Potential Anal (2014) 41:1127–1145
DOI 10.1007/s11118-014-9414-9

Removable Singularities for Anisotropic Elliptic Equations

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Received: 4 November 2013 / Accepted: 30 April 2014 / Published online: 28 May 2014

Abstract We study a class of quasi-linear elliptic equations with model representative $\sum_{i=1}^{n}(|u_{x_i}|^{p_i-2}u_{x_i})_{x_i} = 0$, which solutions have singularities on a smooth manifold. We establish the condition for removability of singularity on a manifold for solutions of such equations.

Mathematics Subject Classifications (2010) 35B45 · 35J60

1 Introduction and Main Result

In this paper we study solutions to quasi-linear equations in the divergence form

$$-\operatorname{div}\mathbf{A}(x,\nabla u) = a_0(x,\nabla u), \quad x \in \Omega \setminus \Gamma, \tag{1.1}$$

where Ω is a domain in $\mathbb{R}^n$, $n \geq 3$ and $\Gamma \subset \Omega$ is a manifold of dimension $1 \leq s \leq n-2$.

Throughout the paper we suppose that the functions $\mathbf{A} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ and $a_0 : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ are such that $\mathbf{A}(\cdot,\xi)$, $a_0(\cdot,\xi)$ are Lebesgue measurable for all $\xi \in \mathbb{R}^n$, and $\mathbf{A}(x,\cdot)$, $a_0(x,\cdot)$ are continuous for almost all $x \in \Omega$, $\mathbf{A} = (a_1, a_2, \dots, a_n)$.

We also assume that the following structure conditions are satisfied:

$$\mathbf{A}(x,\xi)\xi \geq \nu_1 \sum_{i=1}^{n} |\xi_i|^{p_i},$$

$$|a_i(x,\xi)| \leq \nu_2 \left(\sum_{j=1}^{n} |\xi_j|^{p_j} \right)^{1-\frac{1}{p_i}}, \quad i = \overline{1,n}$$

$$|a_0(x,\xi)| \leq \nu_2 \left(\left(\sum_{i=1}^{n} |\xi_i|^{p_i} \right)^{1-\frac{1}{\alpha}} + 1 \right), \tag{1.2}$$

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where ν_1, ν_2 are positive constants and

$$1 < p_1 \le \cdots \le p_{n-s}, \quad \frac{1}{\alpha} = \frac{1}{n-s}\sum_{i=1}^{n-s}\frac{1}{p_i}, \tag{1.3}$$

$$\alpha \le p_{n-s+1} \le \cdots \le p_n, \quad \frac{1}{\beta} = \frac{1}{s}\sum_{i=n-s+1}^{n}\frac{1}{p_i}, \quad \frac{1}{p} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i}, \tag{1.4}$$

$$\tilde{p} = \max(p_{n-s}, p_n) < \min\left(\frac{(n-1)p}{n-p}, \frac{n-s-1}{n-s-\alpha}\alpha\right), \quad \alpha < n-s. \tag{1.5}$$

It is well known that the necessary and sufficient conditions for the harmonic function u to have a removable singularity at x_0 is $u(x) = o(|x-x_0|^{2-n})$ as $x \to x_0$. Until recently such a precise result for quasi-linear equations was known only for positive solutions since the celebrated paper by Serrin [14], under relevant assumptions on the coefficients in terms of L^q-spaces (see [18] for the survey of the relevant results). For the sign changing solutions Serrin's result is expressed in terms of L^q-conditions on the coefficients, and for removability of isolated singularities and singularities on the manifolds it leads to a more restrictive condition. A model example of the isotropic Eq. (1.1) is the following equation involving p-Laplacian

$$-\Delta_p u = gu|u|^{p-2} + f \quad \text{in } \Omega \setminus \Gamma, \; p > 1. \tag{1.6}$$

For $g, f \in L^q(\Omega), \; q > \frac{n}{p}$ Serrin's condition [13, 14] on removability of singularities on manifold Γ with dimension s reduces to

$$u(x) = O((d(x,\Gamma))^{-\frac{n-p-s}{p-1}+\delta}), \quad \delta > 0, \; p < n-s, \tag{1.7}$$

where $d(x,\Gamma)$ is the distance from point x to the manifold Γ. Further analysis of sufficient conditions for removability of singularities of solutions has been made by many authors for different classes of nonlinear elliptic and parabolic equations (c.f., e.g [18] and references therein). The precise condition for the removability of singularity on the manifold Γ for Eq. (1.6) with $g, f \in L^q(\Omega), \; q > \frac{n}{p}$ (and more general quasi-linear equations) has the form

$$u(x) = o((d(x,\Gamma))^{-\frac{n-p-s}{p-1}}), \quad 1 < p < n-s, \tag{1.8}$$

which has been proved in [15]. In the case of an isolated singularity ($s = 0$) an analogous result was obtained in [12].

Equations of the form

$$-\sum_{i=1}^{n}(|u_{x_i}|^{p_i-2}u_{x_i})_{x_i} = gu|u|^{p-2} + f \tag{1.9}$$

have not been much studied.

Examples constructed by Giaquinta [4] and Marcellini [9] show that Eq.(1.9) may have unbounded solutions if $p_i s$ are too far apart. Local boundedness of solutions to Eq. (1.9) has been obtained in [3, 6] under the condition

$$1 < p_1 \le \cdots \le p_n \le \frac{np}{n-p}, \quad p < n. \tag{1.10}$$

This condition is sharp as there are unbounded solutions to Eq. (1.9) if condition Eq.(1.10) is violated (cf. [3, 6]). Local boundedness of the gradient of a solution to Eq. (1.9) was obtained in [8, 10] under condition Eq. (1.10) and sufficient smoothness of the coefficients.

It is worth nothing that the explicit fundamental solution to Eq. (1.9) is unknown. Therefore until recently it has not been clear how a precise condition for the removability of an isolated singularity of a solution to Eq. (1.9) can be stated. This question was successfully answered in [11], where it was proved that a singularity at the point $\{x_0\}$ is removable if $g, f \in L^q(\Omega),\ q > \frac{n}{p}$, and

$$\operatorname*{ess\,sup}_{D(R)\setminus D(r)} |u(x)| = o(r^{-\frac{n-p}{p-1}}), \quad p < n, \tag{1.11}$$

where R is some fixed number and

$$D(r) = \left\{ x \in \Omega : \sum_{i=1}^{n} |x_i - x_i^{(0)}|^{a_i} \le r \right\}, \tag{1.12}$$

$$a_i = \frac{p_i(p-1)}{p(n-1) - p_i(n-p)}, \quad i = 1, \ldots, n, \tag{1.13}$$

$$1 < p_1 \le \cdots \le p_n < \frac{n-1}{n-p} p. \tag{1.14}$$

Existence of the positive fundamental solution to equation Eq. (1.9) was proved in [2] under condition Eq. (1.14).

We are interested here in pointwise conditions on solutions to guarantee that the singularity on Γ is removable, that is, the solution can be extended to Ω. Before formulating the main results, let us remind the reader the definition of a weak solution to Eq. (1.1). Let Γ be a manifold of class C^1 without boundary of dimension s contained in Ω. Without loss of generality assume that $\Gamma \subset \{x_1 = x_2 = \cdots = x_{n-s} = 0\}$. We say that u is a weak solution to Eq. (1.1) in $\Omega \setminus \Gamma$ if for an arbitrary function $\psi \in C^1(\Omega)$, vanishing in a neighborhood of Γ, we have the inclusion $u\psi \in W^{1,p_1,\ldots,p_n}(\Omega)$ and the integral identity

$$\int_{\Omega} \{\mathbf{A}(x, \nabla u)\nabla(\varphi\psi) - a_0(x, \nabla u)\varphi\psi\}\, dx = 0 \tag{1.15}$$

holds for any $\varphi \in \overset{\circ}{W}{}^{1,p_1,\ldots,p_n}(\Omega)$.

We say that a solution $u(x)$ of Eq. (1.1) has a removable singularity on the manifold Γ if $u(x)$ can be extended to Γ so that the extension $\tilde{u}(x)$ of $u(x)$ satisfies Eq. (1.1) in Ω and $u(x) \in W^{1,p_1,\ldots,p_n}(\Omega)$.

Let

$$\begin{aligned} &b_i := \frac{p_i(\alpha - 1)}{\alpha(n-s-1) - p_i(n-s-\alpha)}, \quad i = 1, \ldots, n, \\ &x' = (x_1, \ldots, x_{n-s}), \quad x'' = (x_{n-s+1}, \ldots, x_n), \\ &\rho(x') := \left(\sum_{i=1}^{n-s} |x_i|^{\frac{b_i}{b_1}} \right)^{b_1}, \quad \rho(x'') := \left(\sum_{i=n-s+1}^{n} |x_i|^{\frac{b_i}{b_1}} \right)^{b_1}. \end{aligned} \tag{1.16}$$

For $R_0, H_0 > 0$ set

$$D(R_0, H_0) = \{x : \rho(x') < R_0,\ \rho(x'') < H_0\},$$
$$D_1(R_0) = \{x' : \rho(x') < R_0\}, \quad D_2(H_0) = \{x'' : \rho(x'') < H_0\}.$$

We can assume that R_0, H_0 are sufficiently small such that

$$D(R_0, H_0) \subset \Omega, \quad \Gamma \subset D\Big(R_0, \frac{H_0}{2}\Big) \cap \{x' = 0\}.$$

Next we define the number $M(r)$ characterizing local behaviour of the solution u in the neighborhood of the manifold Γ.

$$M(r) := \operatorname{ess\,sup}\{|u(x)| : x \in D(R_0, H_0) \setminus D(r, H_0)\}.$$

The regularity result from [3, 6] yields that $M(r) < \infty$ for $r > 0$. Now we are ready to formulate our main result.

