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General reiteration theorems for $\mathcal{R}$ and $\mathcal{L}$ classes: mixed interpolation of $\mathcal{R}$ and $\mathcal{L}$-spaces

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Abstract

Given E_0, E_1, F_0, F_1, E rearrangement invariant function spaces, $a_0, a_1, \mathrm{b}_0, \mathrm{b}_1, \mathrm{b}$ slowly varying functions and $0 < \theta_0 < \theta_1 < 1$, we characterize the interpolation spaces

$$(\overline{X}^{\mathcal{R}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1})_{\theta,\mathrm{b},E}, \quad (\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{L}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1})_{\theta,\mathrm{b},E}$$

and

$$(\overline{X}^{\mathcal{R}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{L}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1})_{\theta,\mathrm{b},E}, \quad (\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1})_{\theta,\mathrm{b},E},$$

for all possible values of $\theta \in [0, 1]$. Applications to interpolation identities for grand and small Lebesgue spaces, Gamma spaces and A and B-type spaces are given.

Keywords Real interpolation · K-functional · Reiteration theorems · Slowly varying functions · Rearrangement invariant spaces

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1 Introduction

Reiteration theorems play an important role in Interpolation Theory since, in particular, they allow to identify interpolation spaces for many different couples. The classical results are covered in the monographs [6, 7, 9, 50], but there is also an extensive

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literature concerning explicit reiteration formulas in various special cases, see e.g. [1, 4, 11, 13, 19, 32, 37, 38].

In this paper we continue the study started in [23, 24] about reiteration theorems for couples formed by the various possible combinations of the spaces

$$\overline{X}_{\theta,\mathrm{b},E}, \quad \overline{X}^{\mathcal{R}}_{\theta,\mathrm{b},E,a,F}, \quad \overline{X}^{\mathcal{L}}_{\theta,\mathrm{b},E,a,F}$$

where $\theta \in [0,1]$, a and b are slowly varying functions (SV), and E, F are rearrangement invariant (r.i.) function spaces; see Sect. 2 for precise definitions.

In this occasion we shall focus on the reiteration results for couples formed only by combinations of $\mathcal{R}$ and $\mathcal{L}$-spaces, specifically

$$\big(\overline{X}^{\mathcal{R}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1}\big)_{\theta,\mathrm{b},E}, \ \big(\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{L}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1}\big)_{\theta,\mathrm{b},E} \tag{1.1}$$

and

$$\big(\overline{X}^{\mathcal{R}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{L}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1}\big)_{\theta,\mathrm{b},E}, \ \big(\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1}\big)_{\theta,\mathrm{b},E}, \tag{1.2}$$

when $0 < \theta_0 < \theta_1 < 1$ and $\theta \in [0,1]$.

In particular, we shall show that, when $\theta \in (0,1)$, one can identify all the spaces in (1.1) and (1.2) as classical interpolation spaces of the form

$$\overline{X}_{\tilde{\theta},B_\theta,E}$$

where $\tilde{\theta} = (1-\theta)\theta_0 + \theta\theta_1$ and B_θ depends on all the parameters. On the other hand, in the limiting cases $\theta = 0, 1$, the resulting reiteration spaces are described in terms of the extremal scales

$$\overline{X}^{\mathcal{R},\mathcal{R}}_{\theta,c,E,\mathrm{b},F,a,G}, \quad \overline{X}^{\mathcal{L},\mathcal{L}}_{\theta,c,E,\mathrm{b},F,a,G}, \quad \overline{X}^{\mathcal{R},\mathcal{L}}_{\theta,c,E,\mathrm{b},F,a,G} \quad \text{and} \quad \overline{X}^{\mathcal{L},\mathcal{R}}_{\theta,c,E,\mathrm{b},F,a,G}, \tag{1.3}$$

that were introduced in the earlier works [23, 24]. Since the precise statements are lengthy, we refer the reader to Theorems 4.3 through 4.6 in Sect. 4 below.

As in [23, 24], the present work finds its motivation in several recent applications to the interpolation of *grand* and *small Lebesgue* spaces $L^{p),\alpha}$, $L^{(p,\alpha}$, $1 < p < \infty$, $\alpha > 0$. As observed in [26, 40], these can be identified as $\mathcal{R}$ and $\mathcal{L}$-spaces with respect to the couple (L_1, L_∞) in the following way

$$L^{p),\alpha} = (L_1, L_\infty)^{\mathcal{R}}_{1-\frac{1}{p},\ell^{-\frac{\alpha}{p}}(t),L_\infty,1,L_p} \quad \text{and} \quad L^{(p,\alpha} = (L_1, L_\infty)^{\mathcal{L}}_{1-\frac{1}{p},\ell^{\frac{\alpha}{p'}-1}(t),L_1,1,L_p},$$

where $\ell(t) = 1 + |\log(t)|$, $t \in (0,1)$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Therefore, as a direct application of the theorems in this paper, one obtains an explicit identification of the spaces

$$\big(L^{p_0),\alpha}, L^{p_1),\beta}\big)_{\theta,\mathrm{b},E}, \qquad \big(L^{(p_0,\alpha}, L^{(p_1,\beta}\big)_{\theta,\mathrm{b},E}, \qquad \big(L^{(p_0,\alpha}, L^{p_1),\beta}\big)_{\theta,\mathrm{b},E} \tag{1.4}$$

and

$$\big(L^{p_0),\alpha}, L^{(p_1,\beta}\big)_{\theta,\mathrm{b},E}. \tag{1.5}$$

Namely, when $\theta \in (0, 1)$ these are described as spaces of *Lorentz-Karamata type*, $L_{p,B_\theta,E}$, while for $\theta = 0, 1$ the description involves the extremal spaces in (1.3); see precise statements in Corollaries 5.5, 5.7, 5.9 and 5.10 below.

As a special example, which also illustrates the advantage of using as parameters r.i. spaces, consider the case when E is equal to the *Orlicz space* L_M. Then, we shall obtain in particular the following identification

$$\big(L^{p_0),\alpha}, L^{p_1),\beta}\big)_{\theta,1,L_M} = \Lambda(t^{1/p}\ell^{-A}(t), M), \qquad 0 < \theta < 1,$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $A = \frac{\alpha(1-\theta)}{p_0} + \frac{\beta\theta}{p_1}$, and $\Lambda(t^{1/p}\ell^{-A}(t), M)$ is a *Lorentz-Orlicz space*, as defined by Torchinsky in [49]. This result seems new in the literature and may have an independent interest; see §5.2.1 below for details.

Our general reiteration theorems also apply to other $\mathcal{R}$ and $\mathcal{L}$-families, such as the Gamma spaces (see [28]) and the spaces $A_{p,\alpha,E}$ and $B_{p,\alpha,E}$ defined by Pustylnik in [44]. Some of these applications are new in the literature, and have been carried out in detail in Sects. 5.3 and 5.4 .

In the case $\theta \in (0, 1)$, $\mathrm{b} \in SV$ and $E = L_q$, $0 < q \leq \infty$, a very recent work of Doktorski [14] proved (1.1)-(1.2) using a limiting reiteration result, and characterizes (1.4)-(1.5) as corollaries. Our approach in this paper is slightly different to the one used by Doktorski and has the advantage of being applicable also in the extreme cases $\theta = 0, 1$ and when E belongs to the richer family of r.i. spaces. As in [23, 24], the main points in our proofs are explicit identities for the K-functional of all cases in the form of Holmstedt type formulae (see Sect. 3), a change of variables (Lemma 4.1) and equivalences between norms (Lemma 4.2). Additionally, in the case $\theta \in (0, 1)$ and $E = L_q$, $0 < q \leq \infty$, with the additional condition $\mathrm{b} \equiv 1$, Fiorenza et al. [2, 3, 28] have obtained the interpolation spaces in (1.4) via a direct computation of the K-functional or using certain limiting reiteration formulae.

The paper is organized as follows. In Sect. 2 we recall basic concepts regarding rearrangement invariant spaces and slowly varying functions. We also describe the interpolation methods we shall work with, namely $\overline{X}_{\theta,\mathrm{b},E}$, the $\mathcal{R}$ and $\mathcal{L}$-type spaces and the extremal constructions. Generalized Holmstedt type formulae for the K-functional of the different couples can be found in Sect. 3. The statements and proofs of reiteration theorems appear in Sect. 4. Finally, Sect. 5 contains the applications to grand and small Lebesgue spaces, Gamma spaces and A and B-type spaces.

2 Preliminaries

We refer to the monographs [6, 7, 9, 35, 50] for basic concepts on Interpolation Theory and Banach function spaces. A Banach function space E on $(0, \infty)$ is called *rearrangement invariant* (r.i.) if, for any two measurable fuctions f, g,

$$g \in E \text{ and } f^* \leq g^* \Longrightarrow f \in E \text{ and } \|f\|_E \leq \|g\|_E,$$

where f^* and g^* stand for the non-increasing rearrangements of f and g. Following [6], we assume that every Banach function space E enjoys the *Fatou property*. Under this assumption every r.i. space E is an exact interpolation space with respect to the Banach couple (L_1, L_∞).

Along this paper we will handle two different measures on $(0, \infty)$: the usual Lebesgue measure dt and the homogeneous measure $\frac{dt}{t}$. We use a tilde to denote r.i. spaces with respect to the second measure. For example

$$\|f\|_{\widetilde{L}_1} = \int_0^\infty |f(t)| \frac{dt}{t} \quad \text{and} \quad \|f\|_{\widetilde{L}_\infty} = \|f\|_{L_\infty}.$$

If E is an r.i. obtained by applying the interpolation functor $\mathcal{F}$ to the couple (L_1, L_∞), $E = \mathcal{F}(L_1, L_\infty)$, we will denote by $\widetilde{E}$ the space generated by the same $\mathcal{F}$ acting on the couple $(\widetilde{L}_1, L_\infty)$, $\widetilde{E} = \mathcal{F}(\widetilde{L}_1, L_\infty)$. It should be pointed out that there are simple formulae which directly connect the norms of the spaces E and $\widetilde{E}$, without any reference to the parameter D. For all measurable function $f : (0, \infty) \longrightarrow (0, \infty)$ we have

$$\|f\|_{\widetilde{E}(0,1)} = \|f(e^{-u})\|_E \quad \text{and} \quad \|f\|_{\widetilde{E}(1,\infty)} = \|f(e^u)\|_E.$$

In the rest of the paper we will denote by $\overline{f}$ the measurable function defined by

$$\overline{f}(t) = f(1/t), \quad t > 0.$$

Moreover, given $t > 0$, it is sometimes convenient to divide the interval $(0, \infty)$ into two subintervals $(0, t)$ and (t, ∞), considering the spaces $\widetilde{E}(0, t)$ and $\widetilde{E}(t, \infty)$ separately. Observe that $f \in \widetilde{E}(t, \infty)$ if and only if $\overline{f} \in \widetilde{E}(0, 1/t)$ and

$$\|f\|_{\widetilde{E}(t,\infty)} = \|\overline{f}\|_{\widetilde{E}(0,1/t)}. \tag{2.1}$$

Throughout the paper, given two (quasi-) Banach spaces X and Y, we will write $X = Y$ if $X \hookrightarrow Y$ and $Y \hookrightarrow X$, where the latter means that $Y \subset X$ and the natural embedding is continuous. Similarly, $f \sim g$ means that $f \lesssim g$ and $g \lesssim f$, where $f \lesssim g$ is the abbreviation of $f(t) \leq Cg(t)$, $t > 0$, for some positive constant C independent of f and g.

2.1 Slowly varying functions

In this subsection we recall the definition and basic properties of *slowly varying functions*. See [8, 36].

Definition 2.1 A positive Lebesgue measurable function b, $0 \not\equiv \mathrm{b} \not\equiv \infty$, is said to be *slowly varying* on $(0, \infty)$ (notation $b \in SV$) if, for each $\varepsilon > 0$, the function $t \rightsquigarrow t^\varepsilon \mathrm{b}(t)$

is equivalent to a non-decreasing function on $(0,\infty)$ and $t \rightsquigarrow t^{-\varepsilon}\mathrm{b}(t)$ is equivalent to a non-increasing function on $(0,\infty)$.

Examples of SV-functions include powers of logarithms,

$$\ell^{\alpha}(t) = (1 + |\log t|)^{\alpha}(t), \quad t > 0, \quad \alpha \in \mathbb{R},$$

"broken" logarithmic functions defined as

$$\ell^{(\alpha,\beta)}(t) = \begin{cases} \ell^{\alpha}(t), \ 0 < t \le 1 \\ \ell^{\beta}(t), \ t > 1 \end{cases}, \quad (\alpha, \beta) \in \mathbb{R}^2, \tag{2.2}$$

reiterated logarithms $(\ell \circ \ldots \circ \ell)^{\alpha}$, $\alpha \in \mathbb{R}$, $t > 0$ and also functions as $t \rightsquigarrow \exp(|\log t|^{\alpha})$, $\alpha \in (0,1)$.

Some basic properties of slowly varying functions are summarized in the following lemmas.

Lemma 2.2 *Let* $\mathrm{b}, \mathrm{b}_1, \mathrm{b}_2 \in SV$.

(i) Then $\mathrm{b}_1\mathrm{b}_2 \in SV$, $\overline{\mathrm{b}} \in SV$ *and* $\mathrm{b}^r \in SV$ *for all* $r \in \mathbb{R}$.
(ii) If $\alpha > 0$, *then* $\mathrm{b}(t^{\alpha}\mathrm{b}_1(t)) \in SV$.
(iii) If $\epsilon, s > 0$ *then there are positive constants* c_ϵ *and* C_ϵ *such that*

$$c_\epsilon \min\{s^{-\epsilon}, s^{\epsilon}\}\mathrm{b}(t) \le \mathrm{b}(st) \le C_\epsilon \max\{s^{\epsilon}, s^{-\epsilon}\}\mathrm{b}(t) \quad \textit{for every} \quad t > 0.$$

Lemma 2.3 *Let* E *be an r.i. space and* $\mathrm{b} \in SV$.

(i) If $\alpha > 0$, *then, for all* $t > 0$,

$$\|s^{\alpha}\mathrm{b}(s)\|_{\widetilde{E}(0,t)} \sim t^{\alpha}\mathrm{b}(t) \quad \textit{and} \quad \|s^{-\alpha}\mathrm{b}(s)\|_{\widetilde{E}(t,\infty)} \sim t^{-\alpha}\mathrm{b}(t).$$

(ii) If $\alpha \in \mathbb{R}$, *then, for all* $t > 0$,

$$\|s^{\alpha}\mathrm{b}(s)\|_{\widetilde{E}(t,2t)} \sim t^{\alpha}\mathrm{b}(t).$$

(iii) The following functions are slowly varying

$$\Phi_0(t) := \|\mathrm{b}(s)\|_{\widetilde{E}(0,t)} \quad \textit{and} \quad \Phi_\infty(t) := \|\mathrm{b}(s)\|_{\widetilde{E}(t,\infty)}, \quad t > 0.$$

(iv) For all $t > 0$,

$$\mathrm{b}(t) \lesssim \|\mathrm{b}(s)\|_{\widetilde{E}(0,t)} \quad \textit{and} \quad \mathrm{b}(t) \lesssim \|\mathrm{b}(s)\|_{\widetilde{E}(t,\infty)}.$$

We refer to [20, 32] for the proof of Lemma 2.2 and 2.3, respectively.

Remark 2.4 The property (iii) of Lemma 2.2 implies that if $\mathrm{b} \in SV$ is such that $\mathrm{b}(t_0) = 0$ ($\mathrm{b}(t_0) = \infty$) for some $t_0 > 0$, then $\mathrm{b} \equiv 0$ ($\mathrm{b} \equiv \infty$). Thus, by Lemma 2.3

(iii), if $\|\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty$ then $\|\mathrm{b}\|_{\widetilde{E}(0,t)} < \infty$ for all $t > 0$ and if $\|\mathrm{b}\|_{\widetilde{E}(1,\infty)} < \infty$ then $\|\mathrm{b}\|_{\widetilde{E}(t,\infty)} < \infty$ for all $t > 0$.

Moreover, if $f \sim g$ then, using Definition 2.1 and Lemma 2.2 (iii), one can show that $\mathrm{b} \circ f \sim \mathrm{b} \circ g$ for any $b \in SV$.

2.2 Interpolation methods

Everywhere below $\overline{X} = (X_0, X_1)$ is a *compatible (quasi-) Banach couple*, that is, two (quasi-) Banach spaces continuously embedded in a Hausdorff topological vector space. The *Peetre K-funtional* $K(t, f; X_0, X_1) \equiv K(t, f)$ is defined for $f \in X_0+X_1$ and $t > 0$ by

$$K(t, f) = \inf \left\{ \|f_0\|_{X_0} + t\|f_1\|_{X_1} : \ f = f_0 + f_1, \ f_i \in X_i, \ i = 0, 1 \right\}.$$

It is well-known that for each $f \in X_0 + X_1$, the function $t \rightsquigarrow K(t, f)$ is non-decreasing, while $t \rightsquigarrow t^{-1}K(t, f)$, $t > 0$, is non-increasing. Other important property of the K-functional is the fact that

$$K(t, f; X_0, X_1) = tK(t^{-1}, f; X_1, X_0) \quad \text{for all } t > 0, \tag{2.3}$$

(see [6,Chap. 5, Proposition 1.2]).

We now recall the definition and some properties of the real interpolation method $\overline{X}_{\theta,\mathrm{b},E}$, the limiting $\mathcal{R}$ and $\mathcal{L}$ constructions and the extremal interpolation spaces. We refer to the papers [20, 22–24] for the major part of the results of this subsection, and for more information about these constructions.

