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A Voronovskaja type formula for Soardi's operators

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Abstract

In 1991 Soardi introduced a sequence of positive linear operators β_n associating to each function $f \in C[0,1]$ a polynomial function which is closely related to the Bernstein polynomials on $[-1,+1]$. One of the authors already studied the operators β_n in several papers. This paper is devoted to other properties of Soardi's operators. We introduce a version $\tilde{\beta}_n$ which can be expressed in terms of the classical Bernstein operators and present the relations between β_n and $\tilde{\beta}_n$. We derive Voronovskaja-type results for both β_n and $\tilde{\beta}_n$. Furthermore, rates of convergence for $\tilde{\beta}_n$, respectively β_n, are estimated. Finally, we study the first and second moments of β_n.

Keywords Approximation by positive operators · Rate of convergence · Degree of approximation

Mathematics Subject Classification 41A36 · 41A25

1 Introduction

In 1991 Soardi [8] introduced the sequence of positive linear operators β_n associating to each function $f \in C[0,1]$ the polynomial function

$$(\beta_n f)(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} f\left(\frac{n-2k}{n}\right) \tilde{w}_{n,k}(x),$$

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where

$$\tilde{w}_{n,k}(x) = \frac{n+1-2k}{(n+1)\,2^{n+1}x}\binom{n+1}{k}\left[(1-x)^k(1+x)^{n+1-k} - (1-x)^{n+1-k}(1+x)^k\right].$$

Usually, the operators β_n are given in the form

$$(\beta_n f)(x) = \sum_{k=0}^{m} f\left(\frac{n-2m+2k}{n}\right) w_{n,k}(x),$$

where $m = \lfloor n/2 \rfloor$ and $w_{n,k}(x) = \tilde{w}_{n,m-k}(x)$ are the fundamental polynomials. The definition and the proofs in [8] are based on properties of random walks on hypergroups. Soardi proved that, for each $f \in C[0,1]$, the sequence $(\beta_n f)$ is uniformly convergent to f. Furthermore, by an intensive use of probabilistic tools, Soardi [8, Theorem 2] estimated the rate of convergence of $(\beta_n f)$ in terms of the usual modulus of continuity:

$$\|\beta_n f - f\| \leq \left(55 + \frac{32}{n}\right)\omega\left(f; \frac{1}{\sqrt{n}}\right), \text{ for } f \in C[0,1].$$

Shape preserving properties of the operators β_n were investigated in [5–7]. In particular, if $f \in C[0,1]$ is increasing, then $\beta_n f$ is increasing (see [6,Th. 2.1]; this fact will be used in Sect. 3). Moreover, if $f \in C[0,1]$ is increasing and convex, then $\beta_n f \geq f$ (see [6,Th. 3.1]; this inequality will be instrumental in Sect. 5).

For $x \in (0,1)$ and bounded functions f on $[0,1]$, Raşa [6,Theorem 4.1] proved the Voronovskaja-type formula

$$(\beta_n f)(x) = f(x) + \frac{1}{n}\left[\left(\frac{1}{x} - 1\right) f'(x) + \frac{1-x^2}{2} f''(x)\right] + o(1/n)$$

as $n \to \infty$, provided that $f''(x)$ exists.

This paper is devoted to other properties of Soardi's operators. In Sect. 2 we introduce a version $\tilde{\beta}_n$ which can be expressed in terms of the classical Bernstein operators. The relations between β_n and $\tilde{\beta}_n$ are presented in Sect. 3. Section 4 contains Voronovskaja-type results for both β_n and $\tilde{\beta}_n$. Rates of convergence for $\tilde{\beta}_n$, respectively β_n, are estimated in Sects. 5 and 2. The last two sections are devoted to the first and second moments of β_n.

2 The variant $\tilde{\beta}_n$ and its relation to Bernstein polynomials

In this section we introduce a variant $\tilde{\beta}_n$ of Soardi's operator which seems to be more natural. Replacing $f\left(\frac{n-2k}{n}\right)$ with $f\left(\frac{n+1-2k}{n+1}\right)$ leads to the definition

$$\left(\tilde{\beta}_n f\right)(x) = \sum_{k=0}^{m} f\left(\frac{n+1-2k}{n+1}\right) \tilde{w}_{n,k}(x),$$

where $m = \lfloor n/2 \rfloor$. The index manipulation $k \to n+1-k$ yields

$$\left(\tilde{\beta}_n f\right)(x) = \sum_{k=n+1-m}^{n+1} f\left(-\frac{n+1-2k}{n+1}\right) \tilde{w}_{n,k}(x).$$

For even values of n we have

$$2\left(\tilde{\beta}_n f\right)(x) = \sum_{k=0}^{n+1} f\left(\left|\frac{n+1-2k}{n+1}\right|\right) \tilde{w}_{n,k}(x).$$