Theorem 1.1 *Let u be a weak solution to Eq. (1.1) in $\Omega \setminus \Gamma$. Let conditions Eq. (1.2)–(1.5) be fulfilled. Assume also that*

$$\lim_{r \to 0} r^{\frac{n-s-\alpha}{\alpha-1}} M(r) = 0, \quad 1 \le s \le n-2. \tag{1.17}$$

Then a singularity of $u(x)$ on Γ is removable.

Remark 1.1 In the critical case $\alpha = n - s$ the condition of removability on the manifold Γ takes the form

$$\lim_{r \to 0} m(r) |\ln r|^{-1} = 0 \quad \text{(cf. [11])},$$

where $m(r) = \operatorname{ess\,sup}\{|u(x)| : x \in \tilde{D}(R_0, H_0) \setminus \tilde{D}(r, H_0)\}$, $\tilde{D}(R_0, H_0) = \{x : d(x') < R_0,\ d(x'') < H_0\}$, $d(x') = (\sum_{i=1}^{n-s} |x_i|^{\frac{p_i}{p_1}})^{\frac{p_1}{n}}$, $d(x'') = (\sum_{i=n-s+1}^{n} |x_i|^{\frac{p_i}{p_1}})^{\frac{p_1}{n}}$.

The result analogous to Theorem 1.1 can be proved for this case with respective changes in Lemmas 2.1–2.4 (see Section 2). We will not pursue this issue here.

The main step in proving Theorem 1.1 is the following result.

Theorem 1.2 *Let the conditions of Theorem 1.1 be fulfilled. Then there exist positive constants K_0, c depending only on $\nu_1, \nu_2, s, n, p_1, \ldots, p_n, R_0, H_0$ such that*

$$M(r) \le K_0 r^{-\frac{n-s-\alpha}{\alpha-1}+c}, \quad r > 0. \tag{1.18}$$

We Point out that our approach continues the studies of I. V. Skrypnik [16, 17] on pointwise estimates of nonlinear capacity potentials. The rest of the paper contains the proof of the above theorems.

2 Proof of Theorem 1.2

2.1 Auxiliary propositions

The following lemmas will be used in the sequel. The first one is the well-known embedding lemma (see [1]).

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$ be a bounded domain, $v \in \overset{\circ}{W}{}^{1,1}(\Omega)$ and*

$$\sum_{i=1}^{n} \int_{\Omega} |v|^{\alpha_i} \left|\frac{\partial v}{\partial x_i}\right|^{p_i} dx < \infty, \quad \alpha_i \ge 0,\ p_i \ge 1.$$

If $1 < p < n$*, then* $v \in L^q(\Omega)$, $q = \frac{np}{n-p}(1 + \frac{1}{n}\sum_{i=1}^n \frac{\alpha_i}{p_i})$ *and the following inequality holds*

$$\|v\|_{L^q(\Omega)} \le K_1 \prod_{i=1}^n \left(\int_\Omega |v|^{\alpha_i} \left| \frac{\partial v}{\partial x_i} \right|^{p_i} dx \right)^{\frac{1}{np_i(1+\frac{1}{n}\sum_{k=1}^n \frac{\alpha_k}{p_k})}}, \tag{2.1}$$

where the constant K_1 *depends only on* $n, \alpha_i, p_i,\ i = 1, \dots, n$.

The next lemma is an immediate consequence of Lemma 2.1.

Lemma 2.2 *Let* $v \in \overset{\circ}{W}{}^{1,1}(\Omega)$ *and*

$$\sum_{i=1}^n \int_\Omega |v|^{-\alpha_i} \left| \frac{\partial v}{\partial x_i} \right|^{p_i} dx < \infty, \quad 0 \le \alpha_i < p_i,\ p_i \ge 1. \tag{2.2}$$

If $1 < p < n$*, then* $v \in L^q(\Omega)$, $q = \frac{np}{n-p}(1 - \frac{1}{n}\sum_{i=1}^n \frac{\alpha_i}{p_i})$ *and the following inequality holds*

$$\|v\|_{L^q(\Omega)} \le K_2 \prod_{i=1}^n \left(\int_\Omega |v|^{-\alpha_i} \left| \frac{\partial v}{\partial x_i} \right|^{p_i} dx \right)^{\frac{1}{np_i(1-\frac{1}{n}\sum_{k=1}^n \frac{\alpha_k}{p_k})}}, \tag{2.3}$$

with positive constant K_2 *depending only on* $n, \alpha_i, p_i,\ i = 1, \dots, n$.

In what follows we will frequently use the following lemma [7, Chapter II, Lemma 4.7].

Lemma 2.3 *Let* $\{y_j\}$ *be a sequence of non-negative numbers such that for any* $j = 0, 1, 2, \dots$ *the inequality*

$$y_{j+1} \le C b^j y_j^{1+\varepsilon} \tag{2.4}$$

holds with positive constants $\varepsilon, C > 0,\ b > 1$. *Then the following estimate is true*

$$y_j \le C^{\frac{(1+\varepsilon)^j - 1}{\varepsilon}} b^{\frac{(1+\varepsilon)^j - 1}{\varepsilon^2} - \frac{j}{\varepsilon}} y_0^{(1+\varepsilon)^j}. \tag{2.5}$$

Particularly, if $y_0 \le C^{-\frac{1}{\varepsilon}} b^{-\frac{1}{\varepsilon^2}}$*, then* $\lim_{j\to\infty} y_j = 0$.

2.2 Integral estimates for the gradient of solutions

Let $\tau \in C^\infty(\mathbb{R}^1)$ be such that $\tau(t) = 0$ for $t \le 1$, $\tau(t) = 1$ for $t \ge 2$, $0 \le \tau(t) \le 1$, $0 \le \frac{d\tau(t)}{dt} \le 2,\ t \in \mathbb{R}^1$.

Fix a point $|\xi''| \le \frac{H_0}{2}$, for $r > 0,\ h > 0$ set

$$\psi_r(x') = \tau(r^{-1}\rho(x')), \quad \zeta_h(x'') = 1 - \tau(h^{-1}\rho(x'' - \xi'')).$$

For $0 < r \le R_0$ set

$$u_r = (u - M(r))_+, \quad E(r) = \{x \in D(R_0, H_0) : u(x) > M(r)\}.$$

By the known parameters we understand the numbers $\nu_1, \nu_2, n, s, p_1, \dots, p_n, R_0, H_0, m$, where m is a fixed positive number such that $m \ge 1 + \tilde{p}$. In what follows γ stands for a generic constant that depends on known parameters only and may vary from line to line.

Lemma 2.4 *Let the conditions of Theorem 1.2 be fulfilled. Then there exists a positive constant c_1 depending on the known parameters only such that the inequalities*

$$0 < r < \rho \le R_0, \quad 0 < h \le \frac{H_0}{2}, \quad \rho \le h \tag{2.6}$$

imply that

$$\sum_{i=1}^{n} \int_{E(\rho)} |u_{x_i}|^{p_i} \psi_r^m \zeta_h^m dx \le c_1 M(r) h^{a_s} (\mu(r) + 1) \tag{2.7}$$

where $a_s = s(1 + \frac{n-s-1}{\alpha-1}(\frac{\alpha}{\beta} - 1))$, $\mu(r) = \sum_{i=1}^{n-s} (r^{\frac{n-s-\alpha}{\alpha-1}} M(r))^{p_i - 1}$.

Proof Without loss assume that $\lim_{r\to 0} M(r) = \infty$ and suppose that R_0 satisfies the additional condition $M(R_0) \ge 1$. Testing Eq. (1.15) by $\varphi = u_\rho \psi_r^{m-1} \zeta_h^m, \quad \psi = \psi_r$ and using conditions Eq. (1.2) we have

$$\begin{aligned}
\sum_{i=1}^{n} \int_{E(\rho)} |u_{x_i}|^{p_i} \psi_r^m \zeta_h^m dx \le\; & \gamma \sum_{i=1}^{n-s} \int_{E(\rho)} \left(\sum_{j=1}^{n} |u_{x_j}|^{p_j} \right)^{1-\frac{1}{p_i}} u_\rho \left| \frac{\partial \psi_r}{\partial x_i} \right| \psi_r^{m-1} \zeta_h^m dx \\
& + \gamma \sum_{i=n-s+1}^{n} \int_{E(\rho)} \left(\sum_{j=1}^{n} |u_{x_j}|^{p_j} \right)^{1-\frac{1}{p_i}} u_\rho \left| \frac{\partial \zeta_h}{\partial x_i} \right| \psi_r^m \zeta_h^{m-1} dx \\
& + \gamma \int_{E(\rho)} \left(\left(\sum_{j=1}^{n} |u_{x_j}|^{p_j} \right)^{1-\frac{1}{\alpha}} + 1 \right) u_\rho \psi_r^m \zeta_h^m dx.
\end{aligned}$$

From this using Young's inequality we get

$$\begin{aligned}
\sum_{i=1}^{n} \int_{E(\rho)} |u_{x_i}|^{p_i} \psi_r^m \zeta_h^m dx \le\; & \gamma \sum_{i=1}^{n-s} \int_{E(\rho)\cap K(r)} u_\rho^{p_i} \left| \frac{\partial \psi_r}{\partial x_i} \right|^{p_i} \zeta_h^m dx \\
& + \gamma \sum_{i=n-s+1}^{n} \int_{E(\rho)} u_\rho^{p_i} \left| \frac{\partial \zeta_h}{\partial x_i} \right|^{p_i} \psi_r^m dx \\
& + \gamma \int_{E(\rho)} u_\rho^\alpha \psi_r^m \zeta_h^m dx + \gamma \int_{E(\rho)} \psi_r^m \zeta_h^m dx,
\end{aligned} \tag{2.8}$$

where $K(r) = \{x' : r < \rho(x') < 2r\}$.

