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Localization and Schrödinger Perturbations of Kernels

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Abstract We study iterations of integral kernels satisfying a transience-type condition and we prove exponential estimates analogous to Gronwall's inequality. As a consequence we obtain estimates of Schrödinger perturbations of integral kernels, including Markovian semigroups.

Keywords Kernel · Absorbing set

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1 Introduction

To motivate our results we consider the Gaussian transition density on $\mathbb{R}^d$,

$$p(s,x,t,y) = \begin{cases} [4\pi(t-s)]^{-d/2} \exp \dfrac{-|x-y|^2}{4(t-s)}, & \text{if } s < t, \\ 0, & \text{if } s \geq t, \end{cases}$$

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where $d \geq 1, s, t \in \mathbb{R}$ and $x, y \in \mathbb{R}^d$. Note that $-p$ is a left inverse of $\partial_t + \Delta_y$:

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} p(s, x, t, y) \left[\partial_t \phi(t, y) + \Delta_y \phi(t, y)\right] dydt = -\phi(s, x), \quad \phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d).$$

Let $q(t, y) \geq 0$ be a Borel function on $\mathbb{R} \times \mathbb{R}^d$. Let $p_0 = p$, and for $n = 1, 2, \ldots,$

$$p_n(s, x, t, y) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} p_{n-1}(s, x, u, z) q(u, z) p(u, z, t, y) dudz. \tag{1.1}$$

We define $\tilde{p} = \sum_{n=0}^\infty p_n$. Under appropriate integrability conditions, $-\tilde{p}$ is the left inverse of $\partial_t + \Delta_y + q$ [5]. We call $\tilde{p}$ the Schrödinger perturbation of p by q, because $\partial_t + \Delta_y + q$ is an additive perturbation of $\partial_t + \Delta_y$ by the operator of multiplication by q. We see that $\tilde{p}(\cdot, \cdot, t, y)$ is a power series of iterates of an integral kernel operator applied to $p(\cdot, \cdot, t, y)$, which may be considered as a *control function*.

Estimates of such series for rather general kernels are the main subject of the paper, motivated by the results of [5, 15] on transition densities. The main feature of our approach is majorization of the series by means of a control function, e.g. f in our main result, Theorem 3.2. The assumptions on the kernel involve *local smallness* (Eq. 3.1) and *global boundedness* (Eq. 3.2) with respect to an increasing family of *absorbing sets*, which add a strong transience-type property of the kernel to the picture. A representative application of Theorem 3.2 is given in Example 4.1 for the potential kernel of two 1/2-stable subordinators.

In general we neither assume Chapman–Kolmogorov conditions on the kernel nor any connection between the kernel and the control function. However, for Schrödinger perturbations, these two are related by a multiplication operation, and the setting of *space-time* is of special interest because it includes transition kernels. The setting is dealt with in Theorem 4.6, which is complemented by Example 4.5 and Corollary 4.11, and illustrated by Example 4.13.

Our results are analogues, and a strengthening, of Khasminski's lemma [1, 9], under a transience-type properties of the kernel. They may be regarded as extensions of Gronwall's lemma to the context of kernel operators. The results also apply to Schrödinger perturbations of continuous-time transition densities by *measures*. They may be used in discrete time, in fact in quite general settings, including partially ordered state spaces. In a related paper [7] we use different methods to obtain slightly more specific estimates for Schrödinger perturbations of kernels on space-time by *functions*.

The paper is composed as follows. In Section 2 we consider integral kernels on absorbing sets. In Section 3 we prove estimates of von Neumann series for such kernels in presence of a control function. In Section 4 we give the application to Schrödinger perturbations of the potential kernel of two subordinators. We also discuss the local smallness and global boundedness for continuous-time kernels, with focus on transition kernels and singular perturbations, including perturbations by measures.

2 Kernels and Absorbing Sets

Let $(E, \mathcal{E})$ be a measurable space and let K be a kernel on $(E, \mathcal{E})$ [10]. That is, $K : E \times \mathcal{E} \to [0, \infty]$, each $K(x, \cdot)$ is a measure on $(E, \mathcal{E})$, and each function $K(\cdot, B)$ is

$\mathcal{E}$-measurable. We write $f \in \mathcal{E}^+$ if $f\colon E \to [0,\infty]$ and f is $\mathcal{E}$-measurable. For $f \in \mathcal{E}^+$ we let

$$Kf(x) = \int f(y)\, K(x, dy), \quad x \in E, \tag{2.1}$$

and call this K a kernel operator. The operator is additive, positively homogeneous, and $Kf_n(x) \uparrow Kf(x)$ whenever $f_n \uparrow f$. Conversely, every map from $\mathcal{E}^+$ to $\mathcal{E}^+$ having these properties is of the form 2.1, see [10]. For instance, if $q \in \mathcal{E}^+$, then *the multiplication by* q,

$$qf(x) := q(x)\, f(x), \quad x \in E,\ f \in \mathcal{E}^+,$$

is a kernel operator. This is a simple but ambiguous notation, and it should always be clear from the context which meaning of q we have in mind (the function or the multiplication operator). The composition of kernel operators K and L on $\mathcal{E}^+$ and the composition of kernels, $KL(x, B) = \int L(y, B)K(x, dy)$ on $(E, \mathcal{E})$, agree in the sense of Eq. 2.1, and so the composition of kernels is associative. We will often consider the multiplication by 1_A, the indicator function of $A \in \mathcal{E}$.

A set $A \in \mathcal{E}$ is called *K-absorbing*, if $K(x, A^c) = 0$ for every $x \in A$, that is if $1_A K 1_{A^c} = 0$. Since $1_E = 1_A + 1_{A^c}$ and $1_A 1_{A^c} = 0$, A is K-absorbing if and only if

$$1_A K = 1_A K 1_A \tag{2.2}$$

as kernels. Clearly, $\emptyset$ and E are K-absorbing, and the union and intersection of countably many K-absorbing sets are K-absorbing. If A is K-absorbing, then A is L-absorbing for any kernel $L \le K$.

