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On Riesz Transforms Characterization of H^1 Spaces Associated with Some Schrödinger Operators

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Abstract Let $\mathcal{L}f(x) = -\Delta f(x) + V(x)f(x)$, $V \geq 0$, $V \in L^1_{loc}(\mathbb{R}^d)$, be a non-negative self-adjoint Schrödinger operator on $\mathbb{R}^d$. We say that an L^1-function f is an element of the Hardy space $H^1_{\mathcal{L}}$ if the maximal function

$$\mathcal{M}_{\mathcal{L}} f(x) = \sup_{t>0} |e^{-t\mathcal{L}} f(x)|$$

belongs to $L^1(\mathbb{R}^d)$. We prove that under certain assumptions on V the space $H^1_{\mathcal{L}}$ is also characterized by the Riesz transforms $R_j = \frac{\partial}{\partial x_j}\mathcal{L}^{-1/2}$, $j = 1, ..., d$, associated with $\mathcal{L}$. As an example of such a potential V one can take any $V \geq 0$, $V \in L^1_{loc}$, in one dimension.

Keywords Riesz transforms · Hardy spaces · Schrödinger operators · Semigroups of linear operators

Mathematics Subject Classifications (2010) 42B30 · 42B35 · 35J10

1 Introduction

On $\mathbb{R}^d$ we consider a Schrödinger operator $\mathcal{L} = -\Delta + V(x)$, where $V(x)$ is a locally integrable nonnegative potential, $V \not\equiv 0$. It is well known that $-\mathcal{L}$ generates the

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semigroup $\{T_t\}_{t>0}$ of linear contractions on $L^p(\mathbb{R}^d)$, $1 \le p < \infty$. The Feynman-Kac formula implies that the integral kernels $T_t(x, y)$ of this semigroup satisfy

$$0 \le T_t(x, y) \le P_t(x - y) = (4\pi t)^{-d/2} \exp\left(-|x - y|^2/4t\right). \tag{1.1}$$

We say that an L^1-function f is an element of the Hardy space $H^1_{\mathcal{L}}$ if the maximal function

$$\mathcal{M}_{\mathcal{L}} f(x) = \sup_{t>0} |T_t f(x)|$$

belongs to $L^1(\mathbb{R}^d)$. We set

$$\|f\|_{H^1_{\mathcal{L}}} = \|\mathcal{M}_{\mathcal{L}} f\|_{L^1(\mathbb{R}^d)}.$$

Let $\mathcal{Q} = \{Q_j\}_{j=1}^{\infty}$ be a family of closed cubes in $\mathbb{R}^d$ with disjoint interiors such that $\mathbb{R}^d \setminus \bigcup_{j=1}^{\infty} Q_j$ is of Lebesgue measure zero. We shall always assume that there exist constants $C, \beta > 0$ such that if $Q_i^{****} \cap Q_j^{****} \neq \emptyset$ then $d(Q_i) \le C d(Q_j)$, where $d(Q)$ denotes the diameter of Q, and Q^* is the cube with the same center as Q such that $d(Q^*) = (1 + \beta) d(Q)$. Clearly, there is a constant $C > 0$ such that

$$\sum_{j=1}^{\infty} \mathbf{1}_{Q_j^{****}}(x) \le C. \tag{1.2}$$

In order to state results from [4] we recall the notion of the local Hardy space associated with the collection $\mathcal{Q}$. We say that a function a is an $H^1_{\mathcal{Q}}$-atom if there exists $Q \in \mathcal{Q}$ such that either $a = |Q|^{-1}\mathbf{1}_Q$ or a is the classical atom with support contained in Q^* (that is, there is a cube $Q' \subset Q^*$ such that $\operatorname{supp} a \subset Q'$, $\int a = 0$, $|a| \le |Q'|^{-1}$).

The atomic space $H^1_{\mathcal{Q}}$ is defined by

$$H^1_{\mathcal{Q}} = \left\{ f : f = \sum_j \lambda_j a_j, \ \sum_j |\lambda_j| < \infty \right\}, \tag{1.3}$$

where $\lambda_j \in \mathbb{C}$, a_j are $H^1_{\mathcal{Q}}$-atoms. We set

$$\|f\|_{H^1_{\mathcal{Q}}} = \inf \left\{ \sum_j |\lambda_j| \right\},$$

where the infimum is taken over all representations of f as in Eq. 1.3.

Following [4] we will also impose two additional assumptions on the potential V and the collection $\mathcal{Q}$ of cubes, mainly:

$$(\exists C, \varepsilon > 0) \qquad \sup_{y \in Q^*} \int T_{2^n d(Q)^2}(x, y)\, dx \le C n^{-1-\varepsilon} \quad \text{for } Q \in \mathcal{Q}, n \in \mathbb{N}; \tag{D}$$

$$(\exists C, \delta > 0) \quad \int_0^{2t} (\mathbf{1}_{Q^{***}} V) * P_s(x)\, ds \le C \left(\frac{t}{d(Q)^2} \right)^{\delta} \quad \text{for } x \in \mathbb{R}^d,\ Q \in \mathcal{Q}, t \le d(Q)^2. \tag{K}$$