Definition 2.5 Let E be an r.i. space, $\mathrm{b} \in SV$ and $0 \le \theta \le 1$. The real interpolation space $\overline{X}_{\theta,\mathrm{b},E} \equiv (X_0, X_1)_{\theta,\mathrm{b},E}$ consists of all f in $X_0 + X_1$ for which

$$\|f\|_{\theta,\mathrm{b},E} := \left\| t^{-\theta}\mathrm{b}(t)K(t, f) \right\|_{\widetilde{E}} < \infty.$$

The space $\overline{X}_{\theta,\mathrm{b},E}$ is a (quasi-) Banach space, and it is an intermediate space for the couple $\overline{X}$, that is,

$$X_0 \cap X_1 \hookrightarrow \overline{X}_{\theta,\mathrm{b},E} \hookrightarrow X_0 + X_1,$$

provided that $0 < \theta < 1$, or $\theta = 0$ and $\|\mathrm{b}\|_{\widetilde{E}(1,\infty)} < \infty$ or $\theta = 1$ and $\|\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty$. If none of these conditions holds, the space is trivial, that is $\overline{X}_{\theta,\mathrm{b},E} = \{0\}$.

When $0 < \theta < 1$, $\mathrm{b} \equiv 1$ and $E = L_q$, $1 \le q \le \infty$, the space $\overline{X}_{\theta,\mathrm{b},E}$ coincides with the classical real interpolation space $\overline{X}_{\theta,q}$. When $\theta = 0$ or $\theta = 1$ this scale contains extrapolation spaces in the sense of Milman [37], Gómez and Milman [30] and Astashkin, Lykov and Milman [4]. The spaces $\overline{X}_{\theta,\mathrm{b},L_q}$ have been studied in detail by Gogatishvili, Opic and Trebels in [32]. See also [1, 10, 13, 19, 33, 42, 48], among other references.

Remark 2.6 Here we collect some elementary estimates. Using Lemma 2.3 (i) and the monotonicity of $t \rightsquigarrow t^{-1}K(t,\cdot)$, it is easy to check that for all $t > 0$ and $f \in X_0+X_1$

$$t^{-\theta}\mathrm{b}(t)K(t,f) \lesssim \left\|s^{-\theta}\mathrm{b}(s)K(s,f)\right\|_{\widetilde{E}(0,t)}, \qquad 0 \le \theta < 1, \tag{2.4}$$

and

$$t^{-\theta}\mathrm{b}(t)K(t,f) \lesssim \left\|s^{-\theta}\mathrm{b}(s)K(s,f)\right\|_{\widetilde{E}(t,\infty)}, \qquad 0 < \theta \le 1. \tag{2.5}$$

Definition 2.7 Let E, F be two r.i. spaces, $a, \mathrm{b} \in SV$ and $0 \le \theta \le 1$. The (quasi-) Banach space $\overline{X}^{\mathcal{R}}_{\theta,\mathrm{b},E,a,F} \equiv (X_0, X_1)^{\mathcal{R}}_{\theta,\mathrm{b},E,a,F}$ consists of all $f \in X_0 + X_1$ for which

$$\|f\|_{\mathcal{R};\theta,\mathrm{b},E,a,F} := \left\|\mathrm{b}(t)\|s^{-\theta}a(s)K(s,f)\|_{\widetilde{F}(t,\infty)}\right\|_{\widetilde{E}} < \infty.$$

The space $\mathcal{R}$ is intermediate for the couple $\overline{X}$, that is,

$$X_0 \cap X_1 \hookrightarrow \overline{X}^{\mathcal{R}}_{\theta,\mathrm{b},E,a,F} \hookrightarrow X_0 + X_1$$

provided that any of the following conditions holds:

1. $0 < \theta < 1$ and $\|\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty$ or
2. $\theta = 0$, $\|\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty$ and $\left\|\mathrm{b}(t)\|a\|_{\widetilde{F}(t,\infty)}\right\|_{\widetilde{E}(1,\infty)} < \infty$
3. $\theta = 1$ and $\left\|\mathrm{b}(t)\|a\|_{\widetilde{F}(t,1)}\right\|_{\widetilde{E}(0,1)} < \infty$.

Otherwise, $\overline{X}^{\mathcal{R}}_{\theta,\mathrm{b},E,a,F} = \{0\}$.

Definition 2.8 Let E, F be two r.i. spaces, $a, \mathrm{b} \in SV$ and $0 \le \theta \le 1$. The space $\overline{X}^{\mathcal{L}}_{\theta,\mathrm{b},E,a,F} \equiv (X_0, X_1)^{\mathcal{L}}_{\theta,\mathrm{b},E,a,F}$ consists of all $f \in X_0 + X_1$ for which

$$\|f\|_{\mathcal{L};\theta,\mathrm{b},E,a,F} := \left\|\mathrm{b}(t)\|s^{-\theta}a(s)K(s,f)\|_{\widetilde{F}(0,t)}\right\|_{\widetilde{E}} < \infty.$$

This is a (quasi-) Banach space. Moreover it is intermediate for the couple $\overline{X}$,

$$X_0 \cap X_1 \hookrightarrow \overline{X}^{\mathcal{L}}_{\theta,\mathrm{b},E,a,F} \hookrightarrow X_0 + X_1,$$

provided that

1. $0 < \theta < 1$ and $\|\mathrm{b}\|_{\widetilde{E}(1,\infty)} < \infty$, or
2. $\theta = 0$ and $\left\|\mathrm{b}(t)\|a\|_{\widetilde{F}(1,t)}\right\|_{\widetilde{E}(1,\infty)} < \infty$ or
3. $\theta = 1$, $\|\mathrm{b}\|_{\widetilde{E}(1,\infty)} < \infty$ and $\left\|\mathrm{b}(t)\|a\|_{\widetilde{F}(0,t)}\right\|_{\widetilde{E}(0,1)} < \infty$.

If none of these conditions holds, then $\overline{X}^{\mathcal{L}}_{\theta,\mathrm{b},E,a,F}$ is the trivial space.

The above definitions generalize a previous family of spaces introduced by Evans and Opic [18] in which a, b are broken logarithms. Earlier versions of these spaces appeared in a paper by Doktorskii [13], with a, b powers of logarithms and $\overline{X}$ an ordered couple. These spaces also appear in the work of Gogatishvili, Opic and Trebels [32] and Ahmed *et al.* [1]. In all these papers the spaces E and F remain within the L_q classes.

The spaces $\overline{X}_{\theta,\mathrm{b},E}$, $\overline{X}^{\mathcal{R}}_{\theta,\mathrm{b},E,a,F}$ and $\overline{X}^{\mathcal{L}}_{\theta,\mathrm{b},E,a,F}$ satisfy the following symmetry property:

Lemma 2.9 *Let E, F be r.i. spaces, a, $\mathrm{b} \in SV$ and $0 \le \theta \le 1$. Then*

$$(X_0, X_1)_{\theta,\mathrm{b},E} = (X_1, X_0)_{1-\theta,\overline{\mathrm{b}},E} \quad \textit{and} \quad (X_0, X_1)^{\mathcal{L}}_{\theta,\mathrm{b},E,a,F} = (X_1, X_0)^{\mathcal{R}}_{1-\theta,\overline{\mathrm{b}},E,\overline{a},F}.$$

Next lemma collects two estimates that will be used in the rest of the paper.

Lemma 2.10 *Let E, F be r.i. spaces, $a, \mathrm{b} \in SV$ and $0 \le \theta \le 1$. Then, for all $f \in X_0 + X_1$ and $u > 0$*

$$u^{-\theta} a(u) \|\mathrm{b}\|_{\widetilde{E}(0,u)} K(u, f) \lesssim \left\| \mathrm{b}(t) \|s^{-\theta} a(s) K(s, f)\|_{\widetilde{F}(t,u)} \right\|_{\widetilde{E}(0,u)} \tag{2.6}$$

and

$$u^{-\theta} a(u) \|\mathrm{b}\|_{\widetilde{E}(u,\infty)} K(u, f) \lesssim \left\| \mathrm{b}(t) \|s^{-\theta} a(s) K(s, f)\|_{\widetilde{F}(u,t)} \right\|_{\widetilde{E}(u,\infty)}. \tag{2.7}$$

Proof We refer to [23,Lemma 2.12] for the proof of (2.6). Next we prove (2.7). Let $f \in X_0 + X_1$ and $u > 0$. Using the monotonicity of the K-functional, Lemma 2.2 (iii) and Lemma 2.3 (ii) and (iv), we arrived at

$$\begin{aligned} \left\| \mathrm{b}(t) \|s^{-\theta} a(s) K(s, f)\|_{\widetilde{F}(u,t)} \right\|_{\widetilde{E}(u,\infty)} &\ge \left\| \mathrm{b}(t) \|s^{-\theta} a(s) K(s, f)\|_{\widetilde{F}(u,t)} \right\|_{\widetilde{E}(2u,\infty)} \\ &\ge \left\| \mathrm{b}(t) \|s^{-\theta} a(s) K(s, f)\|_{\widetilde{F}(u,2u)} \right\|_{\widetilde{E}(2u,\infty)} \\ &\sim u^{-\theta} a(u) \|\mathrm{b}\|_{\widetilde{E}(2u,\infty)} K(u, f) \\ &\sim u^{-\theta} a(u) \|\mathrm{b}\|_{\widetilde{E}(u,\infty)} K(u, f). \end{aligned}$$

□

Consequently, when $0 \le \theta \le 1$

$$K(u, f) \lesssim \frac{u^{\theta}}{a(u) \|\mathrm{b}\|_{\widetilde{E}(0,u)}} \|f\|_{\mathcal{R};\theta,\mathrm{b},E,a,F} \tag{2.8}$$

and

$$K(u, f) \lesssim \frac{u^{\theta}}{a(u) \|\mathrm{b}\|_{\widetilde{E}(u,\infty)}} \|f\|_{\mathcal{L};\theta,\mathrm{b},E,a,F} \tag{2.9}$$

for all $u > 0$ and $f \in \overline{X}^{\mathcal{R}}_{\theta,\mathrm{b},E,a,F}$, $f \in \overline{X}^{\mathcal{L}}_{\theta,\mathrm{b},E,a,F}$ respectively.

The extreme spaces are defined in [23, 24] in the following way.

Definition 2.11 Let E, F, G be r.i. spaces, a, b, $c \in SV$ and $0 \leq \theta \leq 1$. The space $\overline{X}^{\mathcal{R},\mathcal{R}}_{\theta,c,E,\mathrm{b},F,a,G} \equiv (X_0, X_1)^{\mathcal{R},\mathcal{R}}_{\theta,c,E,\mathrm{b},F,a,G}$ is the set of all $f \in X_0 + X_1$ such that

$$\|f\|_{\mathcal{R},\mathcal{R};\theta,c,E,\mathrm{b},F,a,G} := \left\| c(u) \left\| \mathrm{b}(t) \|s^{-\theta} a(s) K(s, f)\|_{\widetilde{G}(t,\infty)} \right\|_{\widetilde{F}(u,\infty)} \right\|_{\widetilde{E}} < \infty. \tag{2.10}$$

The space $\overline{X}^{\mathcal{L},\mathcal{L}}_{\theta,c,E,\mathrm{b},F,a,G} \equiv (X_0, X_1)^{\mathcal{L},\mathcal{L}}_{\theta,c,E,\mathrm{b},F,a,G}$ is the set of all $f \in X_0 + X_1$ for which

$$\|f\|_{\mathcal{L},\mathcal{L};\theta,c,E,\mathrm{b},F,a,G} := \left\| c(u) \left\| \mathrm{b}(t) \|s^{-\theta} a(s) K(s, f)\|_{\widetilde{G}(0,t)} \right\|_{\widetilde{F}(0,u)} \right\|_{\widetilde{E}} < \infty. \tag{2.11}$$

The space $\overline{X}^{\mathcal{R},\mathcal{L}}_{\theta,c,E,\mathrm{b},F,a,G} \equiv (X_0, X_1)^{\mathcal{R},\mathcal{L}}_{\theta,c,E,\mathrm{b},F,a,G}$ is the set of all $f \in X_0 + X_1$ for which

$$\|f\|_{\mathcal{R},\mathcal{L};\theta,c,E,\mathrm{b},F,a,G} := \left\| c(u) \left\| \mathrm{b}(t) \|s^{-\theta} a(s) K(s, f)\|_{\widetilde{G}(t,u)} \right\|_{\widetilde{F}(0,u)} \right\|_{\widetilde{E}} < \infty. \tag{2.12}$$

The space $\overline{X}^{\mathcal{L},\mathcal{R}}_{\theta,c,E,\mathrm{b},F,a,G} \equiv (X_0, X_1)^{\mathcal{L},\mathcal{R}}_{\theta,c,E,\mathrm{b},F,a,G}$ is the set of all $f \in X_0 + X_1$ such that

$$\|f\|_{\mathcal{L},\mathcal{R};\theta,c,E,\mathrm{b},F,a,G} := \left\| c(u) \left\| \mathrm{b}(t) \|s^{-\theta} a(s) K(s, f)\|_{\widetilde{G}(u,t)} \right\|_{\widetilde{F}(u,\infty)} \right\|_{\widetilde{E}} < \infty. \tag{2.13}$$

These spaces enjoy the following symmetry property.

Lemma 2.12 *Let E, F, G be r.i. spaces, a,* b*, $c \in SV$ and $0 \leq \theta \leq 1$. Then*

$$(X_0, X_1)^{\mathcal{L},\mathcal{L}}_{\theta,c,E,\mathrm{b},F,a,G} = (X_1, X_0)^{\mathcal{R},\mathcal{R}}_{1-\theta,\overline{c},E,\overline{b},F,\overline{a},G}$$

and

$$(X_0, X_1)^{\mathcal{L},\mathcal{R}}_{\theta,c,E,\mathrm{b},F,a,G} = (X_1, X_0)^{\mathcal{R},\mathcal{L}}_{1-\theta,\overline{c},E,\overline{b},F,\overline{a},G}.$$

3 Generalized Holmstedt type formulae

For parameters $0 < \theta_0 < \theta_1 < 1$, $a_0, a_1, \mathrm{b}_0, \mathrm{b}_1 \in SV$ and E_0, E_1, F_0, F_1 r.i spaces, the couples

$$(\overline{X}^{\mathcal{R}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1}), \quad (\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{L}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1})$$

and

$$(\overline{X}^{\mathcal{R}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{L}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1}), \quad (\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1})$$

are compatible (quasi-) Banach couples. We wish to identify the interpolation spaces generated by the previous couples as interpolated spaces of the original one, $\overline{X}$. Our first stage in that process is to relate the K-functional of the underlying couples through several generalized Holmstedt type formulae.

3.1 The K-functional of the couple $(\overline{X}^{\mathcal{R}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1})$, $0 < \theta_0 < \theta_1 < 1$.

Theorem 3.1 *Let $0 < \theta_0 < \theta_1 < 1$. Let E_0, E_1, F_0, F_1 r.i. spaces and $a_0, a_1, \mathrm{b}_0, \mathrm{b}_1 \in SV$ such that $\|\mathrm{b}_0\|_{\widetilde{E}_0(0,1)} < \infty$ and $\|\mathrm{b}_1\|_{\widetilde{E}_1(0,1)} < \infty$. Then, for every $f \in \overline{X}^{\mathcal{R}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0} + \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1}$ and $u > 0$*

$$\begin{aligned} &K\big(\rho(u), f; \overline{X}^{\mathcal{R}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1}\big) \\ &\quad\sim \Big\| \mathrm{b}_0(t) \|s^{-\theta_0} a_0(s) K(s,f)\|_{\widetilde{F}_0(t,u)} \Big\|_{\widetilde{E}_0(0,u)} \\ &\qquad + \rho(u) \|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)} \cdot \|t^{-\theta_1} a_1(t) K(t,f)\|_{\widetilde{F}_1(u,\infty)} \\ &\qquad + \rho(u) \Big\| \mathrm{b}_1(t) \|s^{-\theta_1} a_1(s) K(s,f)\|_{\widetilde{F}_1(t,\infty)} \Big\|_{\widetilde{E}_1(u,\infty)} \end{aligned} \tag{3.1}$$

where

$$\rho(u) = u^{\theta_1-\theta_0} \frac{a_0(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(0,u)}}{a_1(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}}, \quad u > 0. \tag{3.2}$$

Proof Given $f \in X_0 + X_1$ and $u > 0$, we consider the following (quasi-) norms

$$\begin{aligned} (P_j f)(u) &= \Big\| \mathrm{b}_j(t) \|s^{-\theta_j} a_j(s) K(s,f)\|_{\widetilde{F}_j(t,u)} \Big\|_{\widetilde{E}_j(0,u)}, \\ (R_j f)(u) &= \|\mathrm{b}_j\|_{\widetilde{E}_j(0,u)} \cdot \|t^{-\theta_j} a_j(t) K(t,f)\|_{\widetilde{F}_j(u,\infty)}, \\ (Q_j f)(u) &= \Big\| \mathrm{b}_j(t) \|s^{-\theta_j} a_j(s) K(s,f)\|_{\widetilde{F}_j(t,\infty)} \Big\|_{\widetilde{E}_j(u,\infty)}, \end{aligned} \tag{3.3}$$

and we denote $Y_j = \overline{X}^{\mathcal{R}}_{\theta_j,\mathrm{b}_j,E_j,a_j,F_j}$, $j = 0, 1$. With this notation what we pursue to show is that

$$K(\rho(u), f; Y_0, Y_1) \sim (P_0 f)(u) + \rho(u)[(R_1 f)(u) + (Q_1 f)(u)] \tag{3.4}$$

where ρ is defined by (3.2).