Example 2.1 We will generalize the discussion of the Gaussian kernel from Introduction. Let $(X, \mathcal{M})$ be a measurable space. Let $E = \mathbb{R} \times X$, with the σ-algebra $\mathcal{E}$ generated by the sets $(a, b) \times A$, where $a, b \in \mathbb{R}$, $a < b$ and $A \in \mathcal{M}$. Let $p\colon E \times E \to [0,\infty]$ be $\mathcal{E} \otimes \mathcal{E}$-measurable and satisfy

$$p(s, x, t, y) = 0, \quad \text{whenever} \quad s \ge t. \tag{2.3}$$

Given a measure μ on $(E, \mathcal{E})$, we define the kernel K^μ,

$$K^\mu f(s, x) := \int p(s, x, u, z)\, f(u, z)\, d\mu(u, z), \qquad (s, x) \in E, \quad f \in \mathcal{E}^+. \tag{2.4}$$

We note that, for every $t \in \mathbb{R}$, the "open half-space" $(t, \infty) \times X$ and the "closed half-space" $[t, \infty) \times X$ are absorbing for K^μ. Thus, the first coordinate has a distinguished role for *space-time* $E = \mathbb{R} \times X$, which is the main setting of [7].

In many examples of interest p also satisfies the Chapman–Kolmogorov equations, i.e., there is a measure m on $(X, \mathcal{M})$ such that for all $s < u < t$ and $x, y \in X$,

$$p(s, x, t, y) = \int p(s, x, u, z)p(u, z, t, y)\, dm(z). \tag{2.5}$$

For the Brownian transition density, m is the Lebesgue measure on $\mathbb{R}^d$.

Example 2.2 Let $(T, \mathcal{T}, \rho)$ be a measure space. Let $\{K_t\,, t \in T\}$ be a family of kernels on $(E, \mathcal{E})$ such that $(t, x) \mapsto K_t(x, B)$ is $\mathcal{T} \otimes \mathcal{E}$-measurable for each $B \in \mathcal{E}$. Then $K := \int K_t\, \rho(dt)$ is a kernel. Furthermore, if $A \in \mathcal{E}$ is K_t-absorbing for every $t \in T$, then A is also K-absorbing.

For instance, let $\alpha \in (0, 2)$ and let $p_t(y)$ be the density function of the $\alpha/2$-stable subordinator $(\eta_t,\ \ t > 0)$ on $\mathbb{R}$. Recall that (η_t) is time-homogeneous and has independent increments, and $p_t(y) = 0$ if $y \le 0$. Thus the right half-lines are absorbing for the semigroup $K_t(x, dy) := p_t(y - x)dy$. We have (see, e.g., [2, V.3.4] or [6, (1.38)]),

$$\int_0^\infty p_t(y)\, dt = \Gamma(\alpha/2)^{-1} y^{\alpha/2-1}\ , \qquad y > 0\ .$$

Accordingly, the right half-lines are absorbing for the *potential kernel* of (η_t),

$$K(x, A) = \Gamma(\alpha/2)^{-1} \int_A (y - x)_+^{\alpha/2-1}\, dy,$$

and also for

$$K^\mu(x, A) = \Gamma(\alpha/2)^{-1} \int_A (y - x)_+^{\alpha/2-1}\, \mu(dy),$$

where μ is any Borel measure on $\mathbb{R}$.

Example 2.3 If E is partially ordered and each measure $K(x, dy)$ is concentrated on $\Gamma_x := \{y : x \prec y\}$, then the sets Γ_x are K-absorbing. This is the case, e.g., for the semigroup and the potential operator of a vector of subordinators (see also Example 4.1).

Example 2.4 Let $(\mathcal{X}, \mathcal{W})$ be a balayage space [2, II.4]. Here $\mathcal{X}$ is a locally compact space with countable base, and $\mathcal{W}$ denotes the class of nonnegative hyperharmonic functions on $\mathcal{X}$ [2, III.1]. In particular, each $w \in \mathcal{W}$ is lower semicontinuous. Let r be a continuous real potential on $\mathcal{X}$ [2, II.5] and let K be the potential kernel associated with r in the sense of [2, II.6.17]. Thus, $K1 = r$, and for every bounded Borel measurable function $f \ge 0$ on $\mathcal{X}$, the function Kf is a continuous potential, which is harmonic outside the support of f, see [2, III.6.12]. Let $w \in \mathcal{W}$ and $A = \{x \in \mathcal{X} : w = 0\}$. Then A is closed and K-absorbing. Indeed, let B be a compact in A^c. There exists a number $c > 0$ such that $cw > r$ on B. By the minimum principle [2, III.6.6], $cw \ge K1_B$ everywhere, hence $K1_B = 0$ on A. In [2, V.1] such sets A are called *absorbing*, too, and they have a number of equivalent characterizations, of which we mention two: (a) A is closed and $P_t(x, \mathcal{X} \setminus A) = 0$, for every $t > 0$, $x \in A$, and sub-Markov semigroup $(P_t)_{t>0}$ having $\mathcal{W}$ as excessive functions, and (b) A is closed and $P^x[X_t \in A \cup \{\partial\}] = 1$ for every $t > 0$, $x \in A$, and Markov process $(X_t, P^x)_{t>0, x\in\mathcal{X}}$ having $\mathcal{W}$ as excessive functions and ∂ as the cementary state. The details are given in [2, V.1.2].

Furthermore, if A is any Borel set containing the (fine) superharmonic support of r, then $K1_A = K$ [2, II.6.3], and hence A is K-absorbing.

We will collect a few simple facts about K-absorbing sets.

Lemma 2.5 *Let A be K-absorbing and $m \in \mathbb{N}$. Then*

$$1_A K^m = (1_A K)^m = 1_A K^m 1_A. \tag{2.6}$$

In particular, A is K^m-absorbing. If furthermore $f \in \mathcal{E}^+$ and $c \geq 0$ are such that $Kf \leq cf$ on A, then $K^m f \leq c^m f$ on A.

