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Short note on some geometric inequalities derived from matrix inequalities

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Abstract
Using the connection between ellipsoids and positive semidefinite matrices we provide alternative proofs to some recently proven inequalities concerning the volume of L_2 zonoids as consequences of classical inequalities for matrices.

Keywords Positive semidefinite matrices · Determinant · Ellipsoids · Projection · L_2 zonoids

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1 Introduction and background

Many connections exist between the theory of matrices and the theory of convex bodies. Within the realm of convex geometry, the fundamental Brunn–Minkowski inequality [14], [22, Section 7.1], and the Aleksandrov–Fenchel inequality [22, Section 7.3] have versions for positive semidefinite matrices. Appropriate analogue versions of the classical Bergstrom and Ky-Fan inequalities within the theory of matrices have been studied for convex bodies, as well as inequalities for mixed volumes have been investigated in the context of mixed discriminants (see e.g. [1, 12, 13, 15, 19]). The L_2 Brunn–Minkowski theory of convex bodies provides us with a correspondence

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between positive semidefinite matrices and ellipsoids, which takes the usual sum of matrices to the so-called L_2 sum of ellipsoids into account.

This note aims to observe that some recently obtained inequalities for the particular family of convex bodies known as L_2 zonoids correspond to classical matrix inequalities. On the other hand, a characterization result included in (one of) the mentioned recently proven geometrical inequalities allows to establish directly the equality conditions in one of the classical matrix inequalities, which, to the best of the authors' knowledge, seem not to have been explicitly documented in the literature.

Let $\mathcal{M}^n$ be the vector space of real square symmetric matrices, and let the set of positive semidefinite symmetric ones be denoted by $\mathcal{S}^n_+$. It is well-known (see e.g. [18]) that $\mathcal{S}^n_+$ is a closed, convex cone. Next, we recall the Bergstrom and Ky Fan classical matrix inequalities.

Theorem 1.1 (Bergstrom's inequality) [5–7] *Let A and B be two $n \times n$ positive definite real symmetric matrices, and denote by A_i and B_i the two $(n-1) \times (n-1)$ matrices resulting from A and B by deleting the i-th row and the i-th column. Then we have*

$$\frac{\det(A+B)}{\det(A_i+B_i)} \geq \frac{\det(A)}{\det(A_i)} + \frac{\det(B)}{\det(B_i)}, \tag{1.1}$$

for every $i \in \{1, \ldots, n\}$.

Theorem 1.2 (Ky Fan's inequality) [5, 10] *Let A and B be two $n \times n$ positive definite real symmetric matrices, and denote by $A_{(k)}$ and $B_{(k)}$ the principal $k \times k$ matrices of A and B obtained by taking the first k rows and k columns from A and B, respectively. Then we have*

$$\left(\frac{\det(A+B)}{\det(A_{(k)}+B_{(k)})}\right)^{\frac{1}{n-k}} \geq \frac{\det^{\frac{1}{n-k}}(A)}{\det^{\frac{1}{n-k}}(A_{(k)})} + \frac{\det^{\frac{1}{n-k}}(B)}{\det^{\frac{1}{n-k}}(B_{(k)})}, \tag{1.2}$$

for every $k \in \{1, \ldots, n-1\}$.

Inequalities (1.1) and (1.2) have motivated a number of questions concerning quotients of sums of quermassintegrals of convex bodies, in particular, volume and surface area, see e.g. [12, 13, 15], and the references therein.

The next result is Brunn–Minkowski's (or Minkowski's) inequality for positive semidefinite symmetric matrices.

Theorem 1.3 [19, Theorem 7.8.21] *Let $A, B \in \mathcal{M}^n$ be positive definite matrices. Then*

$$\det((1-\lambda)A + \lambda B)^{1/n} \geq (1-\lambda)\det(A)^{1/n} + \lambda \det(B)^{1/n} \tag{1.3}$$

for any $\lambda \in [0, 1]$, with equality if and only if $A = cB$, for some $c > 0$.

Next, we state the geometric analogue of the latter, namely, the Brunn–Minkowski inequality for the volume of convex bodies. For that, we first need to introduce some further notation.

In the n-dimensional Euclidean space $\mathbb{R}^n$, endowed with the standard inner product $\langle\cdot,\cdot\rangle$ and the associated Euclidean norm $||\cdot||$, we denote by $\mathcal{K}^n$ the set of all convex bodies, i.e., compact convex sets in $\mathbb{R}^n$. We set B_n to be the n-dimensional unit ball, and $\mathbb{S}^{n-1}$ to be its boundary, the unit sphere of $\mathbb{R}^n$. Let $1 \le k \le n$, we denote by $\mathcal{L}_k^n$ the set of all k-dimensional linear subspace of $\mathbb{R}^n$. If $L \in \mathcal{L}_k^n$ and $K \in \mathcal{K}^n$, then we denote by $P_L(K) \subset L$ the orthogonal projection of K onto L, which is also a convex body. The subset of $\mathcal{K}^n$ consisting of all convex bodies containing the origin will be denoted by $\mathcal{K}_0^n$, meanwhile, the subset of $\mathcal{K}_0^n$ consisting of all convex bodies containing the origin in their interior is denoted by $\mathcal{K}_{(o)}^n$. The volume of a measurable set $M \subsetneq \mathbb{R}^n$, i.e., its n-dimensional Lebesgue measure, is denoted by $\mathrm{vol}_n(M)$. If M is contained in a k-dimensional affine subspace of $\mathbb{R}^n$, we will write $\mathrm{vol}_k(K)$ to denote its k-dimensional volume.