Thorem 2.2 of [4] states that if we assume (D) and (K) then we have the following atomic characterization of the Hardy space $H^1_{\mathcal{L}}$:

$$f \in H^1_{\mathcal{L}} \iff f \in H^1_{\mathcal{Q}}. \quad \text{Moreover,} \quad C^{-1}\|f\|_{H^1_{\mathcal{Q}}} \le \|f\|_{H^1_{\mathcal{L}}} \le C\|f\|_{H^1_{\mathcal{Q}}}. \tag{1.4}$$

For $j = 1, \dots, d$, let

$$R_j f(x) = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{1/\varepsilon} \frac{\partial}{\partial x_j} T_t f(x) \frac{dt}{\sqrt{t}}$$

be the Riesz transform $\frac{\partial}{\partial x_j}\mathcal{L}^{-1/2}$ associated with $\mathcal{L}$, where the limit is understood in the sense of distributions (see Section 2). The main result of this paper is to prove that, under these conditions, the operators R_j characterize the space $H^1_{\mathcal{L}}$, that is, the following theorem holds.

Theorem 1.5 *Assume that a potential $V \ge 0$ and a collection of cubes $\mathcal{Q}$ are such that* (D) *and* (K) *hold. Then there exists a constant $C > 0$ such that*

$$C^{-1}\|f\|_{H^1_{\mathcal{L}}} \le \|f\|_{L^1(\mathbb{R}^d)} + \sum_{j=1}^{d} \|R_j f\|_{L^1(\mathbb{R}^d)} \le C\|f\|_{H^1_{\mathcal{L}}}. \tag{1.6}$$

Remark 1.7 For $\ell > 0$ denote by $\mathcal{Q}_\ell(\mathbb{R}^n)$ a partition of $\mathbb{R}^n$ into cubes whose diameters have length ℓ. Assume that for a locally integrable nonnegative potential V_1 on $\mathbb{R}^d$ and a collection $\mathcal{Q}$ of cubes the conditions (D) and (K) hold. Consider the potential $V(x_1, x_2) = V_1(x_1)$, $x_1 \in \mathbb{R}^d$, $x_2 \in \mathbb{R}^n$, and the family $\widetilde{\mathcal{Q}} = \{Q_1 \times Q_2 : \ Q_1 \in \mathcal{Q}, \ Q_2 \in \mathcal{Q}_{d(Q_1)}(\mathbb{R}^n)\}$ of cubes in $\mathbb{R}^{d+n}$. It is easily seen that the pair $(V, \widetilde{\mathcal{Q}})$ fulfils (D) and (K).

Remark 1.8 One can check that Theorem 2.2 of [4] (see Eq. 1.4) and Theorem 1.5 together with their proofs remain true if we replace cubes by rectangles in the definition of atoms and in the conditions (D) and (K), provided the rectangles have side-lengths comparable to their diameters. As a corollary of this observation we obtain that if $V(x_1, x_2) = V_1(x_1) + V_2(x_2)$, $x_1 \in \mathbb{R}^d$, $x_2 \in \mathbb{R}^n$, where V_1 and V_2 satisfy conditions (D) and (K) for certain collections $\mathcal{Q}_1$ and $\mathcal{Q}_2$ of cubes on $\mathbb{R}^d$ and $\mathbb{R}^n$ respectively, then the Hardy space $H^1_{\mathcal{L}}$ associated with the operator $\mathcal{L} = -\Delta + V(x_1, x_2)$ in $\mathbb{R}^{d+n}$ admits the atomic and the Riesz transforms characterizations. Indeed, for any $Q_j \in \mathcal{Q}_1$ and $Q_k \in \mathcal{Q}_2$ we divide the rectangle $Q_j \times Q_k$ into rectangles $Q^s_{j,k}$, $s = 1, 2, ..., s_{j,k}$, with side-lengths comparable to $\min(d(Q_j), d(Q_k))$. It is not difficult to verify that (D) and (K) hold for $V(x_1, x_2)$ and the collection $Q^s_{j,k}$.

Example We finish the section by recalling some examples of nonnegative potentials V considered in [1] and [4] such that the semigroups generated by $\Delta - V$ satisfy (D) and (K) for relevant collections $\mathcal{Q}$ of cubes.

- The Hardy space $H^1_{\mathcal{L}}$ associated with one-dimensional Schrödinger operator $-\mathcal{L}$ was studied in Czaja and Zienkiewicz [1]. It was proved there that for any nonnegative $V \in L^1_{loc}(\mathbb{R})$ the collection $\mathcal{Q}$ of maximal dyadic intervals Q of $\mathbb{R}$ that are defined by the stopping time condition

$$|Q| \int_{16Q} V(y)\, dy \le 1, \tag{1.9}$$

fulfils (D) for certain small $\beta > 0$ (see [1, Lemma 2.2]). The authors also remarked that (K) is satisfied. Indeed,

$$\int_0^{2t} (\mathbf{1}_{Q^{***}} V) * P_s(x)\, ds \leq \int_0^{2t} \|\mathbf{1}_{Q^{***}} V\|_{L^1} \|P_s\|_{L^\infty}\, ds$$
$$\leq \int_0^{2t} |Q|^{-1} \frac{ds}{\sqrt{4\pi s}} \leq C\frac{t^{1/2}}{|Q|},$$

where in the second inequality we have used Eq. 1.9.

