Positivity (2023) 27:38
https://doi.org/10.1007/s11117-023-00989-3

Positivity

Some new $(H_p - L_p)$ type inequalities for weighted maximal operators of partial sums of Walsh–Fourier series

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Received: 26 October 2022 / Accepted: 22 April 2023 / Published online: 16 May 2023

Abstract
In this paper we introduce some new weighted maximal operators of the partial sums of the Walsh–Fourier series. We prove that for some "optimal" weights these new operators indeed are bounded from the martingale Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$, for $0 < p < 1$. Moreover, we also prove sharpness of this result. As a consequence we obtain some new and well-known results.

Keywords Walsh system · Partial sums · Martingale Hardy space · Weighted maximal operators

Mathematics Subject Classification 42C10

1 Introduction

It is well-known that the Walsh system does not form a basis in the space L_1 (see e.g. [2, 27]). Moreover, there exists a function f in the dyadic Hardy space H_1, such that

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the partial sums of f are not bounded in L_1-norm, but the partial sums S_n of the Walsh–Fourier series of a function $f \in L_1$ convergence in measure (see [9, 12]). Uniform and pointwise convergence and some approximation properties of partial sums in L_1 norm were investigated by Onneweer [16], Goginava [10], Goginava and Tkebuchava [11], Nagy [15], Avdispahić and Memić [1], Persson et al. [18]. Fine [6] obtained sufficient conditions for the uniform convergence, which are complete analogies with the Dini-Lipschits conditions. Guličev [13] estimated the rate of uniform convergence of a Walsh–Fourier series by using Lebesgue constants

$$L(n) := \|D_n\|_1$$

and modulus of continuity. These problems for Vilenkin groups were considered by Blahota [3], Fridli [7] and Gát [8].

Above, and in the sequel, all used notations can be found in Sect. 2. For example, the notations D_n and S_n are given in (6).

To study convergence of subsequences of partial sums in the martingale Hardy spaces $H_p(G)$ for $0 < p \leq 1$, the central role plays uniquely expression of any natural number $n \in \mathbb{N}$

$$n = \sum_{k=0}^{\infty} n_j 2^j, \quad n_j \in Z_2 \ (j \in \mathbb{N}),$$

where only a finite numbers of n_j differ from zero and their important characters $[n]$, $|n|$, $\rho(n)$ and $V(n)$ are defined by

$$[n] := \min\{j \in \mathbb{N}, n_j \neq 0\}, \quad |n| := \max\{j \in \mathbb{N}, n_j \neq 0\}, \quad \rho(n) = |n| - [n]$$

and

$$V(n) := n_0 + \sum_{k=1}^{\infty} |n_k - n_{k-1}|, \quad \text{for all} \quad n \in \mathbb{N} \tag{1}$$

In particular, (see [5, 14, 19])

$$\frac{V(n)}{8} \leq \|D_n\|_1 \leq V(n),$$

from which it follows that, for any $F \in L_1(G)$, there exists an absolute constant c such that

$$\|S_n F\|_1 \leq cV(n) \|F\|_1 . \tag{2}$$

Moreover, for any $f \in H_1$,

$$\|S_n F\|_{H_1} \leq cV(n) \|F\|_{H_1} .$$

In [23] and [24] it was proved that if $0 < p < 1$ and $F \in H_p$, then there exists an absolute constant c_p, depending only on p, such that

$$\|S_n F\|_{H_p} \le c_p 2^{\rho(n)(1/p-1)} \|F\|_{H_p}.$$

Moreover, if $0 < p < 1$, $\{n_k : k \ge 0\}$ is any increasing sequence of positive integers such that

$$\sup_{k\in\mathbb{N}} \rho(n_k) = \infty$$

and $\Phi : \mathbb{N}_+ \to [1, \infty)$ is any nondecreasing function, satisfying the condition

$$\overline{\lim_{k\to\infty}} \frac{2^{\rho(n_k)(1/p-1)}}{\Phi(n_k)} = \infty,$$

then there exists a martingale $F \in H_p$, such that

$$\sup_{k\in\mathbb{N}} \left\| \frac{S_{n_k} F}{\Phi(n_k)} \right\|_{\text{weak}-L_p} = \infty.$$

For $0 < p < 1$ in [21, 22] the weighted maximal operator $\tilde{S}^{*,p}$, defined by

$$\tilde{S}^{*,p} F := \sup_{n\in\mathbb{N}} \frac{|S_n F|}{(n+1)^{1/p-1}} \tag{3}$$

was investigated and it was proved that the following estimate holds:

$$\left\| \tilde{S}^{*} F \right\|_p \le c_p \|F\|_{H_p}. \tag{4}$$

Moreover, it was also proved that the rate of the sequence $(n+1)^{1/p-1}$ given in denominator of (3) can not be improved, but it was proved only for the special subsequences.