The Minkowski sum of the convex bodies K, H is defined as $K + H := \{x + y : x \in K, y \in H\}$. Moreover, if $\alpha \ge 0$, then $\alpha K := \{\alpha x : x \in K\}$. For every $K, H \in \mathcal{K}^n$ and $\alpha, \beta \ge 0$, we have that $\alpha K + \beta H$ is again a convex body.

The Brunn–Minkowski inequality provides us with the concavity of the n-th root of the volume with respect to the Minkowski sum. The Brunn–Minkowski inequality is the content of the following theorem, which is a cornerstone of the classical Brunn–Minkowski theory.

Theorem 1.4 [22, Theorem 7.1.1] *Let $K, H \in \mathcal{K}^n$ be two convex bodies. Then, for $\lambda \in [0,1]$*

$$\mathrm{vol}_n\big((1-\lambda)K + \lambda H\big)^{1/n} \ge (1-\lambda)\mathrm{vol}_n(K)^{1/n} + \lambda \mathrm{vol}_n(H)^{1/n}. \tag{1.4}$$

Equality for some $\lambda \in (0,1)$ holds if and only if K and H either lie in parallel hyperplanes or are homothetic.

We refer the interested reader to [14, 22] for details and a wealth of results and contributions to the Brunn–Minkowski theory. Considering other additions of convex bodies, other than the vectorial one, far-reaching extensions of the classical Brunn–Minkowski theory have emerged [22, Chapter 9]. An example of those is the L_p Brunn–Minkowski theory.

The mentioned L_2 sum of convex bodies -which contain the origin- is just a particular case of the more general L_p sum, defined via the support function of a convex body, within the L_p Brunn–Minkowski theory. With the aim of introducing the latter precisely, we need some further background on the theory of convex bodies.

Given a convex body $K \in \mathcal{K}^n$, the support function of K in the direction $x \in \mathbb{R}^n$ is defined as $h(K,x) = \max\{\langle x, y\rangle : y \in K\}$, and it describes K uniquely. If $f : \mathbb{R}^n \to \mathbb{R}$ is a positively 1-homogeneous and sub-additive function in $\mathbb{R}^n$, then there exists a unique convex body $K \in \mathcal{K}^n$, such that $f(x) = h(K,x)$, for every $x \in \mathbb{R}^n$, see [22, Theorem 1.7.1].

Now, for convex bodies K, H containing the origin, the latter ensures that the p-mean of the support functions of K and H provides us with the support function of a new convex body, $K +_p L$, called the L_p sum of K and H. More precisely, for $x \in \mathbb{R}^n$

and $K, H \in \mathcal{K}_0^n$ (see [22, Chapter 9]), the function $f : \mathbb{R}^n \to \mathbb{R}$, given by

$$f(x) := (h(K, x)^p + h(H, x)^p)^{\frac{1}{p}} \tag{1.5}$$

is the support function of $K +_p H$.

Observe that the origin belongs to $K \in \mathcal{K}^n$, i.e., $K \in \mathcal{K}_0^n$, if and only if $h(K, \cdot) \geq 0$. Let furthermore $\lambda \cdot_p K := \lambda^{1/p} K$, i.e., $h^p(\lambda \cdot_p K, x) = \lambda h^p(K, x)$, for $x \in \mathbb{R}^n$. With this notation, the following inequality is known as the L_p Brunn–Minkowski inequality. Although we will only use it in the case $p = 2$, we establish it in the general case $p \geq 1$, for completeness. For $p = 1$ we recover the Brunn–Minkowski inequality above.

Theorem 1.5 (L_p Brunn–Minkowski) [22, Corollary 9.1.5] *Let $K, H \in \mathcal{K}_{(0)}^n$ be two convex bodies containing the origin. Then*

$$\mathrm{vol}_n\big((1-\lambda) \cdot_p K +_p \lambda \cdot_p M\big)^{\frac{p}{n}} \geq (1-\lambda)\mathrm{vol}_n(K)^{\frac{p}{n}} + \lambda \mathrm{vol}_n(H)^{\frac{p}{n}}, \tag{1.6}$$

for $\lambda \in [0, 1]$ and $p \geq 1$. Equality holds if and only if K and H are dilates of each other.

When $p = 2$, using the explicit expression of the support function of an ellipsoid, there is a correspondence between positive definite matrices and ellipsoids, which involves the L_2 sum. In the next, we follow [11] to denote $\mathcal{E}^n$ the set of all ellipsoids centered at the origin in $\mathbb{R}^n$, i.e., $E \in \mathcal{E}^n$ if there is a linear map $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, such that $E = T(B_n) =: TB_n$. Indeed, if $E \in \mathcal{E}^n$ and $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear map such that $E = TB_n$, then the support function of E is given by $h(E, x) = h(TB_n, x) = \max\{\langle y, x\rangle : y \in TB_n\} = \max\{\langle Tb, x\rangle : b \in B_n\}$, for $x \in \mathbb{R}^n$. Thus,

$$\begin{aligned} h(E, x) &= \max\{\langle b, T^T x\rangle : b \in B_n\} = h(B_n, T^T x) = ||T^T x|| \\ &= \langle T^T x, T^T x\rangle^{1/2} = \langle x, TT^T x\rangle^{1/2}. \end{aligned}$$