This representation is valid also in the case of odd integers n since the term $f(0)\,\tilde{w}_{n,\frac{n+1}{2}}(x)$ with $k = \frac{n+1}{2}$ is vanishing. Hence, for all $n \geq 0$,

$$\left(\tilde{\beta}_n f\right)(x) = \frac{1}{2} \sum_{k=0}^{n+1} f\left(\left|\frac{n+1-2k}{n+1}\right|\right) \tilde{w}_{n,k}(x).$$

Writing

$$\begin{aligned}\left(\tilde{\beta}_n f\right)(x) = \frac{1}{2x} \sum_{k=0}^{n+1} \frac{n+1-2k}{n+1} f\left(\left|1 - 2\frac{k}{n+1}\right|\right) \\ \times \binom{n+1}{k} \left[\left(\frac{1-x}{2}\right)^{k} \left(\frac{1+x}{2}\right)^{n+1-k} - \left(\frac{1-x}{2}\right)^{n+1-k} \left(\frac{1+x}{2}\right)^{k}\right]\end{aligned}$$

we obtain the following relation to the classical Bernstein polynomials.

Lemma 1 *For a function f on $[0, 1]$, we have the relation*

$$\left(\tilde{\beta}_n f\right)(x) = \frac{1}{2x} \left[(B_{n+1} g)\left(\frac{1-x}{2}\right) - (B_{n+1} g)\left(\frac{1+x}{2}\right)\right],$$

where

$$g(t) = (1-2t) f(|1-2t|)$$

and $B_n g$ denotes the classical Bernstein polynomial on $[0, 1]$.

3 Relations among the operators β_n and $\tilde{\beta}_n$

Consider the operators $\beta_n : C[0,1] \to C[0,1]$ and $\tilde{\beta}_n : C\left[\frac{1}{n+1}, 1\right] \to C[0,1]$. Let

$$u_n : \left[\frac{1}{n+1}, 1\right] \to [0,1], \; u_n(t) = \frac{(n+1)t - 1}{n}$$
$$v_n : [0,1] \to \left[\frac{1}{n+1}, 1\right], \; v_n(t) = \frac{nt+1}{n+1}.$$

Then, for $n = 1, 2, 3, \ldots$, $v_n = u_n^{-1}$. We have $\beta_n f = \tilde{\beta}_n (f \circ u_n)$, for $f \in C[0,1]$ and $\tilde{\beta}_n g = \beta_n (g \circ v_n)$, for $g \in C\left[\frac{1}{n+1}, 1\right]$. The shape preserving properties of β_n can be translated to $\tilde{\beta}_n$. In particular, let $h \in C^1[0,1]$. Then, the functions $\|h'\| e_1 \pm h$ are monotonically increasing, hence $\|h'\| \beta_n e_1 \pm \beta_n h$ are monotonically increasing. This implies $\|h'\| (\beta_n e_1)' \pm (\beta_n h)' \geq 0$, i.e.,

$$-\|h'\| (\beta_n e_1)' \leq (\beta_n h)' \leq \|h'\| (\beta_n e_1)'.$$

Since $0 \leq (\beta_n e_1)' \leq \frac{n-1}{n}$ (see [6,Theorem 2.1(i) and Rem. 2.3]) we obtain

$$\|(\beta_n h)'\| \leq \frac{n-1}{n} \|h'\|, \text{ for all} h \in C^1[0,1] \tag{1}$$

(see also [4,Ex. 4.1]).

Now let $g \in C^1\left[\frac{1}{n+1}, 1\right]$. Then

$$\left\|\left(\tilde{\beta}_n g\right)'\right\| = \|\beta_n (g \circ v_n)'\| \leq \frac{n-1}{n} \|(g \circ v_n)'\|$$
$$= \frac{n-1}{n} \|g'(v_n) v_n'\| \leq \frac{n-1}{n} \|g'\| \cdot \frac{n}{n+1},$$

i.e.,

$$\left\|\left(\tilde{\beta}_n g\right)'\right\| \leq \frac{n-1}{n+1} \|g'\|, \text{ for all} g \in C^1\left[\frac{1}{n+1}, 1\right]. \tag{2}$$

The inequalities (1) and (2) are instrumental in investigating the asymptotic behaviour of the iterates of β_n and $\tilde{\beta}_n$; see [4].

