(r))dr} Z'_x(s)$. Set $\delta \bar{Y}_x(s) = \bar{Y}_x(s) - \bar{Y}'_x(s)$, $\delta \bar{Z}_x(s) = \bar{Z}_x(s) - \bar{Z}'_x(s)$, $\bar{f}(s, y) = e^{\int_0^s (-J_1 - \alpha)(X(r))dr} f(X(s), e^{\int_0^s (J_1 + \alpha)(X(r))dr} y) + (J_1 + \alpha)(X(s))y$. Then

$$(\gamma_1 - \gamma_2)(\bar{f}(s, y_1) - \bar{f}(s, y_2)) \leq \alpha(y_1 - y_2)^2. \quad (3.46)$$

Choose a sequence of stopping times $\{\tau'_n\}_{n \geq 1}$ that increases to ∞ and such that $\int_0^{t \wedge \tau'_n \wedge \tau_D} \langle \frac{\delta \bar{Y}_x(s)}{|\delta \bar{Y}_x(s)|} \delta \bar{Z}_x(s), dM_s \rangle$ is a uniformly integrable martingale. By (3.45)

$$\begin{aligned} &\sup_n E_x[(e^{\int_0^{\tau'_n \wedge \tau_D} (-J_1 - \alpha)(X(s))ds})^\gamma] \\ &\leq \sup_n \gamma_1^{-1} E_x[E_{X_{\tau'_n \wedge \tau_D}}[e^{\int_0^{\tau_D} \gamma(-J_1 - \alpha)(X(s))ds}] \cdot e^{\int_0^{\tau'_n \wedge \tau_D} \gamma(-J_1 - \alpha)(X(s))ds}] \\ &= \sup_n \gamma_1^{-1} E_x[e^{\int_0^{\tau_D} \gamma(-J_1 - \alpha)(X(s))ds}] \leq \gamma_1^{-1} \gamma_2 < \infty, \end{aligned}$$

which implies that $\{e^{\int_0^{\tau'_n \wedge \tau_D} (-J_1 - \alpha)(X(s))ds}\}_{n \geq 1}$ is P_x -uniformly integrable. Hence $\{\delta \bar{Y}_x(\tau'_n \wedge \tau_D)\}_{n \geq 1}$ is P_x -uniformly integrable because $Y_x - Y'_x$ is bounded. Similarly, we can show that $\{\delta \bar{Y}_x(t \wedge \tau'_n \wedge \tau_D)\}_{n \geq 1}$ is also P_x -uniformly integrable.

Note that

$$\delta \bar{Y}_x(t) = \int_{t \wedge \tau_D}^{\tau_D} (\bar{f}(s, \bar{Y}_x(s)) - \bar{f}(s, \bar{Y}'_x(s)))ds - \int_{t \wedge \tau_D}^{\tau_D} \langle \delta \bar{Z}(s), dM_s \rangle.$$

By Lemma 3.2 and (3.46), it holds that

$$\begin{aligned} &|\delta \bar{Y}_x(t \wedge \tau'_n \wedge \tau_D)| \\ &\leq |\delta \bar{Y}_x(\tau'_n \wedge \tau_D)| + \int_{t \wedge \tau'_n \wedge \tau_D}^{\tau'_n \wedge \tau_D} \frac{\delta \bar{Y}_x(s)}{|\delta \bar{Y}_x(s)|} (\bar{f}(s, \bar{Y}_x(s)) - \bar{f}(s, \bar{Y}'_x(s)))ds \\ &\quad - \int_{t \wedge \tau'_n \wedge \tau_D}^{\tau'_n \wedge \tau_D} \langle \frac{\delta \bar{Y}_x(s)}{|\delta \bar{Y}_x(s)|} \delta \bar{Z}(s), dM_s \rangle \\ &\leq |\delta \bar{Y}_x(\tau'_n \wedge \tau_D)| + \alpha \int_{t \wedge \tau'_n \wedge \tau_D}^{\tau'_n \wedge \tau_D} |\delta \bar{Y}_x(s)|ds \\ &\quad - \int_{t \wedge \tau'_n \wedge \tau_D}^{\tau'_n \wedge \tau_D} \langle \frac{\delta \bar{Y}_x(s)}{|\delta \bar{Y}_x(s)|} \delta \bar{Z}(s), dM_s \rangle, \quad P_x - a.e.. \end{aligned} \quad (3.47)$$

Taking the expectation on the both sides of (3.47) we get

$$E_x[|\delta \bar{Y}_x(t \wedge \tau'_n \wedge \tau_D)|] \leq E_x[|\delta \bar{Y}_x(\tau'_n \wedge \tau_D)|] + \alpha E_x[\int_{t \wedge \tau'_n \wedge \tau_D}^{\tau'_n \wedge \tau_D} |\delta \bar{Y}_x(s)|ds]. \quad (3.48)$$