Proof The case of $m = 1$ follows from Eq. 2.2. If Eq. 2.6 holds for some $m \in \mathbb{N}$, then

$$1_A K^{m+1} = 1_A K^m K = (1_A K)^m 1_A K = (1_A K)^m 1_A K 1_A$$

showing that Eq. 2.6 holds for $m + 1$, and we can use induction. Further, $1_A K f \leq cf$ yields that $1_A K^m f = (1_A K)^m f \leq c^m f$. □

Lemma 2.6 *Let A and B be K-absorbing, $A \subset B$, and $m \in \mathbb{N}$. Then*

$$1_B K^m 1_{B \setminus A} = 1_B (K 1_{B \setminus A})^m = 1_{B \setminus A} (K 1_{B \setminus A})^m. \tag{2.7}$$

Proof Since A is K-absorbing, $1_B K 1_{B\setminus A} = 1_A K 1_{B\setminus A} + 1_{B\setminus A} K 1_{B\setminus A} = 1_{B\setminus A} K 1_{B\setminus A}$. By this and Lemma 2.5 (with B in place of A),

$$\begin{aligned} 1_B K^m 1_{B\setminus A} &= (1_B K)^m 1_{B\setminus A} = (1_B K)^{m-1} 1_{B\setminus A} K 1_{B\setminus A} = \ldots = 1_B (K 1_{B\setminus A})^m \\ &= 1_B K 1_{B\setminus A} (K 1_{B\setminus A})^{m-1} = 1_{B\setminus A} K 1_{B\setminus A} (K 1_{B\setminus A})^{m-1} = 1_{B\setminus A} (K 1_{B\setminus A})^m. \end{aligned}$$

□

The next result is a slight modification of [12, Proposition 7.4].

Proposition 2.7 *Let A be K-absorbing, and let $f \in \mathcal{E}^+$ and $c \geq 1$ be such that $\sum_{m=0}^{\infty} K^m f \leq cf$ on A. Then, for $n = 0, 1, \ldots$, we have*

$$K^n f \leq c(1 - 1/c)^n f \quad \text{on } A. \tag{2.8}$$

Proof Let $g = \sum_{m=0}^{\infty} K^m f$. We see that $g = f + Kg \geq (1/c)g + Kg$ on A, hence $Kg \leq (1 - 1/c)g$ on A. By Lemma 2.5, for every $n \in \mathbb{N}$,

$$K^n f \leq K^n g \leq (1 - 1/c)^n g \leq c(1 - 1/c)^n f \quad \text{on } A.$$

The case of $n = 0$ is trivial. □

Remark 2.8 We note that, conversely, Eq. 2.8 yields that

$$\sum_{n=0}^{\infty} K^n f \leq \sum_{n=0}^{\infty} c(1 - 1/c)^n f = c^2 f \quad \text{on } A.$$

Thus, comparability of $\sum K^n f$ and f is equivalent to exponential decay of $K_n f$.

Remark 2.9 We will consider $f = 1$, the constant function. For every $a \geq 1$, there exist kernels K such that $\sup_{x \in E} K1(x) = a$, but $\sum_{m=0}^{\infty} K^m 1$ is bounded (see [14, Proposition 10.1]). Then the estimate for $K^n 1$ given in Eq. 2.8 is asymptotically better than the more evident upper bound by a^n.

3 Localization on Differences of Absorbing Sets

We first prove a discrete variant of Gronwall's lemma.

Lemma 3.1 *Let $\alpha, \delta \in [0, \infty)$ and $\gamma_1, \ldots, \gamma_k \in \mathbb{R}$ be such that, for $j = 1, \ldots, k$, we have $\gamma_j \le \alpha + \delta \sum_{1 \le i < j} \gamma_i$. Then $\gamma_j \le \alpha(1+\delta)^{j-1}$ for every $j = 1, \ldots, k$.*

Proof We proceed by induction: $\gamma_{k+1} \le \alpha + \delta \sum_{i=1}^{k} \alpha(1+\delta)^{i-1} = \alpha(1+\delta)^k$. □

We fix K-absorbing sets $A_1, \ldots, A_k$ such that

$$A_1 \subset A_2 \subset \cdots \subset A_k.$$

Taking $A_0 := \emptyset$, for $1 \le j \le k$ we define *slices* $S_j := A_j \setminus A_{j-1}$ and operators

$$K_j := K 1_{S_j}.$$

Thus, in Example 2.1 we may choose $-\infty < t_k < \cdots < t_1 < \infty$, and let A_j be the open half-space $(t_j, \infty) \times X$ or the closed half-space $[t_j, \infty) \times X$. Then each slice S_j equals $I_j \times X$, where I_j is an interval, see also Example 4.1 and Fig. 1.

Theorem 3.2 *Let $0 \le \eta < 1$, $\beta \ge 0$, and $f \in \mathcal{E}^+$ be such that*

$$K_j f \le \eta f \quad \text{on } S_j, \qquad j = 1, \ldots, k, \tag{3.1}$$

and

$$K_j f \le \beta f \quad \text{on } A_k, \qquad j = 1, \ldots, k. \tag{3.2}$$

Then, for $j = 1, \ldots, k$,

$$\sum_{m=0}^{\infty} K^m f \le \frac{1}{1-\eta}\left(1 + \frac{\beta}{1-\eta}\right)^{j-1} f \quad \text{on } S_j. \tag{3.3}$$

Proof Let $n \in \mathbb{N}$ and $g_n := \sum_{m=0}^{n} K^m f$. For $j = 1, \ldots, k$, we (recursively) define

$$\gamma_j := \frac{1}{1-\eta}\left(1 + \beta \sum_{1 \le i < j} \gamma_i\right). \tag{3.4}$$

We will prove by induction that $g_n \le \gamma_j f$ on S_j. Let $1 \le j \le k$, and

$$g_n \le \gamma_i f \quad \text{on } S_i, \qquad \text{for every } 1 \le i < j. \tag{3.5}$$

Trivially, this assumption is satisfied for $j = 1$. By Eq. 3.2, $Kf \le k\beta f$ on A_k. By Lemma 2.5 we obtain a rough bound, $g_n \le \sum_{m=0}^{n} (k\beta)^m f$ on A_k. Let $\gamma \ge 0$ be the smallest real number such that $g_n \le \gamma f$ on S_j. If $j < l \le k$, then $1_{S_j} K_l = 0$. By Eqs. 3.5 and 3.2 for all $x \in S_j$ we have,

$$\begin{aligned} g_n(x) &\le f(x) + Kg_n(x) = f(x) + \sum_{i=1}^{j} K_i g_n(x) \\ &\le f(x) + \sum_{i=1}^{j-1} \gamma_i K_i f(x) + \gamma K_j f(x) \le \left(1 + \beta \sum_{i=1}^{j-1} \gamma_i\right) f(x) + \gamma \eta f(x). \end{aligned}$$