Let $f \in C[0,1]$. Then, with $\delta = \sqrt{\frac{3n+1}{(n+1)^2}\left(1 - x^2\right)}$, we obtain from Theorem 2 below

$$|(\beta_n f)(x) - f(x)| = \left|\left(\tilde{\beta}_n (f \circ u_n)\right)(x) - f(x)\right|$$
$$\leq \left|\left(\tilde{\beta}_n (f \circ u_n)\right)(x) - (f \circ u_n)(x)\right| + |(f \circ u_n)(x) - f(x)|$$
$$\leq 2\omega(f \circ u_n; \delta) + |f(u_n(x)) - f(x)|,$$

where

$$\begin{aligned}\omega(f\circ u_n;\delta) &= \sup\left\{|(f\circ u_n)(t_1)-(f\circ u_n)(t_2)| : \frac{1}{n+1}\le t_1,t_2\le 1, |t_1-t_2|\le\delta\right\}\\ &= \sup\left\{|f(u_n(t_1))-f(u_n(t_2))| : \frac{1}{n+1}\le t_1,t_2\le 1, |t_1-t_2|\le\delta\right\}\\ &= \sup\left\{|f(s_1)-f(s_2)| : 0\le s_1,s_2\le 1, |s_1-s_2|\le \frac{n+1}{n}\delta\right\}\\ &= \omega\left(f;\frac{n+1}{n}\delta\right).\end{aligned}$$

Thus

$$|(\beta_n f)(x)-f(x)|\le 2\omega\left(f;\frac{n+1}{n}\delta\right)+\omega\left(f;\frac{|1-x|}{n}\right).$$

Consequently,

$$\begin{aligned}|(\beta_n f)(x)-f(x)| &\le 2\omega\left(f;\frac{1}{n}\sqrt{(3n+1)\left(1-x^2\right)}\right)\\ &+\omega\left(f;\frac{1-x}{n}\right), \text{ for } f\in C[0,1].\end{aligned} \tag{3}$$

In particular,

$$|(\beta_n f)(x)-f(x)|\le 2\omega\left(f;\frac{1}{n}\sqrt{3n+1}\right)+\omega\left(f;\frac{1}{n}\right), \text{ for } f\in C[0,1].$$

See also Soardi's estimate [8, Theorem 2]

$$\|\beta_n f-f\|\le\left(55+\frac{32}{n}\right)\omega\left(f;\frac{1}{\sqrt{n}}\right), \text{ for } f\in C[0,1].$$

4 Voronovskaja-type results for the operators β_n and $\tilde{\beta}_n$

In 2000, Raşa [6, Theorem 4.1] proved the following Voronovskaja-type formula for the operators β_n.

Theorem 1 *Let* $x\in(0,1)$ *and* f *be a bounded function on* $[0,1]$. *If* $f''(x)$ *exists, then*

$$(\beta_n f)(x)=f(x)+\frac{1}{n}\left[\left(\frac{1}{x}-1\right)f'(x)+\frac{1-x^2}{2}f''(x)\right]+o(1/n)$$

as $n\to\infty$.

If $x \neq 0$, i.e., $t \neq 1/2$, you can insert the well-known asymptotic formulas for B_n. One obtains

$$\left(\tilde{\beta}_n f\right)(x) = f(x) + \frac{1-x^2}{n}\left[\frac{1}{x} f'(x) + \frac{1}{2} f''(x)\right] + o(1/n)$$

as $n \to \infty$. In the special case $x = 0$, we can use

$$\lim_{x \to 0} \frac{(1-x)^k (1+x)^{n+1-k} - (1-x)^{n+1-k} (1+x)^k}{x} = 2(n+1-2k)$$

in order to obtain

$$\left(\tilde{\beta}_n f\right)(0) = (n+1)\left(B_{n+1}\hat{g}\right)\left(\frac{1}{2}\right),$$

where

$$\hat{g}(t) = (1-2t)\, g(t) = (1-2t)^2 f(|1-2t|).$$

The asymptotic behaviour can easily be derived if f is an even function which is smooth in $x = 0$. If f is not an even function, $\left(B_{n+1}\hat{g}\right)\left(\frac{1}{2}\right)$ is an unpleasant expression.

The link to Soardi's original operator is given by

$$\beta_n f = \tilde{\beta}_n (f \circ u_n) \tag{4}$$

with $u_n(x) = ((n+1)t - 1)/n$. Therefore,

$$(\beta_n f)(x) = \left(\tilde{\beta}_n f_n\right)(x) = \left(\tilde{\beta}_n f\right)(x) + \frac{x-1}{n}\left(\tilde{\beta}_n f'\right)(x) + o(1/n)$$

as $n \to \infty$. A look into the proof of asymptotic formulas for Bernstein polynomials reveals that the latter formula is valid if f is only locally smooth.

We have

$$(B_n f)(x) \sim f(x) + \sum_{k=1}^{\infty} \frac{c_k(f,x)}{n^k} \quad (n \to \infty)$$

with

$$c_k(f,x) = \sum_{j=k}^{2k} a_{k,j}(x)\, f^{(j)}(x),$$

where $a_{k,j}(x)$ are certain polynomials involving Stirling numbers of the first and the second kind. More precisely, we have

$$(B_n f)(x) = f(x) + \sum_{k=1}^{q} n^{-k} \sum_{s=k}^{2k} \frac{1}{s!} f^{(s)}(x) \sum_{\nu=0}^{s} a(k, s, \nu)\; x^{s-\nu} + o\left(n^{-q}\right)$$

as $n \to \infty$, where

$$a(k, s, \nu) = \sum_{r=\max\{\nu,k\}}^{s} (-1)^{s-r} \binom{s}{r} S(r-\nu, r-k)\; \sigma(r, r-\nu),$$

provided that f is bounded on [0, 1] and admits a derivative of order $2q$ at $x \in [0, 1]$ (see [1, Remark 2]).