For $p = 1$ analogical results for the maximal operator $\tilde{S}^{*}$, defined by

$$\tilde{S}^{*} F := \sup_{n\in\mathbb{N}} \frac{|S_n F|}{\log(n+1)}$$

was proved in [22].

One main aim of this paper is to generalize the estimate (4) for $f \in H_p(G)$, $0 < p < 1$. Our main idea is to investigate much more general maximal operators by replacing the weights $(n+1)^{1/p-1}$ in (3) by more general "optimal" weights

$$2^{\rho(n)(1/p-1)}(\varphi(\rho(n))),$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is any nonnegative and nondecreasing function satisfying the condition

$$\sum_{n=1}^{\infty} 1/\varphi^p(n) < c < \infty$$

and prove that it is bounded from the martingale Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$, for $0 < p < 1$. As a consequence we obtain some new and well-known results. In particular, we prove that the maximal operator $\widetilde{S}^{*,\nabla}$, defined by

$$\widetilde{S}^{*,\nabla,\varepsilon} F := \sup_{n\in\mathbb{N}} \frac{|S_n F|}{2^{\rho(n)(1/p-1)} \left(\log^{1+\varepsilon}(\rho(n))\right)^{1/p}}, \quad \text{where } 0 < p < 1, \ \varepsilon \geq 0,$$

is bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$ for any $\varepsilon > 0$ and is not bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$ when $\varepsilon = 0$.

This paper is organized as follows: In order not to disturb our discussions later on some definitions and notations are presented in Sect. 2. The main results and some of its consequences can be found in Sect. 3. The detailed proofs are given in Sect. 4.

2 Preliminaries

Let $\mathbb{N}_+$ denote the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Denote by Z_2 the discrete cyclic group of order 2, that is $Z_2 := \{0, 1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on Z_2 is given so that the measure of a singleton is 1/2.

Define the group G as the complete direct product of the group Z_2, with the product of the discrete topologies of Z_2's. The elements of G are represented by sequences $x := (x_0, x_1, ..., x_j, ...)$, where $x_k = 0 \vee 1$.

It is easy to give a base for the neighborhood of $x \in G$:

$$I_0(x) := G, \quad I_n(x) := \{y \in G : y_0 = x_0, ..., y_{n-1} = x_{n-1}\} \ (n \in \mathbb{N}).$$

Denote $I_n := I_n(0)$, $\overline{I_n} := G \setminus I_n$ and

$$e_n := (0, ..., 0, x_n = 1, 0, ...) \in G, \quad \text{for} \quad n \in \mathbb{N}.$$

Then it is easy to show that

$$\overline{I_M} = \bigcup_{s=0}^{M-1} I_s \backslash I_{s+1}. \tag{5}$$

The norms (or quasi-norm) of the spaces $L_p(G)$ and weak$-L_p(G)$, $(0 < p < \infty)$ are, respectively, defined by

$$\|f\|_p^p := \int_G |f|^p \, d\mu \quad \text{and} \quad \|f\|_{\text{weak}-L_p}^p := \sup_{\lambda>0} \lambda^p \mu (f > \lambda) .$$

The k-th Rademacher function $r_k(x)$ is defined by

$$r_k(x) := (-1)^{x_k} \qquad (x \in G, \ k \in \mathbb{N}) .$$

Now, define the Walsh system $w := (w_n : n \in \mathbb{N})$ on G by

$$w_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) = r_{|n|}(x) (-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \qquad (n \in \mathbb{N}) .$$

The Walsh system is orthonormal and complete in $L_2(G)$ (see e.g. [19]).