We fix $f \in X_0+X_1$, $u > 0$ and we assume that $(P_0 f)(u)$, $(R_1 f)(u)$ and $(Q_1 f)(u)$ are finite, otherwise the upper estimate of (3.4) holds trivially. As usual (see for example [6] or [19]) we choose a decomposition $f = g + h$ such that $\|g\|_{X_0} + u\|h\|_{X_1} \le 2K(u, f)$. Then, for all $s > 0$,

$$K(s, g) \le 2K(u, f) \quad \text{and} \quad K(s, h) \le 2s\frac{K(u, f)}{u}. \tag{3.5}$$

Therefore, in order to obtain the upper estimate of (3.4) it is enough to prove that

$$\|g\|_{Y_0} + \rho(u)\|h\|_{Y_1} \lesssim (P_0 f)(u) + \rho(u)[(R_1 f)(u) + (Q_1 f)(u)].$$

We start establishing that $\|g\|_{Y_0} \lesssim (P_0 f)(u)$. By the triangle inequality and the (quasi-) subadditivity of the K-functional, we have the inequalities

$$\begin{aligned} \|g\|_{Y_0} &\le (P_0 g)(u) + (R_0 g)(u) + (Q_0 g)(u) \\ &\lesssim (P_0 f)(u) + (P_0 h)(u) + (R_0 g)(u) + (Q_0 g)(u). \end{aligned} \tag{3.6}$$

So, it suffices to estimate $(P_0 h)(u)$, $(R_0 g)(u)$ and $(Q_0 g)(u)$ from above. Using (3.5) and the monotonicity of $s \rightsquigarrow s^{-1}K(s, f)$, $s > 0$, we obtain that

$$\begin{aligned} (P_0 h)(u) &= \Big\| \mathrm{b}_0(t) \| s^{-\theta_0} a_0(s) K(s, h) \|_{\widetilde{F}_0(t,u)} \Big\|_{\widetilde{E}_0(0,u)} \\ &\lesssim \Big\| \mathrm{b}_0(t) \| s^{1-\theta_0} a_0(s) \frac{K(u, f)}{u} \|_{\widetilde{F}_0(t,u)} \Big\|_{\widetilde{E}_0(0,u)} \\ &\le \Big\| \mathrm{b}_0(t) \| s^{-\theta_0} a_0(s) K(s, f) \|_{\widetilde{F}_0(t,u)} \Big\|_{\widetilde{E}_0(0,u)} = (P_0 f)(u). \end{aligned} \tag{3.7}$$

On the other hand, using (3.5), Lemma 2.3 (i) and (2.6), we have

$$\begin{aligned} (R_0 g)(u) &= \|\mathrm{b}_0\|_{\widetilde{E}_0(0,u)} \| t^{-\theta_0} a_0(t) K(t, g) \|_{\widetilde{F}_0(u,\infty)} \\ &\lesssim K(u, f) \|\mathrm{b}_0\|_{\widetilde{E}_0(0,u)} \| t^{-\theta_0} a_0(t) \|_{\widetilde{F}_0(u,\infty)} \\ &\sim u^{-\theta_0} a_0(u) \|\mathrm{b}_0\|_{\widetilde{E}_0(0,u)} K(u, f) \lesssim (P_0 f)(u). \end{aligned} \tag{3.8}$$

Similarly, using also Lemma 2.3 (iv) it follows

$$
\begin{aligned}
(Q_0 g)(u) &= \Big\| \mathrm{b}_0(t) \| s^{-\theta_0} a_0(s) K(s, g) \|_{\widetilde{F}_0(t,\infty)} \Big\|_{\widetilde{E}_0(u,\infty)} \\
&\lesssim K(u, f) \Big\| \mathrm{b}_0(t) \| s^{-\theta_0} a_0(s) \|_{\widetilde{F}_0(t,\infty)} \Big\|_{\widetilde{E}_0(u,\infty)} \\
&\sim u^{-\theta_0} \mathrm{b}_0(u) a_0(u) K(u, f) \lesssim (P_0 f)(u). \qquad (3.9)
\end{aligned}
$$

Hence, inequalities (3.6)-(3.9) give that $\|g\|_{Y_0} \lesssim (P_0 f)(u) < \infty$, which in particular shows that $g \in Y_0$. In a similar way it can be proved that

$$
\|h\|_{Y_1} \lesssim (R_1 f)(u) + (Q_1 f)(u) < \infty,
$$

so $h \in Y_1$. Indeed, by the triangle inequality and the (quasi-) subadditivity of the K-functional, we have

$$
\begin{aligned}
\|h\|_{Y_1} &\le (P_1 h)(u) + (R_1 h)(u) + (Q_1 h)(u) \\
&\lesssim (P_1 h)(u) + (R_1 f)(u) + (R_1 g)(u) + (Q_1 f)(u) + (Q_1 g)(u). \qquad (3.10)
\end{aligned}
$$

Now we estimate $(P_1 h)(u)$. Using (3.5), Lemma 2.3 (i) and (2.5), we obtain

$$
\begin{aligned}
(P_1 h)(u) &= \Big\| \mathrm{b}_1(t) \| s^{-\theta_1} a_1(s) K(s, h) \|_{\widetilde{F}_1(t,u)} \Big\|_{\widetilde{E}_1(0,u)} \\
&\lesssim \frac{K(u, f)}{u} \Big\| \mathrm{b}_1(t) \| s^{1-\theta_1} a_1(s) \|_{\widetilde{F}_1(0,u)} \Big\|_{\widetilde{E}_1(0,u)} \\
&\sim u^{-\theta_1} a_1(u) \|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)} K(u, f) \lesssim (R_1 f)(u). \qquad (3.11)
\end{aligned}
$$

Similarly, $(R_1 g)(u)$ and $(Q_1 g)(u)$ can be bounded from above. In fact, Lemma (3.5), Lemma 2.3 (i) and (2.5) yield

$$
\begin{aligned}
(R_1 g)(u) &= \|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)} \Big\| t^{-\theta_1} a_1(t) K(t, g) \Big\|_{\widetilde{F}_1(u,\infty)} \\
&\lesssim K(u, f) \|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)} \Big\| t^{-\theta_1} a_1(t) \Big\|_{\widetilde{F}_1(u,\infty)} \\
&\sim u^{-\theta_1} a_1(u) \|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)} K(u, f) \lesssim (R_1 f)(u), \qquad (3.12)
\end{aligned}
$$

and also, using Lemma 2.3 (iv),

$$
\begin{aligned}
(Q_1 g)(u) &= \Big\| \mathrm{b}_1(t) \| s^{-\theta_1} a_1(s) K(s, g) \|_{\widetilde{F}_1(t,\infty)} \Big\|_{\widetilde{E}_1(u,\infty)} \\
&\lesssim K(u, f) \Big\| \mathrm{b}_1(t) \| s^{-\theta_1} a_1(s) \|_{\widetilde{F}_1(t,\infty)} \Big\|_{\widetilde{E}_1(u,\infty)} \\
&\lesssim u^{-\theta_1} a_1(u) \mathrm{b}_1(u) K(u, f) \lesssim (R_1 f)(u). \qquad (3.13)
\end{aligned}
$$

Summing up

$$
\|g\|_{Y_0} + \rho(u) \|h\|_{Y_1} \lesssim (P_0 f)(u) + \rho(u) [(R_1 f)(u) + (Q_1 f)(u)],
$$

for any positive function ρ. This concludes the proof of the upper estimate in (3.4).

Next we proceed with the lower estimate in (3.4), that is

$$(P_0 f)(u) + \rho(u)[(R_1 f)(u) + (Q_1 f)(u)] \lesssim K(\rho(u), f; Y_0, Y_1).$$

Suposse that $f \in Y_0 + Y_1$ and fix $u > 0$. Let $f = f_0 + f_1$ be any decomposition of f with $f_0 \in Y_0$ and $f_1 \in Y_1$. The (quasi-) subadditivity of the K-functional and the definition of the norm in Y_0, Y_1 imply

$$\begin{aligned}
(P_0 f)(u) &\lesssim (P_0 f_0)(u) + (P_0 f_1)(u) \leq \|f_0\|_{Y_0} + (P_0 f_1)(u),\\
(R_1 f)(u) &\lesssim (R_1 f_0)(u) + (R_1 f_1)(u) \leq (R_1 f_0)(u) + \|f_1\|_{Y_1},\\
(Q_1 f)(u) &\lesssim (Q_1 f_0)(u) + (Q_1 f_1)(u) \leq (Q_1 f_0)(u) + \|f_1\|_{Y_1}.
\end{aligned}$$

Then

$$\begin{aligned}
&(P_0 f)(u) + \rho(u)[(R_1 f)(u) + (Q_1 f)(u)]\\
&\quad \lesssim \|f_0\|_{Y_0} + \rho(u)\|f_1\|_{Y_1} + (P_0 f_1)(u) + \rho(u)[(R_1 f_0)(u) + (Q_1 f_0)(u)].
\end{aligned}$$

Thus, to finish the proof, it is enough to verify that

$$(P_0 f_1)(u) + \rho(u)[(R_1 f_0)(u) + (Q_1 f_0)(u)] \lesssim \|f_0\|_{Y_0} + \rho(u)\|f_1\|_{Y_1}.$$

We begin with $(P_0 f_1)(u)$. Using (2.8) with $f = f_1 \in Y_1$ and Lemma 2.3 (i) ($\theta_1 - \theta_0 > 0$), we obtain

$$\begin{aligned}
(P_0 f_1)(u) &\lesssim \|f_1\|_{Y_1} \left\| \mathrm{b}_0(t) \left\| s^{\theta_1-\theta_0} \frac{a_0(s)}{a_1(s)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,s)}} \right\|_{\widetilde{F}_0(0,u)} \right\|_{\widetilde{E}_0(0,u)}\\
&\sim u^{\theta_1-\theta_0} \frac{a_0(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(0,u)}}{a_1(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}} \|f_1\|_{Y_1} = \rho(u)\|f_1\|_{Y_1}. \qquad (3.14)
\end{aligned}$$

In a similar way, using (2.8) with $f = f_0 \in Y_0$ and Lemma 2.3 (i) ($\theta_0 - \theta_1 < 0$), we have

$$\begin{aligned}
(R_1 f_0)(u) &\lesssim \|f_0\|_{Y_0} \|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)} \left\| t^{\theta_0-\theta_1} \frac{a_1(t)}{a_0(t)\|\mathrm{b}_0\|_{\widetilde{E}_0(0,t)}} \right\|_{\widetilde{F}_1(u,\infty)}\\
&\sim u^{\theta_0-\theta_1} \frac{a_1(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}}{a_0(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(0,u)}} \|f_0\|_{Y_0} = \frac{1}{\rho(u)}\|f_0\|_{Y_0} \qquad (3.15)
\end{aligned}$$

and

$$\begin{aligned}
(Q_1 f_0)(u) &\lesssim \|f_0\|_{Y_0} \left\| \mathrm{b}_1(t) \left\| s^{\theta_0-\theta_1} \frac{a_1(s)}{a_0(s)\|\mathrm{b}_0\|_{\widetilde{E}_0(0,s)}} \right\|_{\widetilde{F}_1(t,\infty)} \right\|_{\widetilde{E}_1(u,\infty)}\\
&\sim u^{\theta_0-\theta_1} \frac{a_1(u)\mathrm{b}_1(u)}{a_0(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(0,u)}} \|f_0\|_{Y_0} \lesssim \frac{1}{\rho(u)}\|f_0\|_{Y_0}, \qquad (3.16)
\end{aligned}$$

where the last equivalence follows from Lemma 2.3 iv).

Putting together the previous estimates we establish that

$$(P_0 f)(u) + \rho(u)[(R_1 f)(u) + (Q_1 f)(u) \lesssim \|f_0\|_{Y_0} + \rho(u)\|f_1\|_{Y_1}.$$

And taking infimum over all possible decomposition of $f = f_0 + f_1$, with $f_0 \in Y_0$ and $f_1 \in Y_1$, we finish the proof. □

3.2 The K-functional of the couple $(\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{L}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1})$, $0 < \theta_0 < \theta_1 < 1$.

Next theorem follow from Theorem 3.1 by means of a symmetry argument.

Theorem 3.2 *Let* $0 < \theta_0 < \theta_1 < 1$. *Let* E_0, E_1, F_0, F_1 *r.i. spaces and* a_0, a_1, b_0, $\mathrm{b}_1 \in SV$ *such that* $\|\mathrm{b}_0\|_{\widetilde{E}_0(1,\infty)} < \infty$ *and* $\|\mathrm{b}_1\|_{\widetilde{E}_1(1,\infty)} < \infty$. *Then, for every* $f \in \overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0} + \overline{X}^{\mathcal{L}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1}$ *and* $u > 0$

$$\begin{aligned} &K\big(\rho(u), f; \overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{L}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1}\big) \\ &\quad\sim \Big\|\mathrm{b}_0(t)\|s^{-\theta_0}a_0(s)K(s,f)\|_{\widetilde{F}_0(0,t)}\Big\|_{\widetilde{E}_0(0,u)} \\ &\qquad + \|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}\|t^{-\theta_0}a_0(t)K(t,f)\|_{\widetilde{F}_0(0,u)} \\ &\qquad + \rho(u)\Big\|\mathrm{b}_1(t)\|s^{-\theta_1}a_1(s)K(s,f)\|_{\widetilde{F}_1(u,t)}\Big\|_{\widetilde{E}_1(u,\infty)} \end{aligned} \tag{3.17}$$

where

$$\rho(u) = u^{\theta_1-\theta_0}\frac{a_0(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}}{a_1(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(u,\infty)}}, \quad u > 0.$$

Proof Lemma 2.9 and (2.3) yield

$$\begin{aligned} &K\big(\rho(u), f; \overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{L}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1}\big) \\ &\qquad = \rho(u)K\Big(\frac{1}{\rho(u)}, f; (X_1,X_0)^{\mathcal{R}}_{1-\theta_1,\overline{\mathrm{b}}_1,E_1,\overline{a}_1,F_1}, (X_1,X_0)^{\mathcal{R}}_{1-\theta_0,\overline{\mathrm{b}}_0,E_0,\overline{a}_0,F_0}\Big), \end{aligned}$$

where

$$\frac{1}{\rho(u)} = u^{\theta_0-\theta_1}\frac{\overline{a}_1(\frac{1}{u})}{\overline{a}_0(\frac{1}{u})}\frac{\|\overline{\mathrm{b}}_1\|_{\widetilde{E}_1(0,1/u)}}{\|\overline{\mathrm{b}}_0\|_{\widetilde{E}_0(0,1/u)}}, \quad u > 0.$$

Now, applying Theorem 3.1 we obtain the estimate

$$
\begin{aligned}
&K\big(\rho(u), f; \overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{L}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1}\big)\\
&\quad\sim \rho(u)\Big\|\overline{\mathrm{b}}_1(t)\|s^{\theta_1-1}\overline{a}_1(s)K(s, f; X_1, X_0)\|_{\widetilde{F}_1(t,1/u)}\Big\|_{\widetilde{E}_1(0,1/u)}\\
&\qquad+\|\overline{\mathrm{b}}_0\|_{\widetilde{E}_0(0,1/u)}\|t^{\theta_0-1}\overline{a}_0(t)K(t, f; X_1, X_0)\|_{\widetilde{F}_0(1/u,\infty)}\\
&\qquad+\Big\|\overline{\mathrm{b}}_0(t)\|s^{\theta_0-1}\overline{a}_0(s)K(s, f; X_1, X_0)\|_{\widetilde{F}_0(t,\infty)}\Big\|_{\widetilde{E}_0(1/u,\infty)}.
\end{aligned}
$$

The equivalence (3.17) follows using again (2.1) and (2.3). □

3.3 The K-functional of the couple $(\overline{X}^{\mathcal{R}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{L}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1})$, $0 < \theta_0 < \theta_1 < 1$.

Theorem 3.3 *Let $0 < \theta_0 < \theta_1 < 1$. Let E_0, E_1, F_0, F_1 r.i. spaces and a_0, a_1, b_0, $\mathrm{b}_1 \in SV$ such that $\|\mathrm{b}_0\|_{\widetilde{E}_0(0,1)} < \infty$ and $\|\mathrm{b}_1\|_{\widetilde{E}_1(1,\infty)} < \infty$. Then, for every $f \in \overline{X}^{\mathcal{R}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0} + \overline{X}^{\mathcal{L}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1}$ and $u > 0$*

$$
\begin{aligned}
&K\big(\rho(u), f; \overline{X}^{\mathcal{R}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{L}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1}\big)\\
&\quad\sim \Big\|\mathrm{b}_0(t)\|s^{-\theta_0}a_0(s)K(s, f)\|_{\widetilde{F}_0(t,u)}\Big\|_{\widetilde{E}_0(0,u)}\\
&\qquad+\rho(u)\Big\|\mathrm{b}_1(t)\|s^{-\theta_1}a_1(s)K(s, f)\|_{\widetilde{F}_1(u,t)}\Big\|_{\widetilde{E}_1(u,\infty)},
\end{aligned}
$$

where

$$
\rho(u) = u^{\theta_1-\theta_0}\frac{a_0(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(0,u)}}{a_1(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(u,\infty)}}, \quad u > 0. \tag{3.18}
$$

Proof Given $f \in X_0 + X_1$ and $u > 0$, we consider the (quasi-) norms $(P_0 f)(u)$, $(R_0 f)(u)$ and $(Q_0 f)(u)$ defined as in (3.3) and we redefine

$$
\begin{aligned}
(P_1 f)(u) &= \Big\|\mathrm{b}_1(t)\|s^{-\theta_1}a_1(s)K(s, f)\|_{\widetilde{F}_1(0,t)}\Big\|_{\widetilde{E}_1(0,u)},\\
(R_1 f)(u) &= \|\mathrm{b}_1\|_{\widetilde{E}_1(u,\infty)} \cdot \|t^{-\theta_1}a_1(t)K(t, f)\|_{\widetilde{F}_1(0,u)},\\
(Q_1 f)(u) &= \Big\|\mathrm{b}_1(t)\|s^{-\theta_1}a_1(s)K(s, f)\|_{\widetilde{F}_1(u,t)}\Big\|_{\widetilde{E}_1(u,\infty)}.
\end{aligned}
$$

We denote, as usual, $Y_0 = \overline{X}^{\mathcal{R}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}$ and $Y_1 = \overline{X}^{\mathcal{L}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1}$. What we want to show is that

$$
K(\rho(u), f; Y_0, Y_1) \sim (P_0 f)(u) + \rho(u)(Q_1 f)(u) \tag{3.19}
$$

where ρ is defined by (3.18). In order to do that we follow the same steps as in the proof of Theorem 3.1.