Let $f \in C[0, 1]$. We define f on $[-1, +1]$ such that f becomes an even function, i.e., $f(-x) = f(x)$. Put $\varphi(t) = 1 - 2t$. If $x \neq 0$, i.e., $t \neq 1/2$, we have

$$g(t) = \varphi(t) f(\varphi(t))$$

and

$$\begin{aligned} g^{(j)}(t) &= (-2)^j \left[(1-2t) f^{(j)}(\varphi(t)) + j f^{(j-1)}(\varphi(t))\right], \\ g^{(j)}\left(\frac{1-x}{2}\right) &= (-2)^j \left[x f^{(j)}(x) + j f^{(j-1)}(x)\right], \\ g^{(j)}\left(\frac{1+x}{2}\right) &= (-2)^j \left[-x f^{(j)}(-x) + j f^{(j-1)}(-x)\right] \\ &= -2^j \left[x f^{(j)}(x) + j f^{(j-1)}(x)\right]. \end{aligned}$$

Then

$$\left(\tilde{\beta}_{n-1} f\right)(x) = \frac{1}{x} (B_n g) \left(\frac{1-x}{2}\right) \sim f(x) + \sum_{k=1}^{\infty} \frac{x^{-1} c_k\left(g, \frac{1-x}{2}\right)}{n^k} \quad (n \to \infty).$$

5 An estimate of the rate of convergence for the operators $\tilde{\beta}_n$

In this section we derive an estimate for the rate of convergence for the operators $\tilde{\beta}_n$ in terms of the ordinary modulus of continuity $\omega(f, \delta)$.

Put $\tilde{g}(x) = g(1-x)$. Then $(B_n g)\left(\frac{1+x}{2}\right) = (B_n \mathbf{g})\left(\frac{1-x}{2}\right)$ and

$$\left(\tilde{\beta}_{n-1} f\right)(x) = \frac{1}{2x} (B_n (g - \tilde{g})) \left(\frac{1-x}{2}\right).$$

For functions of the form

$$g(t) = (1-2t)\, f(|1-2t|),$$

we have $\tilde{g} = -g$. Hence,

$$\left(\tilde{\beta}_{n-1} f\right)(x) = \frac{1}{x}\,(B_n g)\left(\frac{1-x}{2}\right). \tag{5}$$

Lemma 2 *For all $n \in \mathbb{N}$,*

$$\left(\tilde{\beta}_n e_1\right)(x) \geq x \; (x \in [0,1]).$$

Proof With the notations of Sect. 3 we have

$$\begin{aligned}\tilde{\beta}_n e_1 &= \beta_n (e_1 \circ v_n) = \beta_n v_n = \beta_n \left(\frac{n}{n+1} e_1 + \frac{1}{n+1} e_0\right) \\ &= \frac{n}{n+1} \beta_n e_1 + \frac{1}{n+1} \beta_n e_0.\end{aligned}$$

Since β_n preserves constant functions and $\beta_n f \geq f$, for all increasing and convex functions $f \in C[0,1]$, we obtain

$$\tilde{\beta}_n e_1 \geq \frac{n}{n+1} e_1 + \frac{1}{n+1} e_0 = e_1 + \frac{e_0 - e_1}{n+1} \geq e_1.$$

□

For reals t, x, put $\psi_x(t) = t - x$.

Lemma 3 *For all $n \in \mathbb{N}$, the second central moment of $\tilde{\beta}_n$ satisfies the estimate*

$$\left(\tilde{\beta}_n \psi_x^2\right)(x) \leq \frac{3n+1}{(n+1)^2}\left(1 - x^2\right) \quad (x \in [0,1]).$$

Remark 1 The constant on the right-hand side is best possible on [0, 1] because, for $x = 0$, we have $\left(\tilde{\beta}_n \psi_0^2\right)(0) = \left(\tilde{\beta}_n e_2\right)(0) = (3n+1)/(n+1)^2$.