Using the definition of $M(r)$ we have

$$\begin{aligned}
\sum_{i=1}^{n-s} \int_{E(\rho)\cap K(r)} u_\rho^{p_i} \left| \frac{\partial \psi_r}{\partial x_i} \right|^{p_i} \zeta_h^m dx \le\; & \gamma r^{n-s-\alpha} h^{a_s} \sum_{i=1}^{n-s} M^{p_i}(r) r^{-\frac{n-s-\alpha}{\alpha-1}(\alpha - p_i)} \\
& = \gamma M(r) h^{a_s} \sum_{i=1}^{n-s} (r^{\frac{n-s-\alpha}{\alpha-1}} M(r))^{p_i - 1}.
\end{aligned} \tag{2.9}$$

Using condition Eq. (1.17) and inclusion $E(\rho) \subset D(\rho, H_0)$ we deduce

$$\sum_{i=n-s+1}^{n} \int_{E(\rho)} u_\rho^{p_i} \left|\frac{\partial \zeta_h}{\partial x_i}\right|^{p_i} \psi_r^m dx \leq \gamma M(r) h^{a_s-\alpha} \sum_{i=n-s+1}^{n} h^{-\frac{n-s-\alpha}{\alpha-1}(\alpha-p_i)} \times \int_{r\leq\rho(x')\leq\rho} \rho^{-\frac{n-s-\alpha}{\alpha-1}(p_i-1)}(x')\, dx'.$$

Let introduce new independent variables

$$x_i := y_i^{\frac{\delta}{b_i}} \operatorname{sign} y_i, \quad y_i = 1, \dots, n,\ \delta = 2 \max_{1\leq i\leq n-s}(1, b_i),$$

then after simple computation we get

$$\begin{aligned}\int_{r\leq\rho(x')\leq\rho} \rho^{-\frac{n-s-\alpha}{\alpha-1}(p_i-1)}(x')\, dx' &\leq \gamma \int_{r\leq|y'|^\delta\leq\rho} \left(\sum_{i=1}^{n-s} |y_i|^\delta\right)^{-\frac{n-s-\alpha}{\alpha-1}(p_i-1)} \prod_{i=1}^{n-s} |y_i|^{\frac{\delta}{b_i}-1} dy' \\ &\leq \gamma \int_0^{\rho^{\frac{1}{\delta}}} |y'|^{-\delta\frac{(n-s-\alpha)(p_i-1)}{\alpha-1}+\delta(n-s)-1} d|y'| \\ &\leq \gamma \rho^{n-s-\frac{n-s-\alpha}{\alpha-1}(p_i-1)}.\end{aligned} \tag{2.10}$$

Therefore, using Eq. (1.5) we get

$$\sum_{i=n-s+1}^{n} \int_{E(\rho)} u_\rho^{p_i} \left|\frac{\partial \zeta_h}{\partial x_i}\right|^{p_i} \psi_r^m dx \leq \gamma M(r) h^{a_s} \sum_{i=n-s+1}^{n} \left(\frac{\rho}{h}\right)^{\alpha+\frac{n-s-\alpha}{\alpha-1}(\alpha-p_i)} \leq \gamma M(r) h^{a_s}. \tag{2.11}$$

Similarly

$$\int_{E(\rho)} u_\rho^\alpha \psi_r^m \zeta_h^m dx \leq \gamma M(r) h^{a_s} \int_{r\leq\rho(x')\leq\rho} \rho^{-n+s+\alpha}(x') dx' \leq \gamma M(r) h^{a_s} \rho^\alpha. \tag{2.12}$$

The last term in the right-hand side of Eq. (2.8) we estimate using the inclusion $E(\rho) \subset D(\rho, H_0)$. Thus collecting Eq. (2.8)–(2.12) we arrive at the required Eq. (2.7). □

For $0 < \theta\rho < \rho \leq R_0$ set

$$E(\theta\rho, \rho) = \{x \in E(\rho) : u(x) < M(\theta\rho)\}, \quad u^{(\theta\rho)}(x) = \min\{u_\rho(x),\ M(\theta\rho) - M(\rho)\}.$$

Lemma 2.5 *Let the conditions of Theorem 1.2 be fulfilled. Then there exists a positive constant c_2 depending on the known parameters only such that the inequalities*

$$\theta \in (0, 1),\ 0 < \lambda < \min\left(1, \frac{\alpha(\alpha-1)}{n-s-\alpha}\right),\ 0 < r < \frac{\theta\rho}{2} < \rho \leq R_0,\ \rho \leq h$$

imply that

$$\sum_{i=1}^{n}\int\limits_{E(\theta\rho,\rho)} u_\rho^{\lambda-1}|u_{x_i}|^{p_i}\psi_r^m\zeta_h^m dx \le c_2(M(\theta\rho)-M(\rho))^{\frac{\lambda p_n}{p_n-1}}\sum_{i=1}^{n} \times\int\limits_{E(\theta\rho)} u_\rho^{-1-\frac{\lambda}{p_n-1}}|u_{x_i}|^{p_i}\psi_r^m\zeta_h^m dx + c_2\sum_{i=n-s+1}^{n}\int\limits_{E(\rho)} u_\rho^{p_i-1+\lambda}\left|\frac{\partial\zeta_h}{\partial x_i}\right|^{p_i}\psi_r^m dx + c_2\int\limits_{E(\rho)} u_\rho^{\alpha-1+\lambda}\psi_r^m\zeta_h^m dx + c_2(\theta\rho)^{-\frac{\lambda(n-s-\alpha)}{\alpha-1}}\rho^{n-s}h^{a_s} + c_2\mu_1(r)(\theta\rho)^{-\frac{\lambda(n-s-\alpha)}{\alpha-1}}h^{a_s}, \tag{2.13}$$

where $\mu_1(r)=\sum_{i=1}^{n-s}(r^{\frac{n-s-\alpha}{\alpha-1}}M(r))^{1-\frac{1}{p_i}}$*, and* a_s *was defined in Lemma 2.4.*

Proof Test Eq. (1.15) by $\varphi=(u^{(\theta\rho)})^\lambda\psi_r^{m-1}\zeta_h^m, \quad \psi=\psi_r$. Using Eq. (1.2) we have

$$\sum_{i=1}^{n}\int\limits_{E(\theta\rho,\rho)} u_\rho^{\lambda-1}|u_{x_i}|^{p_i}\psi_r^m\zeta_h^n dx \le \gamma\sum_{i=1}^{n-s}\int\limits_{E(\rho)\cap K(r)}\left(\sum_{j=1}^{n}|u_{x_j}|^{p_j}\right)^{1-\frac{1}{p_i}}(u^{(\theta\rho)})^\lambda\left|\frac{\partial\psi_r}{\partial x_i}\right|\psi_r^{m-1}\zeta_h^m dx + \gamma\sum_{i=n-s+1}^{n}\int\limits_{E(\rho)}\left(\sum_{j=1}^{n}|u_{x_j}|^{p_j}\right)^{1-\frac{1}{p_i}}(u^{(\theta\rho)})^\lambda\left|\frac{\partial\zeta_h}{\partial x_i}\right|\psi_r^m\zeta_h^{m-1}dx + \gamma\int\limits_{E(\rho)}\left(\left(\sum_{j=1}^{n}|u_{x_j}|^{p_j}\right)^{1-\frac{1}{\alpha}}+1\right)(u^{(\theta\rho)})^\lambda\psi_r^m\zeta_h^m dx = I_1+I_2+I_3. \tag{2.14}$$

First we estimate I_1. By the Hölder inequality, Eq. (1.17) and Lemma 2.4 we obtain

$$I_1 \le \gamma M^\lambda(\theta\rho)\sum_{i=1}^{n-s}\left(\sum_{j=1}^{n}\int\limits_{E(\rho)}|u_{x_i}|^{p_i}\psi_r^m\zeta_h^n dx\right)^{1-\frac{1}{p_i}}\left(\int\limits_{E(\rho)}\left|\frac{\partial\psi_r}{\partial x_i}\right|^{p_i}\zeta_h^m dx\right)^{\frac{1}{p_i}} \le \gamma(\theta\rho)^{-\frac{\lambda(n-s-\alpha)}{\alpha-1}}h^{a_s}\sum_{i=1}^{n-s}(r^{\frac{n-s-\alpha}{\alpha-1}}M(r)(1+\mu(r)))^{1-\frac{1}{p_i}} \le \gamma(\theta\rho)^{-\frac{\lambda(n-s-\alpha)}{\alpha-1}}h^{a_s}\mu_1(r). \tag{2.15}$$