The matrix $A = TT^T$ defines uniquely an element in the space of positive semidefinite real symmetric matrices $\mathcal{S}_+^n$. Therefore, for every $x \in \mathbb{R}^n$, $h^2(E, x) = \langle x, Ax\rangle$. By the latter, a centered ellipsoid $E \in \mathcal{E}^n$ determines uniquely a matrix $A \in \mathcal{S}_+^n$. On the other hand, any matrix $A \in \mathcal{S}_+^n$ determines uniquely a centered ellipsoid $E \in \mathcal{E}^n$ via its support function as follows:

$$h^2(E, x) = \langle x, Ax\rangle, \qquad x \in \mathbb{R}^n. \tag{1.7}$$

In the next, we use the notation E_A for the ellipsoid associated with the matrix $A \in \mathcal{S}_+^n$. In this setting, we have also $\dim(E_A) = \mathrm{rank}(A)$ and

$$\mathrm{vol}_n(E_A) = \kappa_n \sqrt{\det(A)}. \tag{1.8}$$

Moreover, as already mentioned, there is a correspondence of the sum of positive semidefinite matrices to the L_2 sum of ellipsoids, which follows directly from (1.5) and reads

$$E_{(1-\lambda)A+\lambda B} = (1-\lambda)\cdot_2 E_A +_2 \lambda \cdot_2 E_B, \tag{1.9}$$

for every $A, B \in \mathcal{S}^n_+$ and $\lambda \in [0,1]$, where $\lambda \cdot_2 E_A = \sqrt{\lambda} E_A$ (see e.g. [11]).

Our first aim in this note is to use the mentioned correspondence of ellipsoids and positive semidefinite matrices, along with (1.9), to observe that the classical determinantal inequalities of Bergstrom and Ky-Fan provide us directly with alternative proofs of the following two results proven in [12].

For those results, we first need to introduce the notion of L_2 zonoid. An L_p zonotope is the L_p sum of centered segments and an L_p zonoid is the limit of L_p zonotopes, where the space of convex bodies has been endowed with the usual Hausdorff metric (see e.g. [22, Section 1.8]). For $p = 2$, an L_2 zonoid is a centered ellipsoid. We provide a short argument of this fact, for completeness.

Remark 1.6 Let $x_1, \dots, x_n \in \mathbb{R}^n$ be points and $[-x_i, x_i]$ denote the centred segment joining $-x_i$ and x_i. We denote by X the $n \times n$ matrix whose columns are $x_1, \dots, x_n$. Then, the support function of the L_2 sum of these segments, $[-x_1, x_1] +_2 \cdots +_2 [-x_n, x_n]$, according to (1.5) satisfies

$$\begin{aligned} h([-x_1, x_1] +_2 \cdots +_2 [-x_n, x_n], u)^2 &= \sum_{i=1}^n h([-x_i, x_i], u)^2 = \sum_{i=1}^n |\langle x_i, u\rangle|^2 \\ &= \sum_{i=1}^n |\langle X e_i, u\rangle|^2 = \sum_{i=1}^n |\langle e_i, X^T u\rangle|^2 = |X^T u|^2 \\ &= h(UB^n, u)^2, \end{aligned}$$

where X is the matrix having $x_1, \dots, x_n$ as columns and $U = (XX^T)^{1/2}$ is the square root of the positive semidefinite real symmetric matrix XX^T. Hence, $[-x_1, x_1] +_2 \cdots +_2 [-x_n, x_n] = UB^n$, which is an ellipsoid.

In the case that the sum consists of $m \neq n$ segments, the same argument proves that the L_2 sum of centered segments is an ellipsoid. We point out that L_2 zonoids may not have interior points. Indeed, if $E \subset \mathbb{R}^n$ is the sum of $m < n$ centered segments, then it is an L_2 zonotope, and it has clearly empty interior (see [12] and the references therein for further aspects of L_p zonotopes and zonoids in this context).

For a vector $u \in \mathbb{S}^{n-1}$, we denote by $u^\perp$ the hyperplane orthogonal to u, i.e., the $(n-1)$-dimensional linear subspace having u as normal vector. Then, as before, for $K \in \mathcal{K}^n$, the orthogonal projection of K onto $u^\perp$ is denoted by $P_{u^\perp} K$.

Theorem 1.7 [12, Theorem 6.2] *Let K, H be a pair of L_2 zonoids in $\mathbb{R}^n$ and let $u \in \mathbb{S}^{n-1}$. Then*

$$\left(\frac{\mathrm{vol}_n(K +_2 H)}{\mathrm{vol}_{n-1}(P_{u^\perp}(K +_2 H))}\right)^2 \geq \left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_{n-1}(P_{u^\perp}(K))}\right)^2 + \left(\frac{\mathrm{vol}_n(H)}{\mathrm{vol}_{n-1}(P_{u^\perp}(H))}\right)^2, \tag{1.10}$$

with equality if and only if K and H have parallel tangent hyperplanes at $\rho_K(u)u$ *and* $\rho_H(u)u$, *where* $\rho_K(u) = \max\{\lambda : \lambda u \in K\}$ *is the radial function of the body* $K \in \mathcal{K}^n$ *at* $u \in \mathbb{S}^{n-1}$.

The following theorem is a generalization of Theorem 1.7.