If $f \in L_1(G)$ we can establish the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels with respect to the Walsh system in the usual manner:

$$\begin{aligned} \widehat{f}(k) &:= \int_G f w_k d\mu \quad (k \in \mathbb{N}) , \\ S_n f &:= \sum_{k=0}^{n-1} \widehat{f}(k) w_k , \\ D_n &:= \sum_{k=0}^{n-1} w_k \quad (n \in \mathbb{N}_+) . \end{aligned} \tag{6}$$

Recall that (see [17, 19])

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n \\ 0, & \text{if } x \notin I_n \end{cases} \tag{7}$$

and

$$D_n = w_n \sum_{k=0}^{\infty} n_k r_k D_{2^k} = w_n \sum_{k=0}^{\infty} n_k \left(D_{2^{k+1}} - D_{2^k} \right), \text{ for } n = \sum_{i=0}^{\infty} n_i 2^i . \tag{8}$$

Moreover, we have the following lower estimate (see [17]):

Lemma 1 *Let $n \in \mathbb{N}$ and $[n] \neq |n|$. Then*

$$|S_n(x)| = \left| S_{n-2^{|n|}}(x) \right| \geq 2^{[n]}, \quad \textit{for} \quad x \in I_{[n]} \backslash I_{[n]+1} .$$

The σ-algebra generated by the intervals $\{I_n(x) : x \in G\}$ will be denoted by ζ_n $(n \in \mathbb{N})$. It is easy to see that

$$I_{n+1}(x) \subset I_n(x) \quad \text{and} \quad I_n(x) = \bigcup_{x_n=0}^{m_n-1} I_{n+1}(x), \quad \text{for any } x \in G \text{ and } n \in \mathbb{N}.$$

It follows that

$$\zeta_n \subset \zeta_{n+1} \, (n \in \mathbb{N}).$$

Denote by $F = (F_n, n \in \mathbb{N})$ the martingale with respect to F_n $(n \in \mathbb{N})$ (for details see e.g. [25]).

The maximal function F^* of a martingale F is defined by

$$F^* := \sup_{n\in\mathbb{N}} |F_n|.$$

In the case $f \in L_1(G)$, the maximal function f^* is given by

$$f^*(x) := \sup_{n\in\mathbb{N}} \frac{1}{\mu(I_n(x))} \left| \int_{I_n(x)} f(u)\, d\mu(u) \right|.$$

For $0 < p < \infty$ the Hardy martingale spaces $H_p(G)$ consists of all martingales for which

$$\|F\|_{H_p} := \left\|F^*\right\|_p < \infty.$$

It is easy to check that for every martingale $F = (F_n, n \in \mathbb{N})$ and every $k \in \mathbb{N}$ the limit

$$\widehat{F}(k) := \lim_{n\to\infty} \int_G F_n(x)\, w_k(x)\, d\mu(x)$$

exists and it is called the k-th Walsh–Fourier coefficients of F.

If $F := (S_{2^n} f : n \in \mathbb{N})$ is a regular martingale, generated by $f \in L_1(G)$, then (for details see e.g. [17, 20] and [25])

$$\widehat{F}(k) = \widehat{f}(k), \ k \in \mathbb{N}.$$

A bounded measurable function a is called p-atom, if there exists a dyadic interval I, such that

$$\int_I a\, d\mu = 0, \quad \|a\|_\infty \le \mu(I)^{-1/p}, \quad \operatorname{supp}(a) \subset I.$$

The dyadic Hardy martingale spaces H_p for $0 < p \leq 1$ have an atomic characterization. Namely, the following theorem holds (see [17, 25, 26]):

Lemma 2 *A martingale $F = (F_n, n \in \mathbb{N})$ belongs to H_p $(0 < p \leq 1)$ if and only if there exists a sequence $(a_k,\ k \in \mathbb{N})$ of p-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that for every $n \in \mathbb{N}$*

$$\sum_{k=0}^{\infty} \mu_k S_{2^n} a_k = F_n, \tag{9}$$

where

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$

Moreover, $\|F\|_{H_p} \backsim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p\right)^{1/p}$, where the infimum is taken over all decomposition of F of the form (9).

3 The main results

Our first main result reads:

Theorem 1 *Let $0 < p < 1$, $f \in H_p(G)$ and $\varphi : \mathbb{N}_+ \to \mathbb{R}_+$ be any nonnegative and nondecreasing function satisfying the condition*

$$\sum_{n=1}^{\infty} \frac{1}{\varphi^p(n)} < c < \infty. \tag{10}$$

Then the weighted maximal operator $\widetilde{S}^{*,\nabla}$, defined by

$$\widetilde{S}^{*,\nabla} F := \sup_{n \in \mathbb{N}} \frac{|S_n F|}{2^{\rho(n)(1/p-1)} \varphi(\rho(n))},$$

is bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$.