To estimate I_2 we decompose $E(\rho)$ as $E(\rho)=E(\theta\rho,\rho)\cup E(\theta\rho)$. By the Young inequality and using the evident inequality $u_\rho^{-1}\le(M(\theta\rho)-M(\rho))^{-1}$ for $x\in E(\theta\rho)$, we have

$$
\begin{aligned}
I_2 - \frac{1}{4}\sum_{i=1}^{n}\int\limits_{E(\theta\rho,\rho)} u_\rho^{\lambda-1}|u_{x_i}|^{p_i}\psi_r^m\zeta_h^m dx \le{}& \gamma\sum_{i=n-s+1}^{n}\int\limits_{E(\theta\rho,\rho)} u_\rho^{p_i-1+\lambda}\left|\frac{\partial\zeta_h}{\partial x_i}\right|^{p_i}\psi_r^m dx\\
&+\gamma\sum_{i=n-s+1}^{n}(M(\theta\rho)-M(\rho))^{\frac{\lambda p_i}{p_i-1}}\\
&\times\int\limits_{E(\theta\rho)} u_\rho^{-1-\frac{\lambda}{p_i-1}}|u_{x_i}|^{p_i}\psi_r^m\zeta_h^m dx\\
&+\gamma\sum_{i=n-s+1}^{n}\int\limits_{E(\theta\rho)} u_\rho^{p_i-1+\lambda}\left|\frac{\partial\zeta_h}{\partial x_i}\right|^{p_i}\psi_r^n dx\\
&\le\gamma(M(\theta\rho)-M(\rho))^{\frac{\lambda p_n}{p_n-1}}\sum_{i=1}^{n}\\
&\times\int\limits_{E(\theta\rho)} u_\rho^{-1-\frac{\lambda}{p_n-1}}|u_{x_i}|^{p_i}\psi_r^m\zeta_h^m dx\\
&+\gamma\sum_{i=n-s+1}^{n}\int\limits_{E(\rho)} u_\rho^{p_i-1+\lambda}\left|\frac{\partial\zeta_h}{\partial x_i}\right|^{p_i}\psi_r^m dx.
\end{aligned}
\tag{2.16}
$$

Similarly to Eq. (2.16) we have

$$
\begin{aligned}
I_3 \le \frac{1}{4}\sum_{i=1}^{n}\int\limits_{E(\theta\rho,\rho)} u_\rho^{\lambda-1}|u_{x_i}|^{p_i}\psi_r^m\zeta_h^m dx \,+\, &\gamma(M(\theta\rho)-M(\rho))^{\frac{\lambda p_n}{p_n-1}}\sum_{i=1}^{n}\\
&\times\int\limits_{E(\theta\rho)} u_\rho^{-1-\frac{\lambda}{p_n-1}}|u_{x_i}|^{p_i}\psi_r^m\zeta_h^m dx\\
&+\gamma\int\limits_{E(\rho)} u_\rho^{\alpha-1+\lambda}\psi_r^m\zeta_h^m dx\\
&+\gamma(\theta\rho)^{-\frac{\lambda(n-s-\alpha)}{\alpha-1}}\rho^{n-s}h^{a_s}.
\end{aligned}
\tag{2.17}
$$

Collecting Eq. (2.14)–(2.17) we arrive at the required Eq. (2.13). □

Lemma 2.6 *Let the conditions of Lemma 2.5 be fulfilled. Then there exists a positive number c_3 depending on the known parameters only such that*

$$
\begin{aligned}
\sum_{i=1}^{n}\int\limits_{E(\theta\rho)} u_\rho^{-1-\frac{\lambda}{p_n-1}}|u_{x_i}|^{p_i}\psi_r^m\zeta_h^m dx \le{}& c_3(M(\theta\rho)-M(\rho))^{-\frac{\lambda p_n}{p_n-1}}\sum_{i=n-s+1}^{n}\\
&\times\int\limits_{E(\rho)} u_\rho^{p_i-1+\lambda}\left|\frac{\partial\zeta_h}{\partial x_i}\right|^{p_i}\psi_r^m dx\\
&+c_3(M(\theta\rho)-M(\rho))^{-\frac{\lambda p_n}{p_n-1}}\int\limits_{E(\rho)} u_\rho^{\alpha-1+\lambda}\psi_r^m\zeta_h^m dx\\
&+c_3(M(\theta\rho)-M(\rho))^{-\frac{\lambda}{p_n-1}}\mu_1(r)+c_3(M(\theta\rho)\\
&\quad-M(\rho))^{-\frac{\lambda}{p_n-1}}\rho^{n-s}h^{a_s},
\end{aligned}
\tag{2.18}
$$

where $\mu_1(s)$ *was defined in Lemma 2.5.*

Proof Test Eq. (1.15) by

$$\varphi = \Big[(M(\theta\rho) - M(\rho))^{-\frac{\lambda}{p_n-1}} - \max{}^{-\frac{\lambda}{p_n-1}}(u_\rho, M(\theta\rho) - M(\rho))\Big]\psi_r^{m-1}\zeta_h^m, \quad \psi = \psi_r,$$

using Eq. (1.2) and the Young inequality we have

$$\begin{aligned}
&\sum_{i=1}^{n}\int\limits_{E(\theta\rho)} u_\rho^{-1-\frac{\lambda}{p_n-1}}|u_{x_i}|^{p_i}\psi_r^m\zeta_h^m dx \\
&\quad\le \gamma(M(\theta\rho)-M(\rho))^{-\frac{\lambda}{p_n-1}}\sum_{i=1}^{n-s}\int\limits_{E(\rho)\cap K(r)}\Big(\sum_{j=1}^{n}|u_{x_j}|^{p_j}\Big)^{1-\frac{1}{p_i}}\Big|\frac{\partial\psi_r}{\partial x_i}\Big|\psi_r^{m-1}\zeta_h^m dx \\
&\quad+\gamma\sum_{i=n-s+1}^{n}(M(\theta\rho)-M(\rho))^{-\frac{\lambda p_i}{p_n-1}}\int\limits_{E(\theta\rho)} u_\rho^{\frac{\lambda(p_i-1)}{p_n-1}+p_i-1}\Big|\frac{\partial\zeta_h}{\partial x_i}\Big|^{p_i}\psi_r^m dx \\
&\quad+\gamma(M(\theta\rho)-M(\rho))^{-\frac{\lambda\alpha}{p_n-1}}\int\limits_{E(\theta\rho)} u_\rho^{\frac{\lambda(\alpha-1)}{p_n-1}+\alpha-1}\psi_r^m\zeta_h^m dx + \gamma(M(\theta\rho)-M(\rho))^{-\frac{\lambda}{p_n-1}} \\
&\qquad\times\int\limits_{E(\theta\rho)}\psi_r^m\zeta_h^m dx.
\end{aligned} \tag{2.19}$$

The first term in the right-hand side of Eq. (2.19) has been estimated in Eq. (2.15), therefore we arrive at the required Eq. (2.18). □

Combining Lemmas 2.5, 2.6 we get

Theorem 2.1 *Let the conditions of Theorem 1.2 be fulfilled. Then there exists a positive constant* c_4 *depending on the known parameters only such that the inequalities* $0 < \theta < 1,\ 0 < \lambda < \min(1, \frac{\alpha(\alpha-1)}{n-s-\alpha}),\ 0 < r < \frac{\theta\rho}{2} < \rho \le R_0,\ \rho \le h$ *imply that*

$$\sum_{i=1}^{n}\int\limits_{E(\theta\rho,\rho)} u_\rho^{\lambda-1}|u_{x_i}|^{p_i}\psi_r^m\zeta_h^m dx \le c_4\sum_{i=n-s+1}^{n}\int\limits_{E(\rho)} u_\rho^{p_i-1+\lambda}\Big|\frac{\partial\zeta_h}{\partial x_i}\Big|^{p_i}\psi_r^m dx \tag{2.20}$$

$$+c_4\int\limits_{E(\rho)} u_\rho^{\alpha-1+\lambda}\psi_r^m\zeta_h^m dx + c_4 G(r,\rho,h), \tag{2.21}$$

$$\tag{2.22}$$

where $G(r,\rho,h) = (\theta\rho)^{-\frac{\lambda(n-s-\alpha)}{\alpha-1}}\rho^{n-s}h^{a_s} + (\theta\rho)^{-\frac{\lambda(n-s-\alpha)}{\alpha-1}}h^{a_s}\mu_1(r).$

2.3 Integral estimates of solutions

Let

$$\frac{n-s}{n-s-(p_n-\alpha)\frac{n-s-\alpha}{\alpha-1}} < q < \frac{n-s}{n-s-\alpha}, \tag{2.23}$$

set

$$I(\rho,h)=\rho^{(n-s)\frac{q-1}{q}}\int\limits_{D_2(H_0)}dx''\Big(\int\limits_{D_1(R_0)}(u_\rho^{\alpha-1+\lambda}\psi_r^m\zeta_h^m)^q dx'\Big)^{\frac{1}{q}}. \tag{2.24}$$

Lemma 2.7 *Let the conditions of Theorem 1.2 be fulfilled and* $0<\lambda<\min(1,\frac{\alpha(\alpha-1)}{n-s-\alpha},\frac{\alpha(n-s-1)}{n-s-\alpha}-p_n)$. *Then there exists a positive constant* c_5 *depending on the known parameters only such that*

$$I(\rho,h)\le 2^{\alpha-1+\lambda}\theta^{-(n-s)\frac{q-1}{q}}I(\theta\rho,h)+c_5\Big(\frac{\rho}{h}\Big)^{\alpha-\frac{n-s-\alpha}{\alpha-1}(p_n-\alpha)}I(\rho,2h)+c_5G_1(r,\rho,h), \tag{2.25}$$

where

$$\begin{aligned}G_1(r,\rho,h)&=(\theta\rho)^{-\frac{\lambda(n-s-\alpha)}{\alpha-1}}\rho^{n-s+\alpha}h^{a_s}\\&+(\theta\rho)^{-\frac{\lambda(n-s-\alpha)}{\alpha-1}}\rho^{\alpha}h^{a_s}\Big(\sum_{i=n-s+1}^{n}(r^{\frac{n-s-\alpha}{\alpha-1}}M(r))^{p_i-1}+\sum_{i=n-s+1}^{n}(r^{\frac{n-s-\alpha}{\alpha-1}}M(r))^{1-\frac{1}{p_i}}\Big)\\&+\rho^{\alpha}h^{a_s-\frac{\alpha(n-s-1)}{\alpha-1}}\sum_{i=n-s+1}^{n}h^{\frac{h-s-\alpha}{\alpha-1}p_i}(r^{\frac{n-s-\alpha}{\alpha-1}}M(r))^{p_i-1+\lambda}\\&+\rho^{\alpha}h^{a_s}(r^{\frac{n-s-\alpha}{\alpha-1}}M(r))^{\alpha-1+\lambda}.\end{aligned}$$