Notice that $\lim_{n \rightarrow \infty} |\delta \bar{Y}_x(\tau'_n \wedge \tau_D)| = e^{\int_0^{\tau_D} (-J_1 - \alpha)(X(s))ds} |\xi - \xi| = 0$, P_x -a.e.. Using the uniformly integrability of $\{\delta \bar{Y}_x(t \wedge \tau'_n \wedge \tau_D)\}_{n \geq 1}$ and $\{\delta \bar{Y}_x(\tau'_n \wedge \tau_D)\}_{n \geq 1}$, letting $n \rightarrow \infty$ in (3.48) we find

$$E_x[|\delta \bar{Y}_x(t)|] \leq \alpha \int_t^\infty E_x[|\delta \bar{Y}_x(s)|] ds.$$

By Gronwall's inequality we conclude $E_x[|\delta \bar{Y}_x(t)|] = 0$. Hence $Y_x = Y'_x$. Furthermore we deduce that $\int_0^t \langle \delta \bar{Z}_x(s), dM_s \rangle = 0$ and hence $Z_x = Z'_x$. The uniqueness is proved. $\square$

4 Dirichlet Boundary Value Problem

4.1 Linear Case

Consider the following second order elliptic operator

$$L_2 u = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial u}{\partial x_j}) + \sum_{i=1}^d b_i(x) \frac{\partial u}{\partial x_i} + q(x). \quad (4.1)$$

The quadratic form generated by L_2 is

$$\begin{cases} \mathcal{E}^q(u, v) = (-L_2 u, v) \\ \quad = \frac{1}{2} \sum_{i,j=1}^d \int_D a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx - \sum_{i=1}^d \int_D b_i(x) \frac{\partial u}{\partial x_i} v(x) dx - \int_D q(x) u(x) v(x) dx, \\ D(\mathcal{E}^q) = W_0^{1,2}(D). \end{cases}$$

Since $|b|^2, q \in L^p$ for $p > \frac{d}{2}$, it is known that $(\mathcal{E}^q, D(\mathcal{E}^q))$ is a coercive closed form and there exist a $\alpha_0 > 0$ such that

$$\lambda_{\alpha_0} \mathcal{E}(u, u) \leq \mathcal{E}_{\alpha_0}^q(u, u) \leq \Lambda_{\alpha_0} \mathcal{E}(u, u)$$

for some constant $\lambda_{\alpha_0}, \Lambda_{\alpha_0}$, where $\mathcal{E}(u, u) = \frac{1}{2} \sum_{i,j=1}^d \int_D \tilde{a}_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx$ and $\mathcal{E}_{\alpha_0}^q(u, u) = \mathcal{E}^q(u, u) + \alpha_0(u, u)$. By Theorem 3.2 in [13] or Lemma 4.1 in [7], for sufficient large $\alpha > 0$, the α -resolvent operator of $(\mathcal{E}^q, D(\mathcal{E}^q))$ is given by

$$G_\alpha^q f(x) = E_x \left[\int_0^{\tau_D} e^{\int_0^t q(X(s))ds - \alpha t} f(X(t)) dt \right], \quad \forall f \in L^2(D), \quad (4.2)$$

where $X = (X_t, P_x, x \in D)$ is the diffusion associated with the operator L_1 in (1.4).

Let F be a measurable function satisfying

$$|F(x)| \leq C + C|q(x)|, \quad (4.3)$$

for some constant C . Take $\varphi \in C(\partial D)$ and consider the boundary value problem

$$\begin{cases} L_2 u(x) = -F(x), & \forall x \in D, \\ u(x)|_{\partial D} = \varphi, & \forall x \in \partial D. \end{cases} \quad (4.4)$$

We have the following result:

Theorem 4.1 Assume there exists a $x_0 \in D$ such that

$$E_{x_0}[\exp(\int_0^{\tau_D} q(X(s))ds)] < \infty.$$

Then there exists a unique continuous weak solution to problem (4.4) which is given by

$$u(x) = E_x[e^{\int_0^{\tau_D} q(X(s))ds} \varphi(X(\tau_D))] + \int_0^{\tau_D} e^{\int_0^t q(X(s))ds} F(X(t))dt. \quad (4.5)$$

Proof It is known from Lemma 2.1 in [5] that $u_1(x) = E_x[\varphi(X(\tau_D))]$ is the unique continuous weak solution of the problem

$$\begin{cases} L_1 u_1(x) = 0, & \forall x \in D, \\ u_1(x)|_{\partial D} = \varphi, & \forall x \in \partial D. \end{cases} \quad (4.6)$$

Let $F_1(x) = F(x) + q(x)u_1(x)$ and $u_2(x) = E_x[\int_0^{\tau_D} e^{\int_0^t q(X(s))ds} F_1(X(t))dt]$. By Lemma 3.1 and (4.3) u_2 is well defined. Let $u(x)$ be defined as (4.5). Then $u = u_1 + u_2$. Hence, to prove that u is a continuous weak solution to (4.4) we only need show that u_2 is a continuous weak solution of the following problem:

$$\begin{cases} L_2 u_2(x) = -F_1, & \forall x \in D, \\ u_2(x)|_{\partial D} = 0, & \forall x \in \partial D. \end{cases} \quad (4.7)$$

Since u_1 is bounded, by (4.3) we have $|F_1(x)| \leq c_1(1 + |q(x)|)$ for some $c_1 > 0$. By Lemma 5.7 in [8] and Lemma 2.1 in [5], we know that the semigroup generated by L_2 is strong Feller. Hence, using the similar proof as that of Theorem 3.18 in [9] we obtain that $u_2 \in C(\overline{D})$.