- $V(x) = \gamma|x|^{-2}$, $d \geq 3$, $\gamma > 0$. Then for $\mathcal{Q}$ being the Whitney decomposition of $\mathbb{R}^d \setminus \{0\}$ that consists of dyadic cubes the conditions (D) and (K) hold (see Theorem 2.8 of [4]).
- $d \geq 3$, V satisfies the reverse Hölder inequality with exponent $q > d/2$, that is,

$$\left(\frac{1}{|B|}\int_B V(y)^q\, dy\right)^{1/q} \leq C\frac{1}{|B|}\int_B V(y)\, dy \quad \text{for every ball } B.$$

Define the family $\mathcal{Q}$ by: $Q \in \mathcal{Q}$ if and only if Q is the maximal dyadic cube for which $d(Q)^2|Q|^{-1}\int_Q V(y)\, dy \leq 1$. Then the conditions (D) and (K) are true (see [4, Section 8]).

Let us finally mention that the Riesz transforms characterization of the Hardy spaces associated with Schrödinger operators with potentials satisfying the reverse Hölder inequality was proved in [2]. For the foundation of the theory of the real Hardy spaces we refer the reader to [6].

2 Auxiliary Estimates

Lemma 2.1 *For every $\alpha > 0$ there exists a constant $C > 0$ (independent of V) such that for $j = 1, \ldots, d$ and $y \in \mathbb{R}^d$ we have*

$$\int_{\mathbb{R}^d} \left|\frac{\partial}{\partial x_j} T_t(x, y)\right|^2 \exp\left(\alpha|x - y|/\sqrt{t}\right)\, dx \leq Ct^{-d/2-1}, \tag{2.2}$$

$$\int_{\mathbb{R}^d} \left|\frac{\partial}{\partial x_j} T_t(x, y)\right| \exp\left(\alpha|x - y|/\sqrt{t}\right)\, dx \leq Ct^{-1/2}. \tag{2.3}$$

The lemma seems to be known. For reader's convenience we give a sketch of a proof in Section 4.

For $\varepsilon > 0$, $j = 1, \ldots, d$, we define the operator

$$R_j^\varepsilon f(x) = \int R_j^\varepsilon(x, y) f(y)\, dy,$$

where $R_j^\varepsilon(x, y) = \int_\varepsilon^{1/\varepsilon} \frac{\partial}{\partial x_j} T_t(x, y) \frac{dt}{\sqrt{t}}$. It is not difficult to see that for $f \in L^1(\mathbb{R}^d)$ the limits $\lim_{\varepsilon\to 0} R_j^\varepsilon f(x)$ exist in the sense of distributions and define tempered distributions which will be denoted by $R_j f$. Moreover, for $\varphi \in \mathcal{S}(\mathbb{R}^d)$ we have

$$|\langle R_j f, \varphi\rangle| \leq C\|f\|_{L^1(\mathbb{R}^d)}\left(\|\varphi\|_{L^2(\mathbb{R}^d)} + \left\|\frac{\partial}{\partial x_j}\varphi\right\|_{L^\infty(\mathbb{R}^d)}\right). \tag{2.4}$$

To see this we write

$$R_j^{\varepsilon *}\varphi(y) = \int_1^{1/\varepsilon}\int_{\mathbb{R}^d} \frac{\partial}{\partial x_j} T_t(x,y)\varphi(x)\,dx\,\frac{dt}{\sqrt{t}} - \int_\varepsilon^1\int_{\mathbb{R}^d} T_t(x,y)\frac{\partial}{\partial x_j}\varphi(x)\,dx\,\frac{dt}{\sqrt{t}}.$$

Since

$$\int_1^\infty \left[\int_{\mathbb{R}^d}\left|\frac{\partial}{\partial x_j}T_t(x,y)\right|^2 dx\right]^{\frac{1}{2}} \frac{dt}{\sqrt{t}} \le C\int_1^\infty t^{-1-\frac{d}{4}}\,dt \le C$$

and

$$\int_{\mathbb{R}^d}\int_0^1 T_t(x,y)\,\frac{dt}{\sqrt{t}}\,dx \le 2$$

(see Lemma 2.1), we conclude that $R_j^{\varepsilon *}\varphi(y)$ converges uniformly, as $\varepsilon \to 0$, to a bounded function which will be denoted by $R_j^*\varphi(y)$, and

$$|R_j^*\varphi(y)| \le C\left(\|\varphi\|_{L^2(\mathbb{R}^d)} + \left\|\frac{\partial}{\partial x_j}\varphi\right\|_{L^\infty(\mathbb{R}^d)}\right).$$

For fixed $Q \in \mathcal{Q}$ and $0 < \varepsilon < 1$, let

$$R_{j,Q,0}^\varepsilon(x,y) = \begin{cases} \int_\varepsilon^{d(Q)^2} \frac{\partial}{\partial x_j}T_t(x,y)\frac{dt}{\sqrt{t}} & \text{if } \varepsilon < d(Q)^2 < 1/\varepsilon; \\ \int_\varepsilon^{1/\varepsilon} \frac{\partial}{\partial x_j}T_t(x,y)\frac{dt}{\sqrt{t}} & \text{if } d(Q)^2 \ge 1/\varepsilon; \\ 0 & \text{if } d(Q)^2 \le \varepsilon; \end{cases}$$

$$R_{j,Q,\infty}^\varepsilon(x,y) = \begin{cases} \int_{d(Q)^2}^{1/\varepsilon} \frac{\partial}{\partial x_j}T_t(x,y)\frac{dt}{\sqrt{t}} & \text{if } \varepsilon < d(Q)^2 < 1/\varepsilon; \\ 0 & \text{if } d(Q)^2 \ge 1/\varepsilon; \\ \int_\varepsilon^{1/\varepsilon} \frac{\partial}{\partial x_j}T_t(x,y)\frac{dt}{\sqrt{t}} & \text{if } d(Q)^2 \le \varepsilon. \end{cases}$$