Proof Let $\chi(E(\theta\rho,\rho))$, $\chi(E(\theta\rho))$ denote the characteristic functions of the sets $E(\theta\rho,\rho)$, $E(\theta\rho)$ respectively. We will estimate $I(\rho,h)$ using the inequality

$$\begin{aligned}u_\rho^{\alpha-1+\lambda}&\le u_\rho^{\alpha-1+\lambda}\chi(E(\theta\rho,\rho))+2^{\alpha-1+\lambda}(u_{\theta\rho}^{\alpha-1+\lambda}+(M(\theta\rho)-M(\rho))^{\alpha-1+\lambda})\chi(E(\theta\rho))\\&\le 2^{\alpha-1+\lambda}u_{\theta\rho}^{\alpha-1+\lambda}\chi(E(\theta\rho))+2^{\alpha-1+\lambda}(u^{(\theta\rho)})^{\alpha-1+\lambda},\quad x\in E(\rho).\end{aligned}$$

Thus

$$I(\rho,h)\le 2^{\alpha-1+\lambda}\theta^{-(n-s)\frac{q-1}{q}}I(\theta\rho,h)+\gamma\rho^{(n-s)\frac{q-1}{q}}I_4, \tag{2.26}$$

where

$$I_4=\int\limits_{D_2(H_0)}dx''\Big(\int\limits_{D_1(R_0)}((u^{(\theta\rho)})^{\alpha-1+\lambda}\psi_r^m\zeta_h^m)^q dx'\Big)^{\frac{1}{q}}. \tag{2.27}$$

Using the Hölder inequality and Lemma 2.2 with $\alpha_1=\cdots=\alpha_{n-s}=1-\lambda$, and choosing m from the condition $m\frac{p_1-1+\lambda}{\alpha-1+\lambda}-\tilde p\ge 1$, we obtain

$$\begin{aligned}I_4&\le\gamma\rho^{-(n-s)\frac{q-1}{q}+\alpha}\int\limits_{D_2(H_0)}dx''\Big(\int\limits_{D_1(R_0)}((u^{(\theta\rho)})^{\alpha-1+\lambda}\psi_r^m\zeta_h^m)^{\frac{n-s}{n-s-\alpha}}dx'\Big)^{\frac{n-s-\alpha}{n-s}}\\&\le\gamma\rho^{-(n-s)\frac{q-1}{q}+\alpha}\sum_{i=1}^{n-s}\int\limits_{E(\theta\rho,\rho)}u_\rho^{\lambda-1}|u_{x_i}|^{p_i}\psi_r^{m\frac{p_1-1+\lambda}{\alpha-1+\lambda}}\zeta_h^{m\frac{p_1-1+\lambda}{\alpha-1+\lambda}}dx\\&+\gamma\rho^{-(n-s)\frac{q-1}{q}+\alpha}\sum_{i=1}^{n-s}\int\limits_{E(\rho)}(u^{(\theta\rho)})^{p_i-1+\lambda}\Big|\frac{\partial\psi_r}{\partial x_i}\Big|^{p_i}\zeta_h^{m\frac{p_1-1+\lambda}{\alpha-1+\lambda}}dx.\end{aligned}$$

By Theorem 2.1

$$\sum_{i=1}^{n-s} \int\limits_{E(\theta\rho,\rho)} u_\rho^{\lambda-1}|u_{x_i}|^{p_i}\psi_r^{m\frac{p_1-1+\lambda}{\alpha-1+\lambda}}\zeta_h^{m\frac{p_1-1+\lambda}{\alpha-1+\lambda}}dx$$

$$\leq \gamma \sum_{i=n-s+1}^{n} \int\limits_{E(\rho)} u_\rho^{p_i-1+\lambda}\left|\frac{\partial \zeta_h}{\partial x_i}\right|^{p_i}\psi_r^{m\frac{p_1-1+\lambda}{\alpha-1+\lambda}}dx + \gamma \int\limits_{E(\rho)} u_\rho^{\alpha-1+\lambda}\psi_r^{m\frac{p_1-1+\lambda}{\alpha-1+\lambda}}\zeta_h^{m\frac{p_1-1+\lambda}{\alpha-1+\lambda}}dx$$

$$+\gamma G(r,\rho,h).$$

From this and form the fact that $\{\zeta_h \neq 0\} \subseteq \{\zeta_{2h} = 1\}$ we obtain

$$I_4 \leq \gamma\rho^{-(n-s)\frac{q-1}{q}+\alpha}\sum_{i=1}^{n-s}\int\limits_{E(\rho)}(u^{(\theta\rho)})^{p_i-1+\lambda}\left|\frac{\partial\psi_r}{\partial x_i}\right|^{p_i}\zeta_h dx + \gamma\rho^{-(n-s)\frac{q-1}{q}+\alpha}\sum_{i=n-s+1}^{n}$$

$$\times\int\limits_{E(\rho)} u_\rho^{p_i-1+\lambda}\left|\frac{\partial\zeta_h}{\partial x_i}\right|^{p_i}\psi_r\zeta_{2h}^m dx$$

$$+\gamma\rho^{-(n-s)\frac{q-1}{q}+\alpha}\int\limits_{E(\rho)}u^{\alpha-1+\lambda}\psi_r\zeta_{2h}^m dx + \gamma\rho^{-(n-s)\frac{q-1}{q}+\alpha}G(r,\rho,h). \tag{2.28}$$

Similarly to Eq. (2.9) we have

$$\sum_{i=1}^{n-s}\int\limits_{E(\rho)}(u^{(\theta\rho)})^{p_i-1+\lambda}\left|\frac{\partial\psi_r}{\partial x_i}\right|^{p_i}\zeta_h dx \leq \gamma(\theta\rho)^{-\frac{\lambda(n-s-\alpha)}{\alpha-1}}h^{a_s}\sum_{i=1}^{n-s}(r^{\frac{n-s-\alpha}{\alpha-1}}M(r))^{p_i-1}. \tag{2.29}$$

Using Eq. (1.17) and the Hölder inequality we obtain

$$\sum_{i=n-s+1}^{n}\int\limits_{E(\rho)}u^{p_i-1+\lambda}\left|\frac{\partial\zeta_h}{\partial x_i}\right|^{p_i}\psi_r^m\zeta_{2h}^m dx$$

$$\leq \gamma h^{a_s-\alpha\frac{n-s-1}{\alpha-1}}\sum_{i=n-s+1}^{n}h^{\frac{n-s-\alpha}{\alpha-1}p_i}(r^{\frac{n-s-\alpha}{\alpha-1}}M(r))^{p_i-1+\lambda}r^{\frac{n-s-1}{\alpha-1}\alpha-\frac{n-s-\alpha}{\alpha-1}(p_n+\lambda)}$$

$$+\gamma\sum_{i=n-s+1}^{n}\int\limits_{E(\rho)\setminus K(r)}\rho^{-\frac{n-s-\alpha}{\alpha-1}(p_i-\alpha)}(x')u_\rho^{\alpha-1+\lambda}\left|\frac{\partial\zeta_h}{\partial x_i}\right|^{p_i}\psi_r^m\zeta_{2h}^m dx$$

$$\leq \gamma h^{a_s-\alpha\frac{n-s-1}{\alpha-1}}\sum_{i=n-s+1}^{n}h^{\frac{n-s-\alpha}{\alpha-1}p_i}(r^{\frac{n-s-\alpha}{\alpha-1}}M(r))^{p_i-1+\lambda}$$

$$+\gamma I(\rho,2h)\sum_{i=n-s+1}^{n}h^{-\alpha-\frac{n-s-\alpha}{\alpha-1}(\alpha-p_i)}\rho^{-\frac{n-s-\alpha}{\alpha-1}(p_i-\alpha)}$$

$$\leq \gamma h^{a_s-\alpha\frac{n-s-1}{\alpha-1}}\sum_{i=n-s+1}^{n}h^{\frac{n-s-\alpha}{\alpha-1}p_i}(r^{\frac{n-s-\alpha}{\alpha-1}}M(r))^{p_i-1+\lambda}+\gamma\rho^{-\alpha}\left(\frac{\rho}{h}\right)^{\alpha-\frac{n-s-\alpha}{\alpha-1}(p_n-\alpha)}I(\rho,2h). \tag{2.30}$$

Similarly to Eq. (2.30) we have

$$\int_{E(\rho)} u_\rho^{\alpha-1+\lambda}\psi_r^m\zeta_{2h}^m dx \le \gamma h^{a_s}(r^{\frac{n-s-\alpha}{\alpha-1}}M(r))^{\alpha-1+\lambda} + \gamma\rho^{(n-s)\frac{q-1}{q}} I(\rho,2h). \tag{2.31}$$

Combining estimates Eq. (2.26)–(2.31) we arrive at the required Eq. (2.25). □

We choose λ such that

$$0<\lambda<\min\Big(1,\frac{\alpha(n-s-1)}{n-s-\alpha}-p_n,\frac{\alpha-1}{n-s-\alpha}\Big(\alpha-(n-s)\frac{q-1}{q}\Big)\Big). \tag{2.32}$$