Thus $\gamma \le \gamma_j$ (see Eq. 3.4), $g_n \le \gamma_j f$ on S_j, and the result follows by Lemma 3.1. □

Remark 3.3 We shall refer to Eq. 3.1 as *local smallness* and to Eq. 3.2 as *global boundedness*. In many important cases, the local smallness already implies the global boundedness with $\beta = \eta$. In particular, it is so in Example 2.4, if $f, 1 \in \mathcal{W}$. This follows from the minimum principle [2, III.6.6] applied to the functions $\eta f - K1_L \min\{f, n\}$, for compacts sets $L \subset S_j$ and $n \in \mathbb{N}$. It is also true in Example 2.1 provided $f = p(\cdot, \cdot, t, y)$, each A_j is a half-space, μ does not charge the "hyperplanes" $\{t\} \times X$, $t \in \mathbb{R}$, and the Chapman–Kolmogorov equations are satisfied, see Lemma 4.9 below.

The following result is motivated by Proposition 2.7 and Remark 2.8.

Corollary 3.4 *Assume $c > 1$ and $N \in \mathbb{N}$ are such that $\eta := c\,(1 - 1/c)^N < 1$. Let $\beta \ge 0$ and $f \in \mathcal{E}^+$ be such that $Kf \le \beta f$ on A_k and*

$$\sum_{m=0}^{\infty} K_j^m f \le cf \quad \text{on } S_j \tag{3.6}$$

for every $1 \le j \le k$. Then, for every $1 \le j \le k$,

$$\sum_{m=0}^{\infty} K^m f \le \left(\sum_{n=0}^{N-1} \beta^n\right) \frac{1}{1-\eta} \left(1 + \frac{\beta}{1-\eta}\right)^{j-1} f \quad \text{on } S_j. \tag{3.7}$$

If Eq. 3.6 holds on A_k for every $1 \le j \le k$, then $Kf \le ckf$ on A_k.

Proof Let $1 \le j \le k$. Since each $K_j^m f$ vanishes on A_{j-1}, Eq. 3.6 means that $\sum_{m=0}^{\infty} K_j^m f \le cf$ on A_j. By a remark following Eq. 2.2, A_j is K_j-absorbing. By Lemma 2.6 and Proposition 2.7,

$$(K^N)(1_{S_j} f) = (K_j)^N f \le \eta f \quad \text{on } A_j.$$

An application of Theorem 3.3 yields that

$$\sum_{m=0}^{\infty} (K^N)^m f \le \frac{1}{1-\eta} \left(1 + \frac{\beta}{1-\eta}\right)^{j-1} f \quad \text{on } S_j.$$

By Lemma 2.5, $\sum_{n=0}^{N-1} K^n f \le \sum_{n=0}^{N-1} \beta^n f$ on A_k. We finally note that

$$\sum_{m=0}^{\infty} K^m f = \sum_{m=0}^{\infty} (K^N)^m \left(\sum_{n=0}^{N-1} K^n f\right) \le \left(\sum_{n=0}^{N-1} \beta^n\right) \sum_{m=0}^{\infty} (K^N)^m f.$$

If Eq. 3.6 holds even on A_k for $1 \le j \le k$, then $Kf \le \sum_{j=1}^{k} K_j f \le ckf$ on A_k, and we can take $\beta = ck$. □

4 Examples and Applications

We may use Theorem 3.2 to estimate Schrödinger-type perturbations of kernels. As a rule, auxiliary estimates of the kernels are needed for such applications.

Example 4.1 For $t > 0$ and $x \in \mathbb{R}$ we define

$$f_t(x) = \begin{cases} (4\pi)^{-1/2}\, t\, x^{-3/2} e^{-t^2/(4x)}, & \text{if } x > 0, \\ 0\,, & \text{else}, \end{cases}$$

the density function of the 1/2-stable subordinator. By [20, Example 2.13],

$$\int_0^\infty f_t(x) e^{-ux} dx = e^{-tu^{1/2}}, \quad u \ge 0\,.$$

For $\phi \in C_c^\infty(\mathbb{R})$ (smooth compactly supported real-valued functions on $\mathbb{R}$) we let

$$P_t\phi(x) = \int_0^\infty \phi(x+z)\, f_t(z) dz\,, \quad x \in \mathbb{R}\,.$$

The generator of the semigroup (P_t) is the Weyl fractional derivative,

$$\begin{aligned} \partial^{1/2}\phi(x) &= \int_0^\infty (4\pi)^{-1/2} z^{-3/2} \left(\phi(x+z) - \phi(x)\right) dz \\ &= \pi^{-1/2} \int_0^\infty z^{-1/2} \phi'(x+z)\, dz\,. \end{aligned}$$

Schrödinger perturbations of ∂^β for $\beta \in (0,1)$ were considered in [7]. We shall discuss those for the generator $L = \partial_s^{1/2} + \partial_x^{1/2}$ of the semigroup of two independent 1/2-stable subordinators,

$$T_t\varphi(s,x) = \int_0^\infty \int_0^\infty \varphi(s+u, x+z)\, f_t(u)\, f_t(z) du dz\,, \quad s, x \in \mathbb{R}\,.$$

Here and below, $\varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R})$. For $s, x \in \mathbb{R}$ we have

$$\varphi(s,x) = -\int_0^\infty \frac{d}{dt} T_t\varphi(s,x)\, dt = -\int_0^\infty T_t\, L\varphi(s,x)\, dt\,. \tag{4.1}$$

In view of Eq. 4.1 we need to calculate the potential kernel $\int_0^\infty T_t dt$. Let

$$\begin{aligned} \kappa(s,x) &= \int_0^\infty f_t(s)\, f_t(x) dt \\ &= \begin{cases} (4\pi)^{-1/2}(s+x)^{-3/2}, & \text{if } s, x > 0, \\ 0\,, & \text{else}, \end{cases} \end{aligned}$$

where the latter formula follows from direct integration. Define

$$\kappa(s,x,u,z) = \kappa(u-s, z-x)\,, \quad s, x, u, z \in \mathbb{R}\,.$$

By Eq. 4.1, we obtain

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \kappa(s,x,u,z)\left(\partial_u^{1/2}+\partial_z^{1/2}\right)\varphi(u,z)\,dudz = -\varphi(s,x)\,,\quad s,x\in\mathbb{R}\,. \tag{4.2}$$