We fix $f \in X_0 + X_1$ and $u > 0$, and we choose a decomposition of $f = g + h$ such that $\|g\|_{X_0} + u\|h\|_{X_1} \leq 2K(u, f)$. We start by showing that

$$\|g\|_{X_0} + \rho(u)\|h\|_{X_1} \lesssim (P_0 f)(u) + \rho(u)(Q_1 f)(u)$$

to establish the upper bound for the K-functional in (3.19). The inequality $\|g\|_{Y_0} \lesssim (P_0 f)(u)$ can be proved exactly as we did through (3.6) to (3.9). Now we proceed with $\|h\|_{Y_1}$. Using the triangular inequality and the (quasi-) subadditivity of the K-functional it follows

$$\begin{aligned}\|h\|_{Y_1} &\leq (P_1 h)(u) + (R_1 h)(u) + (Q_1 h)(u)\\ &\lesssim (P_1 h)(u) + (R_1 h)(u) + (Q_1 f)(u) + (Q_1 g)(u).\end{aligned}$$

Hence, it is enough to estimate $(P_1 h)(u)$, $(R_1 h)(u)$ and $(Q_1 g)(u)$ from above. Using (3.5), Lemma 2.3 (i), (iv) and (2.7), we deduce that

$$\begin{aligned}(P_1 h)(u) &= \left\| \mathrm{b}_1(t) \|s^{-\theta_1} a_1(s) K(s,h)\|_{\widetilde{F}_1(0,t)} \right\|_{\widetilde{E}_1(0,u)}\\ &\lesssim \frac{K(u,f)}{u} \left\| \mathrm{b}_1(t) \|s^{1-\theta_1} a_1(s)\|_{\widetilde{F}_1(0,t)} \right\|_{\widetilde{E}_1(0,u)}\\ &\sim u^{-\theta_1} a_1(u) \mathrm{b}_1(u) K(u,f) \lesssim (Q_1 f)(u)\end{aligned}$$

and

$$\begin{aligned}(R_1 h)(u) &= \|\mathrm{b}_1\|_{\widetilde{E}_1(u,\infty)} \|t^{-\theta_1} a_1(t) K(t,h)\|_{\widetilde{F}_1(0,u)}\\ &\lesssim \frac{K(u,f)}{u} \|\mathrm{b}_1\|_{\widetilde{E}_1(u,\infty)} \|t^{1-\theta_1} a_1(t)\|_{\widetilde{F}_1(0,u)}\\ &\sim u^{-\theta_1} a_1(u) \|\mathrm{b}_1\|_{\widetilde{E}_1(u,\infty)} K(u,f) \lesssim (Q_1 f)(u).\end{aligned}$$

In the same vein

$$\begin{aligned}(Q_1 g)(u) &= \left\| \mathrm{b}_1(t) \|s^{-\theta_1} a_1(s) K(s,g)\|_{\widetilde{F}_1(u,t)} \right\|_{\widetilde{E}_1(u,\infty)}\\ &\lesssim K(u,f) \|\mathrm{b}_1\|_{\widetilde{E}_1(u,\infty)} \|s^{-\theta_1} a_1(s)\|_{\widetilde{F}_1(u,\infty)}\\ &\sim u^{-\theta_1} a_1(u) \|\mathrm{b}_1\|_{\widetilde{E}_1(u,\infty)} K(u,f) \lesssim (Q_1 f)(u).\end{aligned}$$

Then, $\|h\|_{Y_1} \lesssim (Q_1 f)(u)$ and $h \in Y_1$. Summing up, we establish the upper estimate of (3.19), that is

$$K(\rho(u), f; Y_0, Y_1) \le \|g\|_{Y_0} + \rho(u)\|h\|_{Y_1} \lesssim (P_0 f)(u) + \rho(u)(Q_1 f)(u).$$

Let us prove the lower estimate

$$(P_0 f)(u) + \rho(u)(Q_1 f)(u) \lesssim K(\rho(u), f; Y_0, Y_1) \tag{3.20}$$

for any $f \in Y_0 + Y_1$ and $u > 0$. Let $f = f_0 + f_1$ be any decomposition of f with $f_0 \in Y_0$ and $f_1 \in Y_1$. Using the (quasi-) subadditivity of the K-functional and the definition of the norm in Y_0, Y_1, we have

$$(P_0 f)(u) \lesssim (P_0 f_0)(u) + (P_0 f_1)(u) \le \|f_0\|_{Y_0} + (P_0 f_1)(u)$$
$$(Q_1 f)(u) \lesssim (Q_1 f_0)(u) + (Q_1 f_1)(u) \le (Q_1 f_0)(u) + \|f_1\|_{Y_1}.$$

Thus, we have to study the boundedness of $(P_0 f_1)(u)$ and $\rho(u)(Q_1 f_0)(u)$ by $\|f_0\|_{Y_0} + \rho(u)\|f_1\|_{Y_1}$. For the proof of the estimate $(P_0 f_1)(u) \lesssim \rho(u)\|f_1\|_{Y_1}$ one has to argue as in (3.14). Similarly, using (2.8) with $f = f_0$ and Lemma 2.3 (i) ($\theta_0 - \theta_1 < 0$), we obtain

$$\begin{aligned}(Q_1 f_0)(u) &\lesssim \|f_0\|_{Y_0} \|\mathrm{b}_1\|_{\widetilde{E}_1(u,\infty)} \left\| s^{\theta_0-\theta_1} \frac{a_1(s)}{a_0(s)\|\mathrm{b}_0\|_{\widetilde{E}_0(0,s)}} \right\|_{\widetilde{F}_1(u,\infty)} \\ &\sim u^{\theta_0-\theta_1} \frac{a_1(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(u,\infty)}}{a_0(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(0,u)}} \|f_0\|_{Y_0} = \frac{1}{\rho(u)} \|f_0\|_{Y_0}.\end{aligned}$$

Putting together the previous estimates we establish that

$$(P_0 f)(u) + \rho(u)(Q_1 f)(u) \lesssim \|f_0\|_{Y_0} + \rho(u)\|f_1\|_{Y_1}.$$

Taking infimum over all possible decomposition of $f = f_0 + f_1$, with $f_0 \in Y_0$ and $f_1 \in Y_1$, we deduce (3.20). The proof is completed. □

3.4 The K-functional of the couple $(\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1})$, $0 < \theta_0 < \theta_1 < 1$.

Finally we present the last Holmstedt type formula. Although it seems that the result is the symmetric counterpart of Theorem 3.3, this is not the case since the condition $\theta_0 < \theta_1$ is crucial.

Theorem 3.4 *Let* $0 < \theta_0 < \theta_1 < 1$. *Let* E_0, E_1, F_0, F_1 *r.i. spaces and* a_0, a_1, b_0, $\mathrm{b}_1 \in SV$ *such that* $\|\mathrm{b}_0\|_{\widetilde{E}_0(1,\infty)} < \infty$ *and* $\|\mathrm{b}_1\|_{\widetilde{E}_1(0,1)} < \infty$. *Then, for every*

$f \in \overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0} + \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1}$ *and* $u > 0$

$$\begin{aligned} &K\big(\rho(u), f; \overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1}\big) \\ &\quad\sim \Big\| \mathrm{b}_0(t) \|s^{-\theta_0} a_0(s) K(s,f)\|_{\widetilde{F}_0(0,t)} \Big\|_{\widetilde{E}_0(0,u)} \\ &\quad + \|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)} \|t^{-\theta_0} a_0(t) K(t,f)\|_{\widetilde{F}_0(0,u)} \\ &\quad + \rho(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)} \|t^{-\theta_1} a_1(t) K(t,f)\|_{\widetilde{F}_1(u,\infty)} \\ &\quad + \rho(u)\Big\| \mathrm{b}_1(t) \|s^{-\theta_1} a_1(s) K(s,f)\|_{\widetilde{F}_1(t,\infty)} \Big\|_{\widetilde{E}_1(u,\infty)}, \end{aligned}$$

where

$$\rho(u) = u^{\theta_1-\theta_0} \frac{a_0(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}}{a_1(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}}, \quad u > 0. \tag{3.21}$$

Proof Given $f \in X_0 + X_1$ and $u > 0$, we consider the (quasi-) norms $(P_1 f)(u)$, $(R_1 f)(u)$ and $(Q_1 f)(u)$ defined as in (3.3) and we redefine

$$\begin{aligned} (P_0 f)(u) &= \Big\| \mathrm{b}_0(t) \|s^{-\theta_0} a_0(s) K(s,f)\|_{\widetilde{F}_0(0,t)} \Big\|_{\widetilde{E}_0(0,u)}, \\ (R_0 f)(u) &= \|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)} \|t^{-\theta_0} a_0(t) K(t,f)\|_{\widetilde{F}_0(0,u)}, \\ (Q_0 f)(u) &= \Big\| \mathrm{b}_0(t) \|s^{-\theta_0} a_0(s) K(s,f)\|_{\widetilde{F}_0(u,t)} \Big\|_{\widetilde{E}_0(u,\infty)}. \end{aligned}$$

We denote, as usual, $Y_0 = \overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}$, $Y_1 = \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1}$. We want to show that

$$\begin{aligned} K(\rho(u), f; Y_0, Y_1) \sim\, &(P_0 f)(u) + (R_0 f)(u) \\ &+\rho(u)[(R_1 f)(u) + (Q_1 f)(u)] \end{aligned} \tag{3.22}$$

where ρ is defined by (3.21). Again we follow the same steps as in Theorem 3.1. We fix $f \in Y_0 + Y_1$ and $u > 0$, and we choose a decomposition $f = g + h$ such that $\|g\|_{X_0} + u\|h\|_{X_1} \le 2K(u, f)$. Hence, to obtain the upper estimate of (3.22) it is enough to prove that

$$\|g\|_{Y_0} + \rho(u)\|h\|_{Y_1} \lesssim (P_0 f)(u) + (R_0 f)(u) + \rho(u)[(R_1 f)(u) + (Q_1 f)(u)].$$

The inequality $\|h\|_{Y_1} \lesssim (R_1 f)(u)+(Q_1 f)(u)$ can be proved exactly as we did through (3.10) to (3.13). Now, we are going to establish that $\|g\|_{Y_0} \lesssim (P_0 f)(u) + (R_0 f)(u)$. By the triangle inequality and the (quasi-) subadditivity of the K-functional, we have that

$$\begin{aligned} \|g\|_{Y_0} &\le (P_0 g)(u) + (R_0 g)(u) + (Q_0 g)(u) \\ &\lesssim (P_0 f)(u) + (P_0 h)(u) + (R_0 f)(u) + (R_0 h)(u) + (Q_0 g)(u). \end{aligned}$$

For the proof of $(P_0 h)(u) \lesssim (P_0 f)(u)$ proceed as in (3.7). Besides that (3.5), Lemma 2.3 (i) and (2.4) give

$$(R_0 h)(u) \lesssim \frac{K(u,f)}{u} \|b_0\|_{\widetilde{E}_0(u,\infty)} \|t^{1-\theta_0} a_0(t)\|_{\widetilde{F}_0(0,u)}$$
$$\sim u^{-\theta_0} a_0(u) \|b_0\|_{\widetilde{E}_0(u,\infty)} K(u,f) \lesssim (R_0 f)(u).$$

Similarly, it follows

$$(Q_0 g)(u) \lesssim K(u,f) \Big\| b_0(t) \|s^{-\theta_0} a_0(s)\|_{\widetilde{F}_0(u,\infty)} \Big\|_{\widetilde{E}_0(u,\infty)}$$
$$\sim u^{-\theta_0} a_0(u) \|b_0\|_{\widetilde{E}_0(u,\infty)} K(u,f) \lesssim (R_0 f)(u).$$

Hence $\|g\|_{Y_0} \lesssim (P_0 f)(u) + (R_0 f)(u)$ and summing up

$$\|g\|_{Y_0} + \rho(u)\|h\|_{Y_1} \lesssim (P_0 f)(u) + (R_0 f)(u) + \rho(u)[(R_1 f)(u) + (Q_1 f)(u)].$$

This concludes the proof of the upper estimate of (3.22).

Next, we proceed with the lower estimate of (3.22), that is

$$(P_0 f)(u) + (R_0 f)(u) + \rho(u)[(R_1 f)(u) + (Q_1 f)(u)] \lesssim K(\rho(u), f; Y_0, Y_1)$$

for all $f \in Y_0 + Y_1$ and $u > 0$ where ρ is defined by (3.21).

We fix again $u > 0$, $f \in Y_0 + Y_1$. Let $f = f_0 + f_1$ be any decomposition of f with $f_0 \in Y_0$ and $f_1 \in Y_1$. Using the (quasi-) subadditivity of the K-functional and the definition of the norm in Y_0, Y_1, we have

$$(P_0 f)(u) \lesssim (P_0 f_0)(u) + (P_0 f_1)(u) \le \|f_0\|_{Y_0} + (P_0 f_1)(u),$$
$$(R_0 f)(u) \lesssim (R_0 f_0)(u) + (R_0 f_1)(u) \le \|f_0\|_{Y_0} + (R_0 f_1)(u),$$
$$(R_1 f)(u) \lesssim (R_1 f_0)(u) + (R_1 f_1)(u) \le (R_1 f_0)(u) + \|f_1\|_{Y_1},$$
$$(Q_1 f)(u) \lesssim (Q_1 f_0)(u) + (Q_1 f_1)(u) \le (Q_1 f_0)(u) + \|f_1\|_{Y_1}.$$

Thus, it is enough to verify that

$$(P_0 f_1)(u) + (R_0 f_1)(u) + \rho(u)[(R_1 f_0)(u) + (Q_1 f_0)(u)] \lesssim \|f_0\|_{Y_0} + \rho(u)\|f_1\|_{Y_1}. \tag{3.23}$$

Arguing as in (3.14)-(3.16) we can obtain that $(P_0 f_1)(u) \lesssim \rho(u)\|f_1\|_{Y_1}$, $(R_1 f_0)(u) \lesssim \frac{1}{\rho(u)}\|f_0\|_{Y_0}$ and $(Q_1 f_0)(u) \lesssim \frac{1}{\rho(u)}\|f_0\|_{Y_0}$. On the other hand, using (2.8) with $f = f_1$ and Lemma 2.3 (i), we have

$$(R_0 f_1)(u) \lesssim \|f_1\|_{Y_1} \|b_0\|_{\widetilde{E}_0(u,\infty)} \left\| t^{\theta_1-\theta_0} \frac{a_0(t)}{a_1(t)\|b_1\|_{\widetilde{E}_1(0,t)}} \right\|_{\widetilde{F}_0(0,u)}$$
$$\sim u^{\theta_1-\theta_0} \frac{a_0(u)\|b_0\|_{\widetilde{E}_0(u,\infty)}}{a_1(u)\|b_1\|_{\widetilde{E}_1(0,u)}} \|f_1\|_{Y_1} = \rho(u)\|f_1\|_{Y_1}.$$

Putting together the previous equations we establish (3.23). Finally, by taking infimum over all possible decomposition of $f = f_0 + f_1$, with $f_0 \in Y_0$ and $f_1 \in Y_1$, we obtain the desired estimate. □

4 Reiteration formulae

Our objective in this section is to identify the spaces

$$(\overline{X}^{\mathcal{R}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1})_{\theta,\mathrm{b},E}, \quad (\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{L}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1})_{\theta,\mathrm{b},E},$$
$$(\overline{X}^{\mathcal{R}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{L}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1})_{\theta,\mathrm{b},E} \quad \text{and} \quad (\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1})_{\theta,\mathrm{b},E}$$

for all possible values of $\theta \in [0, 1]$. In that process we shall need two lemmas that we collect in next subsection.

4.1 Lemmas

The first lemma is a change of variables and the second one is an equivalence between norms.

Lemma 4.1 *[23,Lemma 4.1] Let E be an r.i. space, $a, \mathrm{b} \in SV$, $0 \leq \theta \leq 1$, $0 < \alpha < 1$ and consider the function $\rho(u) = u^\alpha a(u)$, $u > 0$. Then*

$$\left\| \rho(u)^{-\theta} \mathrm{b}(\rho(u)) K(\rho(u), f) \right\|_{\widetilde{E}} \sim \left\| u^{-\theta} \mathrm{b}(u) K(u, f) \right\|_{\widetilde{E}}$$

for all $f \in X_0 + X_1$, with equivalent constant independent of f.

Lemma 4.2 *Let E, F be r.i. spaces, $a, \mathrm{b} \in SV$ and $\alpha, \beta, \gamma \in \mathbb{R}$ with $\beta < 0$ and $\gamma > 0$. Then, the equivalences*

$$\left\| t^\beta \mathrm{b}(t) \| s^\alpha a(s) f(s) \|_{\widetilde{F}(0,t)} \right\|_{\widetilde{E}} \sim \| t^{\alpha+\beta} a(t) \mathrm{b}(t) f(t) \|_{\widetilde{E}} \tag{4.1}$$

and

$$\left\| t^\gamma \mathrm{b}(t) \| s^\alpha a(s) f(s) \|_{\widetilde{F}(t,\infty)} \right\|_{\widetilde{E}} \sim \| t^{\alpha+\gamma} a(t) \mathrm{b}(t) f(t) \|_{\widetilde{E}} \tag{4.2}$$

hold for any monotone measurable function f on $(0, \infty)$.