Next we show that $u_2 \in W_0^{1,2}(D)$.

Since $q \in L^p$ for $p > \frac{d}{2}$ and since the transition density function of $(X_t, P_x, x \in \mathbb{R}^d)$ has the Gaussian upper bound estimate by (3.9), we can easily deduce that αG_α^q is strongly continuous on L^p .

By the Markov property, it is easy to see that

$$\alpha(u_2 - \alpha G_\alpha^q u_2) = \alpha G_\alpha^q F_1(x). \quad (4.8)$$

Hence we get $\overline{\lim}_{\alpha \rightarrow \infty} \alpha(u_2 - \alpha G_\alpha^q u_2, u_2) \leq \overline{\lim}_{\alpha \rightarrow \infty} \|\alpha G_\alpha^q F_1(x)\|_{L^p} \|u_2\|_{L^{p'}} < \infty$, where $\frac{1}{p} + \frac{1}{p'} = 1$. By Theorem 1.1.4 in [15] we deduce that $u_2 \in W_0^{1,2}(D)$.

Finally we show that u_2 is a continuous weak solution of (4.7). For any $\phi \in C_0^\infty(D)$, by (4.8) we have

$$\mathcal{E}^q(u_2, \phi) - (F_1, \phi) = \lim_{\alpha \rightarrow \infty} (\alpha(u_2 - \alpha G_\alpha^q u_2, F_1) - (F_1, \phi)) = \lim_{\alpha \rightarrow \infty} (\alpha G_\alpha^q F_1, \phi) - (F_1, \phi) = 0.$$

Hence u_2 is a continuous weak solution of (4.7).

Now we prove the uniqueness. If $\tilde{u}$ is another continuous weak solution of (4.4), then $\tilde{u} - u_2$ is a continuous weak solution of (4.6). By the uniqueness of (4.6), we get $\tilde{u} = u$. $\square$

4.2 Semilinear Case

Recall that M^0 is the martingale part of the diffusion process (X_t, P_x^0) generated by L_0 . Define the exponential martingale

$$U(t) = \exp\left(\int_0^t \langle \tilde{a}^{-1} b(X(s)), dM_s^0 \rangle - \frac{1}{2} \int_0^t \langle \tilde{a}^{-1} b(X(s)), b(X(s)) \rangle ds\right).$$

Then we have

$$\frac{dP_x}{dP_x^0}|_{\mathcal{F}_t} = U(t).$$

Let $g(x, y) : D \times \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function such that g is continuous *w.r.t.* y and satisfies

$$\begin{cases} (y_1 - y_2)(g(x, y_1) - g(x, y_2)) \leq -K(x)|y_1 - y_2|^2, \\ |g(x, y)| \leq C(1 + |q(x) - K(x)| + |K_1(x)y|), \end{cases}$$

for some constant C and $K, K_1 \in L^p(D)$.

Consider the following semilinear boundary value problem:

$$\begin{cases} L_2 u(x) = -g(x, u(x)), & \forall x \in D, \\ u(x)|_{\partial D} = \varphi, & \forall x \in \partial D. \end{cases} \quad (4.9)$$

Lemma 4.1 Assume there exists a $x_0 \in D$ such that

$$E_{x_0}^0 \left[\exp \left(\int_0^{\tau_D} \langle \tilde{a}^{-1} b(X(s)), dM_s^0 \rangle - \frac{1}{2} \int_0^{\tau_D} \langle b^* \tilde{a}^{-1} b \rangle(X(s)) ds \right) + \int_0^{\tau_D} (q(X(s)) - K(X(s))) ds \right] < \infty. \quad (4.10)$$

Then there exists a unique continuous weak solution to the problem (4.9).

Proof Set $g_1(x, y) = g(x, y) + q(x)y$. By (4.10) and Corollary 3.3 for any $x \in D$, there exists a unique solution (Y_x, Z_x) to the following BSDE:

$$Y_x(t) = \varphi(X(\tau_D)) + \int_{t \wedge \tau_D}^{\tau_D} g_1(X(s), Y_x(s)) ds - \int_{t \wedge \tau_D}^{\tau_D} \langle Z_x(s), dM_s \rangle, \quad P_x\text{-a.e.} \quad (4.11)$$

Define $u_0(x) := Y_x(0)$, $v_0(x) := Z_x(0)$. By the strong Markov property of X and the uniqueness of the BSDE (4.11), it is easy to see that $P_x\text{-a.e.}$

$$u_0(X(t)) = Y_x(t), \quad v_0(X(t)) = Z_x(t), \quad 0 \leq t \leq \tau_D. \quad (4.12)$$

Consider the following linear problem:

$$\begin{cases} L_1 u(x) = -g_1(x, u_0(x)), & \forall x \in D, \\ u(x)|_{\partial D} = \varphi, & \forall x \in \partial D. \end{cases}$$