We observe a 3G-type inequality: if $s < u < t$ and $x < z < y$, then

$$\kappa(s,x,t,y) \le \kappa(s,x,u,z)\wedge\kappa(u,z,t,y) \le 2\sqrt{2}\,\kappa(s,x,t,y)\,, \tag{4.3}$$

since $t-s+y-x \ge (u-s+z-x)\vee(t-u+y-z) \ge (t-s+y-x)/2$. Thus,

$$\kappa(s,x,u,z)\kappa(u,z,t,y) \le 2\sqrt{2}\,\kappa(s,x,t,y)\left[\kappa(s,x,u,z)\vee\kappa(u,z,t,y)\right], \tag{4.4}$$

where $s < u < t$, $x < z < y$, and this is sharp, since Eq. 4.3 also yields

$$\kappa(s,x,u,z)\kappa(u,z,t,y) \ge \kappa(s,x,t,y)\left[\kappa(s,x,u,z)+\kappa(u,z,t,y)\right]/2\,.$$

For $0 < p < 1/2$ and a number $c > 0$ we let

$$q_0(u,z) = \begin{cases} c(u+z)^{-p}, & \text{if } u,z > 0\,, \\ 0, & \text{else.} \end{cases}$$

We consider $0 \le q \le q_0$ and the kernel

$$Kf(s,x) := \int_{\mathbb{R}^2} \kappa(s,x,u,z)q(u,z)f(u,z)dzdu\,.$$

We will use Theorem 3.2 to compare κ with $\tilde\kappa$ defined as

$$\tilde\kappa = \sum_{m=0}^{\infty}(\kappa q)^m\kappa, \tag{4.5}$$

or, more precisely,

$$\tilde\kappa(s,x,t,y) = \sum_{m=0}^{\infty} K^m f(s,x)\,,$$

where we fix $t, y \in \mathbb{R}$ and denote (the control function)

$$f(s,x) := \kappa(s,x,t,y)\,.$$

We let $s < t$ and $x < y$, because otherwise $\tilde\kappa(s,x,t,y) = 0 = \kappa(s,x,t,y)$. Furthermore, we assume that $t + y > 0$, else $\tilde\kappa(s,x,t,y) = \kappa(s,x,t,y)$. Let $h > 0$ and $k \ge 1$ be such that $(k-1)h \le t+y < kh$ (h is defined later on). For $j = 0,\dots,k$, we let $a_j = (k-j)h$. For $j = 1,\dots,k-1$, we define $A_j = \{(u,z) : u+z \ge a_j\}$. We also let $A_0 = \emptyset$, and $A_k = \mathbb{R}^2$. The sets A_j are increasing and absorbing. For $j = 1,\dots,k$, we define $S_j = A_j \setminus A_{j-1}$, see Fig. 1. We will call $\{(u,z) : u+z = \xi\}$, $\xi \in \mathbb{R}$, the *level lines*. We define $K_j = K1_{S_j}$, as in Theorem 3.2. We have

$$K_j f(s,x)/f(s,x) \le 2\sqrt{2}c \int\limits_{S_j\cap\{s\le u\le t,\ x\le z\le y\}} [(u+z-s-x)^{-3/2} + (t+y-u-z)^{-3/2}](u+z)^{-p}dzdu\,. \tag{4.6}$$

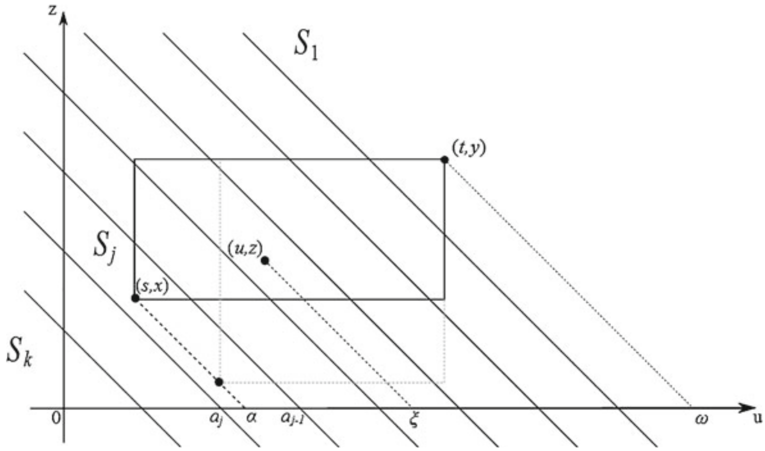

Fig. 1 Notation for Example 4.1

We will estimate the right-hand side of Eq. 4.6. Denote $\alpha = s + x$, $\omega = t + y$ and $\xi = u + z$. Let $\alpha < a_{j-1}$ and $\omega > a_j$ (otherwise the integral is zero). The integrand is constant along the level lines. The integral is the largest when $\{(s, u) \in \mathbb{R}^2 : s \le u \le t,\ x \le z \le y\}$ is a square, because the square's intersections with the level lines have the largest length, namely $\sqrt{2}[(\xi - \alpha) \wedge (\omega - \xi)]$, see Fig. 1. Taking this into account or substituting $\xi = u + z$, $\eta = (u - z)/2$, we bound the integral in Eq. 4.6 by

$$\int_{\alpha \vee a_j}^{\omega \wedge a_{j-1}} (\xi - \alpha)(\xi - \alpha)^{-3/2}\xi^{-p} + (\omega - \xi)(\omega - \xi)^{-3/2}\xi^{-p}\, d\xi$$

$$\le \int_{a_j}^{a_{j-1}} (\xi - a_j)^{-1/2-p} + (a_{j-1} - \xi)^{-1/2}(\xi - a_j)^{-p}\, d\xi$$

$$= \left[B(1/2 - p, 1) + B(1/2, 1 - p)\right](a_{j-1} - a_j)^{1/2-p},$$

where B is the Euler beta function.