Clearly, $R_j^\varepsilon(x,y) = R_{j,Q,0}^\varepsilon(x,y) + R_{j,Q,\infty}^\varepsilon(x,y)$ for every $Q \in \mathcal{Q}$ and $0 < \varepsilon < 1$. For $f \in L^1(\mathbb{R}^d)$ denote

$$R_{j,Q,0}f(x) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^d} R_{j,Q,0}^\varepsilon(x,y)f(y)\,dy,$$

$$R_{j,Q,\infty}f(x) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^d} R_{j,Q,\infty}^\varepsilon(x,y)f(y)\,dy,$$

which of course exist in the sense of distributions.

For $Q \in \mathcal{Q}$ we define

$$\mathcal{Q}'(Q) = \left\{Q' \in \mathcal{Q} : Q^{***} \cap \left(Q'\right)^{***} \ne \emptyset\right\},$$

$$\mathcal{Q}''(Q) = \left\{Q'' \in \mathcal{Q} : Q^{***} \cap \left(Q''\right)^{***} = \emptyset\right\}.$$

Lemma 2.5 *Assume* (D) *holds. Then there exists a constant* $C>0$ *such that for every* $Q\in\mathcal{Q}$ *we have*

$$\int_{\mathbb{R}^d}\sup_{0<\varepsilon<1}|R^{\varepsilon}_{j,Q,\infty}(x,y)|\,dx\le C\quad \textit{for } y\in\bigcup_{Q'\in\mathcal{Q}'(Q)}Q'^*. \tag{2.6}$$

Proof Fix $y\in\bigcup_{Q'\in\mathcal{Q}'(Q)}Q'^*$. Let $Q'\in\mathcal{Q}'(Q)$ be such that $y\in Q'^*$. Denote by S the left-hand side of Eq. 2.6. Then

$$\begin{aligned}S&\le\int_{\mathbb{R}^d}\int_{\min(d(Q),\,d(Q'))^2}^{d(Q')^2}\Big|\frac{\partial}{\partial x_j}T_t(x,y)\Big|\frac{dt}{\sqrt{t}}dx+\int_{\mathbb{R}^d}\int_{d(Q')^2}^{\infty}\Big|\frac{\partial}{\partial x_j}T_t(x,y)\Big|\frac{dt}{\sqrt{t}}dx\\&=S_1+S_2.\end{aligned}$$

Recall that $d(Q)\sim d(Q')$. Using Eq. 2.3, we get

$$S_1\le C\int_{\min(d(Q),\,d(Q'))^2}^{d(Q')^2}t^{-1}\,dt\le C.$$

Applying Eq. 2.3 and (D), we obtain

$$\begin{aligned}S_2&=\sum_{n=0}^{\infty}\int_{\mathbb{R}^d}\int_{2^nd(Q')^2}^{2^{n+1}d(Q')^2}\Big|\frac{\partial}{\partial x_j}T_t(x,y)\Big|\frac{dt}{\sqrt{t}}dx\\&\le C\sum_{n=0}^{\infty}\int\int_{2^nd(Q')^2}^{2^{n+1}d(Q')^2}\int\Big|\frac{\partial}{\partial x_j}T_{t-2^{n-1}d(Q')^2}(x,z)\Big|T_{2^{n-1}d(Q')^2}(z,y)\,dz\frac{dt}{\sqrt{t}}dx\\&\le C\sum_{n=0}^{\infty}\int_{2^nd(Q')^2}^{2^{n+1}d(Q')^2}\int_{\mathbb{R}^d}(2^nd(Q')^2)^{-1/2}T_{2^{n-1}d(Q')^2}(z,y)\,dz\frac{dt}{\sqrt{t}}\\&\le C\sum_{n=0}^{\infty}\int_{\mathbb{R}^d}T_{2^{n-1}d(Q')^2}(z,y)\,dz\le C+C\sum_{n=1}^{\infty}n^{-1-\varepsilon}\le C,\end{aligned}$$

and the lemma is proved. □

For $0\le\varepsilon<d(Q)^2$ let

$$W^{\varepsilon}_{j,Q}(x,y)=\int_{\varepsilon}^{d(Q)^2}\frac{\partial}{\partial x_j}\big(T_t(x,y)-P_t(x-y)\big)\frac{dt}{\sqrt{t}}. \tag{2.7}$$

Set $W^{\varepsilon}_{j,Q}f(x)=\int W^{\varepsilon}_{j,Q}(x,y)f(y)\,dy$, $W_{j,Q}f=W^0_{j,Q}f$.