Theorem 1 can be of special interest even if we restrict to subsequences.

Corollary 1 *Let $0 < p < 1$, $f \in H_p(G)$, $\varphi : \mathbb{N}_+ \to \mathbb{R}_+$ be any nonnegative and nondecreasing function satisfying the condition (10) and $\{n_k : k \geq 0\}$ be any sequence of positive numbers. Then the weighted maximal operator $\widetilde{S}^{*,\nabla}$, defined by*

$$\widetilde{S}^{*,\nabla} F = \sup_{k \in \mathbb{N}} \frac{\left|S_{n_k} F\right|}{2^{\rho(n_k)(1/p-1)} \varphi(\rho(n_k))}, \tag{11}$$

is bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$.

We also prove sharpness of Theorem 1:

Theorem 2 *Let $0 < p < 1$, $\{n_k : k \geq 0\}$ be a sequence of positive numbers and $\varphi : \mathbb{N}_+ \to \mathbb{R}_+$ be any nonnegative and nondecreasing function satisfying the condition*

$$\sum_{n=1}^{\infty} \frac{1}{\varphi^p(n)} = \infty. \tag{12}$$

Then there exists p-atoms a_k, such that

$$\sup_{k\in\mathbb{N}} \frac{\left\| \sup_{n\in\mathbb{N}} \frac{|S_n a_k|}{2^{\rho(n)(1/p-1)}\varphi(\rho(n))} \right\|_p}{\|a_k\|_{H_p}} = \infty.$$

If we take

$$\varphi(n) = \left(n \log^{1+\varepsilon} n\right)^{1/p}, \quad \text{for any } \varepsilon > 0,$$

we get that condition (10) is fulfilled, on the other hand, if we take

$$\varphi(n) = (n \log n)^{1/p},$$

then condition (12) holds true. Hence, Theorems 1 and 2 imply the following sharp result:

Corollary 2 *a) Let $0 < p < 1$ and $f \in H_p(G)$. Then the weighted maximal operator $\widetilde{S}^{*,\nabla,\varepsilon}$, defined by*

$$\widetilde{S}^{*,\nabla,\varepsilon} F := \sup_{n\in\mathbb{N}} \frac{|S_n F|}{2^{\rho(n)(1/p-1)} \left(\rho(n) \log^{1+\varepsilon} \rho(n)\right)^{1/p}}, \quad \varepsilon > 0,$$

is bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$.

b) The weighted maximal operator $\widetilde{S}^{*,\nabla,0}$, defined by

$$\widetilde{S}^{*,\nabla,0} F := \sup_{n\in\mathbb{N}} \frac{|S_n F|}{2^{\rho(n)(1/p-1)} \left(\rho(n) \log(\rho(n))\right)^{1/p}}$$

is not bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$.

Remark 1 Suppose that $\{n_k : k \geq 0\}$ is a sequence of positive numbers, such that

$$\sup_{k\in\mathbb{N}} [n_k] < c < \infty.$$

Then

$$\sup_{k\in\mathbb{N}} \varphi([n_k]) < \varphi(c) < \infty,$$

$$2^{\rho(n_k)(1/p-1)} \sim 2^{|n_k|(1/p-1)} \sim n_k^{1/p-1} \sim (n_k+1)^{1/p-1}$$

and

$$\widetilde{S}^{*,\nabla} F \leq \sup_{k\in\mathbb{N}} \frac{\left|S_{n_k} F\right|}{(n_k+1)^{1/p-1}}.$$

Let

$$\sup_{k\in\mathbb{N}} [n_k] = \infty.$$

Then, the maximal operator (11) can not be estimated by

$$\sup_{k\in\mathbb{N}} \frac{\left|S_{n_k} F\right|}{(n_k+1)^{1/p-1}} \leq \widetilde{S}^{*,\nabla} F.$$

Hence, Theorem 1 and Remark 1 and Theorem proved in [21, 22] follows that if $0 < p < 1$, $f \in H_p(G)$ and $\varphi : \mathbb{N}_+ \to \mathbb{R}_+$ be any nonnegative and nondecreasing function satisfying the condition (10), then the weighted maximal operator $\widetilde{S}^{*,\nabla}$, defined by

$$\widetilde{S}^{*,\nabla} F := \sup_{n\in\mathbb{N}} \frac{|S_n F|}{\min\{2^{\rho(n)(1/p-1)}\varphi(\rho(n)), (n+1)^{1/p-1}\}},$$

is bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$.