By Theorem 3.2, if we let $\eta = \beta = 2\sqrt{2}c[B(1/2 - p, 1) + B(1/2, 1 - p)]h^{1/2-p} < 1$ (the inequality determines h), then

$$\tilde{\kappa}(s, x, t, y) \le \left(\frac{1}{1 - \eta}\right)^j \kappa(s, x, t, y) \quad \text{for} \quad (s, x) \in S_j. \tag{4.7}$$

In fact, $j < k + 1 - (s + x)/h \le (t + y - s - x)/h + 2$. We see that κ and $\tilde{\kappa}$ are locally comparable. We also note that the first coordinate does not play a distinguished role here, in contrast to the examples in [7] and below. Finally, $\tilde{\kappa}$ may be considered a Schrödinger perturbation of κ, because

$$\int_{\mathbb{R}\times\mathbb{R}} \tilde{\kappa}(s, x, u, z)\left[\partial_z^{1/2} + \partial_u^{1/2} + q(u, z)\right]\varphi(u, z)\, dz du = -\varphi(s, x), \tag{4.8}$$

for $s, x \in \mathbb{R}$ and $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R})$. The identity 4.8 is proved by using [7, (31)]. Indeed, the absolute integrability of the integrals in [7, (31)] follows by considering the supports of the involved functions (we leave details to the reader). We also wish to note that if $q_0(u, z)$ depends only on u or $u \wedge z$, then it is more convenient to consider absorbing sets $\{(u, z) \in \mathbb{R}^2 : u > s\}$ or $\{(u, z) \in \mathbb{R}^2 : u > s, z > x\}$, correspondingly.

In the remainder of the paper we shall adopt the setting of Example 2.1. More precisely, we consider the space-time $E = \mathbb{R} \times X$, with the product σ-algebra $\mathcal{E}$, and an $\mathcal{E} \times \mathcal{E}$-measurable function $p \geq 0$ on $E \times E$ such that Eq. 2.3 holds, but we do *not* assume Eq. 2.5. For a measure μ on $(E, \mathcal{E})$ we define kernel K^μ by Eq. 2.4. Motivated by the discussion in Section 1 and Example 2.1, we let

$$p^\mu = \sum_{n=0}^{\infty} p_n^\mu, \tag{4.9}$$

where $p_0^\mu = p$, and the positive functions $p_1^\mu, p_2^\mu \dots$ on $E \times E$ are defined as follows,

$$p_n^\mu(s, x, t, y) := \int p_{n-1}^\mu(s, x, u, z) p(u, z, t, y)\, d\mu(u, z). \tag{4.10}$$

By induction, $p_n^\mu(s, x, t, y) = 0$ for $n \geq 0$, $(s, x), (t, y) \in E$, if $s \geq t$. According to Introduction, we perturb p by the *measure* μ (but see Example 4.5, too). We regard (t, y) as fixed when iteratively transforming $f(s, x) := p(s, x, t, y)$ by K^μ:

$$p_n^\mu(\cdot, \cdot, t, y) = (K^\mu)^n p(\cdot, \cdot, t, y).$$

Remark 4.2 Similar perturbations may be studied for signed measures, say ν. We clearly have $|p^\nu| \leq p^\mu$, where $\mu = \nu_- + \nu_+$ is the variation measure of ν. We will not further concern ourselves with signed kernels or functions in this paper.

In Examples 4.3, 4.4 and 4.5 we will additionally suppose that p is a transition density, that is, the Chapman–Kolmogorov Eq. 2.5 hold with respect to a σ-finite measure m on X.

Example 4.3 Let $\rho \geq 0$ be a Radon measure on $\mathbb{R}$ having no atoms, and let $\mu := \rho \otimes m$. Then, for all $(s, x), (t, y) \in E$ and $n \in \mathbb{N}$, $p_n^\mu(s, x, t, y) = \rho((s, t))^n p(s, x, t, y)/n!$ by induction, and we obtain the transition density

$$p^\mu(s, x, t, y) = e^{\rho((s,t))} p(s, x, t, y). \tag{4.11}$$

Example 4.4 Let $\eta > 0$, $u_0 \in \mathbb{R}$, $\mu := \eta \varepsilon_{u_0} \otimes m$. Here $\varepsilon_{u_0}(f) = f(u_0)$ is the Dirac measure. Then μ is concentrated on the "hyperplane" $\{u_0\} \times E$, and for (s, x), $(t, y) \in E$ we have by Eq. 2.5,

$$p_1^\mu(s, x, t, y) = \int p(s, x, u, z) p(u, z, t, y)\, d\mu(u, z) = \begin{cases} \eta p(s, x, t, y), & \text{if } s < u_0 < t, \\ 0, & \text{otherwise.} \end{cases}$$

For $n = 2, 3, \dots$ and all $(s, x), (t, y) \in E$, we obtain $p_n^\mu(s, x, t, y) = 0$, hence

$$p^\mu(s, x, t, y) := \sum_{n=0}^{\infty} p_n^\mu(s, x, t, y) = \begin{cases} (1 + \eta)\, p(s, x, t, y), & \text{if } s < u_0 < t, \\ p(s, x, t, y), & \text{otherwise.} \end{cases} \tag{4.12}$$

There is, however, an alternative approach to perturbations by such measures.

Example 4.5 Let $u_0 \in \mathbb{R}$ and $\mu := \varepsilon_{u_0} \otimes m$. For $g \in \mathcal{E}^+$ we define

$$Kg(s,x) = \begin{cases} 0\,, & \text{if } s > u_0\,, \\ g(s,x)\,, & \text{if } s = u_0\,, \\ \int_{\mathbb{R}^d} p(s,x,u_0,z)g(u_0,z)\,dm(z)\,, & \text{if } s < u_0\,. \end{cases}$$

Let $t > u_0$ and $y \in \mathbb{R}^d$ be fixed. We consider $f(s,x) = p(s,x,t,y)$, $(s,x) \in E$. By Chapman–Kolmogorov equations, $Kf(s,x) = 1_{s \le u_0}\, p(s,x,t,y)$. By induction, $K^n f(s,x) = 1_{s \le u_0}\, p(s,x,t,y)$, for $n = 1, 2, \ldots$. If $0 < \eta < 1$, then

$$\tilde{p}(s,x,t,y) := \sum_{n=0}^{\infty} (\eta K)^n f(s,x) = \begin{cases} (1-\eta)^{-1} p(s,x,t,y)\,, & \text{for } s \le u_0\,, \\ p(s,x,t,y)\,, & \text{otherwise,} \end{cases} \tag{4.13}$$

whereas $\eta \ge 1$ leads to explosion of $\tilde{p}$. We observe that $\tilde{p}$ satisfies Chapman–Kolmogorov equations, but not p^μ defined in Example 4.4.