Theorem 1.8 [12, Theorem 6.6] *Let n be an integer, then for any* $1 \le k \le n$ *and for every pair of* L^2 *zonoids* K, H *in* $\mathbb{R}^n$ *and any* $(n-k)$*-dimensional subspace* L *of* $\mathbb{R}^n$, *i.e.,* $L \in \mathcal{L}_k^n$, *one has*

$$\left(\frac{\mathrm{vol}_n(K +_2 H)}{\mathrm{vol}_{n-k}(P_L(K +_2 H))}\right)^{\frac{2}{k}} \ge \left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_{n-k}(P_L(K))}\right)^{\frac{2}{k}} + \left(\frac{\mathrm{vol}_n(H)}{\mathrm{vol}_{n-k}(P_L(H))}\right)^{\frac{2}{k}}. \tag{1.11}$$

In [9], Bergstrom and Ky-Fan inequalities are used to obtain *linearized versions* of inequalities within the realm of convex geometry. In particular, they are fundamental to obtain *a linearized version* of the Brunn–Minkowski, and the Aleksandrov–Fenchel inequalities for positive semidefinite matrices, that satisfy certain conditions on the projection onto a subspace. In the last subsection of this note, we investigate connections of some of the results in [9] with other results coming from the context of convex geometry, in the spirit of the previous results.

2 Ellipsoids, positive semidefinite matrices and projections

We start fixing the notation that will be used throughout the paper. Let $A \in \mathcal{M}^n$ be a positive semidefinite matrix, and let $S = \{e_1, \ldots, e_n\}$ denote the standard orthonormal basis of $\mathbb{R}^n$. With some abuse of notation, let $A\colon \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto A \cdot x$, denote the linear map defined by the matrix A, when considered with respect to the standard basis in $\mathbb{R}^n$.

Let L be a linear subspace of $\mathbb{R}^n$, and let B_L, $B_{L^\perp}$, $B_{L,L^\perp} = B_L \dot{\cup} B_{L^\perp}$ be orthonormal bases of L, $L^\perp$, and $\mathbb{R}^n$, respectively. We denote by A_L the matrix of the linear map associated with A, with respect to the bases $B_{L,L^\perp}$.

The inclusion of the subspace L into $\mathbb{R}^n$ will be denoted by $\iota_L\colon L \to \mathbb{R}^n$, and the orthogonal projection of $\mathbb{R}^n$ onto L will be denoted by $P_L\colon \mathbb{R}^n \to L$. The linear map $\iota_L \circ P_L : \mathbb{R}^n \to \mathbb{R}^n$, which *embeds the projection onto* L *into* $\mathbb{R}^n$ will be denoted by P^L.

The following notion of projection of a matrix onto a subspace has been considered in [3, 4], and it is inherited from the definition of the restriction of a quadratic form to a linear subspace (see e.g. [20]).

Let L be a linear subspace of $\mathbb{R}^n$, $A \in \mathcal{M}^n$, and let q_A be the quadratic form on $\mathbb{R}^n$ associated to A, i.e., $q_A(x) = \langle x, Ax\rangle$. The projection of the matrix A onto L is defined as the matrix associated with the restriction of q to the subspace $L \subset \mathbb{R}^n$, and denoted by $P_L(A) \in \mathcal{M}^{\dim L}$. The matrix $P_L(A)$ is well defined, and if A is positive semidefinite, then so is $P_L(A)$.

Proposition 2.1 [3, 4] *Let $A \in \mathcal{M}^n$ be positive semidefinite matrix, and let $L \subseteq \mathbb{R}^n$ be a linear subspace of $\mathbb{R}^n$ of dimension $1 \le k \le n$, i.e., $L \in \mathcal{L}_k^n$. The following statements are equivalent:*

i) *Let $q_A : \mathbb{R}^n \to \mathbb{R}$ be the quadratic form $x \mapsto x^T Ax$. Then, the projection of the matrix A onto the subspace L is the $k \times k$ positive semidefinite real symmetric matrix of the restriction of q_A to the subspace L.*
ii) *The projection of the matrix A onto L is the matrix $P_L(A)$ given by*

$$\begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \end{pmatrix} \cdot A_L \cdot \begin{pmatrix} 1 & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & 1 \\ 0 & \dots & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & 0 \end{pmatrix},$$

where the identity submatrices in the left and right-hand sides are of size $k \times k$. We observe that the left and right matrices in the above product are, indeed, the matrix representations, with respect to B_L, $B_{L^\perp}$, and $B_{L,L^\perp}$, of the inclusion ι_L and the projection P_L. We also remark that $P_L(A)$ is the $k \times k$ principal submatrix of A_L given by the first k columns and rows of A_L.

The next lemma provides us with a connection between the projection of a matrix and its principal submatrices.

Lemma 2.2 [9, Lemma 3.4] *Let $A \in \mathcal{S}_+^n$ be a positive definite matrix and $1 \le i \le n$. Let A_i denote the $(n-1) \times (n-1)$ matrix obtained from A by removing the i-th row and the i-th column, and let $A_{(i)}$ denote the $i \times i$ matrix obtained from A by taking the first i columns and i rows. Then,*

i) $A_i = P_L(A)$ *for* $L = e_i^\perp$,
ii) $A_{(i)} = P_L(A)$ *for* $L = \mathrm{lin}\,\{e_1, \dots, e_i\}$.

The following two remarks will be useful in the next.

Remark 2.3 [3, Proof of Lemma 2.3.1] Let $A \in \mathcal{M}^n$ and let $u \in \mathbb{S}^{n-1}$. Let O be an orthogonal matrix such that $O\,u = e_i$. As O is orthogonal, we also have $O\left(u^\perp\right) = e_i^\perp$. Consequently, $\det\left(P_{u^\perp}(A)\right) = \det\left(P_{e_i^\perp}(OAO^T)\right)$.