Proof Both properties are proved in [21,Theorems 3.6 and 3.7] for non-increasing measurable functions on $(0, \infty)$. We assume now that f is a non-decreasing measurable function. By (2.1) and applying (4.2), with $\overline{f}$ and $\gamma = -\beta < 0$, we obtain (4.1) for f

$$\left\|t^{\beta}\mathrm{b}(t)\|s^{\alpha}a(s)f(s)\|_{\widetilde{F}(0,t)}\right\|_{\widetilde{E}} = \left\|t^{\beta}\mathrm{b}(t)\|s^{-\alpha}\overline{a}(s)\overline{f}(s)\|_{\widetilde{F}(1/t,\infty)}\right\|_{\widetilde{E}}$$
$$= \left\|t^{-\beta}\overline{\mathrm{b}}(t)\|s^{-\alpha}\overline{a}(s)\overline{f}(s)\|_{\widetilde{F}(t,\infty)}\right\|_{\widetilde{E}}$$
$$\sim \|t^{-(\alpha+\beta)}\overline{a}(t)\overline{\mathrm{b}}(t)\overline{f}(t)\|_{\widetilde{E}}$$
$$= \|t^{\alpha+\beta}a(t)\mathrm{b}(t)f(t)\|_{\widetilde{E}}.$$

The proof of (4.2) can be done in a similar way. □

Now, we are in position to establish reiteration theorems for $\mathcal{R}$ and $\mathcal{L}$-classes.

4.2 The space $(\overline{X}^{\mathcal{R}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1})_{\theta,\mathrm{b},E}$, $0<\theta_0<\theta_1<1$ and $0\le\theta\le 1$.

Theorem 4.3 *Let $0<\theta_0<\theta_1<1$. Let E, E_0, E_1, F_0, F_1 r.i. spaces, a_0, a_1, b, b_0, $\mathrm{b}_1 \in SV$ such that $\|\mathrm{b}_0\|_{\widetilde{E}_0(0,1)}<\infty$ and $\|\mathrm{b}_1\|_{\widetilde{E}_1(0,1)}<\infty$ and denote*

$$\rho(u) = u^{\theta_1-\theta_0}\frac{a_0(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(0,u)}}{a_1(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}}, \quad u>0.$$

(a) If $0<\theta<1$, then

$$\left(\overline{X}^{\mathcal{R}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1}\right)_{\theta,\mathrm{b},E} = \overline{X}_{\tilde{\theta},B_\theta,E}$$

where $\tilde{\theta}=(1-\theta)\theta_0+\theta\theta_1$ and

$$B_\theta(u) = \left(a_0(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(0,u)}\right)^{1-\theta}\left(a_1(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}\right)^{\theta}\mathrm{b}(\rho(u)), \quad u>0.$$

(b) If $\theta=0$ and $\|\mathrm{b}\|_{\widetilde{E}(1,\infty)}<\infty$, then

$$\left(\overline{X}^{\mathcal{R}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1}\right)_{0,\mathrm{b},E} = \overline{X}^{\mathcal{R},\mathcal{L}}_{\theta_0,\mathrm{b}\circ\rho,E,\mathrm{b}_0,E_0,a_0,F_0}.$$

(c) If $\theta=1$ and $\|\mathrm{b}\|_{\widetilde{E}(0,1)}<\infty$, then

$$\left(\overline{X}^{\mathcal{R}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1}\right)_{1,\mathrm{b},E} = \overline{X}^{\mathcal{R}}_{\theta_1,B_1,E,a_1,F_1}\cap\overline{X}^{\mathcal{R},\mathcal{R}}_{\theta_1,\mathrm{b}\circ\rho,E,\mathrm{b}_1,E_1,a_1,F_1}$$

where $B_1(u)=\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}\mathrm{b}(\rho(u))$, $u>0$.

Proof Throughout the proof we use the notation $Y_0 = \overline{X}^{\mathcal{R}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}$, $Y_1 = \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1}$ and $\overline{K}(u,f)=K(u,f;Y_0,Y_1)$, $u>0$.

We start with the proof of (a). Let $f \in \overline{Y}_{\theta,\mathrm{b},E}$. The generalized Holmstedt type formula (3.1) and estimate (2.6) give that

$$\begin{aligned}\overline{K}(\rho(u), f) &\gtrsim \left\| \mathrm{b}_0(t) \| s^{-\theta_0} a_0(s) K(s, f) \|_{\widetilde{F}_0(t,u)} \right\|_{\widetilde{E}_0(0,u)} \\ &\gtrsim u^{-\theta_0} a_0(u) \|\mathrm{b}_0\|_{\widetilde{E}_0(0,u)} K(u, f).\end{aligned}$$

Hence, using also Lemma 4.1, we have

$$\begin{aligned}\|f\|_{\overline{Y}_{\theta,\mathrm{b},E}} &\sim \|\rho(u)^{-\theta} \mathrm{b}(\rho(u)) \overline{K}(\rho(u), f)\|_{\widetilde{E}} \\ &\gtrsim \left\| \rho(u)^{-\theta} \mathrm{b}(\rho(u)) u^{-\theta_0} a_0(u) \|\mathrm{b}_0\|_{\widetilde{E}_0(0,u)} K(u, f) \right\|_{\widetilde{E}}.\end{aligned}$$

Observing

$$\rho(u)^{-\theta} \mathrm{b}(\rho(u)) = u^{\theta_0 - \tilde{\theta}} \frac{B_\theta(u)}{a_0(u) \|\mathrm{b}_0\|_{\widetilde{E}_0(0,u)}}, \quad u > 0, \tag{4.3}$$

we obtain that $\|f\|_{\overline{X}_{\tilde{\theta},B_\theta,E}} \lesssim \|f\|_{\overline{Y}_{\theta,\mathrm{b},E}}$. This proves the inclusion $\overline{Y}_{\theta,\mathrm{b},E} \hookrightarrow \overline{X}_{\tilde{\theta},B_\theta,E}$.

Next, we proceed with the reverse inclusion. Let $f \in \overline{X}_{\tilde{\theta},B_\theta,E}$. Using again Lemma 4.1, the generalized Holmstedt type formula (3.1) and the triangular inequality, we have

$$\begin{aligned}\|f\|_{\overline{Y}_{\theta,\mathrm{b},E}} &\sim \left\| \rho(u)^{-\theta} \mathrm{b}(\rho(u)) \overline{K}(\rho(u), f) \right\|_{\widetilde{E}} \\ &\lesssim \left\| \rho(u)^{-\theta} \mathrm{b}(\rho(u)) \left\| \mathrm{b}_0(t) \| s^{-\theta_0} a_0(s) K(s, f) \|_{\widetilde{F}_0(t,u)} \right\|_{\widetilde{E}_0(0,u)} \right\|_{\widetilde{E}} \\ &\quad + \left\| \rho(u)^{1-\theta} \mathrm{b}(\rho(u)) \|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)} \| t^{-\theta_1} a_1(t) K(t, f) \|_{\widetilde{F}_1(u,\infty)} \right\|_{\widetilde{E}} \\ &\quad + \left\| \rho(u)^{1-\theta} \mathrm{b}(\rho(u)) \left\| \mathrm{b}_1(t) \| s^{-\theta_1} a_1(s) K(s, f) \|_{\widetilde{F}_1(t,\infty)} \right\|_{\widetilde{E}_1(u,\infty)} \right\|_{\widetilde{E}} \\ &:= I_1 + I_2 + I_3.\end{aligned}$$

Hence, in order to prove that $f \in \overline{Y}_{\theta,\mathrm{b},E}$ it is suffices to estimate last three expressions, I_1, I_2 and I_3, by $\|u^{-\tilde{\theta}} B_\theta(u) K(u, f)\|_{\widetilde{E}}$. We start with I_1. Using (4.3) and Lemma 4.2, with the K-functional and $\beta = \theta_0 - \tilde{\theta} < 0$, we obtain

$$\begin{aligned}I_1 &= \left\| u^{\theta_0 - \tilde{\theta}} \frac{B_\theta(u)}{a_0(u) \|\mathrm{b}_0\|_{\widetilde{E}_0(0,u)}} \left\| \mathrm{b}_0(t) \| s^{-\theta_0} a_0(s) K(s, f) \|_{\widetilde{F}_0(t,u)} \right\|_{\widetilde{E}_0(0,u)} \right\|_{\widetilde{E}} \\ &\leq \left\| u^{\theta_0 - \tilde{\theta}} \frac{B_\theta(u)}{a_0(u)} \| s^{-\theta_0} a_0(s) K(s, f) \|_{\widetilde{F}_0(0,u)} \right\|_{\widetilde{E}} \\ &\sim \left\| u^{-\tilde{\theta}} B_\theta(u) K(u, f) \right\|_{\widetilde{E}}.\end{aligned} \tag{4.4}$$

Now we estimate I_2. The relation

$$\rho(u)^{1-\theta}\mathrm{b}(\rho(u)) = u^{\theta_1-\tilde{\theta}}\frac{B_\theta(u)}{a_1(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}}, \quad u>0, \tag{4.5}$$

and Lemma 4.2, with the K-functional and $\gamma=\theta_1-\widetilde{\theta}>0$, give that

$$I_2 = \left\| u^{\theta_1-\tilde{\theta}}\tfrac{B_\theta(u)}{a_1(u)}\|t^{-\theta_1}a_1(t)K(t,f)\|_{\widetilde{F}_1(u,\infty)}\right\|_{\widetilde{E}} \sim \|u^{-\tilde{\theta}}B_\theta(u)K(u,f\|_{\widetilde{E}}.$$

Finally, we approach I_3 through I_2. Using (4.5) we have

$$I_3 = \left\| u^{\theta_1-\tilde{\theta}}\tfrac{B_\theta(u)}{a_1(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}}\left\|\mathrm{b}_1(t)\|s^{-\theta_1}a_1(s)K(s,f)\|_{\widetilde{F}_1(t,\infty)}\right\|_{\widetilde{E}_1(u,\infty)}\right\|_{\widetilde{E}}.$$

Since the function $t \rightsquigarrow \|\cdot\|_{\widetilde{F}_1(t,\infty)}$ is monotone and $\theta_1-\tilde{\theta}>0$, we can apply Lemma 4.2 to obtain that

$$I_3 \sim \left\| u^{\theta_1-\tilde{\theta}}\tfrac{B_\theta(u)}{a_1(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}}\mathrm{b}_1(u)\|s^{-\theta_1}a_1(s)K(s,f)\|_{\widetilde{F}_1(u,\infty)}\right\|_{\widetilde{E}}.$$

From Lemma 2.3 (iv) and (4.5) we deduce that $I_3 \lesssim I_2$ and the proof of (a) is completed.

The proof of (b) follows similar steps. The inclusion $\overline{Y}_{0,\mathrm{b},E} \hookrightarrow \overline{X}^{\mathcal{R},\mathcal{L}}_{\theta_0,\mathrm{b}\circ\rho,E,\mathrm{b}_0,E_0,a_0,F_0}$ comes directly from Theorem 3.1 and the lattice property of $\widetilde{E}$. In order to prove the reverse inclusion we use again Lemma 4.1, Theorem 3.1 and the triangular inequality. Indeed, let $f\in\overline{X}^{\mathcal{R},\mathcal{L}}_{\theta_0,\mathrm{b}\circ\rho,E,\mathrm{b}_0,E_0,a_0,F_0}$, then

$$\begin{aligned}
\|f\|_{\overline{Y}_{0,\mathrm{b},E}} &\sim \left\|\mathrm{b}(\rho(u))\overline{K}(\rho(u),f)\right\|_{\widetilde{E}}\\
&\lesssim \left\|\mathrm{b}(\rho(u))\left\|\mathrm{b}_0(t)\|s^{-\theta_0}a_0(s)K(s,f)\|_{\widetilde{F}_0(t,u)}\right\|_{\widetilde{E}_0(0,u)}\right\|_{\widetilde{E}}\\
&\quad+\left\|\rho(u)\mathrm{b}(\rho(u))\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}\|t^{-\theta_1}a_1(t)K(t,f)\|_{\widetilde{F}_1(u,\infty)}\right\|_{\widetilde{E}}\\
&\quad+\left\|\rho(u)\mathrm{b}(\rho(u))\left\|\mathrm{b}_1(t)\|s^{-\theta_1}a_1(s)K(s,f)\|_{\widetilde{F}_1(t,\infty)}\right\|_{\widetilde{E}_1(u,\infty)}\right\|_{\widetilde{E}}\\
&:= I_4+I_5+I_6.
\end{aligned}$$

It is clear that $I_4=\|f\|_{\overline{X}^{\mathcal{R},\mathcal{L}}_{\theta_0,\mathrm{b}\circ\rho,E,\mathrm{b}_0,E_0,a_0,F_0}}$. Let us estimate I_5 by I_4. Lemma 4.2, with the K-functional and $\gamma=\theta_1-\theta_0>0$, and (2.6) guarantee that

$$\begin{aligned}
I_5 &= \left\|u^{\theta_1-\theta_0}\frac{a_0(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(0,u)}}{a_1(u)}\mathrm{b}(\rho(u))\|t^{-\theta_1}a_1(t)K(t,f)\|_{\widetilde{F}_1(u,\infty)}\right\|_{\widetilde{E}}\\
&\sim \left\|u^{-\theta_0}a_0(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(0,u)}\mathrm{b}(\rho(u))K(u,f)\right\|_{\widetilde{E}} \lesssim I_4.
\end{aligned}$$

The third term I_6 is bounded by I_5. Indeed, since the function $t \rightsquigarrow \|\cdot\|_{\widetilde{F}_1(t,\infty)}$ is monotone and $\gamma = \theta_1 - \tilde{\theta} > 0$, Lemma 4.2 gives that

$$I_6 = \left\| u^{\theta_1-\theta_0} \frac{a_0(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(0,u)}}{a_1(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}} \mathrm{b}(\rho(u)) \left\| \mathrm{b}_1(t)\|s^{-\theta_1} a_1(s) K(s,f)\|_{\widetilde{F}_1(t,\infty)} \right\|_{\widetilde{E}_1(u,\infty)} \right\|_{\widetilde{E}}$$
$$\sim \left\| u^{\theta_1-\theta_0} \frac{a_0(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(0,u)}}{a_1(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}} \mathrm{b}(\rho(u))\mathrm{b}_1(u)\|s^{-\theta_1} a_1(s) K(s,f)\|_{\widetilde{F}_1(u,\infty)} \right\|_{\widetilde{E}}.$$

Then, Lemma 2.3 (iv) implies that $I_6 \lesssim I_5$. Summing up,

$$\|f\|_{\overline{Y}_{0,\mathrm{b},E}} \lesssim I_4 + I_5 + I_6 \lesssim I_4 = \|f\|_{\overline{X}^{\mathcal{R},\mathcal{L}}_{\theta_0,\mathrm{b}\circ\rho,E,\mathrm{b}_0,E_0,a_0,F_0}}$$

and the proof of b) is concluded.

Finally, we proceed with the proof of c). Let $f \in \overline{Y}_{1,\mathrm{b},E}$. Lemma 4.1, Theorem 3.1 and the lattice property guarantee that

$$\|f\|_{\overline{Y}_{1,\mathrm{b},E}} \sim \|\rho(u)^{-1}\mathrm{b}(\rho(u))\overline{K}(\rho(u),f)\|_{\widetilde{E}}$$
$$\gtrsim \left\| \mathrm{b}(\rho(u))\, \|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)} \|t^{-\theta_1} a(t) K(t,f)\|_{\widetilde{F}_1(u,\infty)} \right\|_{\widetilde{E}} = \|f\|_{\overline{X}^{\mathcal{R}}_{\theta_1,B_1,E,a_1,F_1}}$$

and

$$\|f\|_{\overline{Y}_{1,\mathrm{b},E}} \gtrsim \left\| \mathrm{b}(\rho(u)) \left\| \mathrm{b}_1(t)\|s^{-\theta_1} a(s) K(s,f)\|_{\widetilde{F}_1(t,\infty)} \right\|_{\widetilde{E}_1(u,\infty)} \right\|_{\widetilde{E}}$$
$$= \|f\|_{\overline{X}^{\mathcal{R},\mathcal{R}}_{\theta_1,\mathrm{b}\circ\rho,E,\mathrm{b}_1,E_1,a_1,F_1}}.$$

Then, $f \in \overline{X}^{\mathcal{R}}_{\theta_1,B_1,E,a_1,F_1} \cap \overline{X}^{\mathcal{R},\mathcal{R}}_{\theta_1,\mathrm{b}\circ\rho,E,\mathrm{b}_1,E_1,a_1,F_1}$.

