A Characterization of Hardy Spaces Associated with Certain Schrödinger Operators

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Abstract Let $\{K_t\}_{t>0}$ be the semigroup of linear operators generated by a Schrödinger operator $-L = \Delta - V(x)$ on $\mathbb{R}^d$, $d \geq 3$, where $V(x) \geq 0$ satisfies $\Delta^{-1}V \in L^\infty$. We say that an L^1-function f belongs to the Hardy space H_L^1 if the maximal function $\mathcal{M}_L f(x) = \sup_{t>0} |K_t f(x)|$ belongs to $L^1(\mathbb{R}^d)$. We prove that the operator $(-\Delta)^{1/2} L^{-1/2}$ is an isomorphism of the space H_L^1 with the classical Hardy space $H^1(\mathbb{R}^d)$ whose inverse is $L^{1/2}(-\Delta)^{-1/2}$. As a corollary we obtain that the space H_L^1 is characterized by the Riesz transforms $R_j = \frac{\partial}{\partial x_j} L^{-1/2}$.

Keywords Hardy spaces · Schrödinger operators

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1 Introduction and Statement of the Result

Let $\mathcal{L} = -\Delta + \mathcal{V}(x)$ be a Schrödinger operator on $\mathbb{R}^d$, where $\mathcal{V}(x)$ is a nonnegative locally integrable potential and let $\{T_t\}_{t>0}$ be the semigroup generated by $-\mathcal{L}$. The action of the semigroup is given by the Feynman-Kac formula

$$T_t f(x) = E^x \left(e^{-\int_0^t \mathcal{V}(X_s)\,ds} f(X_t) \right), \tag{1.1}$$

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where X_t is a Brownian motion associated with the heat semigroup $P_t = e^{t\Delta}$ (see, e.g., Chapter V of [18]). Let $T_t(x, y)$ denote the integral kernel of the semigroup $\{T_t\}_{t>0}$. Since $\mathcal{V}(x)$ is non-negative, Eq. 1.1 implies that

$$0 \leq T_t(x, y) \leq (4\pi t)^{-d/2} e^{-|x-y|^2/4t} =: P_t(x - y). \tag{1.2}$$

It easily follows from Eq. 1.2 that the maximal function

$$\mathcal{M}_{\mathcal{L}} f(x) = \sup_{t>0} |T_t f(x)| \tag{1.3}$$

is a bounded operator on $L^p(\mathbb{R}^d)$ for $1 < p \leq \infty$ and of weak-type (1.1).

The real Hardy space $H^1_{\mathcal{L}}$ associated with $\mathcal{L}$ is defined as

$$H^1_{\mathcal{L}} = \{f \in L^1(\mathbb{R}^d) : \mathcal{M}_{\mathcal{L}} f \in L^1(\mathbb{R}^d)\} \tag{1.4}$$

with the norm

$$\|f\|_{H^1_{\mathcal{L}}} = \|\mathcal{M}_{\mathcal{L}} f\|_{L^1(\mathbb{R}^n)}. \tag{1.5}$$

Let us note that the Hardy space $H^1_{\mathcal{L}}$ coincides with the Hardy spaces $H^1_{\mathcal{L},\max,h}(\mathbb{R}^d)$ considered in Hofmann et al. [14, Chapters 7 and 8], where $H^1_{\mathcal{L},\max,h}(\mathbb{R}^d)$ is defined as the completion of the space

$$\{f \in L^2(\mathbb{R}^d) : \mathcal{M}_{\mathcal{L}} f \in L^1(\mathbb{R}^d)\}$$

in the norm Eq. 1.5. We shall present a proof of this fact using standard arguments in the Appendix.

In the present paper we consider the semigroup $\{K_t\}_{t>0}$ of linear operators on $\mathbb{R}^d$, $d \geq 3$, generated by a Schrödinger operator $-L = \Delta - V(x)$, where $V(x)$ is a non-negative locally integrable function which satisfies

$$\Delta^{-1} V(x) = -c_d \int_{\mathbb{R}^d} \frac{1}{|x-y|^{d-2}} V(y) \, dy \in L^\infty(\mathbb{R}^d). \tag{1.6}$$

Let $K_t(x, y)$ denote the integral kernel of the semigroup $\{K_t\}_{t>0}$. Clearly, the upper Gaussian bounds (1.2) hold for $K_t(x, y)$. It is known, see [17], that for $V(x) \geq 0$ the condition (1.6) is equivalent to the lower Gaussian bounds for $K_t(x, y)$, that is, there are $c, C > 0$ such that

$$ct^{-d/2} e^{-C|x-y|^2/t} \leq K_t(x, y). \tag{1.7}$$

The Hardy spaces H^1_L associated with Schrödinger operators with nonnegative potentials satisfying Eq. 1.6 were studied in [12]. It was proved that the map $f(x) \mapsto w(x) f(x)$ is an isomorphism of H^1_L onto the classical Hardy space $H^1(\mathbb{R}^d)$, where

$$w(x) = \lim_{t \to \infty} \int K_t(x, y) \, dy, \tag{1.8}$$

which in particular means that

$$\|f w\|_{H^1(\mathbb{R}^d)} \sim \|f\|_{H^1_L}, \tag{1.9}$$

see [12, Theorem 1.1]. The function $w(x)$ is L-harmonic, that is, $K_t w = w$, and satisfies $0 < \delta \leq w(x) \leq 1$.

Let us remark that the classical real Hardy space $H^1(\mathbb{R}^d)$ can be thought as the space H^1_L associated with the classical heat semigroup $e^{t\Delta}$, that is, $L = -\Delta + V$ with $V \equiv 0$ in this case. Obviously, the constant functions are the only bounded harmonic functions for Δ.