In a similar manner, let $L \in \mathcal{L}_k^n$ be a k-dimensional linear subspace of $\mathbb{R}^n$, and let O be an orthonormal matrix such that $O(L) = L_k$, where $L_k = \mathrm{lin}\,\{e_1, \cdots, e_k\}$. As before, the orthonormality of O yields $O(L^\perp) = L_k^\perp$ and thus, $\det\left(P_L(A)\right) = \det\left(P_{L_k}(OAO^T)\right)$.

Next, we will describe the existing connection between the projection of a matrix and the ellipsoids associated with the given matrix, and the projection of that matrix.

For any $A \in \mathcal{S}_+^n$, let $E_A \in \mathcal{E}^n$ be the ellipsoid given by A, i.e., $h(E_A, x)^2 = \langle x, Ax \rangle$, for every $x \in \mathbb{R}^n$.

Let $L \in \mathcal{L}_k^n$. Then the projection of A onto L, i.e., $P_L(A)$, is a $k \times k$ symmetric and positive semidefinite matrix. Therefore, there exists a unique ellipsoid $E_{P_L(A)} \in \mathcal{E}^k$, such that $h(E_{P_L(A)}, x)^2 = \langle x, P_L(A)x \rangle$, for every $x \in L$, where the inner product is taken in L and inherited from $\mathbb{R}^n$.

We consider now the projection of the ellipsoid E_A onto L, i.e., the ellipsoid $\iota_L(P_L(E_A)) \subseteq \mathbb{R}^n$. We recall the notation $P^L = \iota_L \circ P_L$ for the projection onto L embedded in $\mathbb{R}^n$, in contrast to P_L, the projection onto L, where L is the ambient space.

As from the very definition $\iota_L(P_L(E_A)) = P^L(E_A) \subseteq \mathbb{R}^n$ is an ellipsoid in $\mathbb{R}^n$, there exists a unique matrix $C \in \mathcal{S}_+^n$ such that $h(P^L(E_A)), x)^2 = \langle x, Cx \rangle$, for every $x \in \mathbb{R}^n$.

We point out that we need to distinguish the projected ellipsoid as a subset of L, being L a k-dimensional space, and as a subset of (L embedded into) $\mathbb{R}^n$, which explains the introduction of the matrix C, of rank k, defining the projected ellipsoid as a subset of $\mathbb{R}^n$. The next proposition establishes the precise relation between $E_{P_L(A)}$ and E_C.

Proposition 2.4 *Let $A \in \mathcal{S}_+^n$, and let $L \in \mathcal{L}_k^n$. Then*

$$E_{P_L(A)} = P_L(E_A) \in \mathcal{E}^k \text{ and } \iota_L(E_{P_L(A)}) = E_C \in \mathcal{E}^n. \tag{2.1}$$

Further,

$$P_L(C) = P_L(A). \tag{2.2}$$

Proof Let $B_L, B_{L^\perp}, B_{L,L^\perp} = B_L \dot{\cup} B_{L^\perp}$ be orthonormal bases of $L, L^\perp$, and $\mathbb{R}^n$, respectively. Further, let $A \in \mathcal{S}_+^n$, and let $L \in \mathcal{L}_k^n$. We denote by $\iota_L : L \longrightarrow \mathbb{R}^n$ both, the inclusion of L into $\mathbb{R}^n$, and the matrix of it w.r.t. B_L, and $B_{L,L^\perp}$, where we are again making some abuse of notation. It is enough to prove that the support functions of $E_{P_L(A)}$ and $P_L(E_A)$ coincide. For that, observe first that $E_{P_L(A)}, P_L(E_A) \subset L$, and, moreover, by definition, $P_L(A)$ is the $k \times k$ matrix satisfying

$$\langle P_L(A)x, x \rangle = \langle A\iota_L x, \iota_L x \rangle,$$

for every $x \in L$. Thus, for $x \in L$,

$$\begin{aligned} h(E_{P_L(A)}, x)^2 &= \langle P_L(A)x, x \rangle \\ &= \langle A\iota_L x, \iota_L x \rangle = h(E_A, \iota_L x)^2 \\ &= h(P_L(E_A), P_L \iota_L x)^2 \\ &= h(P_L(E_A), x)^2. \end{aligned}$$

For the second equality, we first state the following property of the support function: for $K \in \mathcal{K}^n$ and $L \in \mathcal{L}^n_k$, then for every $x \in L$, we have

$$h(K, x) = h(\iota_L(P_L(K)), x). \tag{2.3}$$

Using $E_{P_L(A)} = P_L(E_A)$ and (2.3), we obtain

$$\begin{aligned} h(\iota_L(E_{P_L(A)}), x)^2 &= h(P_L \iota_L(E_{P_L(A)}, P_L x)^2 = h(E_{P_L(A)}, P_L x)^2 \\ &= h(P_L(E_A), P_L x)^2 = h(\iota_L P_L(E_A), \iota_L P_L x)^2 \\ &= h(E_C, x)^2, \end{aligned}$$

for all $x \in \mathbb{R}^n$. Finally, Eq. (2.2) follows from (2.1). Indeed, as

$$E_C = \iota_L(E_{P_L(A)}) = \iota_L P_L(E_A),$$

projecting onto L yields $P_L(E_C) = P_L \iota_L P_L(E_A) = P_L(E_A)$. Therefore, by the definition of C, one gets $P_L(A) = P_L(C)$. □

3 Main results

3.1 Inequalities for L_2 zonoids via determinantal inequalities

In this section, we provide proofs for Theorems 1.7 and 1.8, alternative to those in [12], based on classical inequalities for matrices. We remark here, that we do not provide a proof of the equality case, stated in [12]. Instead, we use the equality case of Theorem 1.7 proven in [12] to provide Bergstrom's inequality with a characterization of the equality case.