Next, we study the reverse inclusion. Let $f \in \overline{X}^{\mathcal{R}}_{\theta_1,B_1,E,a_1,F_1} \cap \overline{X}^{\mathcal{R},\mathcal{R}}_{\theta_1,\mathrm{b}\circ\rho,E,\mathrm{b}_1,E_1,a_1,F_1}$. Using again Lemma 4.1, Theorem 3.1 and the triangular inequality, we have that

$$\|f\|_{\overline{Y}_{1,\mathrm{b},E}} \sim \|\rho(u)^{-1}\mathrm{b}(\rho(u))\overline{K}(\rho(u),f)\|_{\widetilde{E}}$$
$$\lesssim \left\| \rho(u)^{-1}\mathrm{b}(\rho(u)) \left\| \mathrm{b}_0(t)\|s^{-\theta_0} a_0(s) K(s,f)\|_{\widetilde{F}_0(t,u)} \right\|_{\widetilde{E}_0(0,u)} \right\|_{\widetilde{E}}$$
$$+ \left\| \mathrm{b}(\rho(u))\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)} \|t^{-\theta_1} a_1(t) K(t,f)\|_{\widetilde{F}_1(u,\infty)} \right\|_{\widetilde{E}}$$
$$+ \left\| \mathrm{b}(\rho(u)) \left\| \mathrm{b}_1(t)\|s^{-\theta_1} a_1(s) K(s,f)\|_{\widetilde{F}_1(t,\infty)} \right\|_{\widetilde{E}_1(u,\infty)} \right\|_{\widetilde{E}}$$
$$:= I_7 + I_8 + I_9.$$

Since $I_8 = \|f\|_{\overline{X}^{\mathcal{R}}_{\theta_1,B_1,E,a_1,F_1}}$ and $I_9 = \|f\|_{\overline{X}^{\mathcal{R},\mathcal{R}}_{\theta_1,\mathrm{b}\circ\rho,E,\mathrm{b}_1,E_1,a_1,F_1}}$, it is enough to estimate I_7. Using Lemma 4.2, with the K-functional and $\beta = \theta_0 - \theta_1 < 0$, and estimate (2.5), we obtain

$$\begin{aligned} I_7 &= \left\| u^{\theta_0-\theta_1} \frac{a_1(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}}{a_0(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(0,u)}} \mathrm{b}(\rho(u)) \left\| \|\mathrm{b}_0(t)\| s^{-\theta_0} a_0(s) K(s,f)\|_{\widetilde{F}_0(t,u)} \right\|_{\widetilde{E}_0(0,u)} \right\|_{\widetilde{E}} \\ &\le \left\| u^{\theta_0-\theta_1} \frac{a_1(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}}{a_0(u)} \mathrm{b}(\rho(u)) \|s^{-\theta_0} a_0(s) K(s,f)\|_{\widetilde{F}_0(0,u)} \right\|_{\widetilde{E}} \\ &\sim \left\| u^{-\theta_1} a_1(u) \|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)} \mathrm{b}(\rho(u)) K(u,f) \right\|_{\widetilde{E}} \\ &\lesssim \left\| \mathrm{b}(\rho(u)) \|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)} \|t^{-\theta_1} a_1(t) K(t,f)\|_{\widetilde{F}_1(u,\infty)} \right\|_{\widetilde{E}} = I_8 = \|f\|_{\overline{X}^{\mathcal{R}}_{\theta_1,B_1,E,a_1,F_1}}. \end{aligned}$$

This completes the proof. □

4.3 The space $(\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{L}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1})_{\theta,\mathrm{b},E}$, $0 < \theta_0 < \theta_1 < 1$ and $0 \le \theta \le 1$.

Next theorem can be proved similarly, except that Holmstedt formula (3.17) has to be used in place of (3.1). However, since this approach is lenghty, we prove it using Theorem 4.3 and a symmetry argument.

Theorem 4.4 *Let* $0 < \theta_0 < \theta_1 < 1$. *Let* E, E_0, E_1, F_0, F_1 *r.i. spaces,* a_0, a_1, b, b_0, $\mathrm{b}_1 \in SV$ *such that* $\|\mathrm{b}_0\|_{\widetilde{E}_0(1,\infty)} < \infty$ *and* $\|\mathrm{b}_1\|_{\widetilde{E}_1(1,\infty)} < \infty$ *and denote*

$$\rho(u) = u^{\theta_1-\theta_0} \frac{a_0(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}}{a_1(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(u,\infty)}}, \quad u > 0.$$

(a) If $0 < \theta < 1$, *then*

$$(\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{L}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1})_{\theta,\mathrm{b},E} = \overline{X}_{\tilde{\theta},B_\theta,E}$$

where $\tilde{\theta} = (1-\theta)\theta_0 + \theta\theta_1$ *and*

$$B_\theta(u) = \left(a_0(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}\right)^{1-\theta} \left(a_1(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(u,\infty)}\right)^{\theta} \mathrm{b}(\rho(u)), \quad u > 0.$$

(b) If $\theta = 0$ *and* $\|\mathrm{b}\|_{\widetilde{E}(1,\infty)} < \infty$, *then*

$$(\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{L}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1})_{0,\mathrm{b},E} = \overline{X}^{\mathcal{L}}_{\theta_0,B_0,E,a_0,F_0} \cap \overline{X}^{\mathcal{L},\mathcal{L}}_{\theta_0,\mathrm{b}\circ\rho,E,\mathrm{b}_0,E_0,a_0,F_0}$$

where $B_0(u) = \|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}\mathrm{b}(\rho(u))$, $u > 0$.

(c) If $\theta = 1$ *and* $\|\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty$, *then*

$$(\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{L}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1})_{1,\mathrm{b},E} = \overline{X}^{\mathcal{L},\mathcal{R}}_{\theta_1,\mathrm{b}\circ\rho,E,\mathrm{b}_1,E_1,a_1,F_1}. \tag{4.6}$$

Proof We prove (c). Applying both identities of Lemma 2.9 we have

$$
\begin{aligned}
&(\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{L}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1})_{1,\mathrm{b},E} \\
&\quad = \big(\overline{X}^{\mathcal{L}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1}, \overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}\big)_{0,\overline{\mathrm{b}},E} \\
&\quad = \big((X_1, X_0)^{\mathcal{R}}_{1-\theta_1,\overline{\mathrm{b}}_1,E_1,\overline{a}_1,F_1}, (X_1, X_0)^{\mathcal{R}}_{1-\theta_0,\overline{\mathrm{b}}_0,E_0,\overline{a}_0,F_0}\big)_{0,\overline{\mathrm{b}},E}.
\end{aligned}
$$

Then, by Theorem 4.3 b) we obtain

$$
\big(\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{L}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1}\big)_{1,\mathrm{b},E} = (X_1, X_0)^{\mathcal{R},\mathcal{L}}_{1-\theta_1,\overline{\mathrm{b}}\circ\rho^{\#},E,\overline{\mathrm{b}}_1,E_1,\overline{a}_1,F_1}
$$

where

$$
\rho^{\#}(u) = u^{\theta_1-\theta_0}\frac{\overline{a}_1(u)\|\overline{\mathrm{b}}_1\|_{\widetilde{E}_1(0,u)}}{\overline{a}_0(u)\|\overline{\mathrm{b}}_0\|_{\widetilde{E}_0(0,u)}}, \quad u > 0.
$$

Equality (4.6) follows from Lemma 2.12 and that $\overline{\overline{\mathrm{b}} \circ \rho^{\#}}(u) = \mathrm{b} \circ \rho(u)$, $u > 0$.

The proofs of (a) and (b) can be carried out using similar arguments and Theorem 4.3 (a), (c). □

4.4 The space $(\overline{X}^{\mathcal{R}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{L}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1})_{\theta,\mathrm{b},E}$, $0 < \theta_0 < \theta_1 < 1$ and $0 \leq \theta \leq 1$.

The following theorems deal with reiteration formulae with mixed $\mathcal{R}$ and $\mathcal{L}$-spaces. Both theorems have to be proved independently, since the symmetry argument cannot be applied.

Theorem 4.5 *Let* $0 < \theta_0 < \theta_1 < 1$. *Let* E, E_0, E_1, F_0, F_1 *r.i. spaces,* a_0, a_1, b, b_0, $\mathrm{b}_1 \in SV$ *such that* $\|\mathrm{b}_0\|_{\widetilde{E}_0(0,1)} < \infty$ *and* $\|\mathrm{b}_1\|_{\widetilde{E}_1(1,\infty)} < \infty$ *and denote*

$$
\rho(u) = u^{\theta_1-\theta_0}\frac{a_0(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(0,u)}}{a_1(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(u,\infty)}}, \quad u > 0.
$$

(a) If $0 < \theta < 1$, *then*

$$
\big(\overline{X}^{\mathcal{R}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{L}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1}\big)_{\theta,\mathrm{b},E} = \overline{X}_{\tilde{\theta},B_\theta,E}
$$

where $\tilde{\theta} = (1-\theta)\theta_0 + \theta\theta_1$ *and*

$$
B_\theta(u) = \big(a_0(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(0,u)}\big)^{1-\theta}\big(a_1(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(u,\infty)}\big)^{\theta}\,\mathrm{b}(\rho(u)), \quad u > 0.
$$

(b) If $\theta = 0$ *and* $\|\mathrm{b}\|_{\widetilde{E}(1,\infty)} < \infty$, *then*

$$
\big(\overline{X}^{\mathcal{R}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{L}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1}\big)_{0,\mathrm{b},E} = \overline{X}^{\mathcal{R},\mathcal{L}}_{\theta_0,\mathrm{b}\circ\rho,E,\mathrm{b}_0,E_0,a_0,F_0}.
$$

(c) If $\theta = 1$ *and* $\|\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty$*, then*

$$(\overline{X}^{\mathcal{R}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{L}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1})_{1,\mathrm{b},E} = \overline{X}^{\mathcal{L},\mathcal{R}}_{\theta_1,\mathrm{b}\circ\rho,E,\mathrm{b}_1,E_1,a_1,F_1}.$$

Proof The proof follows essentially the same steps as the proof of Theorem 4.3, although some estimates are different. Let us start by denoting as usual $Y_0 = \overline{X}^{\mathcal{R}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}$, $Y_1 = \overline{X}^{\mathcal{L}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1}$ and $\overline{K}(u, f) = K(u, f; Y_0, Y_1)$, $u > 0$.

The inclusion $\overline{Y}_{\theta,\mathrm{b},E} \hookrightarrow \overline{X}_{\tilde{\theta},B_\theta,E}$ follows from Lemma 4.1, Theorem 3.3 and inequality (2.6) as we did in the proof of Theorem 4.3 (a). Let now $f \in \overline{X}_{\tilde{\theta},B_\theta,E}$. Using Lemma 4.1, Theorem 3.3 and the triangular inequality, we have

$$\begin{aligned}\|f\|_{\overline{Y}_{\theta,\mathrm{b},E}} &\sim \left\|\rho(u)^{-\theta}\mathrm{b}(\rho(u))\overline{K}(\rho(u), f)\right\|_{\widetilde{E}} \\ &\lesssim \left\|\rho(u)^{-\theta}\mathrm{b}(\rho(u))\left\|\mathrm{b}_0(t)\|s^{-\theta_0}a_0(s)K(s, f)\|_{\widetilde{F}_0(t,u)}\right\|_{\widetilde{E}_0(0,u)}\right\|_{\widetilde{E}} \\ &\quad + \left\|\rho(u)^{1-\theta}\mathrm{b}(\rho(u))\left\|\mathrm{b}_1(t)\|s^{-\theta_1}a_1(s)K(s, f)\|_{\widetilde{F}_1(u,t)}\right\|_{\widetilde{E}_1(u,\infty)}\right\|_{\widetilde{E}} \\ &:= I_{10} + I_{11}.\end{aligned}$$

The expression I_{10} can be estimate by $\|f\|_{\overline{X}_{\tilde{\theta},B_\theta,E}}$ as we did with I_1 in (4.4). We also observe

$$\begin{aligned}I_{11} &= \left\|u^{\theta_1-\tilde{\theta}}\frac{B_\theta(u)}{a_1(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(u,\infty)}}\left\|\mathrm{b}_1(t)\|s^{-\theta_1}a_1(s)K(s, f)\|_{\widetilde{F}_1(u,t)}\right\|_{\widetilde{E}_1(u,\infty)}\right\|_{\widetilde{E}} \\ &\le \left\|u^{\theta_1-\tilde{\theta}}\frac{B_\theta(u)}{a_1(u)}\|s^{-\theta_1}a_1(s)K(s, f)\|_{\widetilde{F}_1(u,\infty)}\right\|_{\widetilde{E}}.\end{aligned}$$

Applying, as usual, Lemma 4.2 with the K-functional and $\gamma = \theta_1 - \widetilde{\theta} > 0$, we obtain that $I_{11} \lesssim \|f\|_{\overline{X}_{\tilde{\theta},B_\theta,E}}$. This completes the proof of (a).

Next we proceed with the proof of (b). The inclusion $\overline{Y}_{0,\mathrm{b},E} \hookrightarrow \overline{X}^{\mathcal{R},\mathcal{L}}_{\theta_0,\mathrm{b}\circ\rho,E,\mathrm{b}_0,E_0,a_0,F_0}$ follows directly from Lemma 4.1, Theorem 3.3 and the lattice property of $\widetilde{E}$. In order to prove the reverse inclusion we use again Lemma 4.1, Theorem 3.3 and the triangular inequality to obtain that

$$\begin{aligned}\|f\|_{\overline{Y}_{0,\mathrm{b},E}} &\sim \left\|\mathrm{b}(\rho(u))\overline{K}(\rho(u), f)\right\|_{\widetilde{E}} \\ &\lesssim \left\|\mathrm{b}(\rho(u))\left\|\mathrm{b}_0(t)\|s^{-\theta_0}a_0(s)K(s, f)\|_{\widetilde{F}_0(t,u)}\right\|_{\widetilde{E}_0(0,u)}\right\|_{\widetilde{E}} \\ &\quad + \left\|\rho(u)\mathrm{b}(\rho(u))\left\|\mathrm{b}_1(t)\|s^{-\theta_1}a_1(s)K(s, f)\|_{\widetilde{F}_1(u,t)}\right\|_{\widetilde{E}_1(u,\infty)}\right\|_{\widetilde{E}} \\ &:= I_{12} + I_{13}.\end{aligned}$$

It is clear that I_{12} is equal to $\|f\|_{\overline{X}^{\mathcal{R},\mathcal{L}}_{\theta_0,\mathrm{b}\circ\rho,E,\mathrm{b}_0,E_0,a_0,F_0}}$. Besides that, arguing as we did with I_{11} and using (2.6) we estimate I_{13} by I_{12}. Indeed,

$$\begin{aligned} I_{13} &\lesssim \left\| u^{-\theta_0} a_0(u) \|\mathrm{b}_0\|_{\widetilde{E}_0(0,u)} \mathrm{b}(\rho(u)) K(u, f) \right\|_{\widetilde{E}} \\ &\lesssim \left\| \mathrm{b}(\rho(u)) \left\| \mathrm{b}_0(t) \| s^{-\theta_0} a_0(s) K(s, f) \|_{\widetilde{F}_0(t,u)} \right\|_{\widetilde{E}_0(0,u)} \right\|_{\widetilde{E}} = I_{12}. \end{aligned}$$

This completes the proof of (b). Similar steps using (2.7) instead of (2.6) lead to the proof of (c). □

4.5 The space $(\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1})_{\theta,\mathrm{b},E}$, $0 < \theta_0 < \theta_1 < 1$ and $0 \le \theta \le 1$.

Theorem 4.6 *Let $0 < \theta_0 < \theta_1 < 1$. Let E, E_0, E_1, F_0, F_1 r.i. spaces, a_0, a_1, b, b_0, $\mathrm{b}_1 \in SV$ such that $\|\mathrm{b}_0\|_{\widetilde{E}_0(1,\infty)} < \infty$ and $\|\mathrm{b}_1\|_{\widetilde{E}_1(0,1)} < \infty$ and denote*

$$\rho(u) = u^{\theta_1-\theta_0} \frac{a_0(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}}{a_1(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}}, \quad u > 0.$$

(a) If $0 < \theta < 1$, then

$$\left(\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1}\right)_{\theta,\mathrm{b},E} = \overline{X}_{\tilde{\theta},B_\theta,E}$$

where $\tilde{\theta} = (1-\theta)\theta_0 + \theta\theta_1$ and

$$B_\theta(u) = \left(a_0(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}\right)^{1-\theta} \left(a_1(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}\right)^{\theta} \mathrm{b}(\rho(u)), \quad u > 0.$$

(b) If $\theta = 0$ and $\|\mathrm{b}\|_{\widetilde{E}(1,\infty)} < \infty$, then

$$\left(\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1}\right)_{0,\mathrm{b},E} = \overline{X}^{\mathcal{L}}_{\theta_0,B_0,E,a_0,F_0} \cap \overline{X}^{\mathcal{L},\mathcal{L}}_{\theta_0,\mathrm{b}\circ\rho,E,\mathrm{b}_0,E_0,a_0,F_0}$$

where $B_0(u) = \|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}\mathrm{b}(\rho(u))$, $u > 0$.

(c) If $\theta = 1$ and $\|\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty$, then

$$\left(\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1}\right)_{1,\mathrm{b},E} = \overline{X}^{\mathcal{R}}_{\theta_1,B_1,E,a_1,F_1} \cap \overline{X}^{\mathcal{R},\mathcal{R}}_{\theta_1,\mathrm{b}\circ\rho,E,\mathrm{b}_1,E_1,a_1,F_1}$$

where $B_1(u) = \|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}\mathrm{b}(\rho(u))$, $u > 0$.