The present paper is a continuation of [12]. Our goal is to study the mappings

$$L^{1/2}(-\Delta)^{-1/2} \quad \text{and} \quad (-\Delta)^{1/2} L^{-1/2}$$

which turn out to be bounded on $L^1(\mathbb{R}^d)$ (see Lemma 2.6). Our main result is the following theorem, which states another characterization of H_L^1.

Theorem 1.10 *Assume that $L = -\Delta + V(x)$ is a Schrödinger operator on $\mathbb{R}^d$, $d \geq 3$, with a locally integrable non-negative potential $V(x)$ satisfying* Eq. 1.6. *Then the mapping $f \mapsto (-\Delta)^{1/2} L^{-1/2} f$ is an isomorphism of H_L^1 onto the classical Hardy space $H^1(\mathbb{R}^d)$, that is, there is a constant $C > 0$ such that*

$$\|(-\Delta)^{1/2} L^{-1/2} f\|_{H^1(\mathbb{R}^d)} \leq C \|f\|_{H_L^1}, \tag{1.11}$$

$$\|L^{1/2} (-\Delta)^{-1/2} f\|_{H_L^1} \leq C \|f\|_{H^1(\mathbb{R}^d)}. \tag{1.12}$$

As a corollary we immediately obtain the following Riesz transform characterization of H_L^1.

Corollary 1.13 *Under the assumptions of Theorem 1.10 an L^1-function f belongs to the space H_L^1 if and only if $R_j f = \frac{\partial}{\partial x_j} L^{-1/2} f$ belong to $L^1(\mathbb{R}^d)$ for $j = 1, 2, ..., d$. Moreover, there is a constant $C > 0$ such that*

$$C^{-1} \|f\|_{H_L^1} \leq \|f\|_{L^1(\mathbb{R}^d)} + \sum_{j=1}^{d} \|R_j f\|_{L^1(\mathbb{R}^d)} \leq C \|f\|_{H_L^1}. \tag{1.14}$$

Example 1 It is not hard to see that if for a function $V(x) \geq 0$ defined on $\mathbb{R}^d$, $d \geq 3$, there is $\varepsilon > 0$ such that $V \in L^{d/2-\varepsilon}(\mathbb{R}^d) \cap L^{d/2+\varepsilon}(\mathbb{R}^d)$, then V satisfies Eq. 1.6.

Example 2 Assume that Eq. 1.6 holds for a function $V : \mathbb{R}^d \to [0, \infty)$, $d \geq 3$. Then $V(x_1, x_2) := V(x_1)$ defined on $\mathbb{R}^d \times \mathbb{R}^n$, $n \geq 1$, fulfils Eq. 1.6.

The reader interested in other results concerning Hardy spaces associated with semigroups of linear operators, and in particular semigroups generated by Schrödinger operators, is referred to [1, 2, 6–10, 14].

2 Boundedness on L^1

We define the operators:

$$(-\Delta)^{-1} f(x) = \int_0^\infty P_t f(x) \, dt = c_d \int \frac{f(y)}{|x-y|^{d-2}} \, dy =: \int \Gamma_0(x-y) f(y) \, dy,$$

$$L^{-1} f(x) = \int_0^\infty K_t f(x) \, dt =: \int \Gamma(x, y) f(y) \, dy,$$

$$(-\Delta)^{-1/2} f = c_1 \int_0^\infty P_t f \frac{dt}{\sqrt{t}} = c_d' \int \frac{1}{|x-y|^{d-1}} f(y) \, dy =: \int \tilde{\Gamma}_0(x-y) f(y) \, dy,$$

$$L^{-1/2} f = c_1 \int_0^\infty K_t f \frac{dt}{\sqrt{t}} =: \int \tilde{\Gamma}(x, y) f(y) \, dy,$$

where $c_1 = \Gamma(1/2)^{-1}$. Clearly, by Eq. 1.2,

$$0 \leq \tilde{\Gamma}(x, y) \leq c_d' |x-y|^{-d+1}, \quad 0 < \Gamma(x, y) \leq c_d |x-y|^{-d+2}. \tag{2.1}$$

The perturbation formula asserts that

$$P_t(x - y) = K_t(x, y) + \int_0^t \int P_{t-s}(x - z)V(z)K_s(z, y)\,dz\,ds$$

$$= K_t(x, y) + \int_0^t \int K_{t-s}(x, z)V(z)P_s(z - y)\,dz\,ds. \tag{2.2}$$

Multiplying the second inequality in Eq. 2.2 by $w(x)$ and integrating with respect to dx we get

$$\int P_t(x - y)w(x)\,dx = w(y) + \int_{\mathbb{R}^d} \int_0^t w(z)V(z)P_s(z, y)\,ds\,dx, \tag{2.3}$$

since w is L-harmonic. The left-hand side of Eq. 2.3 tends to a harmonic function, which is bounded from below by δ and above by 1, as t tends to infinity. Thus there is a constant $0 < c_w \le 1$ such that

$$c_w = w(y) + \int_{\mathbb{R}^d} w(z)V(z)\Gamma_0(z - y)\,dz. \tag{2.4}$$

Similarly, integrating the first equation in Eq. 2.2 with respect to x and taking limit as t tends to infinity, we obtain

$$1 = w(y) + \int_{\mathbb{R}^d} V(z)\Gamma(z, y)\,dz. \tag{2.5}$$

For a reasonable function f the following operators are well defined in the sense of distributions:

$$(-\Delta)^{1/2}f = c_2 \int_0^\infty (P_t f - f)\frac{dt}{t^{3/2}}, \quad c_2 = \Gamma(-1/2)^{-1},$$