Since u_0 is bounded by Corollary 3.3, by Theorem 4.1 the above equation admits a unique continuous weak solution $u \in W^{1,2}(D)$. On the other hand, by the Fukushima's decomposition we have for *q.e.* $x \in D$, $P_x\text{-a.e.}$

$$\begin{aligned} u(X(t)) - u(x) &= \int_0^t \langle \nabla u(X(s)), dM_s \rangle + \int_0^t g_1(X(s), u_0(X(s))) ds \\ &= \int_0^t \langle \nabla u(X(s)), dM_s \rangle + \int_0^t g_1(X(s), Y_x(s)) ds, \quad \forall t < \tau_D. \end{aligned} \quad (4.13)$$

Choose a sequence of stopping times $\{T_n\}_{n \geq 1}$ that increases to ∞ and such that $\int_0^{t \wedge T_n \wedge \tau_D} \langle Z_x(s), dM_s \rangle$ and $\int_0^{t \wedge T_n \wedge \tau_D} \langle \nabla u(X(s)), dM_s \rangle$ are P_x -uniformly integrable martingales. It follows from (4.11) and (4.13) that

$$\begin{aligned} Y_x(t \wedge T_n \wedge \tau_D) - Y_x(T_n \wedge \tau_D) &= \int_{t \wedge T_n \wedge \tau_D}^{T_n \wedge \tau_D} g_1(X(s), Y_x(s)) ds - \int_{t \wedge T_n \wedge \tau_D}^{T_n \wedge \tau_D} \langle Z_x(s), dM_s \rangle, \\ u(X(t \wedge T_n \wedge \tau_D)) - u(X(T_n \wedge \tau_D)) &= \int_{t \wedge T_n \wedge \tau_D}^{T_n \wedge \tau_D} g_1(X(s), Y_x(s)) ds - \int_{t \wedge T_n \wedge \tau_D}^{T_n \wedge \tau_D} \langle \nabla u(X(s)), dM_s \rangle. \end{aligned}$$

Taking the conditional expectations respect to $\mathcal{F}_t$ on both sides of the above equations, we get

$$u(X(t \wedge T_n \wedge \tau_D)) - E_x[u(X(T_n \wedge \tau_D))|\mathcal{F}_t] = Y_x(t \wedge T_n \wedge \tau_D) - E_x[Y_x(T_n \wedge \tau_D)|\mathcal{F}_t]. \quad (4.14)$$

Since $u \in C(\overline{D})$ and since $Y_x(t)$ is bounded (see (4.12)), let $T_n \rightarrow \infty$ in (4.14) to obtain

$$u(X(t \wedge \tau_D)) - E_x[u((X(\tau_D))|\mathcal{F}_t] = Y_x(t \wedge \tau_D) - E_x[\varphi(\tau_D)|\mathcal{F}_t].$$

Since $u(x)|_{\partial D} = \varphi$, letting $t = 0$ we deduce that $u(x) = Y_x(0) = u_0(x)$ for $q.e. x \in D$. Hence u is a weak solution to (4.9).

Now we proceed to prove the uniqueness of the boundary value problem (4.9). Suppose u_1 is another continuous weak solution of (4.9). Then by Fukushima's decomposition $(u_1(X(t)), \nabla u_1(X(t)))$ is also a solution of BSDE (4.11) for $q.e. x \in D$. Since $u(X(t))$ and $u_1(X(t))$ are bounded, by Corollary 3.3(ii) we have $u_1(X(t)) = u(X(t))$. In particular, $|u_1(x) - u(x)| = E_x[|u_1(X(0)) - u(X(0))|] = 0$ for $q.e. x \in D$. By the continuity of u_1 and u , it holds for every $x \in D$. The uniqueness is proved. $\square$

Let $f(x, y) : D \times \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function that is continuous *w.r.t.* y and satisfies

$$\begin{cases} (y_1 - y_2)(f(x, y_1) - f(x, y_2)) \leq -J(x)|y_1 - y_2|^2, \\ |f(x, y)| \leq C(1 + J_1(x)|y|), \end{cases} \quad (4.15)$$

for some constant C and $J, J_1 \in L^p(D)$. Consider the following semilinear elliptic PDEs:

$$\begin{cases} \mathcal{A}u(x) = -f(x, u(x)), & \forall x \in D, \\ u(x)|_{\partial D} = \varphi, & \forall x \in \partial D. \end{cases} \quad (4.16)$$

Now we can state the main result in this section.