Proof As usual, we denote $Y_0 = \overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}$, $Y_1 = \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1}$ and $\overline{K}(u, f) = K(u, f; Y_0, Y_1)$, $u > 0$. Lemma 4.1 establishes the equivalence

$$\|f\|_{\overline{Y}_{\theta,\mathrm{b},E}} = \|u^{-\theta}\mathrm{b}(u)\overline{K}(u, f)\|_{\widetilde{E}} \sim \|\rho(u)^{-\theta}\mathrm{b}(\rho(u))\overline{K}(\rho(u), f)\|_{\widetilde{E}}.$$

Then, in order to obtain (a) it is sufficient to prove that

$$\|\rho(u)^{-\theta}\mathrm{b}(\rho(u))\overline{K}(\rho(u), f)\|_{\widetilde{E}} \sim \left\|u^{-\tilde{\theta}}B_\theta(u)K(u, f)\right\|_{\widetilde{E}}. \tag{4.7}$$

Theorem 3.4 and (2.4) give

$$\begin{aligned}\overline{K}(\rho(u), f) &\gtrsim \|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}\|s^{-\theta_0}a_0(s)K(s, f)\|_{\widetilde{F}_0(0,u)} \\ &\gtrsim u^{-\theta_0}a_0(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}K(u, f).\end{aligned}$$

Hence, the inequality " $\lesssim$ " of (4.7) follows using the relation

$$\rho(u)^{-\theta}\mathrm{b}(\rho(u)) = u^{\theta_0-\tilde{\theta}}\frac{B_\theta(u)}{a_0(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}}, \quad u > 0.$$

Next, we proceed with the reverse inequality " $\lesssim$ " of (4.7). By Theorem 3.4 and the triangular inequality, we have

$$\begin{aligned}&\left\|\rho(u)^{-\theta}\mathrm{b}(\rho(u))\overline{K}(\rho(u), f)\right\|_{\widetilde{E}} \\ &\quad\lesssim \left\|\rho(u)^{-\theta}\mathrm{b}(\rho(u))\left\|\mathrm{b}_0(t)\|s^{-\theta_0}a_0(s)K(s, f)\|_{\widetilde{F}_0(0,t)}\right\|_{\widetilde{E}_0(0,u)}\right\|_{\widetilde{E}} \\ &\qquad+ \left\|\rho(u)^{-\theta}\mathrm{b}(\rho(u))\|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}\left\|t^{-\theta_0}a_0(t)K(t, f)\right\|_{\widetilde{F}_0(0,u)}\right\|_{\widetilde{E}} \\ &\qquad+ \left\|\rho(u)^{1-\theta}\mathrm{b}(\rho(u))\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}\|t^{-\theta_1}a_1(t)K(t, f)\|_{\widetilde{F}_1(u,\infty)}\right\|_{\widetilde{E}} \\ &\qquad+ \left\|\rho(u)^{1-\theta}\mathrm{b}(\rho(u))\left\|\mathrm{b}_1(t)\|s^{-\theta_1}a_1(s)K(s, f)\|_{\widetilde{F}_1(t,\infty)}\right\|_{\widetilde{E}_1(u,\infty)}\right\|_{\widetilde{E}} \\ &\quad:= I_{14} + I_{15} + I_{16} + I_{17}.\end{aligned}$$

Let us estimate the four expressions by $\|u^{-\tilde{\theta}}B_\theta(u)K(u, f)\|_{\widetilde{E}}$. Lemma 4.2, with the monotone function $t \rightsquigarrow \|\cdot\|_{\widetilde{F}_0(0,t)}$, $\beta = \theta_0 - \tilde{\theta} < 0$ and Lemma 2.3 (iv) yield that

$$\begin{aligned}I_{14} &\sim \left\|u^{\theta_0-\tilde{\theta}}\frac{B_\theta(u)}{a_0(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}}\mathrm{b}_0(u)\|s^{-\theta_0}a_0(s)K(s, f)\|_{\widetilde{F}_0(0,u)}\right\|_{\widetilde{E}} \\ &\lesssim \left\|u^{\theta_0-\tilde{\theta}}\frac{B_\theta(u)}{a_0(u)}\|s^{-\theta_0}a_0(s)K(s, f)\|_{\widetilde{F}_0(0,u)}\right\|_{\widetilde{E}}.\end{aligned}$$

Lemma 4.2 again, with the K-functional, implies that $I_{14} \lesssim \|u^{-\tilde{\theta}}B_\theta(u)K(u, f)\|_{\widetilde{E}}$. Similarly, using Lemma 4.2 only once, we establish that $I_{15} \sim \|u^{-\tilde{\theta}}B_\theta(u)K(u, f)\|_{\widetilde{E}}$. Having in mind that

$$\rho(u)^{1-\theta}\mathrm{b}(\rho(u)) = u^{\theta_1-\tilde{\theta}}\frac{B_\theta(u)}{a_1(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}}, \quad u > 0$$

and arguing as we did with I_2 in Theorem 4.3, we have that $I_{16} \sim \|u^{\tilde\theta} B_\theta(u) K(u, f)\|_{\widetilde{E}}$. The estimate of the term I_{17} can be carried out, arguing as in I_{14}, using Lemma 4.2 twice, once with the monotone function $t \rightsquigarrow \|\cdot\|_{\widetilde{F}_1(t,\infty)}$, the second one with the K-functional, $\gamma = \theta_1 - \tilde\theta > 0$ and Lemma 2.3 (iv). In fact,

$$\begin{aligned}
I_{17} &= \left\| u^{\theta_1-\tilde\theta} \frac{B_\theta(u)}{a_1(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}} \left\| \mathrm{b}_1(t) \|s^{-\theta_1} a_1(s) K(s, f)\|_{\widetilde{F}_1(t,\infty)} \right\|_{\widetilde{E}_1(u,\infty)} \right\|_{\widetilde{E}} \\
&\sim \left\| u^{\theta_1-\tilde\theta} \frac{B_\theta(u)}{a_1(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}} \mathrm{b}_1(u) \|s^{-\theta_1} a_1(s) K(s, f)\|_{\widetilde{F}_1(u,\infty)} \right\|_{\widetilde{E}} \\
&\lesssim \left\| u^{\theta_1-\tilde\theta} \frac{B_\theta(u)}{a_1(u)} \|s^{-\theta_1} a_1(s) K(s, f)\|_{\widetilde{F}_1(u,\infty)} \right\|_{\widetilde{E}} \\
&\sim \left\| u^{\tilde\theta} B_\theta(u) K(u, f) \right\|_{\widetilde{E}}.
\end{aligned}$$

This conclues the proof of (a).

To prove (b) it suffices to show that

$$\|\mathrm{b}(\rho(u))\overline{K}(\rho(u), f)\|_{\widetilde{E}} \sim \max\left\{ \|f\|_{\overline{X}^{\mathcal{L}}_{\theta_0, B_0, E, a_0, F_0}}, \|f\|_{\overline{X}^{\mathcal{L},\mathcal{L}}_{\theta_0, \mathrm{b}\circ\rho, E, \mathrm{b}_0, E_0, a_0, F_0}} \right\}. \quad (4.8)$$

Theorem 3.4 guarantees that

$$\overline{K}(\rho(u), f) \gtrsim \|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)} \|t^{-\theta_0} a_0(t) K(t, f)\|_{\widetilde{F}_0(0,u)}$$

and

$$\overline{K}(\rho(u), f) \gtrsim \left\| \mathrm{b}_0(t) \|s^{-\theta_0} a_0(s) K(s, f)\|_{\widetilde{F}_0(0,t)} \right\|_{\widetilde{E}_0(0,u)}.$$

Hence the inequality "$\gtrsim$" of (4.8) follows. On the other hand, Theorem 3.4 and triangular inequality give that

$$\begin{aligned}
\left\|\mathrm{b}(\rho(u))\overline{K}(\rho(u), f)\right\|_{\widetilde{E}} \lesssim & \left\| \mathrm{b}(\rho(u)) \left\| \mathrm{b}_0(t) \|s^{-\theta_0} a_0(s) K(s, f)\|_{\widetilde{F}_0(0,t)} \right\|_{\widetilde{E}_0(0,u)} \right\|_{\widetilde{E}} \\
&+ \left\| \mathrm{b}(\rho(u)) \|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)} \left\| t^{-\theta_0} a_0(t) K(t, f) \right\|_{\widetilde{F}_0(0,u)} \right\|_{\widetilde{E}} \\
&+ \left\| \rho(u) \mathrm{b}(\rho(u)) \|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)} \|t^{-\theta_1} a_1(t) K(t, f)\|_{\widetilde{F}_1(u,\infty)} \right\|_{\widetilde{E}} \\
&+ \left\| \rho(u) \mathrm{b}(\rho(u)) \left\| \mathrm{b}_1(t) \|s^{-\theta_1} a_1(s) K(s, f)\|_{\widetilde{F}_1(t,\infty)} \right\|_{\widetilde{E}_1(u,\infty)} \right\|_{\widetilde{E}} \\
&:= I_{18} + I_{19} + I_{20} + I_{21}.
\end{aligned}$$

It is clear that $I_{18} = \|f\|_{\overline{X}^{\mathcal{L},\mathcal{L}}_{\theta_0,\mathrm{b}\circ\rho,E,\mathrm{b}_0,E_0,a_0,F_0}}$ and $I_{19} = \|f\|_{\overline{X}^{\mathcal{L}}_{\theta_0,B_0,E,a_0,F_0}}$. Besides, Lemma 4.2, with the K-functional and $\gamma = \theta_1 - \theta_0 > 0$, and inequality (2.4) yield

$$\begin{aligned} I_{20} &= \left\| u^{\theta_1-\theta_0} \frac{a_0(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}}{a_1(u)} \mathrm{b}(\rho(u)) \|t^{-\theta_1} a_1(t) K(t,f)\|_{\widetilde{F}_1(u,\infty)} \right\|_{\widetilde{E}} \\ &\sim \left\| u^{-\theta_0} a_0(u) \|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)} \mathrm{b}(\rho(u)) K(u,f) \right\|_{\widetilde{E}} \\ &\lesssim \left\| \|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)} \mathrm{b}(\rho(u)) \|t^{-\theta_0} a_0(t) K(t,f)\|_{\widetilde{F}_0(0,u)} \right\|_{\widetilde{E}} = \|f\|_{\overline{X}^{\mathcal{L}}_{\theta_0,B_0,E,a_0,F_0}}. \end{aligned}$$

The estimate of the last term I_{21} requires, as with I_{17}, the use of Lemma 4.2 twice, with $\gamma = \theta_1 - \theta_0 > 0$, and Lemma 2.3 (iv). Indeed,

$$\begin{aligned} I_{21} &= \left\| u^{\theta_1-\theta_0} \frac{a_0(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}}{a_1(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}} \mathrm{b}(\rho(u)) \left\| \mathrm{b}_1(t) \|s^{-\theta_1} a_1(s) K(s,f)\|_{\widetilde{F}_1(t,\infty)} \right\|_{\widetilde{E}_1(u,\infty)} \right\|_{\widetilde{E}} \\ &\sim \left\| u^{\theta_1-\theta_0} \frac{a_0(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}}{a_1(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}} \mathrm{b}(\rho(u)) \mathrm{b}_1(u) \|s^{-\theta_1} a_1(s) K(s,f)\|_{\widetilde{F}_1(u,\infty)} \right\|_{\widetilde{E}} \\ &\lesssim \left\| u^{\theta_1-\theta_0} \frac{a_0(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}}{a_1(u)} \mathrm{b}(\rho(u)) \|s^{-\theta_1} a_1(s) K(s,f)\|_{\widetilde{F}_1(u,\infty)} \right\|_{\widetilde{E}} \\ &\sim \left\| u^{-\theta_0} a_0(u) \|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)} \mathrm{b}(\rho(u)) K(u,f) \right\|_{\widetilde{E}}. \end{aligned}$$

Finally, using (2.4) we have that $I_{21} \lesssim \|f\|_{\overline{X}^{\mathcal{L}}_{\theta_0,B_0,E,a_0,F_0}}$. This establishes b).

The proof of c) can be done using similar arguments to Theorem 4.3 c). □

5 Applications

For simplicity, we apply our results to ordered (quasi)-Banach couples $\overline{X} = (X_0, X_1)$, in the sense that $X_1 \hookrightarrow X_0$. The most classical example of an ordered couple is $(L_1(\Omega,\mu), L_\infty(\Omega,\mu))$ when Ω is a finite measure space.

5.1 Ordered couples

We briefly review how our definitions adapt to this simpler setting of ordered couples. When $X_1 \hookrightarrow X_0$, the real interpolation $\overline{X}_{\theta,\mathrm{b},E}$ can be equivalently defined as

$$\overline{X}_{\theta,\mathrm{b},E} = \left\{ f \in X_0 : \|t^{-\theta} \mathrm{b}(t) K(t,f)\|_{\widetilde{E}(0,1)} < \infty \right\}$$

where $0 \le \theta \le 1$, E is an r.i. space and $\mathrm{b} \in SV(0,1)$. The replacement of $(0,\infty)$ by $(0,1)$ in Definition 2.1 yields the definition of the class $SV(0,1)$.

Similarly, given a real parameter $0 \le \theta \le 1$, $a, \mathrm{b}, c \in SV(0,1)$ and r.i. spaces E, F, G, the spaces $\overline{X}^{\mathcal{L}}_{\theta,\mathrm{b},E,a,F}$, $\overline{X}^{\mathcal{L},\mathcal{L}}_{\theta,c,E,\mathrm{b},F,a,G}$, $\overline{X}^{\mathcal{R},\mathcal{L}}_{\theta,c,E,\mathrm{b},F,a,G}$ are defined just as in Definitions 2.8 and 2.11; the only change being that $\widetilde{E}(0,\infty)$ must be replaced by

$\widetilde{E}(0,1)$. Likewise, the spaces $\overline{X}^{\mathcal{R}}_{\theta,\mathrm{b},E,a,F}$, $\overline{X}^{\mathcal{R},\mathcal{R}}_{\theta,c,E,\mathrm{b},F,a,G}$ and $\overline{X}^{\mathcal{L},\mathcal{R}}_{\theta,c,E,\mathrm{b},F,a,G}$ are defined as

$$\overline{X}^{\mathcal{R}}_{\theta,\mathrm{b},E,a,F} = \Big\{ f \in X_0 : \big\| \mathrm{b}(t) \| s^{-\theta} a(s) K(s,f) \|_{\widetilde{F}(t,1)} \big\|_{\widetilde{E}(0,1)} < \infty \Big\},$$
$$\overline{X}^{\mathcal{R},\mathcal{R}}_{\theta,c,E,\mathrm{b},F,a,G} = \Big\{ f \in X_0 : \Big\| c(u) \big\| \mathrm{b}(t) \| s^{-\theta} a(s) K(s,f) \|_{\widetilde{G}(t,1)} \big\|_{\widetilde{F}(u,1)} \Big\|_{\widetilde{E}(0,1)} < \infty \Big\}$$

and

$$\overline{X}^{\mathcal{L},\mathcal{R}}_{\theta,c,E,\mathrm{b},F,a,G} = \Big\{ f \in X_0 : \Big\| c(u) \big\| \mathrm{b}(t) \| s^{-\theta} a(s) K(s,f) \|_{\widetilde{G}(u,t)} \big\|_{\widetilde{F}(u,1)} \Big\|_{\widetilde{E}(0,1)} < \infty \Big\}.$$

Of course, all the results in the paper remain true if we work with an ordered couple and use slowly varying functions on $(0, 1)$ as parameters. In these cases all assumptions concerning the interval $(1, \infty)$ must be omitted.

Moreover, if the couple is ordered, then the $\mathcal{R}$ and $\mathcal{L}$-scales are also ordered.

Lemma 5.1 *Let $\overline{X}$ be an ordered (quasi-) Banach couple, E_0, E_1, F_0, F_1 r.i. spaces and $a_0, a_1, \mathrm{b}_0, \mathrm{b}_1 \in SV(0,1)$. If $0 < \theta_0 < \theta_1 < 1$, then*

$$\overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1} \hookrightarrow \overline{X}^{\mathcal{R}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \qquad \overline{X}^{\mathcal{L}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1} \hookrightarrow \overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0},$$
$$\overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1} \hookrightarrow \overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}, \qquad \overline{X}^{\mathcal{L}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1} \hookrightarrow \overline{X}^{\mathcal{R}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0},$$

assuming, if it is necessary, the condition $\|\mathrm{b}_0\|_{\widetilde{E}_0(0,1)} < \infty$ such that the right hand side of each inclusion is not the trivial space.

Proof Let $f \in \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a_1,F_1}$ and assume that $\|\mathrm{b}_0\|_{\widetilde{E}_0(0,1)} < \infty$ and $\|\mathrm{b}_1\|_{\widetilde{E}_1(0,1)} < \infty$, otherwise the first inclusion is trivial. Using (2.8), that the function $t \rightsquigarrow \|\cdot\|_{\widetilde{F}_0(t,1)}$ is non-increasing and Lemma 2.3 (i) with $\alpha = \theta_1 - \theta_0 > 0$, we obtain

$$\begin{aligned}
\|f\|_{\mathcal{R};\theta_0,\mathrm{b}_0,E_0,a_0,F_0} &\lesssim \|f\|_{\mathcal{R};\theta_1,\mathrm{b}_1,E_1,a_1,F_1} \Big\| \mathrm{b}_0(t) \Big\| s^{\theta_1-\theta_0} \frac{a_0(s)}{a_1(s)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,s)}} \Big\|_{\widetilde{F}_0(t,1)} \Big\|_{\widetilde{E}_0(0,1)} \\
&\leq \|f\|_{\mathcal{R};\theta_1,\mathrm{b}_1,E_1,a_1,F_1} \|\mathrm{b}_0\|_{\widetilde{E}_0(0,1)} \Big\| s^{\theta_1-\theta_0} \frac{a_0(s)}{a_1(s)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,s)}} \Big\|_{\widetilde{F}_0(0,1)} \\
&\sim \|f\|_{\mathcal{R};\theta_1,\mathrm{b}_1,E_1,a_1,F_1}
\end{aligned}$$

which gives that $f \in \overline{X}^{\mathcal{R}}_{\theta_0,\mathrm{b}_0,E_0,a_0,F_0}$. The proof of the other three inclusions can be carried out using similar arguments, using (2.9) in place of (2.8) if it is necessary. □

5.2 Interpolation between grand and small Lebesgue spaces

In this subsection we shall describe the interpolation spaces between grand and small Lebesgue spaces as Lorentz-Karamata type spaces $L_{p,\mathrm{b},E}$.

For convenience we will consider (Ω, μ) a finite measure space with $\mu(\Omega) = 1$ and we will denote the function spaces as L_q, $L_{p,q}$, etc..., dropping the dependence with respect to the domain (Ω, μ).