$$L^{1/2} = c_2 \int_0^\infty (K_t f - f)\frac{dt}{t^{3/2}}.$$

Lemma 2.6 *There is a constant $C > 0$ such that*

$$\|(-\Delta)^{1/2}L^{-1/2}f\|_{L^1} \le C\|f\|_{L^1}, \tag{2.7}$$

$$\|L^{1/2}(-\Delta)^{-1/2}f\|_{L^1} \le C\|f\|_{L^1}. \tag{2.8}$$

Proof From the perturbation formula (2.2) we have

$$(-\Delta)^{1/2}L^{-1/2}f(x) = c_2 \int_0^\infty (P_t - I)L^{-1/2}f(x)\frac{dt}{t^{3/2}}$$

$$= c_2 \int_0^\infty (P_t - K_t)L^{-1/2}f(x)\frac{dt}{t^{3/2}} + c_2 \int_0^\infty (K_t - I)L^{-1/2}f(x)\frac{dt}{t^{3/2}}$$

$$= c_2 \int_0^\infty \int_0^t \iint P_{t-s}(x - z)V(z)K_s(z, y)L^{-1/2}f(y)\,dy\,dz\,ds\frac{dt}{t^{3/2}} + f(x). \tag{2.9}$$

Consider the integral kernel $W(x, u)$ of the operator

$$f \mapsto \int_0^\infty \int_0^t \iint P_{t-s}(x - z)V(z)K_s(z, y)L^{-1/2}f(y)\,dy\,dz\,ds\frac{dt}{t^{3/2}},$$

that is,

$$W(x, u) = \int_0^\infty \int_0^t \iint P_{t-s}(x - z)V(z)K_s(z, y)\tilde{\Gamma}(y, u)\, dy\, dz\, ds\, \frac{dt}{t^{3/2}}.$$

Clearly $0 \leq W(x, u)$. Integration of $W(x, u)$ with respect to dx leads to

$$\begin{aligned}
\int W(x, u)\, dx &= \int_0^\infty \int_0^t \iint V(z)K_s(z, y)\tilde{\Gamma}(y, u)\, dy\, dz\, ds\, \frac{dt}{t^{3/2}} \\
&= 2\int_0^\infty \iint V(z)K_s(z, y)\widetilde{\Gamma}(y, u)\, dy\, dz\, \frac{ds}{\sqrt{s}} \\
&\leq 2c_1^{-1} \iint V(z)\widetilde{\Gamma}(z, y)\widetilde{\Gamma}(y, u)\, dy\, dz \\
&= 2c_1^{-1} \int V(z)\Gamma(z, u)dz.
\end{aligned} \tag{2.10}$$

Using Eq. 2.1 we see that $\int W(x, u)\, dx \leq 2c_1^{-1}\|\Delta^{-1}V\|_{L^\infty}$, which completes the proof of Eq. 2.7. The proof of Eq. 2.8 goes in the same way. We skip the details. $\square$

We finish this section by proving the following two lemmas, which will be used in the sequel.

Lemma 2.11 *Assume that $f \in L^1(\mathbb{R}^d)$. Then*

$$\int (-\Delta)^{1/2}L^{-1/2}f(x)\, dx = \int f(x)w(x)\, dx. \tag{2.12}$$

Proof From Eqs. 2.9 and 2.10 we conclude that

$$\begin{aligned}
\int (-\Delta)^{1/2}L^{-1/2}f(x)\, dx &= c_2 \int \int W(x, u)f(u)\, dudx + \int f(x)\, dx \\
&= 2c_2c_1^{-1} \int V(z)\Gamma(z, u)f(u)\, dz\, du + \int f(x)\, dx \\
&= \int (w(u) - 1)f(u)\, du + \int f(x)\, dx,
\end{aligned}$$

where in the last equality we have used Eq. 2.5. $\square$

Lemma 2.13 *Assume that $f \in L^1(\mathbb{R}^d)$. Then*

$$\int (L^{1/2}(-\Delta)^{-1/2}f)(x)w(x)\, dx = c_w \int f(x)\, dx. \tag{2.14}$$

Proof The proof is similar to that of Lemma 2.11. Indeed, by the perturbation formula (2.2) we have

$$\int (L^{1/2}(-\Delta)^{-1/2}f)(x)w(x)\,dx$$

$$= c_2 \int \int_0^\infty (K_t - P_t)((-\Delta)^{-1/2})f)(x)\frac{dt}{t^{3/2}}w(x)\,dx$$

$$+ c_2 \int \int_0^\infty (P_t - I)((-\Delta)^{-1/2})f)(x)\frac{dt}{t^{3/2}}w(x)\,dx$$

$$= -c_2 \int \int_0^\infty \int_0^t \int\int w(x)K_{t-s}(x,z)V(z)$$

$$\times P_s(z-y)((-\Delta)^{-\frac{1}{2}}f)(y)\,dy\,dz\,ds\frac{dt}{t^{3/2}}\,dx$$

$$+ \int w(x)f(x)\,dx$$

$$= -c_2 \int_0^\infty \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} w(z)V(z)P_s(z-y)((-\Delta)^{-1/2}f)(y)$$

$$\times dy\,dz\,ds\frac{dt}{t^{3/2}} + \int w(x)f(x)\,dx,$$

where in the last equality we have used that w is L-harmonic. Integrating with respect to dt and then with respect to ds yields

$$\int (L^{1/2}(-\Delta)^{-1/2}f)(x)w(x)\,dx$$

$$= -\frac{2c_2}{c_1} \int \int w(z)V(z)\widetilde{\Gamma}_0(z-y)((-\Delta)^{-1/2}f)(y)\,dy\,dz + \int f(x)w(x)\,dx$$

$$= \int w(z)V(z)\Gamma_0(z-u)f(u)\,du\,dz + \int f(x)w(x)\,dx$$

$$= \int c_w f(x)\,dx - \int w(y)f(y)\,dy + \int f(x)w(x)\,dx,$$

where in the last equality we have used Eq. 2.4. $\qquad\square$

3 Atoms and Molecules

Fix $1 < q \leq \infty$. We say that a function a is an $(1, q, w)$-atom if there is a ball $B \subset \mathbb{R}^d$ such that $\operatorname{supp} a \subset B$, $\|a\|_{L^q(\mathbb{R}^d)} \leq |B|^{\frac{1}{q}-1}$, $\int a(x)w(x)\,dx = 0$. The atomic norm $\|f\|_{H^1_{\mathrm{at},q,w}}$ is defined by

$$\|f\|_{H^1_{\mathrm{at},q,w}} = \inf \left\{ \sum_{j=1}^\infty |\lambda_j| \right\}, \tag{3.1}$$

where the infimum is taken over all representations $f = \sum_{j=1}^\infty \lambda_j a_j$, where $\lambda_j \in \mathbb{C}$, a_j are $(1, q, w)$-atoms.