We prove first Theorem 1.7 for the particular case of $u = e_i$, $1 \le i \le n$, i.e., when u is one of the vectors of the orthonormal canonical basis of $\mathbb{R}^n$, as a direct application of the Bergstrom's inequality.

Theorem 3.1 *Let K, H be two L_2 zonoids in $\mathbb{R}^n$, let $1 \le i \le n$, and let e_i, be the i-th vector of the canonical orthonormal basis of $\mathbb{R}^n$. Then,*

$$\left(\frac{\mathrm{vol}_n(K +_2 H)}{\mathrm{vol}_{n-1}(P_{e_i^\perp}(K +_2 H))}\right)^2 \ge \left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_{n-1}(P_{e_i^\perp}(K))}\right)^2 + \left(\frac{\mathrm{vol}_n(H)}{\mathrm{vol}_{n-1}(P_{e_i^\perp}(H))}\right)^2. \tag{3.1}$$

Proof Let K and H be two L_2 zonoids. From Remark 1.6 we know that K and H are two centered ellipsoids. Hence, there exist $A, B \in \mathcal{S}^n_+$ such that $K = E_A$ and $H = E_B$. Using (1.8) we have

$$\mathrm{vol}_n^2(K) = \kappa_n^2 \det(A) \text{ and } \mathrm{vol}_n^2(H) = \kappa_n^2 \det(B).$$

Taking now Proposition 2.4 into account we have

$$P_{e_i^\perp}(K) = E_{P_{e_i^\perp}(A)} \in \mathcal{E}^{n-1} \text{ and } P_{e_i^\perp}(H) = E_{P_{e_i^\perp}(B)} \in \mathcal{E}^{n-1}.$$

Moreover, Lemma 2.2 i) yields

$$P_{e_i^\perp}(K) = A_i \text{ and } P_{e_i^\perp}(H) = B_i.$$

From Proposition 2.1 and Lemma 2.2 i), we know that A_i and B_i are two $(n-1)\times(n-1)$ positive semidefinite real symmetric matrices, and determine uniquely the ellipsoids $P_{e_i^\perp}(K)$ and $P_{e_i^\perp}(H)$. Hence by (1.8) we have

$$\kappa_{n-1}^2 \det(A_i) = \mathrm{vol}_{n-1}^2(P_{e_i^\perp}(K)) \text{ and } \kappa_{n-1}^2 \det(B_i) = \mathrm{vol}_{n-1}^2(P_{e_i^\perp}(M)).$$

Using (1.9)

$$\mathrm{vol}_n^2(K +_2 H) = \kappa_n^2 \det(A + B) \text{ and } \mathrm{vol}_{n-1}^2(P_{e_i^\perp}(K +_2 H) = \kappa_{n-1}^2 \det(A_i + B_i).$$

Inserting all the previous equalities in (1.10) yields that (3.1) holds if and only if

$$\frac{\det(A+B)}{\det(A_i+B_i)} \geq \frac{\det(A)}{\det(A_i)} + \frac{\det(B)}{\det(B_i)}, \tag{3.2}$$

holds, which corresponds to Bergstrom's inequality (1.1) and finishes the proof. □

The proof of Theorem 1.7 is now a direct consequence of Theorem 3.1 and Remark 2.3.

Proof of Theorem 1.7 Let K, H be two L_2 zonoids in $\mathbb{R}^n$, and let $u \in \mathbb{S}^{n-1}$. Theorem 1.7 yields the validity of (3.1) for every e_i, $1 \leq i \leq n$. Using now Remark 2.3, and an orthogonal matrix O such that $Ou = e_i$, we have that $\det(P_{u^\perp}A) = \det\left(P_{e_i^\perp}(OAO^T)\right)$, and $\det(P_{u^\perp}B) = \det\left(P_{e_i^\perp}(OBO^T)\right)$. Since, clearly, $\det(A) = \det(OAO^T)$, $\det(B) = \det(OBO^T)$, a direct application of Theorem 3.1 for OAO^T, OBO^T, and $(OAO^T)_i$, and $(OBO^T)_i$, yields the result. □

As already mentioned, we are not proving the equality case, but we will be using the equality case characterization proven in [12, Theorem 6.2] to state a characterization of the equality case of Bergstrom's inequality. Equality in the inequality established in [12, Theorem 6.2], i.e., in Theorem 1.7, holds for some $u \in \mathbb{S}^{n-1}$ if and only if the L_2 zonoids K and H have parallel tangent hyperplanes at the boundary points $\rho_K(u)u \in K$ and $\rho_H(u)u \in H$. The following remark is also established in [12], in connection to the equality case of [12, Theorem 6.2].

Remark 3.2 [12, Remark 6.3] Let $K = T_1 B_n$ and $H = T_2 B_n$ for$T_1, T_2 \in \mathcal{S}_+^n$. Then, the condition of equality in Theorem 1.7 is equivalent to the fact that there is $\lambda > 0$ such that $(T_1^{-2} - \lambda T_2^{-2})u = 0$, or simply that u is an eigenvector of $T_1^2 T_2^{-2}$.