More generally, for an arbitrary Radon measure ρ on $\mathbb{R}$, we let

$$Kg(s,x) = \rho(\{s\})g(s,x) + \int_{(s,\infty)} \int_X p(s,x,u,z)g(u,z)dm(z)\rho(du).$$

We note that $K = K^{\rho \otimes m}$ (see Eq. 2.4), if ρ has no atoms. On one hand this motivates our interest in K^μ later in this section. On the other hand, atoms are intrinsically related to the estimates obtained in [7, 15] and in Theorem 4.6 below, because they produce inflation of mass very close to that given by the estimates. Indeed, let us fix numbers $u_1 < u_2 < \ldots < u_k$, and let $\rho = \varepsilon_{u_1} + \varepsilon_{u_2} + \ldots + \varepsilon_{u_k}$. Assume that $u_k < t$. We have $Kf(s,x) = L(s)p(s,x,t,y)$, with f as before and

$$L(s) := \#\{1 \le i \le k : \; u_i \ge s\}.$$

By induction we verify that

$$\begin{aligned} K^n f(s,x) &= \#\{(i_1,\ldots,i_n) : \; s \le u_{i_1} \le \ldots \le u_{i_n}\} p(s,x,t,y) \\ &= \binom{L(s)+n-1}{n} p(s,x,t,y). \end{aligned} \tag{4.14}$$

Notably, a similar combinatorics is triggered by gradient perturbation series in [17, Lemma 5]. If $0 < \eta < 1$, then, by Taylor series expansion [15, p. 51],

$$\tilde{p}(s,x,t,y) := \sum_{n=0}^{\infty} (\eta K)^n f(s,x,t,y) = \left(\frac{1}{1-\eta}\right)^{L(s)} p(s,x,t,y). \tag{4.15}$$

This should be compared with Theorem 4.6 below.

We now return to functions p as specified before Eq. 4.9, i.e., we do not assume Chapman–Kolmogorov conditions, unless we explicitly say otherwise.

Let $I \subset \mathbb{R}$ be an interval and let

$$\mu_I(A) := \mu(A \cap (I \times X)), \quad A \in \mathcal{E}.$$

For $n = 0, 1, 2, \ldots$, we denote (see above in this section)

$$p_n := p_n^\mu \quad \text{and} \quad p_n^I := p_n^{\mu_I}.$$

We also note that $p_n(s,x,t,y) = p_n^{(s,t)}(s,x,t,y)$, which follows by induction.

The half-spaces $(t, \infty) \times X$ and $[t, \infty) \times X$ are K^{μ}-absorbing for $t \in \mathbb{R}$. The differences of such sets are of the form $I \times X$, where I is an interval. For $I, J \subset \mathbb{R}$, we write $I \prec J$, if $s < t$ for all $s \in I$ and $t \in J$.

Theorem 4.6 *Let $-\infty < r < t < \infty$, $y \in X$, $\eta \in [0, 1)$. Suppose that $[r, t)$ is the union of intervals $I_k \prec \cdots \prec I_1$, such that for all $j = 1, \ldots, k$, and $x \in X$,*

$$\int_{I_j \times X} p(s, x, u, z) p(u, z, t, y)\, d\mu(u, z) \le \eta\, p(s, x, t, y), \quad r \le s < t. \tag{4.16}$$

Then, for $j = 1, \ldots, k$ and $x \in X$,

$$p^{\mu}(s, x, t, y) := \sum_{n=0}^{\infty} p_n(s, x, t, y) \le \left(\frac{1}{1-\eta}\right)^j p(s, x, t, y), \quad s \in I_j. \tag{4.17}$$

Proof We may apply Theorem 3.2 to $f(s, x) := p(s, x, t, y)$, $A_j = (I_j \cup \ldots \cup I_1) \times X$, $K_j := K^{\mu_{I_j}}$, and $\beta := \eta$, since Eq. 4.16 implies both Eqs. 3.1 and 3.2. □

Corollary 4.7 *Let $-\infty < r < t < \infty$, $y \in X$, $\beta \ge 0$ and $c \ge 1$. Suppose that*

$$p_1(s, x, t, y) \le \beta\, p(s, x, t, y), \quad \textit{for all } s > r,\ x \in X, \tag{4.18}$$

and $[r, t)$ is a union of disjoint intervals $I_1, I_2, \ldots, I_k$ satisfying,

$$\sum_{n=0}^{\infty} p_n^{I_j}(s, x, t, y) \le c\, p(s, x, t, y), \quad \textit{for } s \in I_j,\ x \in X \quad (1 \le j \le k). \tag{4.19}$$

Then there exists a constant C such that $\sum_{n=0}^{\infty} p_n(s, x, t, y) \le C\, p(s, x, t, y)$ for all $s \ge r$ and $x \in X$.

Proof We proceed as in the proof of Theorem 4.6, using Corollary 3.4. We let $C = \left(\sum_{n=0}^{N-1} \beta^n\right) \left[1 + \beta/(1-\eta)\right]^{k-1} /(1-\eta)$, where $\eta = c(1 - 1/c)^N < 1$. □

Remark 4.8 If the inequality in Eq. 4.19 holds on $[r, \infty) \times X$, for $1 \le j \le k$, then

$$p_1(s, x, t, y) = \sum_{j=1}^{k} p_1^{I_j}(s, x, t, y) \le kc\, p(s, x, t, y), \quad s \ge r,\ x \in X,$$

and Eq. 4.18 holds with $\beta = kc$.

If p satisfies Eq. 2.5, then we can *localize* Eq. 4.16 as follows.

Lemma 4.9 *Suppose that p satisfies the Chapman–Kolmogorov equations. Let $(t, y) \in E$, $\eta \ge 0$, and let an interval $I \subset (-\infty, t)$ satisfy, for all $(s, x) \in I \times X$,*

$$\int p(s, x, u, z) p(u, z, t, y)\, d\mu_I(u, z) \le \eta\, p(s, x, t, y). \tag{4.20}$$

Then Eq. 4.20 holds for all $(s, x) \in E$.