Theorem 4.2 *Under the Assumption I, if there exists a $x_0 \in D$ such that*

$$E_{x_0}^0[\exp(\int_0^{\tau_D} (\tilde{a}^{-1}(b - \hat{b})(X(s)), dM_s^0) - \frac{1}{2} \int_0^{\tau_D} (b - \hat{b})^* \tilde{a}^{-1}(b - \hat{b})(X(s)) ds + N_{\tau_D}^v + \int_0^{\tau_D} (q(X(s)) - J(X(s))) ds)] < \infty, \quad (4.17)$$

where $E_{x_0}^0$ stands for the expectation under $P_{x_0}^0$, N_t^v is the zero-energy part of the Fukushima's decomposition defined in (2.2). Then there exists a unique continuous weak solution to problem (4.16). Moreover, if $f(x, y) = F(x)$, the solution is given by

$$\begin{aligned} u(x) = & E_x^0[\exp(\int_0^{\tau_D} (\tilde{a}^{-1}(b - \hat{b})(X(s)), dM_s^0) - \frac{1}{2} \int_0^{\tau_D} ((b - \hat{b})^* \tilde{a}^{-1}(b - \hat{b}))(X(s)) ds \\ & + N_{\tau_D}^v + \int_0^{\tau_D} q(X(s)) ds) \varphi(X(\tau_D))] \\ & + E_x^0[\int_0^{\tau_D} \exp(\int_0^t (\tilde{a}^{-1}(b - \hat{b})(X(s)), dM_s^0) - \frac{1}{2} \int_0^t ((b - \hat{b})^* \tilde{a}^{-1}(b - \hat{b}))(X(s)) ds \\ & + N_t^v + \int_0^t q(X(s)) ds) F(X(t)) dt]. \end{aligned}$$

Proof Introduce the following operator on $L^2(D)$:

$$\begin{aligned} \hat{\mathcal{A}} = & \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial}{\partial x_j}) + \sum_{i=1}^d [b_i - \hat{b}_i - (\tilde{a} \nabla v)_i] \frac{\partial}{\partial x_i} \\ & + \frac{1}{2} (\nabla v)^* \tilde{a} \nabla v - (b - \hat{b})^* \nabla v + q. \end{aligned} \quad (4.18)$$

Set $h = e^v$ and consider the following semi-linear elliptic PDE:

$$\begin{cases} \hat{\mathcal{A}}\hat{u}(x) = -h(x)f(x, h^{-1}(x)\hat{u}(x)), & \forall x \in D, \\ \hat{u}(x)|_{\partial D} = h(x)\varphi(x), & \forall x \in \partial D. \end{cases} \quad (4.19)$$

Note that:

$$\begin{aligned}
 & \int_0^{\tau_D} \langle \tilde{a}^{-1}(b - \hat{b} - \tilde{a} \nabla v)(X(s)), dM_s^0 \rangle - \frac{1}{2} \int_0^{\tau_D} ((b - \hat{b} - \tilde{a} \nabla v)^* \tilde{a}^{-1}(b - \hat{b} - \tilde{a} \nabla v))(X(s)) ds \\
 & + \int_0^{\tau_D} (\frac{1}{2} (\nabla v)^* \tilde{a} \nabla v - (b - \hat{b})^* \nabla v + q - J)(X(s)) ds \\
 & = \int_0^{\tau_D} \langle \tilde{a}^{-1}(b - \hat{b})(X(s)), dM_s^0 \rangle - \int_0^{\tau_D} \langle \tilde{a} \nabla v, dM_s^0 \rangle - \frac{1}{2} \int_0^{\tau_D} ((b - \hat{b})^* \tilde{a}^{-1}(b - \hat{b}))(X(s)) ds \\
 & + \frac{1}{2} \int_0^{\tau_D} (b - \hat{b})^* (\nabla v)(X(s)) ds + \frac{1}{2} \int_0^{\tau_D} (\nabla v)^* (b - \hat{b} - \tilde{a} \nabla v)(X(s)) ds \\
 & + \int_0^{\tau_D} (\frac{1}{2} (\nabla v)^* \tilde{a} \nabla v - (b - \hat{b})^* \nabla v + q - J)(X(s)) ds \quad (4.20) \\
 & = \int_0^{\tau_D} \langle \tilde{a}^{-1}(b - \hat{b})(X(s)), dM_s^0 \rangle - \frac{1}{2} \int_0^{\tau_D} ((b - \hat{b})^* \tilde{a}^{-1}(b - \hat{b}))(X(s)) ds - M_{\tau_D}^v \\
 & + \int_0^{\tau_D} (q - J)(X(s)) ds \\
 & = \int_0^{\tau_D} \langle \tilde{a}^{-1}(b - \hat{b})(X(s)), dM_s^0 \rangle - \frac{1}{2} \int_0^{\tau_D} ((b - \hat{b})^* \tilde{a}^{-1}(b - \hat{b}))(X(s)) ds \\
 & + N_{\tau_D}^v + v(X(0)) - v(X(\tau_D)) + \int_0^{\tau_D} (q - J)(X(s)) ds.
 \end{aligned}$$

Thus condition (4.17) is equivalent to

$$\begin{aligned}
 E_{x_0} [& \exp(\int_0^{\tau_D} \langle \tilde{a}^{-1}(b - \hat{b} - \tilde{a} \nabla v)(X(s)), dM_s^0 \rangle \\
 & - \frac{1}{2} \int_0^{\tau_D} ((b - \hat{b} - \tilde{a} \nabla v)^* \tilde{a}^{-1}(b - \hat{b} - \tilde{a} \nabla v))(X(s)) ds \\
 & + \int_0^{\tau_D} (\frac{1}{2} (\nabla v)^* \tilde{a} \nabla v - (b - \hat{b})^* \nabla v + q - J)(X(s)) ds)] < \infty. \quad (4.21)
 \end{aligned}$$