Proof We have

$$\left(\tilde{\beta}_n \psi_x^2\right)(x) = \left(\tilde{\beta}_n e_2\right)(x) - 2x\left(\tilde{\beta}_n e_1\right)(x) + x^2\left(\tilde{\beta}_n e_0\right)(x) \leq \left(\tilde{\beta}_n e_2\right)(x) - x^2$$

on [0, 1], where we used the inequality of Lemma 2. The desired estimate now follows from

$$\left(\tilde{\beta}_{n-1} e_2\right)(x) = x^2 + \left(1 - x^2\right)(3n-2)/n^2.$$

□

Theorem 2 *Let* $f : C[0,1]$. *For all* $n \in \mathbb{N}$, *and* $\delta > 0$,

$$\left|\left(\tilde{\beta}_n f\right)(x) - f(x)\right| \leq \left(1 + \frac{1}{\delta}\sqrt{\frac{3n+1}{(n+1)^2}\left(1-x^2\right)}\right)\omega(f,\delta) \quad (x \in [0,1]).$$

Proof of Theorem 2 The estimate follows from Lemma 3 by standard arguments (see, e.g., [2,Theorem5.1.2]). □

Putting $\delta = \sqrt{3/(n+1)}$ immediately yields the following consequence.

Corollary 1 *For all* $n \in \mathbb{N}$,

$$\left|\left(\tilde{\beta}_n f\right)(x) - f(x)\right| \leq \left(1 + \sqrt{1-x^2}\right)\omega\left(f, \sqrt{\frac{3}{n+1}}\right) \quad (x \in [0,1]).$$

6 An estimate of rate of convergence for the Soardi operator

As already mentioned in the introduction Soardi [8,Theorem 2] estimated the rate of convergence of the operators β_n in terms of the ordinary modulus of continuity:

$$\|\beta_n f - f\| \leq \left(55 + \frac{32}{n}\right)\omega\left(f; \frac{1}{\sqrt{n}}\right), \text{for} f \in C[0,1].$$

In this section we improve this estimate considerably by diminishing the absolute constant.

Theorem 3 *Let* $f : C[0,1]$. *For all* $n \in \mathbb{N}$, *Soardi's operator* β_n *satisfies the estimate*

$$\|\beta_n f - f\| \leq \left(1 + \sqrt{3 + \frac{2}{n}}\right)\omega\left(f; \frac{1}{\sqrt{n}}\right) \quad (n \in \mathbb{N}).$$

Remark 2 In particular, we have

$$\|\beta_n f - f\| \leq c \cdot \omega\left(f; \frac{1}{\sqrt{n}}\right) \quad (n \in \mathbb{N}),$$

where $c = \left(1 + \sqrt{5}\right) \approx 3.236$.

The essential ingredient of the proof is the following estimate of the second central moment of the operators β_n.

Lemma 4 *For all* $n \in \mathbb{N}$, *the second central moment of* β_n *satisfies the estimate*

$$\left(\beta_n \psi_x^2\right)(x) \leq \frac{3}{n}\left(1 - x^2\right) + \frac{2}{n^2}(1-x) \quad (x \in [0,1]).$$

Remark 3 Since $1-x \le 1-x^2$ on $[0,1]$, we have

$$\left(\beta_n \psi_x^2\right)(x) \le \frac{5}{n}\left(1-x^2\right) \quad (x \in [0,1]) .$$

Furthermore, for each $\varepsilon > 0$, there is an index n_0 such that for each $n > n_0$,

$$\left(\beta_n \psi_x^2\right)(x) \le \frac{3+\varepsilon}{n}\left(1-x^2\right) \quad (x \in [0,1]) .$$

Proof of Lemma 4 Using the relation $\beta_n f = \tilde{\beta}_n \left(f \circ u_n\right)$ from Sect. 3 with $u_n(x) = ((n+1)t-1)/n$ we obtain

$$\begin{aligned}\beta_n \psi_x^2 &= \left(\frac{n+1}{n}\right)^2 \tilde{\beta}_n \left(e_1 - \frac{nx+1}{n+1} e_0\right)^2 \\ &= \left(\frac{n+1}{n}\right)^2 \left[\tilde{\beta}_n e_2 - 2\frac{nx+1}{n+1}\tilde{\beta}_n e_1 + \left(\frac{nx+1}{n+1}\right)^2 e_0\right].\end{aligned}$$

By Lemma 2,

$$\begin{aligned}\left(\beta_n \psi_x^2\right)(x) &\le \left(\frac{n+1}{n}\right)^2 \left[x^2 + \left(1-x^2\right)\frac{3n+1}{(n+1)^2} - 2\frac{nx+1}{n+1}x + \left(\frac{nx+1}{n+1}\right)^2\right] \\ &= \frac{3n+1}{n^2}\left(1-x^2\right) + \left(\frac{n+1}{n}\right)^2 \left(x - \frac{nx+1}{n+1}\right)^2 \\ &= \frac{3n+1}{n^2}\left(1-x^2\right) + \left(\frac{x-1}{n}\right)^2 \\ &= \frac{3}{n}\left(1-x^2\right) + \frac{1-x^2+(x-1)^2}{n^2} \\ &= \frac{3}{n}\left(1-x^2\right) + \frac{2}{n^2}(1-x),\end{aligned}$$

which is the desired estimate. □

Proof of Theorem 3 By Lemma 4, it holds

$$\left(\beta_n \psi_x^2\right)(x) \le \frac{3}{n} + \frac{2}{n^2} \quad (x \in [0,1]) .$$