Proof If $s \in I$ or s is to the right of I, then Eq. 4.20 clearly holds, see Eq. 2.3. If s is to the left of I, $a \in I$, $J := [a, \infty) \cap I$, and $x \in E$, then by Eqs. 2.5 and 4.20,

$$\int p(s,x,u,z)p(u,z,t,y)\,d\mu_J(u,z)$$
$$= \int\int p(s,x,a,w)p(a,w,u,z)p(u,z,t,y)\,dm(w)\,d\mu_J(u,z)$$
$$\leq \eta \int p(s,x,a,w)p(a,w,t,y)\,dm(w) = \eta\, p(s,x,t,y).$$

So Eq. 4.20 holds, if $\inf I \in I$ (take $a = \inf I$). If not, it follows by monotone convergence, by letting $a \in I$ approach $\inf I$. □

Lemma 4.10 *Suppose that* p *satisfies the Chapman–Kolmogorov equations. Let* $\eta \geq 0$ *and an interval* I *be such that, for all* $s, t \in I$ *and* $x, y \in X$,

$$\int p(s,x,u,z)p(u,z,t,y)\,d\mu_I(u,z) \leq \eta\, p(s,x,t,y). \tag{4.21}$$

Then Eq. 4.21 *holds for* all $(s, x), (t, y) \in E$.

Proof Let us fix $(t, y) \in E$. By Eq. 2.3 we may replace I by $I \cap (-\infty, t)$. An application of Lemma 4.9 finishes the proof. □

Corollary 4.11 *Suppose that* p *satisfies the Chapman–Kolmogorov equations. Let* $-\infty < r < t < \infty$, $y \in X$ *and* $\eta \in [0, 1)$. *Let* $[r, t)$ *be the union of intervals* $I_k \prec \cdots \prec I_1$. *Assume that for* $j = 1, \ldots, k$ *and* $I := I_j$, *Eq.* 4.21 *holds for all* $s \in I_j$ *and* $x \in X$. *Then Eq.* 4.17 *holds for* $j = 1, \ldots, k$ *and* $x \in X$.

Proof The result follows from Theorem 4.6 and Lemma 4.10. □

Remark 4.12 To prove comparability of p and p^μ under Eq. 2.5 in specific situations, it is enough to choose intervals I_j such that $\mu(E \setminus (I_1 \cup \ldots \cup I_k) \times X) = 0$, and for all $s, t \in I$, $x, y \in X$, $j = 1, \ldots, k$,

$$\int_{I_j} p(s,x,u,z)p(u,z,t,y)\,d\mu(z) \leq \eta\, p(s,x,t,y). \tag{4.22}$$

If Eq. 4.21 fails, then p^μ may be much bigger than p, see Example 4.5.

Our last example is essentially from [5].

Example 4.13 We consider the Cauchy transition density on $\mathbb{R}^d$, i.e. we let

$$p(s,x,t,y) = \begin{cases} c_d(t-s)\big[(t-s)^2 + |y-x|^2\big]^{-(d+1)/2}, & \text{if } s < t, \\ 0, & \text{if } s \geq t. \end{cases}$$

We observe the following power-type asymptotics of p:

$$p(s,x,t,y) \approx \frac{t-s}{|y-x|^{d+1}} \wedge (t-s)^{-d}, \quad x, y \in \mathbb{R}^d,\ s < t, \tag{4.23}$$

where $L \approx R$ means that L/R is bounded away from zero and infinity. In consequence, there is a constant c depending only on d, such that

$$p(s,x,u,z) \wedge p(u,z,t,y) \leq c\, p(s,x,t,y)\,, \quad x,z,y \in \mathbb{R}^d\,,\ s,u,t \in \mathbb{R}\,, \tag{4.24}$$

see the 3P Theorem in [3]. For numbers $a, b \geq 0$ we have $ab = (a \vee b)(a \wedge b)$ and $a \vee b \leq a + b$. Therefore Eq. 4.24 yields the following variant:

$$p(s,x,u,z)p(u,z,t,y) \leq c\, p(s,x,t,y)\big[p(s,x,u,z) + p(u,z,t,y)\big]\,, \tag{4.25}$$

and we obtain

$$p_1(s,x,t,y) \leq c\, p(s,x,t,y) \int_{\mathbb{R}^d}\int_s^t \big[p(s,x,u,z) + p(u,z,t,y)\big]\, d\mu(u,z)\,.$$

Assume that μ is of Kato class, to wit,

$$k(h) := \sup_{x,y\in\mathbb{R}^d,\ s<t\leq s+h} \int_{\mathbb{R}^d}\int_s^t \big[p(s,x,u,z) + p(u,z,t,y)\big]\, d\mu(u,z) \to 0 \quad \text{as } h \to 0\,.$$

Let $h > 0$ and $\eta := ck(h) < 1$. If $s + (j-1)h < t \leq s + jh$, where j is a natural number, then, by Corollary 4.11, for all $x, y \in \mathbb{R}^d$,

$$p^{\mu}(s,x,t,y) \leq \left(\frac{1}{1-\eta}\right)^{j} p(s,x,t,y) \leq \left(\frac{1}{1-\eta}\right)^{1+(t-s)/h} p(s,x,t,y)\,.$$

This is a special case of [15, Theorem 1]. In particular, if $d > 1$, then, by Eq. 4.23,

$$\int_s^t p(s,x,u,z)du \approx |z-x|^{1-d} \wedge \big[(t-s)^2|z-x|^{-d-1}\big]\,, \quad x,y \in \mathbb{R}^d\,,\ s<t\,,$$

and if $|d\mu(u,z)| \leq |z|^{-1+\varepsilon} dz du$ for some $\varepsilon \in (0,1]$, then μ is of Kato class.

We refer the reader to [5] for a comparison of different Kato conditions. We also refer to [1] for a discussion of discontinuous multiplicative functionals of Markov processes, which bring some analogies with Example 4.5. We also wish to mention recent results [4, 8] for non-local Schrödinger-type perturbations (see [18] and [21], too). Schrödinger perturbations of the Gaussian transition density are studied in [19, 22], see also [11]. We refer to [3, 5, 13, 14, 16] for further instances, applications and forms of the 3P (or 3G) inequality 4.24. In a related paper [7] we present a more specialized approach to Schrödinger perturbations by functions for transition densities, transition probabilities and general integral kernels in continuous time.

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