Lemma 2.8 *Assuming* (K) *there exists a constant* $C>0$ *such that for every* $Q\in\mathcal{Q}$ *one has*

$$\sup_{y\in Q^*}\int_{\mathbb{R}^d}\int_0^{d(Q)^2}\Big|\frac{\partial}{\partial x_j}\big(T_t(x,y)-P_t(x,y)\big)\Big|\frac{dt}{\sqrt{t}}\,dx\le C.$$

Proof The proof borrows some ideas from [1, Lemma 2.3]. Fix $j \in \{1, \dots, d\}$ and denote

$$J_Q(x,y) = \int_0^{d(Q)^2} \left| \frac{\partial}{\partial x_j}\big(T_t(x,y) - P_t(x,y)\big)\right| \frac{dt}{\sqrt{t}}.$$

The perturbation formula asserts that

$$T_t - P_t = -\int_0^t P_{t-s} V T_s \, ds.$$

Therefore

$$\begin{aligned} J_Q(x,y) &\le \int_0^{d(Q)^2} \int_0^{t/2} \int_{\mathbb{R}^d} \left| \frac{\partial}{\partial x_j} P_{t-s}(x-z)\right| V_1(z) T_s(z,y)\, dz\, ds \frac{dt}{\sqrt{t}} \\ &\quad + \int_0^{d(Q)^2} \int_{t/2}^{t} \int_{\mathbb{R}^d} \left| \frac{\partial}{\partial x_j} P_{t-s}(x-z)\right| V_1(z) T_s(z,y)\, dz\, ds \frac{dt}{\sqrt{t}} \\ &\quad + \int_0^{d(Q)^2} \int_0^{t} \int_{\mathbb{R}^d} \left| \frac{\partial}{\partial x_j} P_{t-s}(x-z)\right| V_2(z) T_s(z,y)\, dz\, ds \frac{dt}{\sqrt{t}} \\ &= J_1'(x,y) + J_1''(x,y) + J_2(x,y), \end{aligned}$$

where $V_1(x) = V(x)\mathbf{1}_{Q^{***}}$, $V_2(x) = V(x) - V_1(x)$.

To evaluate J_1' observe that

$$\int_{\mathbb{R}^d} \left| \frac{\partial}{\partial x_j} P_{t-s}(x-y)\right| dx \le C t^{-1/2} \quad \text{for } 0 < s < t/2.$$

Thus, using (K), we get

$$\begin{aligned} \int_{Q^{**}} J_1'(x,y)\, dx &\le C \int_0^{d(Q)^2} \int_0^{t/2} \int_{\mathbb{R}^d} t^{-1/2} V_1(z) P_s(z-y)\, dz\, ds \frac{dt}{\sqrt{t}} \\ &\le C \int_0^{d(Q)^2} t^{-1/2} \left(\frac{t}{d(Q)^2}\right)^{\delta} \frac{dt}{\sqrt{t}} \le C. \end{aligned}$$

Similarly,

$$\begin{aligned} \int_{Q^{**}} J_1''(x,y)\, dx &\le C \int_0^{d(Q)^2} \int_{t/2}^{t} \int_{\mathbb{R}^d} (t-s)^{-1/2} V_1(z) P_t(z-y)\, dz\, ds \frac{dt}{\sqrt{t}} \\ &= C' \int_0^{d(Q)^2} \int_{\mathbb{R}^d} V_1(z) P_t(z-y)\, dz\, dt \le C. \end{aligned}$$

In order to estimate J_2 we notice that

$$\left| \frac{\partial}{\partial x_j} P_{t-s}(x-z)\right| \le C d(Q)^{-d-1} e^{-c(|x-z|/d(Q))^2} \tag{2.9}$$

for $0 < s < t < d(Q)^2,\ z \notin Q^{***},\ x \in Q^{**}$. Lemma 3.10 of [4] asserts that

$$\sup_{y\in\mathbb{R}^d}\int_0^\infty\int_{\mathbb{R}^d} V(z)T_s(z,y)\,dz\,ds \le C.$$

Hence, by Eq. 2.9, we obtain

$$\begin{aligned}\int_{Q^{**}} J_2(x,y)\,dx &\le C\,d(Q)^{-1}\int_0^{d(Q)^2}\int_0^t\int_{\mathbb{R}^d} V_2(z)T_s(z,y)\,dz\,ds\,\frac{dt}{\sqrt{t}}\\ &\le C\,d(Q)^{-1}\int_0^{d(Q)^2}\int_0^t\int_{\mathbb{R}^d} V(z)T_s(z,y)\,dz\,ds\,\frac{dt}{\sqrt{t}}\\ &\le Cd(Q)^{-1}\int_0^{d(Q)^2}\frac{dt}{\sqrt{t}} \le C.\end{aligned}$$

We now turn to estimate $J_Q(x,y)$ for $x \notin Q^{**}$ and $y \in Q^*$. Clearly,

$$\begin{aligned}\int_{Q^{**c}} J_Q(x,y)\,dx \le &\int_{Q^{**c}}\int_0^{d(Q)^2}\left|\frac{\partial}{\partial x_j}T_t(x,y)\right|\frac{dt}{\sqrt{t}}\,dx\\ &+\int_{Q^{**c}}\int_0^{d(Q)^2}\left|\frac{\partial}{\partial x_j}P_t(x-y)\right|\frac{dt}{\sqrt{t}}\,dx = \mathcal{J}'_Q+\mathcal{J}''_Q.\end{aligned}$$