Since $h(x) > 0$, (4.15) implies

$$\begin{cases} (y_1 - y_2)(h(x)f(x, h^{-1}(x)y_1) - h(x)f(x, h^{-1}(x)y_2)) \leq -J(x)|y_1 - y_2|^2, \\ |(h(x)f(x, h^{-1}(x)y))| \leq C(\sup_{x \in D} h(x) + |J_1(x)y|). \end{cases} \quad (4.22)$$

By (4.21), (4.22) and Lemma 4.1, we conclude that there exists a unique continuous weak solution $\hat{u}$ to problem (4.19). Let $u(x) = h^{-1}(x)\hat{u}(x)$. Now following the same arguments as in the proof of Theorem 5.1 in [18], we can show that u is the continuous weak solution of problem (4.16). We omit the details here.

Let us now consider the special case, $f(x, y) = F(x)$. Define $\hat{U}(t) = \exp(\int_0^t \langle \tilde{a}^{-1}(b - \hat{b} - \tilde{a} \nabla v)(X(s)), dM_s^0 \rangle - \frac{1}{2} \int_0^t ((b - \hat{b} - \tilde{a} \nabla v)^* \tilde{a}^{-1}(b - \hat{b} - \tilde{a} \nabla v))(X(s)) ds)$. Since $b - \hat{b} - \tilde{a} \nabla v \in L^2 p(D)$, by Novikov's criterion $\hat{U}(t)$ is a P_x^0 -exponential martingale. Define

$$\frac{d\hat{P}_x}{dP_x^0} |_{\mathcal{F}_t} = \hat{U}(t). \quad (4.23)$$

Then it follows from Theorem 4.1 that the solution to problem (4.19) admits the representation:

$$\begin{aligned}
 \hat{u}(x) &= \hat{E}_x [e^{\int_0^{\tau_D} (\frac{1}{2} (\nabla v)^* \tilde{a} \nabla v - (b - \hat{b})^* \nabla v + q)(X(s)) ds} \varphi(X(\tau_D))] \\
 &+ \hat{E}_x [\int_0^{\tau_D} e^{\int_0^t (\frac{1}{2} (\nabla v)^* \tilde{a} \nabla v - (b - \hat{b})^* \nabla v + q)(X(s)) ds} h(X(t)) F(X(t)) dt] \\
 &:= \hat{u}_1(x) + \hat{u}_2(x),
 \end{aligned}$$

where $\hat{E}_x$ is the expectation under $\hat{P}_x$. By the calculations in (4.20), we have

$$\begin{aligned}
 \hat{u}_1(x) &= E_x^0 [\hat{U}_{\tau_D} e^{\int_0^{\tau_D} (\frac{1}{2} (\nabla v)^* \tilde{a} \nabla v - (b - \hat{b})^* \nabla v + q)(X(s)) ds} \varphi(X(\tau_D))] \\
 &= E_x^0 [\exp(\int_0^{\tau_D} \langle \tilde{a}^{-1}(b - \hat{b})(X(s)), dM_s^0 \rangle - \frac{1}{2} \int_0^{\tau_D} ((b - \hat{b})^* \tilde{a}^{-1}(b - \hat{b}))(X(s)) ds \\
 &\quad + N_{\tau_D}^v + \int_0^{\tau_D} q(X(s)) ds) \varphi(X(\tau_D))] e^{v(x)},
 \end{aligned}$$

and

$$\begin{aligned}\hat{u}_2(x) &= E_x^0[\int_0^{\tau_D} \hat{U}(t) e^{\int_0^t (\frac{1}{2}(\nabla v)^* \tilde{a} \nabla v - (b - \hat{b})^* \nabla v + q)(X(s)) ds} h(X(t)) F(X(t)) dt] \\ &= E_x^0[\int_0^{\tau_D} \exp(\int_0^t \tilde{a}^{-1}(b - \hat{b})(X(s)), dM_s^0) - \frac{1}{2} \int_0^t ((b - \hat{b})^* \tilde{a}^{-1}(b - \hat{b}))(X(s)) ds \\ &\quad + N_t^v + v(X(0)) - v(X(t)) + \int_0^t q(X(s)) ds h(X(t)) F(X(t)) dt] \\ &= E_x^0[\int_0^{\tau_D} \exp(\int_0^t \tilde{a}^{-1}(b - \hat{b})(X(s)), dM_s^0) - \frac{1}{2} \int_0^t ((b - \hat{b})^* \tilde{a}^{-1}(b - \hat{b}))(X(s)) ds \\ &\quad + N_t^v + \int_0^t q(X(s)) ds F(X(t)) dt] e^{v(x)}.\end{aligned}$$