Theorem 2.2 *Let the conditions of Theorem 1.2 be fulfilled. Then there exist positive numbers c_6, c_7 depending on the known parameters only such that the inequalities*

$$0<r<2\rho\le R_0,\quad \rho\le c_6 h \tag{2.33}$$

imply that

$$I(\rho,h)\le c_7 h^{n+a_s-s}\rho^{\alpha-\frac{\lambda(n-s-\alpha)}{\alpha-1}} + c_7 h^{a_s}\rho^{n-s+\alpha-\frac{\lambda(n-s-\alpha)}{\alpha-1}} + c_7 G_2(r,\rho,h), \tag{2.34}$$

where

$$G_2(r,\rho,h)=\rho^{\alpha-\frac{\lambda(n-s-\alpha)}{\alpha-1}}h^{a_s}\Big\{\sum_{i=1}^{n-s}(r^{\frac{n-s-\alpha}{\alpha-1}}M(r))^{p_i-1}+\sum_{i=1}^{n-s}(r^{\frac{n-s-\alpha}{\alpha-1}}M(r))^{1-\frac{1}{p_i}}$$
$$+\sum_{i=n-s1}^{n} h^{-\alpha\frac{n-s-1}{\alpha-1}+\frac{n-s-\alpha}{\alpha-1}p_i}(r^{\frac{n-s-\alpha}{\alpha-1}}M(r))^{p_i-1+\lambda}+(r^{\frac{n-s-\alpha}{\alpha-1}}M(r))^{\alpha-1+\lambda}\Big\}.$$

Proof Let $A=2^{\alpha-1+\lambda}\theta^{-(n-s)\frac{q-1}{q}}$, $B=c_5c_6^{\alpha-\frac{n-s-\alpha}{\alpha-1}(p_n-\alpha)}$ and choose integers N_1, N_2 such that

$$2r<\rho\theta^{N_1}\le\frac{2r}{\theta},\quad \frac{H_0}{2}<2^{N_2}h\le H_0. \tag{2.35}$$

Thus the inequality Eq. (2.25) can be rewritten in the form

$$I(\rho,h)\le AI(\theta\rho,h)+BI(\rho,2h)+\gamma G_1(r,\rho,h).$$

From this we deduce

$$I(\rho,h)\le(2A)^{N_1}\sum_{j=0}^{N_2-1}(2B)^jI(2r,2^jh)+(2B)^{N_2}\sum_{l=0}^{N_1-1}(2A)^lI(\theta^l\rho,H_0)$$
$$+\sum_{l=0}^{N_1-1}\sum_{j=0}^{N_2-1}A^lB^jG_1(r,\theta^l\rho,2^jh). \tag{2.36}$$

Let us estimate the terms in the right-hand side of Eq. (2.36). By Eq. (1.17) we have

$$I(2r,2^jh)\le\gamma(2^jh)^{a_s}M^{\alpha-1+\lambda}(r)r^{n-s}\le\gamma(2^jh)^{a_s}r^{\alpha-\frac{\lambda(n-s-\alpha)}{\alpha-1}}(r^{\frac{n-s-\alpha}{\alpha-1}}M(r))^{\alpha-1+\lambda},$$

choose $c_6 < 1$ such that

$$2^{a_s+1}B = 2^{a_s+1}c_5 c_6^{\alpha-\frac{n-s-\alpha}{\alpha-1}(p_n-\alpha)} \le \frac{1}{2}, \tag{2.37}$$

hence Eq. (2.37) yields

$$(2A)^{N_1}\sum_{j=0}^{N_2-1}(2B)^j I(2r,2^j h) \le \gamma (2A)^{N_1} h^{a_s} r^{\alpha-\frac{\lambda(n-s-\alpha)}{\alpha-1}}(r^{\frac{n-s-\alpha}{\alpha-1}}M(r))^{\alpha-1+\lambda}. \tag{2.38}$$

By Eq. (2.35) we have

$$(2A)^{N_1} \le \gamma\left(\frac{\rho}{r}\right)^{(n-s)\frac{q-1}{q}+\frac{\alpha+\lambda}{\log_2\frac{1}{\theta}}},$$

choosing $\theta \in (0,1)$ from the condition

$$\frac{\alpha+\lambda}{\log_2\frac{1}{\theta}} \le \alpha-(n-s)\frac{q-1}{q}-\frac{\lambda(n-s-\alpha)}{\alpha-1}, \tag{2.39}$$

we conclude from Eq. (2.38) that

$$(2A)^{N_1}\sum_{j=0}^{N_2-1}(2B)^j I(2r,2^j h) \le \gamma h^{a_s}\rho^{\alpha-\frac{\lambda(n-s-\alpha)}{\alpha-1}}(r^{\frac{n-s-\alpha}{\alpha-1}}M(r))^{\alpha-1+\lambda}. \tag{2.40}$$

Using Eq. (1.17) we have

$$\begin{aligned} I(\theta^l\rho, H_0) &\le \gamma(\theta^l\rho)^{(n-s)\frac{q-1}{q}} \int\limits_{D_2(H_0)} dx''\Bigg(\int\limits_{\rho(x')\le\theta^l\rho} \rho^{-\frac{n-s-\alpha}{\alpha-1}(\alpha-1+\lambda)q}(x')\,dx'\Bigg)^{\frac{1}{q}} \\ &\le \gamma(\theta^l\rho)^{\alpha-\frac{\lambda(n-s-\alpha)}{\alpha-1}}. \end{aligned}$$

The last inequality ensures that

$$\begin{aligned} (2B)^{N_2}\sum_{l=0}^{N_1-1}(2A)^l I(\theta^l\rho, H_0) &\le \gamma(2B)^{N_2}\rho^{\alpha-\frac{\lambda(n-s-\alpha)}{\alpha-1}} \\ &\quad\times\sum_{l=0}^{N_1-1}2^{(\alpha+\lambda)l}\theta^{l(\alpha-(n-s)\frac{q-1}{q}-\frac{\lambda(n-s-\alpha)}{\alpha-1})}. \end{aligned} \tag{2.41}$$

Choosing θ, c_6 small enough so that

$$\frac{\alpha+\lambda+1}{\log_2\frac{1}{\theta}} \le \alpha-(n-s)\frac{q-1}{q}-\frac{\lambda(n-s-\alpha)}{\alpha-1}, \tag{2.42}$$

$$2B = 2c_5 c_6^{\alpha-\frac{n-s-\alpha}{\alpha-1}(p_n-\alpha)} \le 2^{-n-a_s+s}, \tag{2.43}$$

we conclude from Eq. (2.41) that

$$(2B)^{N_2}\sum_{l=0}^{N_1-1}(2A)^l I(\theta^l\rho, H_0) \le \gamma h^{n+a_s-s}\rho^{\alpha-\frac{\lambda(n-s-\alpha)}{\alpha-1}}. \tag{2.44}$$

Finally from the condition Eq. (1.17) we have

$$A^l B^j G_1(r,\theta^l h,2^j h) \le \gamma A^l B^j (2^j h)^{a_s} (\theta^l \rho)^{\alpha} \Bigg\{ (\theta^l \rho)^{n-s-\frac{\lambda(n-s-\alpha)}{\alpha-1}}$$
$$+ (\theta^l \rho)^{-\frac{\lambda(n-s-\alpha)}{\alpha-1}} \left(\sum_{i=1}^{n-s} (r^{\frac{n-s-\alpha}{\alpha-1}} M(r))^{p_i-1} + \sum_{i=1}^{n-s} (r^{\frac{n-s-\alpha}{\alpha-1}} M(r))^{1-\frac{1}{p_i}} \right)$$
$$+ \sum_{i=n-s+1}^{n} (2^j h)^{-\frac{\alpha(n-s-1)}{\alpha-1}+\frac{n-s-\alpha}{\alpha-1}p_i} (r^{\frac{n-s-\alpha}{\alpha-1}} M(r))^{p_i-1+\lambda} + (r^{\frac{n-s-\alpha}{\alpha-1}} M(r))^{\alpha-1+\lambda} \Bigg\}. \quad (2.45)$$

First, choose c_6 from the condition

$$2^{a_s} B = 2^{a_s} c_5 c_6^{\alpha-\frac{n-s-\alpha}{\alpha-1}(p_n-1)} \le 2^{\alpha\frac{n-s-1}{\alpha-1}-\frac{n-s-\alpha}{\alpha-1}p_n-1}, \quad (2.46)$$

next choose θ from the condition

$$2^{\alpha+\lambda}\theta^{\alpha-(n-s)\frac{q-1}{q}} \{\theta^{n-s-\frac{\lambda(n-s-\alpha)}{\alpha-1}} + \theta^{-\frac{\lambda(n-s-\alpha)}{\alpha-1}} + 1\} \le \frac{1}{2}, \quad (2.47)$$

we conclude from Eq. (2.45) that

$$\sum_{l=0}^{N_1-1}\sum_{j=0}^{N_2-1} A^l B^j G_1(r,\theta^l h,2^j h) \le \gamma G_2(r,\rho,h) + \gamma h^{a_s} \rho^{n-s+\alpha-\frac{\lambda(n-s-\alpha)}{\alpha-1}}. \quad (2.48)$$

Combining estimates Eq. (2.36)–(2.48) we arrive at the required Eq. (2.34). □

2.4 Pointwise estimates of solutions

Fix $\rho > 0$, for $j = 1, 2, \ldots, \; J = [\frac{\ln \frac{R_0}{\rho}}{\ln \frac{1}{\theta}}] + 1$ set $\rho_j = R_0\theta^j$. Let x_0 be an arbitrary point in $D(R_0, H_0) \setminus D(\rho_j, H_0)$. For $l = 0, 1, 2, \ldots$ set $R_l = (1-\theta)\rho_j(1-\frac{1}{2}+\frac{1}{2^{l+1}})$, $\overline{R}_l = \frac{1}{2}(R_l + R_{l+1})$, $B_{R_l}(x_0) = \{x : \rho(x'-x_0') \le R_l,\ \rho(x''-x_0'') \le c_6^{-1} R_l\}$, $k_l = 2k_0 - \frac{k_0}{2^l}$, $A_{k,R} = \{x \in B_R(x_0) : u_{\rho_{j-1}} \ge k\}$, here θ, c_6 were defined in Theorem 2.2, k_0 is a positive number depending on the known parameters only, which will be specified later.