Proposition 3.3 *Let A and B be two $n \times n$ positive definite real symmetric matrices, let $1 \leq i \leq n$, and let A_i and B_i the $(n-1) \times (n-1)$ matrices given by A and B deleting the i-th row and the i-th column. Then, there is equality in Bergstrom's inequality* (1.1)*:*

$$\frac{\det(A+B)}{\det(A_i+B_i)} = \frac{\det(A)}{\det(A_i)} + \frac{\det(B)}{\det(B_i)}, \tag{3.3}$$

if and only if $\operatorname{lin}\{A^{-1}e_i\} = \operatorname{lin}\{B^{-1}e_i\}$.

Proof Let $A, B \in \mathcal{S}_+^n$, be two $n \times n$ positive definite real symmetric matrices, such that equality holds in Bergstrom's inequality (1.1) for some $i \in \{1, \cdots, n\}$. Let further T_1 and T_2 be the unique square root of A and B, respectively, i.e., $A = T_1^2$ and $B = T_2^2$. It is well-known that $T_1, T_2 \in \mathcal{S}_+^n$ (see e.g. [19, Exercise 1.3.P7 and Theorem 2.6.3]). Let us consider E_A and E_B the centered ellipsoids defined by A and B, respectively. Thus, by (1.7), we have that $E_A = T_1 B_n$ and $E_B = T_2 B_n$. The latter considerations yield equality in (3.1) for E_A and E_B. Therefore, by Remark 3.2, $T_1^2 T_2^{-2} e_i = \lambda e_i$, which implies $\operatorname{lin}\{A^{-1}e_i\} = \operatorname{lin}\{B^{-1}e_i\}$. □

3.2 Other inequalities

In the last part of this note, we consider other results within convex geometry, which have found a sort of counterpart in matrix theory, in particular involving the projection of a matrix. We are mostly looking at refinements of inequalities of the type of Brunn–Minkowski inequality (1.4). We start recalling the following result.

Theorem 3.4 [22] *Let $K, H \in \mathcal{K}^n$ be convex bodies such that there exists a direction $u \in \mathbb{S}^{n-1}$ with $\operatorname{vol}_{n-1}(P_{u^\perp}(K)) = \operatorname{vol}_{n-1}(P_{u^\perp}(H))$. Then*

$$\operatorname{vol}_n\big((1-\lambda)K + \lambda H\big) \geq (1-\lambda)\operatorname{vol}_n(K) + \lambda \operatorname{vol}_n(H), \tag{3.4}$$

for all $\lambda \in [0, 1]$.

We point out that inequality (3.4) refines the Brunn–Minkowski inequality (see [14, Section 10]), as the inequality

$$[(1-\lambda)\operatorname{vol}_n(K) + \lambda \operatorname{vol}_n(H) \geq \Big((1-\lambda)\operatorname{vol}_n(K)^{1/n} + \lambda \operatorname{vol}_n(H)^{1/n}\Big)^n,$$

holds for every $K, H \in \mathcal{K}^n$, $\lambda \in [0, 1]$, see [14, Section 10]. We refer to [22, Section 7] and to [8, 16, 17] for results in the direction of Theorem 3.4 within the theory of convex geometry.

In the next result, using the connection of positive semidefinite real symmetric matrices and ellipsoids, we obtain, as a corollary of Theorem 3.4, a refinement of the Brunn–Minkowski inequality for matrices, assuming that the determinants of the projections of the two matrices onto a hyperplane coincide.

Corollary 3.5 *Let* $A, B \in \mathcal{S}_+^n$ *be positive definite matrices, and let* $u \in \mathbb{S}^{n-1}$ *be such that* $\det\left(P_{u^\perp}(A)\right) = \det\left(P_{u^\perp}(B)\right)$*. Then,*

$$\det((1-\lambda)A + \lambda B)^{\frac{1}{2}} \geq (1-\lambda)\det(A)^{\frac{1}{2}} + \lambda \det(B)^{\frac{1}{2}}, \tag{3.5}$$

for every $\lambda \in [0, 1]$.

Proof Let $A, B \in \mathcal{S}_+^n$ be positive definite matrices, and let E_A and E_B the associated ellipsoids. We observe first, that he assumption $\det\left(P_{u^\perp}(A)\right) = \det\left(P_{u^\perp}(B)\right)$ is equivalent to the fact that $\mathrm{vol}_{n-1}(P_{u^\perp}(E_A)) = \mathrm{vol}_{n-1}(P_{u^\perp}(E_B))$ by means of (1.8), and Proposition 2.4. Then, taking (1.9) into account, direct application of (1.8) yields the result. □

However, under the same assumptions, a sharper inequality is known to hold. In [9], the authors established *a linear refinement* inequality of the Brunn–Minkowski inequality for the determinant, inequality (1.3), under the assumption that the matrices involved share equal determinant of their projection onto a common hyperplane.

Theorem 3.6 [9, Theorem 5.7] *Let* $A, B \in \mathcal{S}_+^n$ *be positive definite matrices, and let* $u \in \mathbb{S}^{n-1}$*. Assume that* $\det\left(P_{u^\perp}(A)\right) = \det\left(P_{u^\perp}(B)\right)$*. Then,*

$$\det((1-\lambda)A + \lambda B) \geq (1-\lambda)\det(A) + \lambda \det(B), \tag{3.6}$$

for every $\lambda \in [0, 1]$.