Hence the solution of (4.16) in this case is given by

$$\begin{aligned}u(x) &= h^{-1}(x) \hat{u}_1(x) + h^{-1}(x) \hat{u}_2(x) \\ &= E_x^0[\exp(\int_0^{\tau_D} \tilde{a}^{-1}(b - \hat{b})(X(s)), dM_s^0) - \frac{1}{2} \int_0^{\tau_D} ((b - \hat{b})^* \tilde{a}^{-1}(b - \hat{b}))(X(s)) ds \\ &\quad + N_{\tau_D}^v + \int_0^{\tau_D} q(X(s)) ds \varphi(X(\tau_D))] \\ &\quad + E_x^0[\int_0^{\tau_D} \exp(\int_0^t \tilde{a}^{-1}(b - \hat{b})(X(s)), dM_s^0) - \frac{1}{2} \int_0^t ((b - \hat{b})^* \tilde{a}^{-1}(b - \hat{b}))(X(s)) ds \\ &\quad + N_t^v + \int_0^t q(X(s)) ds F(X(t)) dt].\end{aligned}$$

□

Example 4.3 If $q - J - \operatorname{div} \hat{b} \leq g$ for some $g \in L^1(D)$ in the sense of distribution and there exists a $x_0 \in D$ such that

$$E_{x_0}^0[\exp(\int_0^{\tau_D} \tilde{a}^{-1}(b - \hat{b})(X(s)), dM_s^0) - \frac{1}{2} \int_0^{\tau_D} ((b - \hat{b})^* \tilde{a}^{-1}(b - \hat{b}))(X(s)) ds + \int_0^{\tau_D} g(X(s)) ds] < \infty.$$

Then condition (4.17) in Theorem 4.2 is satisfied.

Proof First we show that $P_{x_0}^0$ -a.e., $N_t^v + \int_0^t (q - J)(X(s)) ds \leq \int_0^t g(X(s)) ds$, $\forall t > 0$.

Take an even, nonnegative function $\psi \in C_0^\infty(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \psi(x) dx = 1$. Set $\psi_k(x) := k\psi(kx)$ and $\hat{b}_k^i(x) := \int_{\mathbb{R}^d} \hat{b}_i(y) \psi_k(x - y) dy$. Then $\hat{b}_k^i \in C_0^\infty(\mathbb{R}^d)$. Set $(g_k - q_k + J_k)(x) := \int_{\mathbb{R}^d} (g - q + J)(y) \psi_k(x - y) dy$. Then it's easy to see for any $x \in D$

$$-\operatorname{div} \hat{b}^k(x) \leq (g_k - q_k + J_k)(x). \quad (4.24)$$

Set $f_k(x) := (g - q + J)(x) - (g_k - q_k + J_k)(x)$, $P_t^0 f_k(x) := E_x^0[f_k(X(t))]$. Let $\hat{P}_t^0$ be the dual operator of P_t^0 in $L^2(D)$. Since $g_k - q_k + J_k$ converges to $g - q + J$ in $L^1(D)$,

$$\int_D E_x^0[\int_0^t |f_k|(X(s)) ds] dx = \int_0^t (P_s^0 f_k, I_D) ds = \int_0^t (f_k, \hat{P}_s^0 I_D) ds \leq t \|f_k\|_{L^1} \rightarrow 0$$

as $k \rightarrow \infty$. Hence $\int_0^t |f_k|(X(s)) ds$ converges to 0 in P_m^0 where $P_m^0(\cdot) = \int_D P_x^0(\cdot) dx$. Then by the diagonal method there exists a subsequence $\{f_{k_l}\}$ such that $\int_0^t |f_{k_l}|(X(s)) ds$ converges to 0 uniformly in t on any compact set of $[0, \infty)$ P_m^0 -a.e.. For any $\delta \geq 0$, set

$$\Lambda_\delta := \{\omega : \text{for any } n \in \mathbb{N}, \lim_{k_l \rightarrow \infty} \int_\delta^n |f_{k_l}|(X(s, \omega)) ds = 0\}.$$

It's easy to show that $\Lambda_0 = \theta_\delta(\Lambda_\delta)$. Set $h(x) := E_x^0[\Lambda_0]$. Then $h(x) = 0$ m -a.e.. Since (X_t, P_x^0) has a continuous transition density function, we have for any $x \in D$

$$\begin{aligned}P_x^0[\Lambda_\delta] &= E_x^0[I_{\Lambda_0}(\theta_\delta(\omega))] = E_x^0[E_x^0[I_{\Lambda_0}(\theta_\delta(\omega)) | \mathcal{F}_\delta]] \\ &= E_x^0[E_{X(\delta)}^0[I_{\Lambda_0}]] = E_x^0[h(X(\delta))] = 0, \quad \forall \delta > 0.\end{aligned}$$

Set $\Lambda := \{\omega : \text{for any } n \in \mathbb{N}, \lim_{k_l \rightarrow \infty} \int_{\frac{1}{n}}^n |f_{k_l}|(X(s, \omega))ds = 0\}$. Then $\Lambda = \bigcap_{\delta > 0} \Lambda_\delta$ and $P_x^0(\Lambda) = 0$. Hence P_x^0 -a.e.

$$\lim_{k_l \rightarrow \infty} \int_{\frac{1}{n}}^t (g_{k_l} - q_{k_l} + J_{k_l})(X(s))ds = \int_{\frac{1}{n}}^t (g - q + J)(X(s))ds, \quad \forall t > 0, n \in \mathbb{N}. \quad (4.25)$$