Clearly, if $w_0(x) \equiv 1$, then the $(1, q, w_0)$-atoms coincide with the classical $(1, q)$-atoms for the Hardy space $H^1(\mathbb{R}^d)$, which can be thought as $H^1_{-\Delta}$.

As a direct consequence of Theorem 1.1 of [12] (see Eq. 1.9) and the results about atomic decompositions of the classical real Hardy spaces (see, e.g., [3, 15, 19]), we obtain that

the space H_L^1 admits atomic decomposition into $(1, q, w)$-atoms, that is, there is a constant $C_q > 0$ such that

$$C_q^{-1}\|f\|_{H_{\text{at},q,w}^1} \leq \|f\|_{H_L^1} \leq C_q\|f\|_{H_{\text{at},q,w}^1}. \tag{3.2}$$

Let $\varepsilon > 0$, $1 < q < \infty$. We say that a function b is a $(1, q, \varepsilon, w)$-molecule associated with a ball $B = B(x_0, r)$ if

$$\left(\int_B |b(x)|^q \, dx\right)^{\frac{1}{q}} \leq |B|^{\frac{1}{q}-1}, \quad \left(\int_{2^k B \setminus 2^{k-1} B} |b(x)|^q \, dx\right)^{\frac{1}{q}} \leq |2^k B|^{\frac{1}{q}-1} 2^{-\varepsilon k} \tag{3.3}$$

and

$$\int b(x) w(x) \, dx = 0. \tag{3.4}$$

Obviously every $(1, q, w)$-atom is a $(1, q, \varepsilon, w)$-molecule. It is also not hard to see that for fixed $q > 1$ and $\varepsilon > 0$ there is a constant $C > 0$ such that every $(1, q, \varepsilon, w)$ molecule b can be decomposed into a sum

$$b(x) = \sum_{n=1}^{\infty} \lambda_n a_n, \quad \sum_{n=1}^{\infty} |\lambda_n| \leq C,$$

where $\lambda_n \in \mathbb{C}$, a_n are $(1, q, w)$-atoms.

The following lemma is easy to prove.

Lemma 3.5 *Let $1 < q < \infty$, $\delta, \varepsilon > 0$ be such that $\delta > d\left(1 - \frac{1}{q}\right) + \varepsilon$. Then there is a constant $C > 0$ such that if $b(x)$ satisfies* Eq. 3.4 *and*

$$\left(\int \left|b(x)\left(1 + \frac{|x - y_0|}{r}\right)^{\delta}\right|^q dx\right)^{1/q} \leq \frac{r^{-d+d/q}}{C}, \tag{3.6}$$

then b is a $(1, q, \varepsilon, w)$-molecule associated with $B(y_0, r)$.

In order to prove Theorem 1.10 we shall use general results about Hardy spaces associated with Schrödinger operators $-\mathcal{L} = \Delta - \mathcal{V}(x)$ with non-negative locally integrable potentials $\mathcal{V}(x)$ which were proved in [11]. We say that a function $\mathbf{a}$ is a generalized $(1, \infty, \mathcal{L})$-atom for the Hardy space $H_{\mathcal{L}}^1$ if there is a ball $B = B(y_0, r)$ and a function $\mathbf{b}$ such that

$$\text{supp}\,\mathbf{b} \subset B, \quad \|\mathbf{b}\|_{L^\infty} \leq |B|^{-1}, \quad \mathbf{a} = (I - T_{r^2})\mathbf{b}.$$

Then we say that $\mathbf{a}$ is associated with the ball $B(y_0, r)$. It was proved in Section 6 of [11] that the space $H_{\mathcal{L}}^1$ admits atomic decomposition with the generalized $(1, \infty, \mathcal{L})$-atoms, that is, $\|f\|_{H_{\mathcal{L}}^1} \sim \|f\|_{H_{\text{at},\infty,\mathcal{L}}^1}$, where the norm $\|f\|_{H_{\text{at},\infty,\mathcal{L}}^1}$ is defined as in Eq. 3.1 with $a_j(x)$ replaced by the general $(1, \infty, \mathcal{L})$-atoms $\mathbf{a}_j(x)$.

Lemma 3.7 *There is a constant $C > 0$ such that for every $\mathbf{a}$ being a generalized $(1, \infty, \mathcal{L})$ atom associated with $B(y_0, r)$ one has*

$$|\mathcal{L}^{-1/2}\mathbf{a}(y)| \leq Cr^{1-d}\left(1 + \frac{|y - y_0|}{r}\right)^{-d}.$$