Using the latter, we can now write the *linear Brunn–Minkowski inequality for matrices*, inequality (3.6), by means of (1.8), as a linear refinement of the L_2 Brunn–Minkowski inequality (1.5), in the case of L_2 zonoids. That is the content of the following proposition.

Proposition 3.7 *Let* K, H *be two* L_2 *zonoids in* $\mathbb{R}^n$ *and let* $u \in \mathbb{S}^{n-1}$*. If there exists a direction* $u \in \mathbb{S}^{n-1}$*, such that* $\mathrm{vol}_{n-1}(P_{u^\perp}(K)) = \mathrm{vol}_{n-1}(P_{u^\perp}(H))$*, then*

$$\mathrm{vol}_n\left((1-\lambda)\cdot_2 K +_2 \lambda \cdot_2 H\right)^2 \geq (1-\lambda)\mathrm{vol}_n(K)^2 + \lambda \mathrm{vol}_n(H)^2, \tag{3.7}$$

for all $\lambda \in [0, 1]$.

We point out the following result on the equality case of the linear refinement of Brunn–Minkowski inequality for the determinant.

Theorem 3.8 [21, Theorem 5.1] *Equality in inequality* (3.6) *holds if and only if there exists a matrix* R *of rank at most* 1*, such that* $B = A + R$.

We observe that there is also equality in (3.5) in the case $B = A + R$, namely, when the condition in Theorem 3.8 holds.
Further, we observe that Theorem 3.8 allows us to establish a characterization of equality in (3.7).

Corollary 3.9 *Equality holds in* (3.7) *if and only if there exists a segment* S *in* $\mathbb{R}^n$*, i.e., a* 1*-dimensional* L_2 *zonoid, such that* $K = H +_2 S$.

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References

1. Artstein-Avidan, S., Floretin, D., Ostrover, Y.: Remarks about mixed discriminants and volumes. In: Communications in Contemporary Mathematics, vol. 16 (2014)
2. Bapat, R.B.: Mixed discriminants of positive semidefinite matrices. Linear Algebra Appl. **126**, 107–124 (1989)
3. Barvinok, A.: Computing mixed discriminants, mixed volumes, and permanents. Discrete Comput. Geom. **18**(2), 205–237 (1997)
4. Barvinok, A.: Concentration of the mixed discriminant of well-conditioned matrices. Linear Algebra Appl. **493**, 120–133 (2016)
5. Beckenbach, E.F., Bellman, R.: Inequalities. Springer, Berlin (1971)
6. Bellman, R.: Notes on matrix theory-IV: an inequality due to Bergstrom. Am. Math. Monthly **62**, 172–173 (1955)
7. Bergstrom, H.: A triangle inequality of matrices. Den Elfte Skandinaviski Matimatiker-Kongress, Trondheim, 1949. Oslo: Johan Grundt Tanums Forlag (1952)
8. Colesanti, A., Saorín Gómez, E., Yepes Nicolás, J.: On a linear refinement of the Prékopa-Leindler inequality. Canad. J. Math. **68**, 762–783 (2016)
9. de Vries, C., Lombardi, N., Gómez, E. Saorín: Notes on mixed discriminant and related linear inequalities (2022)
10. Fan, K.: Some inequalities concerning positive-definite Hermitian matrices. Math. Proc. Camb. Philos. Soc. **51**(3), 414–421 (1955)
11. Florentin, D., Milman, V.D., Schneider, R.: A characterization of the mixed discriminant. Proc. Am. Math. Soc. **144**(5), 2197–2204 (2016)
12. Fradelizi, M., Madiman, M., Meyer, M., Zvavitch, A.: On the volume of the Minkowski sum of zonoids
13. Fradelizi, M., Giannopoulos, A., Meyer, M.: Some inequalities about mixed volumes. Israel J. Math. **135**, 157–179 (2003)
14. Gardner, R.J.: The Brunn–Minkowski inequality. Bull. Am. Math. Soc. **39**(3), 355–405 (2002)
15. Giannopoulos, A., Hartzoulaki, M., Paouris, G.: On a local version of the Aleksandrov–Fenchel inequality for the quermassintegrals of a convex body. Proc. Am. Math. Soc. **130**, 2403–2412 (2002)
16. Hernández Cifre, M.A., Yepes Nicolás, J.: Refinements of the Brunn-Minkowski inequality. J. Convex Anal. **3**, 1–17 (2014)
17. Hernández Cifre, M.A., Yepes Nicolás, J.: Brunn–Minkowski and Prékopa–Leindler's inequalities under projection assumptions. J. Math. Anal. Appl. **445**, 1257–1271 (2017)

18. Hiriart-Urruty, J.B., Lemaréchal, C.: Convex Analysis and Minimization Algorithms I & II. Springer, New York (1993)
19. Horn, R.A., Johnson, C.R.: Matrix Analysis. Cambridge University Press, Cambridge, New York (1985)
20. Maddocks, J.H.: Restricted quadratic forms, inertia theorems, and the Schur complement. Linear Algebra Appl. **108**, 1–36 (1988)
21. Saorín Gómez, E., Yepes Nicolás, J.: Linearity of the volume. Looking for a characterization of sausages. J. Math. Anal. Appl. **421**(2), 1081–1100 (2015)
22. Schneider, R.: Convex Bodies: The Brunn–Minkowski Theory. Second expanded edition, Cambridge University Press, Cambridge (2014)