Using Eq. 2.2 combined with the Cauchy-Schwarz inequality we get

$$\begin{aligned}\mathcal{J}'_Q &\le \int_0^{d(Q)^2}\left(\int_{Q^{**c}}\left|\frac{\partial}{\partial x_j}T_t(x,y)\right|^2 e^{2\frac{|x-y|}{\sqrt{t}}}\,dx\right)^{1/2}\left(\int_{Q^{**c}} e^{-2\frac{|x-y|}{\sqrt{t}}}\,dx\right)^{1/2}\frac{dt}{\sqrt{t}}\\ &\le C\int_0^{d(Q)^2} t^{-d/4-1/2}\left(\int_{Q^{**c}}\left(\frac{\sqrt{t}}{|x-y|}\right)^N dx\right)^{1/2}\frac{dt}{\sqrt{t}} \le C. \qquad (2.10)\end{aligned}$$

The estimates for $\mathcal{J}''_Q$ go in the same way. Hence

$$\sup_{y\in Q^*}\int_{Q^{**c}} J_Q(x,y)\,dx \le C,$$

which completes the proof of Lemma 2.8. □

Let $\{\phi_Q\}_{Q\in\mathcal{Q}}$ be a family of smooth functions that form a resolution of identity associated with $\{Q^*\}_{Q\in\mathcal{Q}}$, that is, $\phi_Q \in C_c^\infty(Q^*)$, $0 \le \phi_Q \le 1$, $|\nabla\phi_Q(x)| \le Cd(Q)^{-1}$, $\sum_{Q\in\mathcal{Q}}\phi_Q(x) = 1$ a.e.

The following corollary follows easily from Lemma 2.8.

Corollary 2.11 *For $f \in L^1(\mathbb{R}^d)$ we have*

$$\lim_{\varepsilon\to 0}\|W^{\varepsilon}_{j,Q}(\phi_Q f) - W_{j,Q}(\phi_Q f)\|_{L^1(\mathbb{R}^d)} = 0,$$

$$\|W_{j,Q}(\phi_Q f)\|_{L^1(\mathbb{R}^d)} \le C\|\phi_Q f\|_{L^1(\mathbb{R}^d)}$$

with C independent of Q and f.

Lemma 2.12 *There exists a constant* $C > 0$ *such that for every* $Q \in \mathcal{Q}$ *and every* $f \in L^1(\mathbb{R}^d)$ *such that supp* $f \subset \tilde{Q} = \bigcup_{Q' \in \mathcal{Q}'(Q)} Q'^*$ *we have*

$$\|R_j(\phi_Q f) - \phi_Q R_j f\|_{L^1(\mathbb{R}^d)} \leq C\|f\|_{L^1(\tilde{Q})}. \tag{2.13}$$

Proof Note that

$$\begin{aligned} &R_j(\phi_Q f)(x) - \phi_Q(x) R_j f(x) \\ &= \lim_{\varepsilon \to 0} \int_\varepsilon^{1/\varepsilon} \int \left(\frac{\partial}{\partial x_j} T_t(x, y)\right) \left(\phi_Q(y) - \phi_Q(x)\right) f(y)\, dy \frac{dt}{\sqrt{t}}. \end{aligned}$$

From Eq. 2.2 we conclude

$$\begin{aligned} &\int_{\mathbb{R}^d} \int_0^{d(Q)^2} \left| \left(\frac{\partial}{\partial x_j} T_t(x, y)\right) \left(\phi_Q(y) - \phi_Q(x)\right) \right| \frac{dt}{\sqrt{t}}\, dx \\ &\leq \frac{C}{d(Q)} \int_{\mathbb{R}^d} \int_0^{d(Q)^2} \left| \frac{\partial}{\partial x_j} T_t(x, y) \right| \frac{|x - y|}{\sqrt{t}}\, dt\, dx \\ &\leq \frac{C}{d(Q)} \int_{\mathbb{R}^d} \int_0^{d(Q)^2} \left| \frac{\partial}{\partial x_j} T_t(x, y) \right| e^{|x-y|/\sqrt{t}}\, dt\, dx \leq C. \end{aligned} \tag{2.14}$$

Now Eq. 2.13 follows from Eqs. 2.6 and 2.14. □

The following lemma is motivated by [4, Lemma 3.8].

Lemma 2.15 *There exists a constant* $C > 0$ *such that*

$$\sum_{Q \in \mathcal{Q}} \left\| \mathbf{1}_{Q^{***}} R_j \Big(\sum_{Q'' \in \mathcal{Q}''(Q)} \phi_{Q''} f \Big) \right\|_{L^1(\mathbb{R}^d)} \leq C\|f\|_{L^1(\mathbb{R}^d)}. \tag{2.16}$$

Proof Let S denote the left-hand side of Eq. 2.16. Applying Eq. 1.2, we have

$$\begin{aligned} S &\leq \sum_{Q \in \mathcal{Q}} \sum_{Q'' \in \mathcal{Q}''(Q)} \left\| \mathbf{1}_{Q^{***}} R_j(\phi_{Q''} f) \right\|_{L^1(\mathbb{R}^d)} \\ &= \sum_{Q'' \in \mathcal{Q}} \sum_{Q \in \mathcal{Q}''(Q'')} \ldots \leq C \sum_{Q'' \in \mathcal{Q}} \left\| R_j(\phi_{Q''} f) \right\|_{L^1((Q'')^{**c})} \\ &\leq C \sum_{Q'' \in \mathcal{Q}} \left(\left\| R_{j,Q'',0}(\phi_{Q''} f) \right\|_{L^1((Q'')^{**c})} + \left\| R_{j,Q'',\infty}(\phi_{Q''} f) \right\|_{L^1((Q'')^{**c})} \right). \end{aligned} \tag{2.17}$$