Proof The proof follows from functional calculi (see, e.g., [13]). Note that $\mathcal{L}^{-1/2}\mathbf{a} = m_{(r)}(\mathcal{L})\mathbf{b}$ with $m_{(r)}(\lambda) = r(r^2\lambda)^{-1/2}(e^{-r^2\lambda} - 1)$ and $\mathbf{b}$ such that $\mathrm{supp}\,\mathbf{b} \subset B(y_0, r)$, $\|\mathbf{b}\|_{L^\infty} \leq |B(y_0, r)|^{-1}$. From [13] we conclude that there is a constant $C > 0$ such that for every $r > 0$ one has

$$m_{(r)}(\mathcal{L})f(x) = \int_{\mathbb{R}^d} m_{(r)}(x, y) f(y) \, dy,$$

with $m_{(r)}(x, y)$ satisfying

$$|m_{(r)}(x, y)| \leq Cr^{1-d}\left(1 + \frac{|x - y|}{r}\right)^{-d}. \tag{3.8}$$

Now the lemma can be easily deduced from Eq. 3.8 and the size and support property of $\mathbf{b}$. $\square$

4 Proof of Theorem 1.10

For real numbers $n > 2$, $\beta > 0$ let

$$g(x) = (1 + |x|)^{-n-\beta}, \quad g_s(x) = s^{-n/2}g\left(\frac{x}{\sqrt{s}}\right).$$

One can easily check that

$$\int_0^t g_s(x) \, ds \leq C|x|^{2-n}\left(1 + \frac{|x|}{\sqrt{t}}\right)^{-2-\beta}; \tag{4.1}$$

$$\int_{r^2}^\infty g_s(x) \, ds \leq Cr^{2-n}\left(1 + \frac{|x|}{r}\right)^{-n+2} \quad \text{for } r > 0. \tag{4.2}$$

Moreover, it is easily to verify that for $1 < q < \infty$, $d\left(1 - \frac{1}{q}\right) < \alpha \leq d$, $\beta > 0$ one has

$$\left\| |x|^{\alpha-d}\left(1 + \frac{|x|}{\sqrt{t}}\right)^{-d-\beta} \right\|_{L^q(\mathbb{R}^d, dx)} = C_{\alpha,\beta}t^{(\alpha-d+d/q)/2} \tag{4.3}$$

and

$$\int |z - y|^{2-d}\left(1 + \frac{|z - y|}{r}\right)^{-\beta}\left(1 + \frac{|y|}{r}\right)^{-d+\gamma} dy \leq Cr^2\left(1 + \frac{|z|}{r}\right)^{-d+\gamma+2-\beta} \tag{4.4}$$

for $0 < \gamma < \beta < 2$.

Lemma 4.5 *Assume that $V(x)$ satisfies the assumptions of Theorem 1.10. Then for $0 < \gamma \leq 2$ and $r > 0$ one has*

$$\int_{\mathbb{R}^d} V(z)\left(1 + \frac{|z - y|}{r}\right)^{-d+\gamma} dz \leq c_d^{-1}r^{d-2}\|\Delta^{-1}V\|_{L^\infty}. \tag{4.6}$$

Proof The left-hand side of Eq. 4.6 is bounded by

$$\int_{|z-y|\leq r} V(z)\left(\frac{r}{|z - y|}\right)^{d-2} dz + \int_{|z-y|>r} V(z)\left(\frac{|z - y|}{r}\right)^{-d+2} dz$$

$$\leq c_d^{-1}r^{d-2}\|\Delta^{-1}V\|_{L^\infty}.$$

$\square$

Proof of Theorem 1.10 We already have known that the operators $(-\Delta)^{1/2}L^{-1/2}$ and $L^{1/2}(-\Delta)^{-1/2}$ are bounded on $L^1(\mathbb{R}^d)$. It suffices to prove Eqs. 1.11 and 1.12. Set $\gamma = \frac{1}{10}$ and fix $q > 1$ and $\varepsilon > 0$ such that $\gamma > d\left(1 - \frac{1}{q}\right) + \varepsilon$. Set $w_0(x) \equiv 1$. According to the atomic and molecular decompositions (see Section 3) the proof of Eq. 1.11 will be done if we verify that $(-\Delta)^{1/2}L^{-1/2}\mathbf{a}$ is a multiple of a $(1, q, \varepsilon, w_0)$-molecule for every generalized $(1, \infty, L)$-atom $\mathbf{a}$ with a multiple constant independent of $\mathbf{a}$. Identical arguments can be then applied to show that $L^{1/2}(-\Delta)^{-1/2}\mathbf{a}$ is a multiple of a $(1, q, \varepsilon, w)$-molecule for $\mathbf{a}$ being a generalized atom for the classical Hardy space $H^1(\mathbb{R}^d) = H^1_{-\Delta}$ with a multiple constant independent of $\mathbf{a}$.

Let $\mathbf{a} = (I - K_{r^2})\mathbf{b}$ be a generalized $(1, \infty, L)$-atom for H^1_L associated with $B(y_0, r)$. By Lemma 2.11, since $\int w(x)\mathbf{a}(x)\,dx = 0$, we have

$$\int (-\Delta)^{1/2}L^{-1/2}\mathbf{a}(x)\,dx = 0.$$

Set

$$
\begin{aligned}
J(x) &= \int_0^\infty \int_0^t \iint P_{t-s}(x-z)V(z)K_s(z,y)(L^{-1/2}\mathbf{a})(y)\,dy\,dz\,ds\,\frac{dt}{t^{3/2}} \\
&= \int_0^{r^2} \int_0^t \iint \cdots + \int_{r^2}^\infty \int_0^{t/2} \iint + \int_{r^2}^\infty \int_{t/2}^t \iint \cdots \\
&= J_1(x) + J_2(x) + J_3(x).
\end{aligned}
\tag{4.7}
$$

Thanks to Eq. 2.9 and Lemma 3.5 it suffices to show that there is a constant $C_q > 0$, independent of $\mathbf{a}(x)$ such that

$$\left\| \left(1 + \frac{|x - y_0|}{r}\right)^\gamma J(x) \right\|_{L^q(\mathbb{R}^d)} \le C_q r^{-d+d/q}.
\tag{4.8}$$