Now, we formulate a result proved in [22], which follows from Theorems 1 and 2:

Corollary 3 *a) Let $0 < p \leq 1$ and $\{\alpha_k, k \in \mathbb{N}\}$ be a subsequence of positive numbers such that $\sup_{k\in\mathbb{N}} \rho(\alpha_k) < \infty$. Then the maximal operator $\widetilde{S}^{*,\Delta}$ defined by*

$$\widetilde{S}^{*,\Delta} f := \sup_{k\in\mathbb{N}} \left|S_{\alpha_k} f\right| \tag{13}$$

is bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$.

b) Let $0 < p < 1$ and $(\alpha_k, k \in \mathbb{N})$ be a subsequence of positive numbers satisfying the condition $\sup_{k\in\mathbb{N}} \rho(\alpha_k) = \infty$. Then the maximal operator $\widetilde{S}^{*,\Delta}$ defined by (13) is not bounded from the Hardy space $H_p(G)$ to the Lebesgue space $L_p(G)$.

4 Proofs of the Theorems

Proof of Theorem 1. By using Lemma 2 the proof of Theorem 1 will be complete, if we prove that

$$\int_G \left|\widetilde{S}^{*,\nabla} a(x)\right|^p d\mu(x) \leq c < \infty, \tag{14}$$

for every p-atom a, with support I and $\mu(I) = 2^{-M}$. We may assume that this arbitrary p-atom a has support $I = I_M$. It is easy to see that $S_n a(x) = 0$, when $n \le 2^M$. Therefore, we can suppose that $n > 2^M$. Since $\|a\|_\infty \le 2^{M/p}$ we find that

$$\begin{aligned}
&\left|\frac{S_n a(x)}{2^{(1/p-1)\rho(n)}\varphi(\rho(n))}\right| \\
&\quad\le \frac{1}{2^{(1/p-1)\rho(n)}\varphi(\rho(n))}\|a\|_\infty \int_{I_M} |D_n(x+t)|\, d\mu(t) \\
&\quad\le \frac{1}{2^{(1/p-1)\rho(n)}\varphi(\rho(n))} 2^{M/p} \int_{I_M} |D_n(x+t)|\, d\mu(t). \qquad (15)
\end{aligned}$$

Let $x \in I_M$. Since $V(n) \le \rho(n) + 2$, by applying (2) we get that

$$\begin{aligned}
\left|\frac{S_n a(x)}{2^{(1/p-1)\rho(n)}\varphi(\rho(n))}\right| &\le 2^{M/p} \frac{1}{2^{(1/p-1)\rho(n)}\varphi(\rho(n))} V(n) \\
&\le 2^{M/p}\rho(n) \frac{1}{2^{(1/p-1)\rho(n)}\varphi(\rho(n))} \le c2^{M/p}
\end{aligned}$$

so that

$$\int_{I_M}\left(\sup_{n\in\mathbb{N}}\left|\frac{S_n a(x)}{2^{(1/p-1)\rho(n)}\varphi(\rho(n))}\right|\right)^p d\mu(x) < c_p < \infty. \qquad (16)$$

Let $t \in I_M$ and $x \in I_s \backslash I_{s+1},\ 0 \le s \le M-1 < [n]$ or $0 \le s < [n] \le M-1$. Then $x + t \in I_s \backslash I_{s+1}$. By using (8) we get that $D_n(x+t) = 0$ and

$$\left|\frac{S_n a(x)}{2^{(1/p-1)\rho(n)}\varphi(\rho(n))}\right| = 0. \qquad (17)$$

Let $x \in I_s \backslash I_{s+1},\ [n] \le s \le M-1$. Then $x + t \in I_s \backslash I_{s+1}$, for $t \in I_M$. By using (8) we find that

$$|D_n(x+t)| \le \sum_{j=0}^{s} n_j 2^j \le c2^s.$$

Hence, by applying (15) we get that

$$\begin{aligned}
&\left|\frac{S_n a(x)}{2^{(1/p-1)\rho(n)}\varphi(\rho(n))}\right| \\
&\quad\le \frac{c}{2^{(1/p-1)\rho(n)}\varphi(\rho(n))} 2^{M/p} \frac{2^s}{2^M} \le \frac{c2^{M(1/p-1)}}{2^{|n|(1/p-1)}} \frac{2^{[n](1/p-1)}2^s}{\varphi(\rho(n))}. \qquad (18)
\end{aligned}$$