Using [2,(5.1.5)], we obtain

$$\begin{aligned}\left|\left(\beta_n f\right)(x) - f(x)\right| &\le \left(1 + \sqrt{n\left(\beta_n \psi_x^2\right)(x)}\right)\omega\left(f; \frac{1}{\sqrt{n}}\right) \\ &\le \left(1 + \sqrt{3 + \frac{2}{n}}\right)\omega\left(f; \frac{1}{\sqrt{n}}\right).\end{aligned}$$

This completes the proof. □

7 The second moment of β_n

We have

$$\begin{aligned}(\beta_n e_2)(x) &= \left(\tilde{\beta}_n u_n^2\right)(x) = \left(\tilde{\beta}_n \left(\frac{n+1}{n} e_1 - \frac{1}{n} e_0\right)^2\right)(x) \\ &= \left(\frac{n+1}{n}\right)^2 \left(\tilde{\beta} e_2\right)(x) + \frac{1}{n^2} - 2\frac{n+1}{n^2}\left(\tilde{\beta}_n e_1\right)(x).\end{aligned}$$

Since

$$\left(\tilde{\beta} e_2\right)(x) = x^2 + \frac{3n+1}{(n+1)^2}\left(1 - x^2\right) \text{and } \left(\tilde{\beta}_n e_1\right)(x) \geq x$$

we obtain

$$\begin{aligned}(\beta_n e_2)(x) &\leq \left(\frac{n+1}{n}\right)^2 \left(x^2 + \frac{3n+1}{(n+1)^2}\left(1 - x^2\right)\right) + \frac{1}{n^2} - 2\frac{n+1}{n^2} x \\ &\leq \left(\frac{n+1}{n}\right)^2 x^2 + \frac{3n+1}{n^2} - \frac{3n+1}{n^2} x^2 + \frac{1}{n^2} - \frac{2n+2}{n^2} x \\ &= x^2 + \frac{1}{n^2}\left(-nx^2 - (2n+2)x + 3n + 2\right) \\ &= x^2 + \frac{1-x}{n^2}(nx + 3n + 2).\end{aligned}$$

It follows

$$0 \leq (\beta_n e_2)(x) - x^2 \leq \frac{nx + 3n + 2}{n^2}(1 - x).$$

8 The value $(\beta_n e_1)(0)$ of the first moment

The operator β_n does not reproduce the function $e_1(x) = x$, $x \in [0, 1]$. But $\beta_n e_1$ is increasing and convex ([6,Th. 2.1]), $\beta_n e_1 \geq e_1$ ([6,Th. 3.1]), and $\beta_n e_1(1) = 1$. Consequently,

$$0 \leq \beta_n e_1(x) - x \leq \beta_n e_1(0)(1 - x), \text{for} x \in [0, 1].$$

So, we need a good control on $\beta_n e_1(0)$. This is our aim in what follows.

By Eq. (3) , we infer that

$$x \leq (\beta_n e_1)(x) \leq x + \frac{2}{n}\sqrt{(3n+1)\left(1-x^2\right)} + \frac{1-x}{n}, \text{for} \quad x \in [0,1].$$

In particular, it follows that

$$0 \leq (\beta_n e_1)(0) \leq \frac{1+2\sqrt{3n+1}}{n} \sim 2\sqrt{\frac{3}{n}} \quad (n \to \infty).$$

In the next section we derive closed expressions for $(\beta_n e_1)(0)$ and study its asymptotic behaviour as n tends to infinity. We prove that the exact asymptotic rate of convergence is

$$(\beta_n e_1)(0) \sim \frac{2\sqrt{2}}{\sqrt{\pi n}} \quad (n \to \infty).$$

Note that $2\sqrt{3} \approx 3.4641$ and $2\sqrt{2/\pi} \approx 1.59577$.