Using Eqs. 2.6 and 1.2, we get

$$\sum_{Q'' \in \mathcal{Q}} \left\| R_{j,Q'',\infty}(\phi_{Q''} f) \right\|_{L^1((Q'')^{**c})} \leq C \sum_{Q'' \in \mathcal{Q}} \|\phi_{Q''} f\|_{L^1(\mathbb{R}^d)} \leq C'\|f\|_{L^1(\mathbb{R}^d)}. \tag{2.18}$$

Similarly to Eq. 2.10, for $y \in (Q'')^*$, we have

$$\int_{(Q'')^{**c}} \int_0^{d(Q'')^2} \left| \frac{\partial}{\partial x_j} T_t(x, y) \right| \frac{dt}{\sqrt{t}}\, dx \leq C,$$

which implies

$$\sum_{Q''\in\mathcal{Q}} \left\| R_{j,Q'',0}(\phi_{Q''} f)\right\|_{L^1((Q'')^{**c})} \leq C \sum_{Q''\in\mathcal{Q}} \|\phi_{Q''} f\|_{L^1(\mathbb{R}^d)} \leq C\|f\|_{L^1(\mathbb{R}^d)}. \tag{2.19}$$

The lemma is a consequence of Eqs. 2.17–2.19. □

3 Proof of Theorem 1.5

In order to prove the second inequality of Eq. 1.6 it suffices by Eqs. 2.4 and 1.4 to verify that there exists a constat $C > 0$ such that

$$\|R_j a\|_{L^1(\mathbb{R}^d)} \leq C \tag{3.1}$$

for every $H^1_{\mathcal{Q}}$-atom a and $j = 1, \ldots, d$. Assume that a is an $H^1_{\mathcal{Q}}$-atom supported by a cube Q^*, $Q \in \mathcal{Q}$. Then

$$\begin{aligned} R_j a(x) &= \lim_{\varepsilon\to 0} \left(R^{\varepsilon}_{j,Q,0} a(x) + R^{\varepsilon}_{j,Q,\infty} a(x)\right) \\ &= \lim_{\varepsilon\to 0} \left(W^{\varepsilon}_{j,Q} a(x) + H^{\varepsilon}_{j,Q} a(x) + R^{\varepsilon}_{j,Q,\infty} a(x)\right), \end{aligned}$$

where $H^{\varepsilon}_{j,Q} a(x) = \int_{\varepsilon}^{d(Q)^2} \frac{\partial}{\partial x_j}(a * P_t)(x)\frac{dt}{\sqrt{t}}$. Similarly to Eq. 2.4, the limit

$$H_{j,Q} a(x) = \lim_{\varepsilon\to 0} H^{\varepsilon}_{j,Q} a(x)$$

exists in the sense of distributions. Moreover, by the boundedness of the local Riesz transforms on the local Hardy spaces (see [7]), we have $\|H_{j,Q} a\|_{L^1(\mathbb{R}^d)} \leq C$ with C independent of a. Using Lemmas 2.5 and 2.8, we obtain Eq. 3.1.

We now turn to prove the first inequality of Eq. 1.6. To this end, by the local Riesz transform characterization of the local Hardy spaces (see [7, Section 2]), it suffices to show that

$$\sum_{Q\in\mathcal{Q}} \|H_{j,Q}(\phi_Q f)\|_{L^1(Q^{**})} \leq C\left(\|f\|_{L^1(\mathbb{R}^d)} + \|R_j f\|_{L^1(\mathbb{R}^d)}\right), \quad j = 1, ..., d. \tag{3.2}$$

Clearly,

$$H_{j,Q}(\phi_Q f) = -W_{j,Q}(\phi_Q f) + R_{j,Q,0}(\phi_Q f).$$

Lemma 2.8 together with Eq. 1.2 implies

$$\sum_{Q\in\mathcal{Q}} \|W_{j,Q}(\phi_Q f)\|_{L^1(\mathbb{R}^d)} \leq C \sum_{Q\in\mathcal{Q}} \|\phi_Q f\|_{L^1(\mathbb{R}^d)} \leq C\|f\|_{L^1(\mathbb{R}^d)}. \tag{3.3}$$

Note that

$$\begin{aligned} R_{j,Q,0}(\phi_Q f) = &\left[R_j\left(\phi_Q \sum_{Q'\in\mathcal{Q}'(Q)} (\phi_{Q'} f)\right) - \phi_Q R_j\left(\sum_{Q'\in\mathcal{Q}'(Q)} (\phi_{Q'} f)\right)\right] \\ &- R_{j,Q,\infty}(\phi_Q f) + \phi_Q R_j f - \phi_Q R_j\left(\sum_{Q''\in\mathcal{Q}''(Q)} (\phi_{Q''} f)\right). \end{aligned} \tag{3.4}$$

Lemmas 2.12, 2.5, and 2.15 combined with Eq. 3.4 imply

$$\begin{aligned}
&\sum_{Q\in\mathcal{Q}} \|R_{j,Q,0}(\phi_Q f)\|_{L^1(Q^{**})} \\
&\quad \le C\sum_{Q\in\mathcal{Q}}\left(\sum_{Q'\in\mathcal{Q}'(Q)} \|\phi_{Q'} f\|_{L^1(\mathbb{R}^d)} + \|\phi_Q f\|_{L^1(\mathbb{R}^d)} + \|\phi_Q R_j f\|_{L^1(\mathbb{R}^d)}\right) + C\|f\|_{L^1(\mathbb{R}^d)} \\
&\quad \le C\left(\|f\|_{L^1(\mathbb{R}^d)} + \|R_j f\|_{L^1(\mathbb{R}^d)}\right). \qquad (3.5)
\end{aligned}$$

Now Eq. 3.2 follows from Eqs. 3.3 and 3.5.