Let $\xi_l \in C_0^\infty(B_{\overline{R}_l})$ be such that $\xi_l \equiv 1$ for $x \in B_{R_{l+1}}(x_0)$, $|\frac{\partial \xi_l}{\partial x_i}| \le \gamma 2^l \rho_j^{-\frac{\alpha}{p_i}-\frac{n-s-\alpha}{\alpha-1}(\frac{\alpha}{p_i}-1)}$, $i = 1, \ldots, n$.

Testing Eq. (1.15) by $\varphi = (u_{\rho_{j-1}} - k_{l+1})_+^\lambda \xi_l^{m-1}$, $\psi = \xi_l$, using Eq. (1.2) and the Young inequality we have

$$\sum_{i=1}^{n} \int_{A_{k_{l+1},\overline{R}_l}} (u_{\rho_{j-1}} - k_{l+1})^{\lambda-1} |u_{x_i}|^{p_i} \xi_l^m dx \le \gamma 2^{\gamma l} \sum_{i=1}^{n} \rho^{-\alpha-\frac{n-s-\alpha}{\alpha-1}(\alpha-p_i)}$$
$$\times \int_{A_{k_{l+1},\overline{R}_l}} (u_{\rho_{j-1}} - k_{l+1})^{p_i-1+\lambda} dx$$
$$+\gamma \int_{A_{k_{l+1},\overline{R}_l}} (u_{\rho_{j-1}} - k_{l+1})^{\alpha-1+\lambda} dx + \gamma |A_{k_{l+1},\overline{R}_l}|. \quad (2.49)$$

Using Eq. (1.17) we conclude from Eq. (2.49) that

$$\sum_{i=1}^{n}\int\limits_{A_{k_{l+1},\overline{R}_l}}(u_{\rho_{j-1}}-k_{l+1})^{\lambda-1}|u_{x_i}|^{p_i}\xi_l^m dx\le\gamma 2^{\gamma l}\rho_j^{s-n-\frac{\lambda(n-s-\alpha)}{\alpha-1}}|A_{k_{l+1},\overline{R}_l}|. \tag{2.50}$$

Using Eq. (1.5), the Hölder inequality and Lemma 2.2 with $\alpha_1=\cdots=\alpha_n=1-\lambda$, we obtain

$$\begin{aligned}
&\int\limits_{A_{k_{l+1},R_{l+1}}}(u_{\rho_{j-1}}-k_{l+1})^{\alpha-1+\lambda}dx\le\int\limits_{A_{k_{l+1},\overline{R}_l}}(u_{\rho_{j-1}}-k_{l+1})^{\alpha-1+\lambda}\xi_l^{m(\alpha-1+\lambda)}dx\\
&\le\gamma\Bigg(\int\limits_{A_{k_{l+1},\overline{R}_l}}((u_{\rho_{j-1}}-k_{l+1})\xi_l^m)^{\frac{(p-1+\lambda)n}{n-p}}dx\Bigg)^{\frac{(\alpha-1+\lambda)(n-p)}{(p-1+\lambda)n}}|A_{k_{l+1},\overline{R}_l}|^{1-\frac{(\alpha-1+\lambda)(n-p)}{(p-1+\lambda)n}}\\
&\le\gamma\Bigg(\sum_{i=1}^{n}\int\limits_{A_{k_{l+1},\overline{R}_l}}(u_{\rho_{j-1}}-k_{l+1})^{\lambda-1}\xi_l^{m(\lambda-1)}\Big|\frac{\partial}{\partial x_i}((u_{\rho_{j-1}}-k_{l+1})\xi_l^m)\Big|^{p_i}dx\Bigg)^{\frac{\alpha-1+\lambda}{p-1+\lambda}}|A_{k_{l+1},\overline{R}_l}|^{1-\frac{(\alpha-1+\lambda)(n-p)}{(p-1+\lambda)n}}\\
&\le\gamma 2^{\gamma l}\rho_j^{(s-n-\frac{\lambda(n-s-\alpha)}{\alpha-1})\frac{\alpha-1+\lambda}{p-1+\lambda}}|A_{k_{l+1},\overline{R}_l}|^{1+\frac{\alpha-1+\lambda}{p-1+\lambda}\frac{p}{n}}.
\end{aligned} \tag{2.51}$$

Using the evident inequality

$$|A_{k_{l+1},\overline{R}_l}|\le 2^{l(\alpha-1+\lambda)}k_0^{-(\alpha-1+\lambda)}\int\limits_{A_{k_l,R_l}}(u_{\rho_{j-1}}-k_l)^{\alpha-1+\lambda}dx$$

and setting $y_l=\int_{A_{k_l,R_l}}(u_{\rho_{j-1}}-k_l)^{\alpha-1+\lambda}dx$ we obtain

$$y_{l+1}\le\gamma 2^{\gamma l}\rho_j^{(s-n-\frac{\lambda(n-s-\alpha)}{\alpha-1})\frac{\alpha-1+\lambda}{p-1+\lambda}}k_0^{-(\alpha-1+\lambda)(1+\frac{\alpha-1+\lambda}{p-1+\lambda}\frac{p}{n})}y_l^{1+\frac{\alpha-1+\lambda}{p-1+\lambda}\frac{p}{n}},\quad l=0,1,2,\ldots$$

Due to Lemma 2.3 this inequality implies that $y_l\to 0$ as $l\to\infty$ if k_0 satisfies the following condition

$$y_0=\gamma\rho^{-(s-n-\frac{\lambda(n-s-\alpha)}{\alpha-1})\frac{n}{p}}k_0^{(\alpha-1+\lambda)(\frac{n}{p}\frac{p-1+\lambda}{\alpha-1+\lambda}+1)}.$$

From this we obtain that

$$\big(\operatorname{ess\,sup}\{u_{\rho_{j-1}}(x):x\in B_{\frac{1-\theta}{2}\rho_j}(x_0)\}\big)^{\alpha-1+\lambda+\frac{n}{p}(p-1+\lambda)}\le\gamma\rho_j^{(s-n-\frac{\lambda(n-s-\alpha)}{\alpha-1})\frac{n}{p}}\int\limits_{E(\rho_{j-1})}u_{\rho_{j-1}}^{\alpha-1+\lambda}dx. \tag{2.52}$$

Since x_0 is an arbitrary point in $D(R_0,H_0)\setminus D(\rho_j,H_0)$, from Eq. (2.52) it follows

$$(M(\rho_j)-M(\rho_{j-1}))^{\alpha-1+\lambda+\frac{n}{p}(p-1+\lambda)}\le\gamma\rho_j^{(s-n-\frac{\lambda(n-s-\alpha)}{\alpha-1})\frac{n}{p}}\int\limits_{E(\rho_{j-1})}u_{\rho_{j-1}}^{\alpha-1+\lambda}dx.$$

Using Theorem 2.2 with $\rho=\rho_{j-1},\ h=c_6^{-1}\rho_{j-1}$ and the Hölder inequality we obtain

$$(M(\rho_j)-M(\rho_{j-1}))^{\alpha-1+\lambda+\frac{n}{p}(p-1+\lambda)}\le\gamma\rho_j^{(s-n-\frac{\lambda(n-s-\alpha)}{\alpha-1})\frac{n}{p}}\{\rho_j^{a_s+n-s+\alpha-\frac{\lambda(n-s-\alpha)}{\alpha-1}}+G_2(r,\rho_{j-1},c_6^{-1}\rho_{j-1})\}. \tag{2.53}$$

Passing in Eq. (2.53) to the limit $r \to 0$, by Eq. (1.17) we obtain

$$(M(\rho_j) - M(\rho_{j-1}))^{\alpha-1+\lambda+\frac{n}{p}(p-1+\lambda)} \le \gamma \rho_j^{n+\alpha-\frac{\lambda(n-s-\alpha)}{\alpha-1}+(s-n-\frac{\lambda(n-s-\alpha)}{\alpha-1})\frac{n}{p}}. \tag{2.54}$$

In order to complete the proof of Theorem 1.2 we sum up Eq. (2.54) with respect to j from 1 to J

$$M(\rho) \le M(\rho_J) \le M(R_0) + \gamma \rho_J^{-\frac{n-s-\alpha}{\alpha-1}+c} \le M(R_0) + \gamma \rho^{-\frac{n-s-\alpha}{\alpha-1}+c} \tag{2.55}$$

where $c = \frac{\frac{\alpha}{\beta}(n-s-1)-n+s+\alpha+(n-s-1)(n-1-\frac{n-p}{p}\alpha)}{(\alpha-1)(\alpha+1+\lambda+\frac{n}{p}(p-1+\lambda))} > 0$ by Eq. (1.5).

From this the required Eq. (1.18) follows, which proves Theorem 1.2.