Applying Lemma 3.7 and Eq. 4.1 with $n = d + 1$, we obtain

$$
\begin{aligned}
|J_1(x)| &= \left| \int_0^{r^2} \int_0^t \iint P_{t-s}(x-z)V(z)K_s(z,y)(L^{-1/2}\mathbf{a})(y)\,dy\,dz\,ds\,\frac{dt}{t^{3/2}} \right| \\
&\le C \int_0^{r^2} \int_0^t \int P_{t-s}(x-z)V(z)r^{1-d}\left(1 + \frac{|z - y_0|}{r}\right)^{-d}\,dz\,ds\,\frac{dt}{t^{3/2}} \\
&\le C \int_0^{r^2} \int P_s(x-z)V(z)r^{1-d}\left(1 + \frac{|z - y_0|}{r}\right)^{-d}\,dz\,\frac{ds}{\sqrt{s}} \\
&\le C_N \int |x - z|^{1-d}\left(1 + \frac{|x - z|}{r}\right)^{-N} V(z)r^{1-d}\left(1 + \frac{|z - y_0|}{r}\right)^{-d}\,dz.
\end{aligned}
\tag{4.9}
$$

Consequently,

$$
\begin{aligned}
&|J_1(x)|\left(1 + \frac{|x - y_0|}{r}\right)^\gamma \\
&\le C_N r^{1-d} \int |x - z|^{-d+1}\left(1 + \frac{|x - z|}{r}\right)^{-N+\gamma} V(z)\left(1 + \frac{|z - y_0|}{r}\right)^{-d+\gamma}\,dz.
\end{aligned}
\tag{4.10}
$$

Therefore, using the Minkowski integral inequality together with Eqs. 4.3 and 4.6, we get

$$\left\| J_1(x) \left(1 + \frac{|x - y_0|}{r} \right)^\gamma \right\|_{L^q(dx)} \leq Cr^{-d+d/q}. \tag{4.11}$$

In order to estimate $J_2(x)$ we use Lemma 3.7 and Eq. 4.1 with $n = d$ to obtain

$$
\begin{aligned}
|J_2(x)| &\left(1 + \frac{|x - y_0|}{r} \right)^\gamma \\
&\leq C \int_{r^2}^\infty \left(1 + \frac{|x - y_0|}{r} \right)^\gamma \int_0^{t/2} \iint t^{-d/2} e^{-c|x-z|^2/t} V(z) \\
&\qquad \times K_s(z, y) r^{1-d} \left(1 + \frac{|y - y_0|}{r} \right)^{-d} dy \, dz \, ds \frac{dt}{t^{3/2}} \\
&\leq C \int_{r^2}^\infty \iint t^{(2\gamma-d-3)/2} e^{-c|x-z|^2/t} V(z) \\
&\qquad \times |z - y|^{2-d} \left(1 + \frac{|z - y|}{\sqrt{t}} \right)^{-N+\gamma} r^{1-d-2\gamma} \\
&\qquad \times \left(1 + \frac{|y - y_0|}{r} \right)^{-d+\gamma} dy \, dz \, dt. \tag{4.12}
\end{aligned}
$$

Setting $N = \beta + \gamma$ with $0 < \gamma < \beta < 2$ and applying the Minkowski integral inequality together with Eqs. 4.4 and 4.6 we conclude that

$$
\begin{aligned}
&\left\| J_2(x) \left(1 + \frac{|x - y_0|}{r} \right)^\gamma \right\|_{L^q(dx)} \\
&\leq C \int_{r^2}^\infty \iint t^{-(d+3-2\gamma-d/q)/2} V(z) \\
&\qquad \times |z - y|^{2-d} \left(1 + \frac{|z - y|}{\sqrt{t}} \right)^{-\beta} r^{1-d-2\gamma} \left(1 + \frac{|y - y_0|}{r} \right)^{-d+\gamma} dy \, dz \, dt \\
&\leq C \int_{r^2}^\infty \iint t^{-(d+3-2\gamma-d/q)/2} V(z) \\
&\qquad \times |z - y|^{2-d} \left(1 + \frac{|z - y|}{r} \right)^{-\beta} \left(\frac{\sqrt{t}}{r} \right)^\beta r^{1-d-2\gamma} \left(1 + \frac{|y - y_0|}{r} \right)^{-d+\gamma} dy \, dz \, dt \\
&\leq C \int r^{-2d+2+d/q} V(z) \left(1 + \frac{|z - y_0|}{r} \right)^{-d+2+\gamma-\beta} dz \\
&\leq Cr^{-d+d/q}. \tag{4.13}
\end{aligned}
$$

By Lemma 3.7 and Eq. 4.1 with $n = d$, we have

$$|J_3(x)| \leq C \int_{r^2}^{\infty} \int_{\frac{t}{2}}^{t} \iint P_{t-s}(x-z)V(z)t^{-\frac{d}{2}}e^{-\frac{c|z-y|^2}{t}}$$

$$\times \left(1 + \frac{|y-y_0|}{r}\right)^{-d} r^{1-d}\, dy\, dz\, ds\, \frac{dt}{t^{\frac{3}{2}}}$$

$$\leq C_N \int_{r^2}^{\infty} \iint |x-z|^{2-d} \left(1 + \frac{|x-z|}{\sqrt{t}}\right)^{-N} V(z)$$

$$\times t^{-\frac{d}{2}} e^{-c|z-y|^2/t} \left(1 + \frac{|y-y_0|}{r}\right)^{-d} r^{1-d}\, dy\, dz\, \frac{dt}{t^{3/2}}. \qquad (4.14)$$