By now using (18) for $0 < [n] < s/2$ we can conclude that

$$\left|\frac{S_n a(x)}{2^{(1/p-1)\rho(n)}\varphi(\rho(n))}\right| \leq \frac{c2^{M(1/p-1)}}{2^{|n|(1/p-1)}}\frac{2^{[n](1/p-1)}2^s}{\varphi(\rho(n))} \leq \frac{2^{(s/2)(1/p-1)}2^s}{\varphi(\rho(n))} \leq 2^{(s/2)(1/p+1)}. \tag{19}$$

Moreover, according to (18) for $s/2 \leq [n] \leq s$ we have that

$$\left|\frac{S_n a(x)}{2^{(1/p-1)\rho(n)}\varphi(\rho(n))}\right| \leq \frac{c2^{M(1/p-1)}}{2^{|n|(1/p-1)}}\frac{2^{s/p}}{\varphi(\rho(n))} \leq \frac{2^{s/p}}{\varphi(\rho(n))} \leq \frac{2^{s/p}}{\varphi(M-s)}. \tag{20}$$

By combining (17), (19) and (20), for all $x \in I_s \backslash I_{s+1},\ 0 \leq s \leq M-1$ we get that

$$\widetilde{S}^{*,\nabla}a(x) = \sup_{n\in\mathbb{N}}\left|\frac{S_n a(x)}{2^{(1/p-1)\rho(n)}\varphi(\rho(n))}\right| \leq c2^{(s/2)(1/p+1)} + \frac{c2^{s/p}}{\varphi(M-s)}. \tag{21}$$

By now combining (5) and (21) we obtain that

$$\begin{aligned}
&\int_{\overline{I_M}}\left|\widetilde{S}^{*,\nabla}a(x)\right|^p d\mu(x) \\
&\leq c_p\sum_{s=0}^{M-1}\int_{I_s\backslash I_{s+1}}\left|2^{(s/2)(1/p+1)} + \frac{2^{s/p}}{\varphi(M-s)}\right|^p d\mu(x) \\
&\leq c_p\sum_{s=0}^{M-1}\int_{I_s\backslash I_{s+1}}\left|2^{(s/2)(1/p+1)}\right|^p d\mu(x) + c_p\sum_{s=0}^{M-1}\int_{I_s\backslash I_{s+1}}\left|\frac{2^{s/p}}{\varphi(M-s)}\right|^p d\mu(x) \\
&\leq c_p\sum_{s=0}^{M-1}\frac{1}{2^s}2^{(s/2)(1+p)} + c_p\sum_{s=0}^{M-1}\frac{1}{2^s}\frac{2^s}{\varphi^p(M-s)} \\
&\leq c_p\sum_{s=0}^{M-1}2^{(s/2)(p-1)} + c_p\sum_{s=0}^{M-1}\frac{1}{\varphi^p(M-s)} \leq c_p < \infty.
\end{aligned} \tag{22}$$

By combining (16) and (22) we obtain that (14) holds and the proof is complete. □

Proof of Theorem 2. In view of the condition (12) we have that

$$\left(\sum_{s=0}^{n_k-1}\frac{1}{\varphi^p(s)}\right)^{1/p} \to \infty, \quad \text{as} \quad k \to \infty. \tag{23}$$

Set

$$f_{n_k}(x) = D_{2^{n_k+1}}(x) - D_{2^{n_k}}(x), \qquad n_k \geq 3.$$

It is evident that

$$\widehat{f}_{n_k}(i)=\begin{cases}1, & \text{if } i=2^{n_k},...,2^{n_k+1}-1,\\ 0, & \text{otherwise.}\end{cases}$$

Then we easily can derive that

$$S_i f_{n_k}(x)=\begin{cases}D_i(x)-D_{2^{n_k}}(x), & \text{if } i=2^{n_k},...,2^{n_k+1}-1,\\ f_{n_k}(x), & \text{if } i\geq 2^{n_k+1},\\ 0, & \text{otherwise.}\end{cases} \tag{24}$$