Theorem 4 *At* $x = 0$*, the first moment of Soardi's operator has the explicit representation*

$$(\beta_{2n} e_1)(0) = \frac{1}{2^{2n}}\left(2 + \frac{1}{2n}\right)\binom{2n}{n} - \frac{1}{2n},$$
$$(\beta_{2n-1} e_1)(0) = \frac{1}{2^{2n-1}}\left(1 + \frac{1}{2n-1}\right)\binom{2n}{n} - \frac{1}{2n-1}$$

and satisfies the asymptotic relation

$$(\beta_n e_1)(0) = \frac{2\sqrt{2}}{\sqrt{\pi n}} - \frac{1}{n} + O\left(n^{-3/2}\right) \quad (n \to \infty).$$

Proof Since

$$\lim_{x \to 0} \frac{1}{x}\left[(1-x)^k (1+x)^{n+1-k} - (1-x)^{n+1-k}(1+x)^k\right] = 2(n+1-2k),$$

we have

$$\tilde{w}_{n,k}(0) = \binom{n+1}{k}\frac{2(n+1-2k)^2}{(n+1)2^{n+1}}$$

and

$$(\beta_n e_r)(0) = \frac{1}{2^n (n+1) n^r}\sum_{k=0}^{m}\binom{n+1}{k}(n+1-2k)^2 (n-2k)^r.$$

Although one can calculate it for arbitrary $r \in \mathbb{N}$, we restrict ourselves to $r = 1$. Let us first consider the case of even parameters $2n$:

$$\left(\beta_{2n} e_1\right)(0) = \frac{1}{2^{2n}\,(2n+1)\,n} \sum_{k=0}^{n} \binom{2n+1}{k} (2n+1-2k)^2\,(n-k)\,.$$

Writing

$$\begin{aligned}(2n+1-2k)^2\,(n-k) &= -4k^{\underline{3}} + 4\,(3n-2)\,k^{\underline{2}} \\ &+ \left(-12n^2+4n-1\right) k + n\,(2n+1)^2\end{aligned}$$

we obtain

$$\begin{aligned}&\sum_{k=0}^{n} \binom{2n+1}{k} (2n+1-2k)^2\,(n-k) \\ &= -4\,(2n+1)^{\underline{3}} \sum_{k=3}^{n} \binom{2n-2}{k-3} + 4\,(3n-2)\,(2n+1)^{\underline{2}} \sum_{k=2}^{n} \binom{2n-1}{k-2} \\ &\quad + \left(-12n^2+4n-1\right)(2n+1) \sum_{k=1}^{n} \binom{2n}{k-1} + n\,(2n+1)^2 \sum_{k=0}^{n} \binom{2n+1}{k} \\ &= -4\,(2n+1)^{\underline{3}} \sum_{k=0}^{n-3} \binom{2n-2}{k} + 4\,(3n-2)\,(2n+1)^{\underline{2}} \sum_{k=0}^{n-2} \binom{2n-1}{k} \\ &\quad + \left(-12n^2+4n-1\right)(2n+1) \sum_{k=0}^{n-1} \binom{2n}{k} + n\,(2n+1)^2 \sum_{k=0}^{n} \binom{2n+1}{k} \\ &=: A + B + C + D.\end{aligned}$$

Now

$$\begin{aligned}A &= -4\,(2n+1)^{\underline{3}}\, \frac{2^{2n-2} - \binom{2n-2}{n-2} - \binom{2n-2}{n-1} - \binom{2n-2}{n}}{2} \\ &= -2^{2n-1}\,(2n+1)^{\underline{3}} + 4\,(2n+1)\left[n\,(n-1)\binom{2n}{n} + \frac{n^2}{2}\binom{2n}{n}\right] \\ &= -2^{2n-1}\,(2n+1)^{\underline{3}} + 2n\,(2n+1)\,(3n-2)\binom{2n}{n}, \\ B &= 4\,(3n-2)\,(2n+1)^{\underline{2}}\, \frac{2^{2n-1} - \binom{2n-1}{n-1} - \binom{2n-1}{n}}{2} \\ &= 2^{2n}\,(3n-2)\,(2n+1)^{\underline{2}} - 2\,(3n-2)\,(2n+1)^{\underline{2}}\binom{2n}{n}, \\ C &= \left(-12n^2+4n-1\right)(2n+1)\, \frac{2^{2n} - \binom{2n}{n}}{2},\end{aligned}$$

$$D = 2^{2n} n\,(2n+1)^2\,.$$

Finally,

$$\begin{aligned}(\beta_{2n}e_1)(0) &= \frac{1}{2^{2n}(2n+1)\,n}(A+B+C+D)\\ &= -(2n-1)+(6n-4)+\left(-6n+2-\frac{1}{2n}\right)+(2n+1)\\ &\quad+\frac{1}{2^{2n}}\binom{2n}{n}\left[2(3n-2)-4(3n-2)+\left(6n-2+\frac{1}{2n}\right)\right]\\ &= -\frac{1}{2n}+\frac{1}{2^{2n}}\binom{2n}{n}\left(2+\frac{1}{2n}\right).\end{aligned}$$

The well-known asymptotic behaviour of the central binomial coefficient (cf. Catalan constant $\frac{1}{n+1}\binom{2n}{n}$)

$$\binom{2n}{n} = \frac{4^n}{\sqrt{\pi n}}\left(1+O\left(n^{-1}\right)\right) \quad (n\to\infty)$$

leads to the asymptotic formula

$$(\beta_{2n}e_1)(0) = \frac{2}{\sqrt{\pi n}} - \frac{1}{2n} + O\left(n^{-3/2}\right) \quad (n\to\infty)\,.$$