Hence,

$$|J_3(x)| \left(1 + \frac{|x-y_0|}{r}\right)^{\gamma} \leq C \int_{r^2}^{\infty} \iint |x-z|^{2-d} \left(1 + \frac{|x-z|}{\sqrt{t}}\right)^{-N+\gamma} t^{\gamma} V(z)$$

$$\times t^{-\frac{d}{2}} e^{-c'|z-y|^2/t} \left(1 + \frac{|y-y_0|}{r}\right)^{-d+\gamma} r^{1-d-2\gamma}\, dy\, dz\, \frac{dt}{t^{3/2}}.$$

By Minkowski's integral inequality combined with Eq. 4.3 we arrive to

$$\left\| J_3(x) \left(1 + \frac{|x-y_0|}{r}\right)^{\gamma} \right\|_{L^q(dx)}$$

$$\leq \int_{r^2}^{\infty} \iint t^{(-d+2+d/q)/2+\gamma-3/2} V(z)$$

$$\times t^{-d/2} e^{-c'|z-y|^2/t} \left(1 + \frac{|y-y_0|}{r}\right)^{-d+\gamma} r^{1-d-2\gamma}\, dy\, dz\, dt.$$

Application of Eq. 4.2 with $n = 2d + 1 - \frac{d}{q} - 2\gamma$ and then Eq. 4.6 yields

$$\left\| J_3(x) \left(1 + \frac{|x-y_0|}{r}\right)^{\gamma} \right\|_{L^q(dx)}$$

$$\leq C \iint r^{2-3d+d/q} V(z) \left(1 + \frac{|z-y|}{r}\right)^{-2d+1+d/q+2\gamma} \left(1 + \frac{|y-y_0|}{r}\right)^{-d+\gamma}\, dy\, dz$$

$$\leq \int r^{2-2d+d/q} V(z) \left(1 + \frac{|z-y_0|}{r}\right)^{-2d+1+d/q+3\gamma}\, dz$$

$$\leq C r^{-d+d/q}.$$

The above inequality together with Eqs. 4.11 and 4.13 gives desired Eq. 4.8 and, consequently, the proof of Eq. 1.11 is complete.

Let us note that in the proof Eq. 1.11 we use only Lemmas 2.11, 3.7, and the upper Gaussian bounds for the kernels. The proof of Eq. 1.12 goes identically to that of Eq. 1.11 by replacing Lemma 2.11 by Lemma 2.13. □

5 Proof of the Riesz Transform Characterization of H_L^1

Proof Proof of Corollary 1.13 Assume that $f \in H_L^1$. Then, thanks to Theorem 1.10, there is $g \in H^1(\mathbb{R}^d)$ such that $f = L^{1/2}(-\Delta)^{-1/2}g$. By the characterization of the classical Hardy

space $H^1(\mathbb{R}^d)$ by the Riesz transforms we have

$$\frac{\partial}{\partial x_j} L^{-1/2} f = \frac{\partial}{\partial x_j} L^{-1/2} L^{1/2} (-\Delta)^{-1/2} g = \frac{\partial}{\partial x_j} (-\Delta)^{-1/2} g \in L^1(\mathbb{R}^d). \qquad (5.1)$$

Conversely, assume that for $f \in L^1(\mathbb{R}^d)$ we have $\frac{\partial}{\partial x_j} L^{-1/2} f \in L^1(\mathbb{R}^d)$ for $j = 1, 2, ..., d$. Set $g = (-\Delta)^{1/2} L^{-1/2} f$. Then by Lemma 2.6, $g \in L^1(\mathbb{R}^d)$ and

$$\frac{\partial}{\partial x_j} (-\Delta)^{-1/2} g = \frac{\partial}{\partial x_j} (-\Delta)^{-1/2} (-\Delta)^{1/2} L^{-1/2} f = \frac{\partial}{\partial x_j} L^{-1/2} f \in L^1(\mathbb{R}^d), \qquad (5.2)$$

which implies that $g \in H^1(\mathbb{R}^d)$. Consequently, by Theorem 1.10, $f \in H_L^1$. Finally Eq. 1.14 can be deduced from Eqs. 5.1, 5.2, and Theorem 1.10. $\qquad\qquad\square$

Appendix

In the appendix we shall prove that the spaces $H_{\mathcal{L}}^1$ and $H_{\mathcal{L},\max,h}^1(\mathbb{R}^d)$ defined in the introduction coincide. The proof goes by standard arguments and we do not use atomic decompositions. Let us note that the proof works in more general settings, e.g. for semigroups satisfying Gaussian bounds on spaces of homogeneous type in the sense of Coifman-Weiss [4].

Since $\{T_t\}_{t>0}$ is a strongly continuous semigroup in e.g. $L^2(\mathbb{R}^d)$, one can easily deduce from Eq. 1.2 and the semigroup property that $\{T_t\}_{t>0}$ is a pointwise approximate of the identity, that is,

$$\lim_{t\to 0} T_t f(x) = f(x) \ a.e. \ \text{for} \ f \in L^1(\mathbb{R}^d) + L^\infty(\mathbb{R}^d). \qquad (6.1)$$

Consequently,

$$|f(x)| \le \mathcal{M}_{\mathcal{L}} f(x) \quad a.e. \ \text{for} \ f \in L^1(\mathbb{R}^d) + L^\infty(\mathbb{R}^d). \qquad (6.2)$$

Let $f_n \in L^2(\mathbb{R}^d)$ be a Cauchy sequence in the norm Eq. 1.5 and let $q = \lim_{n\to\infty} \|f_n\|_{H_{\mathcal{L}}^1}$. By virtue of Eq. 6.2 we have $\|f_n - f_m\|_{L^1(\mathbb{R}^d)} \le \|f_n - f_m\|_{H_{\mathcal{L}}^1}$. Hence the sequence f_n converges to a unique function f in $L^1(\mathbb{R}^d)$. We shall prove that $\mathcal{M}_{\mathcal{L}} f \in L^1(\mathbb{R}^d)$ and the convergence of f_n to f is also in the $\|\cdot\|_{H_{\mathcal{L}}^1}$-norm. To see this take a subsequence n_k such that

$$\|f_{n_k} - f_{n_j}\|_{H_{\mathcal{L}}^1} \le 3^{-k} \quad \text{for} \ j \ge k \ge 1 \qquad (6.3)$$

and write

$$f = f_{n_k} + \sum_{j=k}^{\infty} (f_{n_{j+1}} - f_{n_j}) \ \text{(convergence in the } L^1 - \text{norm).} \qquad (6.4)$$