Since

$$D_{j+2^{n_k}}(x)-D_{2^{n_k}}(x)=w_{2^{n_k}}D_j(x), \qquad j=1,2,..,2^{n_k},$$

from (7) it follows that

$$\begin{aligned}\|f_{n_k}\|_{H_p}&=\left\|\sup_{n\in\mathbb{N}} S_{2^n} f_{n_k}\right\|_p=\left\|D_{2^{n_k+1}}-D_{2^{n_k}}\right\|_p\\ &=\|D_{2^{n_k}}\|_p\leq 2^{n_k(1-1/p)}.\end{aligned} \tag{25}$$

Let $q_{n_k}^s\in\mathbb{N}$ be such that

$$2^{n_k}\leq q_{n_k}^s\leq 2^{n_k+1} \quad \text{and} \quad [q_{n_k}^s]=s, \quad \text{where} \quad 0\leq s<n_k.$$

By applying (24) we can conclude that

$$\left|S_{q_{n_k}^s} f_{n_k}(x)\right|=\left|D_{q_{n_k}^s}(x)-D_{2^{n_k}}(x)\right|=\left|D_{q_{n_k}^s-2^{n_k}}(x)\right|$$

Let $x\in I_s\backslash I_{s+1}$. By using Lemma 1 we obtain that

$$\left|S_{q_{n_k}^s} f_{n_k}(x)\right|\geq c2^s$$

and

$$\frac{\left|S_{q_{n_k}^s} f_{n_k}(x)\right|}{2^{(1/p-1)\rho\left(q_{n_k}^s\right)}\varphi\left(\rho\left(q_{n_k}^s\right)\right)}\geq\frac{c_p 2^{s/p}}{2^{n_k(1/p-1)}\varphi(n_k-s)}.$$

Hence,

$$\begin{aligned}&\int_G\left(\sup_{k\in\mathbb{N}}\left|\frac{\left|S_{q_{n_k}^s} f_{n_k}(x)\right|}{2^{(1/p-1)\rho\left(q_{n_k}^s\right)}\varphi\left(\rho\left(q_{n_k}^s\right)\right)}\right|\right)^p d\mu(x)\\ &\geq c_p\sum_{s=0}^{n_k-1}\int_{I_s\backslash I_{s+1}}\left(\frac{2^{s/p}}{2^{n_k(1/p-1)}\varphi(n_k-s)}\right)^p d\mu(x)\\ &\geq c_p\sum_{s=0}^{n_k-1}\frac{1}{2^s}\frac{2^s}{2^{n_k(1-p)}\varphi^p(n_k-s)}\end{aligned}$$

$$\geq \frac{C_p}{2^{n_k(1-p)}} \sum_{s=1}^{n_k} \frac{1}{\varphi^p(s)}.$$

Finally, by combining (23) and (25) we find that

$$\frac{\left(\int_G \left(\sup_{k\in\mathbb{N}} \sup_{0\leq s<n_k} \left| \frac{\left|S_{q_{n_k}^s} f_{n_k}(x)\right|}{2^{(1/p-1)\rho\left(q_{n_k}^s\right)} \varphi\left(\rho\left(q_{n_k}^s\right)\right)} \right| \right)^p d\mu(x)\right)^{1/p}}{\left\| f_{n_k} \right\|_{H_p}}$$

$$\geq \frac{\left(\frac{c_p}{2^{n_k(1-p)}} \sum_{s=1}^{n_k} \frac{1}{\varphi^p(s)}\right)^{1/p}}{2^{n_k(1/p-1)}}$$

$$\geq c_p \left(\sum_{s=1}^{n_k} \frac{1}{\varphi^p(s)}\right)^{1/p} \to \infty, \text{ as } k \to \infty.$$

The proof is complete. □

Acknowledgements The research was supported by Shota Rustaveli National Science Foundation grant no. PHDF-21-1702. We thank the referee for some good suggestions, which have improved this paper.

Author contributions DB and GT gave the idea and initiated the writing of this paper. LEP followed up on this with some complementary ideas. All authors read and approved the final manuscript.

Funding Open access funding provided by UiT The Arctic University of Norway (incl University Hospital of North Norway) The publication charges for this article have been funded by a grant from the publication fund of UiT The Arctic University of Norway.

Declarations

Conflict of interest The authors declare that they have no competing interests.

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