Now we consider the case of odd parameters $2n-1$:

$$(\beta_{2n-1}e_1)(0) = \frac{1}{2^{2n-2}n\,(2n-1)}\sum_{k=0}^{n-1}\binom{2n}{k}(n-k)^2(2n-1-2k)\,.$$

Writing

$$(n-k)^2(2n-1-2k) = -2k^{\underline{3}} + (6n-7)\,k^{\underline{2}} + \left(-6n^2+8n-3\right)k + n^2(2n-1)$$

we obtain

$$\begin{aligned}&\sum_{k=0}^{n-1}\binom{2n}{k}(n-k)^2(2n-1-2k)\\ &= -2\,(2n)^{\underline{3}}\sum_{k=3}^{n-1}\binom{2n-3}{k-3} + (6n-7)\,(2n)^{\underline{2}}\sum_{k=2}^{n-1}\binom{2n-2}{k-2}\\ &\quad+\left(-6n^2+8n-3\right)(2n)\sum_{k=1}^{n-1}\binom{2n-1}{k-1} + n^2(2n-1)\sum_{k=0}^{n-1}\binom{2n}{k}\\ &=: A+B+C+D.\end{aligned}$$

Now

$$A = -2\,(2n)^{\underline{3}} \sum_{k=0}^{n-4} \binom{2n-3}{k} = -2\,(2n)^{\underline{3}}\, \frac{2^{2n-3} - 2\binom{2n-3}{n-3} - 2\binom{2n-3}{n-2}}{2}$$

$$= -\,(2n)^{\underline{3}}\, 2^{2n-3} + 2\binom{2n}{n} n\,(n-1)\,(n-2) + 2\binom{2n}{n} n^2\,(n-1)$$

$$= -\,(2n)^{\underline{3}}\, 2^{2n-3} + 4n\,(n-1)^2 \binom{2n}{n},$$

$$B = (6n-7)\,(2n)^{\underline{2}} \sum_{k=0}^{n-3} \binom{2n-2}{k} = (6n-7)\,(2n)^{\underline{2}}\, \frac{2^{2n-2} - 2\binom{2n-2}{n-2} - \binom{2n-2}{n-1}}{2}$$

$$= (6n-7)\left[(2n)^{\underline{2}}\, 2^{2n-3} - \binom{2n}{n} n\,(n-1) - \frac{1}{2}\binom{2n}{n} n^2\right]$$

$$= (6n-7)\,(2n)^{\underline{2}}\, 2^{2n-3} - \frac{1}{2}\,(6n-7)\,n\,(3n-2)\binom{2n}{n},$$

$$C = \left(-6n^2 + 8n - 3\right)(2n) \sum_{k=0}^{n-2} \binom{2n-1}{k}$$

$$= \left(-6n^2 + 8n - 3\right)(2n)\, \frac{2^{2n-1} - 2\binom{2n-1}{n-1}}{2}$$

$$= \left(-6n^2 + 8n - 3\right) n \left(2^{2n-1} - \binom{2n}{n}\right),$$

$$D = n^2\,(2n-1)\,\frac{1}{2}\left(2^{2n} - \binom{2n}{n}\right).$$

Finally,

$$\begin{aligned}
&(\beta_{2n-1} e_1)\,(0) \\
&= \frac{1}{2^{2n-2} n\,(2n-1)}\,(A + B + C + D) \\
&= -\,(2n-2) + (6n-7) + \frac{2\left(-6n^2 + 8n - 3\right)}{2n-1} + \frac{2n\,(2n-1)}{2n-1} \\
&\quad + \frac{1}{2^{2n-2}\,(2n-1)}\binom{2n}{n} \\
&\quad \left[4\,(n-1)^2 - \frac{1}{2}\,(6n-7)\,(3n-2) + \left(6n^2 - 8n + 3\right) - \frac{1}{2} n\,(2n-1)\right] \\
&= 4n - 5 + \frac{-8n^2 + 14n - 6}{2n-1} + \frac{n}{2^{2n-2}\,(2n-1)}\binom{2n}{n} \\
&= \frac{-1}{2n-1} + \frac{2\,(2n-1) + 2}{2^{2n}\,(2n-1)}\binom{2n}{n} = \frac{-1}{2n-1} + \frac{1}{2^{2n}}\binom{2n}{n}\left(2 + \frac{2}{2n-1}\right).
\end{aligned}$$

This proves the explicit representation for odd values of the parameter. As above we obtain the asymptotic formula

$$(\beta_{2n-1}e_1)(0) = \frac{2}{\sqrt{\pi n}} - \frac{1}{2n-1} + O\left(n^{-3/2}\right) \quad (n \to \infty).$$

This completes the proof. □

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