By Theorem 1 in [3] there exists a unique $v_k \in W_0^{1,2p}(D)$ such that

$$\int_D \langle a(x) \nabla v_k(x), \nabla \phi(x) \rangle dx = -2 \int_D \langle \hat{b}^k(x), \nabla \phi(x) \rangle dx, \quad \forall \phi \in W_0^{1,2}(D), \quad (4.26)$$

where $\hat{b}^k = (b_1^k, b_2^k, \dots, b_d^k)^*$. Moreover

$$\|\nabla v - \nabla v_k\|_{L^{2p}} \leq \vartheta \|\hat{b} - \hat{b}^k\|_{L^{2p}}. \quad (4.27)$$

where ϑ is independent of $\hat{b}^k$. For every $k \geq 1$, we have the Fukushima's decomposition $v_k(X(t)) - v_k(X(0)) = M_t^{v_k} + N_t^{v_k}$ P_x^0 -a.e. for any $x \in D$ according to Theorem 1 in [11]. Note that for any $\phi \in W_0^{1,2}(D)$, $\mathcal{E}^0(v_k, \phi) = \int_D \operatorname{div} \hat{b}^k(x) \phi(x) dx$ by (4.26). So $N_t^{v_k} = -\int_0^t \operatorname{div} \hat{b}^k(X(s)) ds$. Since $|\hat{b}| \in L^{2p}$, $\hat{b}^k = (b_1^k, b_2^k, \dots, b_d^k)$ converges to $\hat{b}$ in $L^{2p}(D)$ as $k \rightarrow \infty$. Furthermore, by Sobolev embedding theorem, $\sup_D |v - v_k| \leq \vartheta_0 \|\nabla v - \nabla v_k\|_{L^{2p}} \leq \vartheta_0 \vartheta \|\hat{b} - \hat{b}^k\|_{L^{2p}}$ where ϑ_0 only depend on d, p, D . So for any $x \in D$, $v_k(X(t))$ converges to $v(X(t))$ uniformly on $[0, \infty)$ P_x^0 -a.e..

Replacing E_x by E_x^0 in (3.6), we have the same estimate for some new constant σ'' . By (1.3) and (4.27), for any $x \in D$,

$$\begin{aligned} E_x^0[\sup_{s \geq 0} (M_s^v - M_s^{v_k})^2] &\leq 4 \sup_{s \geq 0} E_x^0[(M_s^v - M_s^{v_k})^2] \\ &= 4 E_x^0[\int_0^{\tau_D} \langle \tilde{a}(\nabla v - \nabla v_k)(X(s)), (\nabla v - \nabla v_k)(X(s)) \rangle ds] \\ &\leq \frac{4}{\lambda} E_x^0[\int_0^{\tau_D} |\nabla v - \nabla v_k|^2(X(s)) ds] \\ &\leq \frac{4\sigma''}{\lambda} \|\nabla v - \nabla v_k\|_{L^{2p}}^2 \leq \frac{4\sigma''\vartheta^2}{\lambda} \|\hat{b} - \hat{b}^k\|_{L^{2p}}^2 \rightarrow 0 \end{aligned}$$

as $k \rightarrow \infty$. So there exists a subsequence of $\{M_t^{v_{k_l}}\}_{l \geq 1}$ such that for any $x \in D$, it converges to M_t^v uniformly on $[0, \infty)$ P_x^0 -a.e.. We still denote this subsequence by $\{M_t^{v_{k_l}}\}_{l \geq 1}$ for simplicity. Hence $N_t^{v_{k_l}} = -\int_0^t \operatorname{div} \hat{b}^{k_l}(X(s)) ds$ converges to N_t^v uniformly on $[0, \infty)$ P_x^0 -a.e.. By (4.24)

$$N_t^{v_{k_l}} - N_{\frac{1}{n}}^{v_{k_l}} \leq \int_{\frac{1}{n}}^t (g_{k_l} - q_{k_l} + J_{k_l})(X(s)) ds. \quad (4.28)$$

By (4.25) and (4.28), taking $k_l \rightarrow \infty$ and $n \rightarrow \infty$ in (4.28) we have P_x^0 -a.e.

$$N_t^v \leq \int_0^t (g - q + J)(X(s)) ds, \quad \forall t > 0.$$

Thus we have

$$\begin{aligned} &E_{x_0}^0[\exp(\int_0^{\tau_D} (\tilde{a}^{-1}(b - \hat{b})(X(s)), dM_s^0) - \frac{1}{2} \int_0^{\tau_D} (b - \hat{b})^* \tilde{a}^{-1}(b - \hat{b})(X(s)) ds \\ &\quad + N_{\tau_D}^v + \int_0^{\tau_D} (q - J)(X(s)) ds)] \\ &\leq E_{x_0}^0[\exp(\int_0^{\tau_D} (\tilde{a}^{-1}(b - \hat{b})(X(s)), dM_s^0) - \frac{1}{2} \int_0^{\tau_D} (b - \hat{b})^* \tilde{a}^{-1}(b - \hat{b})(X(s)) ds \\ &\quad + \int_0^{\tau_D} g(X(s)) ds)] < \infty. \end{aligned}$$

Hence (4.17) is satisfied. $\square$

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