3 Proof of Theorem 1.1

3.1 Boundedness of the solutions

For $j = 0, 1, 2, \dots$ set $\rho_j = R_0\Big(1 - \frac{1}{2} + \frac{1}{2^{j+1}}\Big)$, $\bar\rho_j = \frac{1}{2}(\rho_j + \rho_{j+1})$, $h_j = H_0\Big(1 - \frac{1}{4} + \frac{1}{2^{j+2}}\Big)$, $\bar h_j = \frac{1}{2}(h_j + h_{j+1})$, $k_j = 2k_0 - \frac{k_0}{2^j}$, $A_{k_j,\rho_j,h_j} = \{x \in D(\rho_j, h_j) : u \ge k_j\}$, where k_0 is a positive number depending on the known parameters only, which will be specified later. Let $\varphi_j(x') \in C_0^\infty(D_1(\bar\rho_j))$ be such that $\varphi_j(x') \equiv 1$ for $x' \in D_1(\rho_{j+1})$, $|\frac{\partial \varphi_j(x')}{\partial x_i}| \le \gamma 2^j$, $i = 1, \dots, n-s$; $\zeta_j(x'') \in C_0^\infty(D_2(\bar h_j))$ be such that $\zeta_j(x'') \equiv 1$ for $x'' \in D_2(h_{j+1})$, $|\frac{\partial \zeta_j(x'')}{\partial x_i}| \le \gamma 2^j$, $i = n-s+1, \dots, n$. Set $\xi_j(x) = \varphi_j(x')\zeta_j(x'')$. Test (1.15) by $\varphi(u-k_{j+1})_+^\varepsilon \xi_j^m \psi_r^{m-1}$, $\quad \psi = \psi_r$, where ε depending on the known parameters only is small enough to be determined later. Using Eq. (1.2) and the Young inequality we have

$$\sum_{i=1}^{n} \int_{A_{k_{j_1},\bar\rho_j,\bar h_j}} (u-k_{j+1})^{\varepsilon-1}|u_{x_i}|^{p_i}\xi_j^m\psi_r^m dx \le \gamma \sum_{i=1}^{n-s} \int_{A_{k_{j+1},\bar\rho_j,\bar h_j}} (u-k_{j+1})^{p_i-1+\varepsilon}\left|\frac{\partial\psi_r}{\partial x_i}\right|^{p_i}\xi_j^m dx$$
$$+\gamma 2^{j\gamma}\sum_{i=1}^{n}\int_{A_{k_{j+1},\bar\rho_j,\bar h_j}}(u-k_{j+1})^{p_i-1+\varepsilon}\psi_r^m\xi_j^{m-\bar p}dx+\gamma\int_{A_{k_{j+1},\bar\rho_j,\bar h_j}}(u-k_{j+1})^{\alpha-1+\varepsilon}\psi_r^m\xi_j^m dx+\gamma|A_{k_{j+1},\bar\rho_j,\bar h_j}|. \tag{3.1}$$

Let us estimate the first term in the right-hand side of Eq. (3.1). By Theorem 1.2 we obtain

$$\sum_{i=1}^{n-s} \int_{A_{k_{j+1},\bar\rho_j,\bar h_j}} (u-k_{j+1})^{p_i-1+\varepsilon}\left|\frac{\partial\psi_r}{\partial x_i}\right|^{p_i}\xi_j^m dx$$
$$\le \gamma \sum_{i=1}^{n-s} r^{n-s-\alpha-\frac{n-s-\alpha}{\alpha-1}(\alpha-p_i)} M^{p_i-1+\varepsilon}(r) \le \gamma \sum_{i=1}^{n-s} r^{c(p_i-1+\varepsilon)-\varepsilon\frac{n-s-\alpha}{\alpha-1}} \le \gamma r^{c(p_1-1)-\varepsilon\frac{n-s-\alpha}{\alpha-1}}, \tag{3.2}$$

where $c > 0$ was defined in Eq. (2.55). Choosing $\varepsilon > 0$ small enough so that

$$\varepsilon = \frac{1}{2}\min\Big(1, \frac{c(p_1-1)(\alpha-1)}{n-s-\alpha}\Big) \tag{3.3}$$

and passing to the limit $r \to 0$, by Eq. (3.1), Eq. (3.2) we obtain

$$\sum_{i=1}^{n} \int\limits_{A_{k_{j+1},\bar\rho_j,\bar h_j}} (u-k_{j+1})^{\varepsilon-1}|u_{x_i}|^{p_i}\xi_j^m dx \le \gamma 2^{j\gamma}\sum_{i=1}^{n} \int\limits_{A_{k_{j+1},\bar\rho_j,\bar h_j}} (u-k_{j+1})^{p_i-1+\varepsilon}\xi_j^{m-\tilde p}dx$$
$$+\gamma \int\limits_{A_{k_{j+1},\bar\rho_j,\bar h_j}} (u-k_{j+1})^{\alpha-1+\varepsilon}\xi_j^m dx + \gamma|A_{k_{j+1},\bar\rho_j,\bar h_j}|. \tag{3.4}$$

Choose $q > 1$ such that

$$\frac{\tilde p-1+\varepsilon}{\alpha-1+\varepsilon} < q < \min\left\{\frac{(p-1+\varepsilon)n}{(\alpha-1+\varepsilon)(n-p)}, \frac{n-s}{n-s-\alpha}\right\}, \tag{3.5}$$

using the Young inequality from Eq. (3.4), Eq. (3.5) we obtain

$$\sum_{i=1}^{n} \int\limits_{A_{k_{j+1},\bar\rho_j,\bar h_j}} (u-k_{j+1})^{\varepsilon-1}|u_{x_i}|^{p_i}\xi_j^m dx \le \gamma 2^{j\gamma}\Bigg(\int\limits_{A_{k_{j+1},\rho_j,h_j}} (u-k_j)^{(\alpha-1+\varepsilon)q}dx + |A_{k_{j+1}\bar\rho_j,\bar h_j}|\Bigg). \tag{3.6}$$

Similarly to Eq. (2.51) we have

$$\int\limits_{A_{k_{j+1},\rho_{j+1},h_{j+1}}} (u-k_{j+1})^{(\alpha-1+\varepsilon)q}dx$$
$$\le \gamma 2^{j\gamma}\Bigg(\int\limits_{A_{k_j,\rho_j,h_j}} (u-k_j)^{(\alpha-1+\varepsilon)q}dx + |A_{k_{j+1}\bar\rho_j,\bar h_j}|\Bigg)^{\frac{(\alpha-1+\varepsilon)q}{p-1+\varepsilon}} |A_{k_{j+1}\bar\rho_j,\bar h_j}|^{1-\frac{(\alpha-1+\varepsilon)q(n-p)}{(p-1+\varepsilon)n}}. \tag{3.7}$$

Using the evident inequality

$$|A_{k_{j+1}\bar\rho_j,\bar h_j}| \le 2^{j(\alpha-1+\varepsilon)q}k_0^{-(\alpha-1+\varepsilon)} \int\limits_{A_{k_j\rho_j,h_j}} (u-k_j)^{(\alpha-1+\varepsilon)q}dx$$

and setting $y_j = \int_{A_{k_j,\rho_j,h_j}}(u-k_j)^{(\alpha-1+\varepsilon)q}dx$ we obtain

$$y_{j+1} \le \gamma 2^{j\gamma}\left(k_0^{1-\frac{(\alpha-1+\varepsilon)q(n-p)}{(p-1+\varepsilon)n}} + k_0^{1+\frac{(\alpha-1+\varepsilon)pq}{(p-1+\varepsilon)n}}\right)^{-(\alpha-1+\varepsilon)q} y_j^{1+\frac{(\alpha-1+\varepsilon)pq}{(p-1+\varepsilon)n}}, \quad j = 0, 1, 2, \ldots \tag{3.8}$$

Due to Lemma 2.3 this inequality implies that $y_j \to 0$ as $j \to \infty$ if k_0 satisfies the following condition

$$y_0 = \gamma k_0^{(p-1+\varepsilon)\frac{n}{p}}\left(k_0^{(\alpha-1+\varepsilon)q} + k_0^{-(\alpha-1+\varepsilon)q\frac{n-p}{n}}\right). \tag{3.9}$$

From Eq. (3.9) we get

$$\operatorname{ess\,sup}\left\{|u(x)| : x \in D\Big(\frac{R_0}{2}, \frac{3H_0}{4}\Big)\right\} \le \gamma + \gamma\Bigg(\int\limits_{D(R_0,H_0)} |u|^{(\alpha-1+\varepsilon)q}dx\Bigg)^{\frac{1}{(p-1+\varepsilon)\frac{n}{p}-(\alpha-1+\varepsilon)q\frac{n-p}{p}}}, \tag{3.10}$$

this completes the proof of the boundedness of u in the whole of $D(\frac{R_0}{4}, \frac{3H_0}{4})$.

3.2 End of the proof of Theorem 1.1

Let K be a compact subset of domain Ω. Let $\eta \in C_0^\infty(\Omega)$ be such that $\eta(x)Eq. \equiv 1$ for $x \in K$.

Testing Eq. (1.15) by $\varphi = u\eta^m \psi_r^{m-1},\ \psi = \psi_r$ using Eq. (1.2) the Young inequality, the boundedness of u and passing to the limit $r \to 0$ we get

$$\sum_{i=1}^{n} \int_K \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx \le \gamma. \tag{3.11}$$

Let $\varphi \in \overset{\circ}{W}{}^{1,\bar{p}}(\Omega)$. Test Eq. (1.15) by $\varphi\psi_r$, using Eq. (3.11) and the boundedness of the solution we pass to the limit $r \to 0$. So we obtain the required integral identity with an arbitrary $\varphi \in \overset{\circ}{W}{}^{1,\bar{p}}(\Omega)$ and $\psi \equiv 1$. Thus Theorem 1.1 is proved.

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