4 Proof of Lemma 2.1

The proof is based on estimates of the semigroup T_t acting on weighted L^2 spaces. This technique was utilize e.g. in [3, 5, 8].

Fix $y_0 \in \mathbb{R}^d$ and $\alpha > 0$. The semigroup $\{T_t\}_{t>0}$ acting on $L^2(e^{\alpha|x-y_0|}dx)$ has the unique extension to a holomorphic semigroup T_ζ, $\zeta \in \{\zeta \in \mathbb{C} : |\mathrm{Arg}\,\zeta| < \pi/4\}$ such that

$$\|T_\zeta\|_{L^2(e^{\alpha|x-y_0|}dx)\to L^2(e^{\alpha|x-y_0|}dx)} \le Ce^{c'\alpha^2\Re\zeta} \qquad (4.1)$$

with C and c' independent of V and y_0 (see, e.g., [3, Section 6]). Let $-\mathcal{L}_\alpha$ denote the infinitesimal generator of $\{T_t\}_{t>0}$ considered on $L^2(e^{\alpha|x-y_0|}dx)$. The quadratic form $\mathbf{Q} = \mathbf{Q}_{\alpha,y_0}$ associated with $\mathcal{L}_\alpha$ is given by

$$\begin{aligned}
\mathbf{Q}(f,g) &= \sum_{j=1}^d \int_{\mathbb{R}^d} \frac{\partial}{\partial x_j} f(x) \frac{\partial}{\partial x_j}\overline{g(x)} e^{\alpha|x-y_0|}\,dx + \int_{\mathbb{R}^d} V(x) f(x)\overline{g(x)} e^{\alpha|x-y_0|}\,dx \\
&\quad + \sum_{j=1}^d \int_{\mathbb{R}^d} f(x)\overline{g(x)} \frac{\partial}{\partial x_j} e^{\alpha|x-y_0|}\,dx,
\end{aligned}$$

$$D(\mathbf{Q}) = \{f:\ f(x),\ V(x)^{1/2} f(x),\ \frac{\partial}{\partial x_j} f(x) \in L^2(e^{\alpha|x-y_0|}dx),\ j=1,...,d\}.$$

Note that

$$\left|\frac{\partial}{\partial x_j} e^{\alpha|x-y_0|}\right| \le C\alpha e^{\alpha|x-y_0|} \quad \text{for } x \ne y_0.$$

Clearly,

$$|\mathbf{Q}(f,g)| \le C_\alpha \|f\|_{\mathbf{Q}} \|g\|_{\mathbf{Q}}$$

with C_α independent of y_0 and V, where

$$\|f\|_{\mathbf{Q}}^2 = \int_{\mathbf{R}^d} \left(\sum_{j=1}^d \left|\frac{\partial}{\partial x_j} f(x)\right|^2 + V(x)|f(x)|^2 + |f(x)|^2\right) e^{\alpha|x-y_0|}dx.$$

Moreover, there exists a constant $C > 0$ independent of V and y_0 such that

$$\|f\|_{\mathbf{Q}}^2 \le C\mathbf{Q}(f, f). \tag{4.2}$$

The holomorphy of the semigroup T_t combined with Eq. 4.1 imply

$$\|\mathcal{L}_\alpha T_t g\|_{L^2(e^{\alpha|x-y|}dx)} \le C' t^{-1} e^{c'' t\alpha^2} \|g\|_{L^2(e^{\alpha|x-y_0|}dx)} \tag{4.3}$$

with constants C' and c'' independent of V and y_0. Setting $g(x) = T_{1/2}(x, y_0)$, $f(x) = T_{1/2}g(x) = T_1(x, y_0)$ and using Eqs. 4.2, 4.3, 4.1, and 1.1, we get

$$\begin{aligned} \left\| \frac{\partial}{\partial x_j} T_1(x, y_0) \right\|_{L^2(e^{\alpha|x-y_0|}dx)}^2 &\le \|f\|_{\mathbf{Q}}^2 \\ &\le C\mathbf{Q}(f, f) \\ &\le C\|\mathcal{L}_\alpha f\|_{L^2(e^{\alpha|x-y_0|}dx)} \|f\|_{L^2(e^{\alpha|x-y_0|}dx)} \\ &\le C''\|g\|_{L^2(e^{\alpha|x-y_0|}dx)}^2 \le C''' \end{aligned} \tag{4.4}$$

with C''' independent of y_0 and V. Since $T_t(x, y) = t^{-d/2}\tilde{T}_1(x/\sqrt{t}, y/\sqrt{t})$, where $\{\tilde{T}_s\}_{s>0}$ is the semigroup generated by $\Delta - tV(\sqrt{t}x)$, we get Eq. 2.2 from Eq. 4.4, because C''' is independent of V and y_0. Now Eq. 2.3 follows from Eq. 2.2 and the Cauchy-Schwarz inequality.

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