Obviously, $\|\mathcal{M}_{\mathcal{L}} f\|_{L^1(\mathbb{R}^d)} \le \|f_{n_k}\|_{H_{\mathcal{L}}^1} + \sum_{j=k}^{\infty} \|f_{n_{j+1}} - f_{n_j}\|_{H_{\mathcal{L}}^1}$, which gives $\|\mathcal{M}_{\mathcal{L}} f\|_{L^1(\mathbb{R}^d)} \le q$. Further, by Eqs. 6.4 and 6.3, we have

$$\|\mathcal{M}_{\mathcal{L}}(f - f_{n_k})\|_{L^1(\mathbb{R}^d)} \le \sum_{j=k}^{\infty} \|f_{n_{j+1}} - f_{n_j}\|_{H_{\mathcal{L}}^1} \to 0 \ \text{as} \ k \to \infty.$$

Thus we have proved that $H^1_{\mathcal{L},\max,h}(\mathbb{R}^d) \subset H^1_{\mathcal{L}}$.

Assume now that $f \in H^1_{\mathcal{L}}$, that is, $f \in L^1(\mathbb{R}^d)$ and $\mathcal{M}_{\mathcal{L}}f \in L^1(\mathbb{R}^d)$. By Eq. 1.2 for $t > 0$ we have $T_t f \in L^2(\mathbb{R}^d)$, and so $T_t f \in H^1_{\mathcal{L},\max,h}(\mathbb{R}^d)$. We shall prove that

$$\lim_{t \to 0^+} \|\mathcal{M}_{\mathcal{L}}(T_t f - f)\|_{L^1(\mathbb{R}^d)} = 0. \tag{6.5}$$

It is well-known that there are constants $C, c > 0$ such that

$$|\partial_t T_t(x,y)| \leq Ct^{-1}t^{-d/2}\exp(-c|x-y|^2/t) \tag{6.6}$$

(see e.g., [5], [16, Theorem 6.17], [7] and references therein). We claim that there exists a constant $C_1 > 0$ such that for every $t > 0$ and $A > 1$ one has

$$\left\|\sup_{s > At}|T_{t+s}f(x) - T_s f(x)|\right\|_{L^1(\mathbb{R}^d)} \leq C_1 A^{-1}\|f\|_{L^1(\mathbb{R}^d)}. \tag{6.7}$$

To prove the claim, we note that for $s > At$, thanks to Eq. 6.6, we have

$$|T_{t+s}(x,y) - T_s(x,y)| = \left|\int_0^t \partial_u T_{s+u}(x,y)\,du\right|$$

$$\leq C\int_0^t (t+s)^{-1-d/2}\exp(-c'|x-y|^2/(t+s))\,du|$$

$$\leq C\int_0^t s^{-1-d/2}\exp(-c'|x-y|^2/s)\,du|$$

$$\leq Cts^{-1-d/2}\exp(-c'|x-y|^2/s).$$

Hence,

$$\sup_{s > At}|T_{t+s}(x,y) - T_s(x,y)| \leq \begin{cases} Ct(At)^{-1-d/2} & \text{if } |x-y| \leq \sqrt{At}, \\ Ct|x-y|^{-2-d} & \text{if } |x-y| > \sqrt{At}, \end{cases}$$

and, consequently,

$$\int_{\mathbb{R}^n}\left(\sup_{s > At}|T_{t+s}(x,y) - T_s(x,y)|\right)dx \leq C_1 A^{-1},$$

which implies Eq. 6.7.

We are now in a position to complete the proof of Eq. 6.5. Using Eq. 6.7 we obtain

$$\|T_t f - f\|_{H^1_{\mathcal{L}}} \leq \left\|\sup_{s > At}|T_{t+s}f - T_s f|\right\|_{L^1(\mathbb{R}^d)} + \left\|\sup_{s \leq At}|T_{t+s}f - T_s f|\right\|_{L^1(\mathbb{R}^d)}$$

$$\leq C_1 A^{-1}\|f\|_{L^1} + C\left\|\sup_{s \leq At}|T_{t+s}f - f|\right\|_{L^1(\mathbb{R}^d)} + C\left\|\sup_{s \leq At}|T_s f - f|\right\|_{L^1(\mathbb{R}^d)}$$

$$\leq C_1 A^{-1}\|f\|_{L^1(\mathbb{R}^d)} + 2C\left\|\sup_{s \leq (A+1)t}|T_s f - f|\right\|_{L^1(\mathbb{R}^d)}. \tag{6.8}$$

Fix $\varepsilon > 0$ and then take $A = \varepsilon^{-1}$. Clearly,

$$\sup_{s \leq (A+1)t}|T_s f(x) - f(x)| \leq 2\mathcal{M}_{\mathcal{L}}f(x) \in L^1(\mathbb{R}^d). \tag{6.9}$$

Since $\lim_{t \to 0}\sup_{s < (A+1)t}|T_s f(x) - f(x)| = 0$ a.e., we get Eq. 6.5 from Eqs. 6.8 and 6.9 by applying the Lebesgue dominated convergence theorem.

Let us finally remark that thanks to the subordination formula and results of [14, Chapter 8] we can adapt the above arguments to prove the equivalence of the definitions of Hardy spaces given by means of the Poisson semigroup $e^{-t\sqrt{\mathcal{L}}}$.

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