Definition 5.2 Let $1 < p < \infty$, $\mathrm{b} \in SV$ and E an r.i. space. The *Lorentz-Karamata type* space $L_{p,\mathrm{b},E}$ is defined as the set of all $f \in \mathcal{M}(\Omega, \mu)$ such that

$$\|f\|_{L_{p,\mathrm{b},E}} = \left\|t^{1/p}\mathrm{b}(t) f^*(t)\right\|_{\widetilde{E}(0,1)} < \infty.$$

The Lorentz-Karamata type spaces comprise a big family of r.i. spaces, including the L_q spaces, the Lorentz spaces $L_{p,q}$, the Lorentz-Zygmund spaces $L^{p,q}(\log L)^\alpha$ and the *Lorentz-Karamata* spaces $L_{p,q;\mathrm{b}}$ (see [32, 39, 43]). Moreover, these spaces are particular examples of the *ultrasymmetric spaces* $L_{\varphi,E}$ studied by E. Pustylnik [45], where it was shown that a r.i. space is ultrasymmetric if and only if its norm is equivalent to

$$\|f\|_{L_{\varphi,E}} = \left\|\varphi(t)\, f^*(t)\right\|_{\widetilde{E}(0,1)} \tag{5.1}$$

for some parameter space $\widetilde{E}$.

Using Peetre's well-known formula [6, 41]

$$K(t, f; L_1, L_\infty) = \int_0^t f^*(s)\, ds = t f^{**}(t), \quad t > 0,$$

and the equivalence $\|t^{1/p}\mathrm{b}(t) f^{**}(t)\|_{\widetilde{E}(0,1)} \sim \|t^{1/p}\mathrm{b}(t) f^*(t)\|_{\widetilde{E}(0,1)}$, $1 < p < \infty$ (see, e.g. [12,Lemma 2.16]), it follows that the Lorentz-Karamata type spaces are interpolation spaces between L_1 and L_∞

$$L_{p,\mathrm{b},E} = (L_1, L_\infty)_{1-\frac{1}{p},\mathrm{b},E}. \tag{5.2}$$

Analogously it can be proved that

$$(L_1, L_\infty)^{\mathcal{L}}_{1-\frac{1}{p},\mathrm{b},E,a,F} = L^{\mathcal{L}}_{p,\mathrm{b},E,a,F}$$

where

$$L^{\mathcal{L}}_{p,\mathrm{b},E,a,F} := \left\{ f \in \mathcal{M}(\Omega, \mu) : \left\| \mathrm{b}(t) \|s^{1/p} a(s) f^*(s)\|_{\widetilde{F}(0,t)} \right\|_{\widetilde{E}(0,1)} < \infty \right\}$$

if E, F are r.i. spaces, $a, \mathrm{b} \in SV$ and $1 < p < \infty$ (see [32] for the case $E = L_r$, $F = L_q$, $0 < q, r \leq \infty$). Working a litte more, it can be stated that

$$(L_1, L_\infty)^{\mathcal{R}}_{1-\frac{1}{p},\mathrm{b},E,a,F} = L^{\mathcal{R}}_{p,\mathrm{b},E,a,F}$$

where

$$L^{\mathcal{R}}_{p,\mathrm{b},E,a,F} := \left\{ f \in \mathcal{M}(\Omega,\mu) : \left\| \mathrm{b}(t) \| s^{1/p} a(s) f^*(s) \|_{\widetilde{F}(t,1)} \right\|_{\widetilde{E}(0,1)} < \infty \right\}$$

(see [16,Lemma 6.6] or [17]). Also,

$$\begin{aligned}
(L_1, L_\infty)^{\mathcal{R},\mathcal{R}}_{1-\frac{1}{p},c,E,\mathrm{b},F,a,G} &= L^{\mathcal{R},\mathcal{R}}_{p,c,E,\mathrm{b},F,a,G},\\
(L_1, L_\infty)^{\mathcal{L},\mathcal{L}}_{1-\frac{1}{p},c,E,\mathrm{b},F,a,G} &= L^{\mathcal{L},\mathcal{L}}_{p,c,E,\mathrm{b},F,a,G},\\
(L_1, L_\infty)^{\mathcal{R},\mathcal{L}}_{1-\frac{1}{p},c,E,\mathrm{b},F,a,G} &= L^{\mathcal{R},\mathcal{L}}_{p,c,E,\mathrm{b},F,a,G} \quad \text{and}\\
(L_1, L_\infty)^{\mathcal{L},\mathcal{R}}_{1-\frac{1}{p},c,E,\mathrm{b},F,a,G} &= L^{\mathcal{L},\mathcal{R}}_{p,c,E,\mathrm{b},F,a,G}
\end{aligned}$$

where the spaces $L^{\mathcal{R},\mathcal{R}}_{p,c,E,\mathrm{b},F,a,G}$, $L^{\mathcal{L},\mathcal{L}}_{p,c,E,\mathrm{b},F,a,G}$, $L^{\mathcal{R},\mathcal{L}}_{p,c,E,\mathrm{b},F,a,G}$ and $L^{\mathcal{L},\mathcal{R}}_{p,c,E,\mathrm{b},F,a,G}$ are defined as the set of all $f \in \mathcal{M}(\Omega,\mu)$ for which (2.10)-(2.13) are satisfied changing $s^{-\theta} a(s) K(s, f) = s^{1/p} a(s) f^{**}(s)$ by $s^{1/p} a(s) f^*(s)$, respectively (see [17] for the case $E = L_r$, $F = L_q$, $0 < q, r \le \infty$).

Following the paper by Fiorenza and Karadzhov [26] we give the next definition:

Definition 5.3 Let $1 < p < \infty$ and $\alpha > 0$. The grand Lebesgue space $L^{p),\alpha}$ is the set of all $f \in \mathcal{M}(\Omega,\mu)$ such that

$$\|f\|_{p),\alpha} = \left\| \ell^{-\frac{\alpha}{p}}(t) \left\| s^{1/p} f^*(s) \right\|_{\widetilde{L}_p(t,1)} \right\|_{L_\infty(0,1)} < \infty.$$

The small Lebesgue space $L^{(p,\alpha}$ is the set of all $f \in \mathcal{M}(\Omega,\mu)$ such that

$$\|f\|_{(p,\alpha} = \left\| \ell^{\frac{\alpha}{p'}-1}(t) \left\| s^{1/p} f^*(s) \right\|_{\widetilde{L}_p(0,t)} \right\|_{\widetilde{L}_1(0,1)} < \infty$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

The classical grand Lebesgue space $L^{p)} := L^{p),1}$ was introduced by Iwaniec and Sbordone in [34] in connection with the integrability properties of the Jacobian under minimal hypothesis. The classical small Lebesgue space $L^{(p} := L^{(p,1}$ was characterized by Fiorenza in [25] as the associated space to $L^{p')}$; that is $(L^{(p})' = L^{p')}$. For more information about these spaces, generalizations and applications to PDE's see the recent paper [27] and the references therein.

As observed in [26, 40], these spaces can be characterized as $\mathcal{R}$ and $\mathcal{L}$-spaces in the following way

$$L^{p),\alpha} = L^{\mathcal{R}}_{p,\ell^{-\alpha/p}(u),L_\infty,1,L_p} \quad \text{and} \quad L^{(p,\alpha} = L^{\mathcal{L}}_{p,\ell^{\alpha/p'-1}(u),L_1,1,L_p}.$$

Thus, we can apply the results from §4.3 to identify the interpolation of grand and small Lebesgue spaces. In order to do that we need the following technical lemma.

Lemma 5.4 *[19,Lemma 6.1]*

i) If $\sigma + \frac{1}{q} < 0$ *with* $1 \le q < \infty$ *or* $q = \infty$ *and* $\sigma \le 0$*, then*

$$\|\ell^{\sigma}(t)\|_{\widetilde{L}_q(0,u)} \sim \ell^{\sigma+\frac{1}{q}}(u), \quad u \in (0,1). \tag{5.3}$$

ii) If $\sigma + \frac{1}{q} > 0$ *with* $1 \le q < \infty$*, or* $q = \infty$ *and* $\sigma \ge 0$*, then*

$$\|\ell^{\sigma}(t)\|_{\widetilde{L}_q(u,1)} \sim \ell^{\sigma+\frac{1}{q}}(u), \quad u \in (0,1/2). \tag{5.4}$$

Notice also that if $\mathrm{b}(t) \sim a(t)$ for all $t \in (0,1/2)$, then the monotonicity properties of the K-functional and the properties of the slowly varying functions imply that

$$\overline{X}_{\theta,\mathrm{b},E} = \overline{X}_{\theta,a,E}.$$

Thus, under conditions of Lemma 5.4 ii)

$$\overline{X}_{\theta,\|\ell^{\sigma}(t)\|_{\widetilde{L}_q(u,1)},E} = \overline{X}_{\theta,\ell^{\sigma+\frac{1}{q}}(u),E}$$

for all $0 < \theta < 1$ and all r.i. space E. A similar identity holds for $\mathcal{R}$, $\mathcal{L}$-spaces and for extreme constructions.

Using Theorem 4.3 and (5.3) with $q = \infty$, we can state the interpolation formulae for a couple formed by two grand Lebesgue spaces. Similar results in the no-limiting cases $0 < \theta < 1$ appear in [3,Theorem 8.3], [14,Corollary 51] and [28,Theorem 1.2]. The result also completes [23,Corollary 5.7].

Corollary 5.5 *Let* E *be an r.i. space,* $\mathrm{b} \in SV(0,1)$*,* $1 < p_0 < p_1 < \infty$ *and* $\alpha, \beta > 0$*. Consider the function* $\rho(u) = u^{\frac{1}{p_0}-\frac{1}{p_1}} \ell^{\frac{\beta}{p_1}-\frac{\alpha}{p_0}}(u)$*,* $u \in (0,1)$*.*

(a) If $0 < \theta < 1$*, then*

$$\left(L^{p_0),\alpha}, L^{p_1),\beta}\right)_{\theta,\mathrm{b},E} = L_{p,B_\theta,E},$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *and* $B_\theta(u) = \ell^{-\left[\frac{\alpha(1-\theta)}{p_0}+\frac{\beta\theta}{p_1}\right]}(u)\mathrm{b}(\rho(u))$*,* $u \in (0,1)$*.*

(b) If $\theta = 0$*, then*

$$\left(L^{p_0),\alpha}, L^{p_1),\beta}\right)_{0,\mathrm{b},E} = L^{\mathcal{R},\mathcal{L}}_{p_0,\mathrm{b}\circ\rho,E,\ell^{-\alpha/p_0}(u),L_\infty,1,L_{p_0}}.$$

(c) If $\theta = 1$ *and* $\|\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty$*, then*

$$\left(L^{p_0),\alpha}, L^{p_1),\beta}\right)_{1,\mathrm{b},E} = L^{\mathcal{R}}_{p_1,B_1,E,1,L_{p_1}} \cap L^{\mathcal{R},\mathcal{R}}_{p_1,\mathrm{b}\circ\rho,E,\ell^{-\beta/p_1}(u),L_\infty,1,L_{p_1}} \tag{5.5}$$

where $B_1(u) = \ell^{\frac{-\beta}{p_1}}(u)\mathrm{b}(\rho(u))$*,* $u \in (0,1)$*.*

Remark 5.6 Comparing (5.5) with Corollary 5.7 c) from [23], we observe that

$$\begin{aligned} &L^{\mathcal{R}}_{p_1,B_1,E,1,L_{p_1}} \cap (L^{p_0),\alpha}, L_{p_1})^{\mathcal{R}}_{1,\mathrm{b}\circ\rho^{\#},E,\ell^{-\beta/p_1}(u),L_\infty} \\ &\quad = L^{\mathcal{R}}_{p_1,B_1,E,1,L_{p_1}} \cap L^{\mathcal{R},\mathcal{R}}_{p_1,\mathrm{b}\circ\rho,E,\ell^{-\beta/p_1}(u),L_\infty,1,L_{p_1}} \end{aligned}$$

where $\rho^{\#}(u) = u\ell^{\frac{\beta}{p_1}}(u)$, $u \in (0,1)$.

Theorem 4.4 and (5.4) with $q = 1$, enables us to state the following interpolation formulae for a couple formed by two small Lebesgue spaces. The result completes [14,Corollary 49], [23,Corollary 5.12] and [28,Theorem 3.4].

Corollary 5.7 *Let E be an r.i. space, $\mathrm{b} \in SV(0,1)$, $1 < p_0 < p_1 < \infty$ and $\alpha, \beta > 0$. Consider the function $\rho(u) = u^{\frac{1}{p_0}-\frac{1}{p_1}} \ell^{\frac{\alpha}{p_0'}-\frac{\beta}{p_1'}}(u)$, $u \in (0,1)$.*

(a) If $0 < \theta < 1$, then

$$\left(L^{(p_0,\alpha}, L^{(p_1,\beta}\right)_{\theta,\mathrm{b},E} = L_{p,B_\theta,E},$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $B_\theta(u) = \ell^{\frac{\alpha(1-\theta)}{p_0'}+\frac{\beta\theta}{p_1'}}(u)\mathrm{b}(\rho(u))$, $u \in (0,1)$.

(b) If $\theta = 0$, then

$$\left(L^{(p_0,\alpha}, L^{(p_1,\beta}\right)_{0,\mathrm{b},E} = L^{\mathcal{L}}_{p_0,B_0,E,1,L_{p_0}} \cap L^{\mathcal{L},\mathcal{L}}_{p_0,\mathrm{b}\circ\rho,E,\ell^{\alpha/p_0'-1}(u),L_1,1,L_{p_0}} \tag{5.6}$$

where $B_0(u) = \ell^{\frac{\alpha}{p_0'}}(u)\mathrm{b}(\rho(u))$, $u \in (0,1)$.

(c) If $\theta = 1$ and $\|\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty$, then

$$\left(L^{(p_0,\alpha}, L^{(p_1,\beta}\right)_{1,\mathrm{b},E} = L^{\mathcal{L},\mathcal{R}}_{p_1,\mathrm{b}\circ\rho,E,\ell^{\beta/p_1'-1}(u),L_1,1,L_{p_1}}.$$

Remark 5.8 Comparing (5.6) with Corollary 5.12 (b) from [23], we observe that

$$\begin{aligned} &L^{\mathcal{L}}_{p_0,B_0,E,1,L_{p_0}} \cap \left(L_{p_0}, L^{(p_1,\beta}\right)^{\mathcal{L}}_{0,\mathrm{b}\circ\rho^{\#},E,\ell^{\alpha/p_0'-1}(u),L_1} \\ &\quad = L^{\mathcal{L}}_{p_0,B_0,E,1,L_{p_0}} \cap L^{\mathcal{L},\mathcal{L}}_{p_0,\mathrm{b}\circ\rho,E,\ell^{\alpha/p_0'-1}(u),L_1,1,L_{p_0}} \end{aligned}$$

where $\rho^{\#}(u) = u\ell^{\alpha/p_0'}(u)$, $u \in (0,1)$.

Additionally, using Theorems 4.5, 4.6 and estimates (5.3), (5.4), we identify the interpolation space between grand and small Lebesgue spaces. Corollary 5.9 completes [14,Corollary 52] with the limiting cases $\theta = 0, 1$, while Corollary 5.10 completes [3,Theorem 6.5], [14,Corollary 50], [24,Theorem 5.7] and [28,Theorem 5.1].

Corollary 5.9 *Let E be an r.i. space,* $\mathrm{b} \in SV(0,1)$*,* $1 < p_0 < p_1 < \infty$ *and* $\alpha, \beta > 0$*. Consider the function* $\rho(u) = u^{\frac{1}{p_0}-\frac{1}{p_1}} \ell^{-[\frac{\alpha}{p_0}+\frac{\beta}{p_1'}]}(u)$*,* $u \in (0,1)$*.*

(a) If $0 < \theta < 1$*, then*

$$\big(L^{p_0),\alpha}, L^{(p_1,\beta}\big)_{\theta,\mathrm{b},E} = L_{p,B_\theta,E},$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *and* $B_\theta(u) = \ell^{-\frac{\alpha(1-\theta)}{p_0}+\frac{\beta\theta}{p_1'}}(u)\mathrm{b}(\rho(u))$*,* $u \in (0,1)$*.*

(b) If $\theta = 0$*, then*

$$\big(L^{p_0),\alpha}, L^{(p_1,\beta}\big)_{0,\mathrm{b},E} = L^{\mathcal{R},\mathcal{L}}_{p_0,\mathrm{b}\circ\rho,E,\ell^{-\alpha/p_0}(u),L_\infty,1,L_{p_0}}.$$

(c) If $\theta = 1$ *and* $\|\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty$*, then*

$$\big(L^{p_0),\alpha}, L^{(p_1,\beta}\big)_{1,\mathrm{b},E} = L^{\mathcal{L},\mathcal{R}}_{p_1,\mathrm{b}\circ\rho,E,\ell^{\beta/p_1'-1}(u),L_1,1,L_{p_1}}.$$

Corollary 5.10 *Let E be an r.i. space,* $\mathrm{b} \in SV(0,1)$*,* $1 < p_0 < p_1 < \infty$ *and* $\alpha, \beta > 0$*. Consider the function* $\rho(u) = u^{\frac{1}{p_0}-\frac{1}{p_1}} \ell^{\frac{\alpha}{p_0'}+\frac{\beta}{p_1}}(u)$*,* $u \in (0,1)$*.*

(a) If $0 < \theta < 1$*, then*

$$\big(L^{(p_0,\alpha}, L^{p_1),\beta}\big)_{\theta,\mathrm{b},E} = L_{p,B_\theta,E},$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *and* $B_\theta(u) = \ell^{\frac{\alpha(1-\theta)}{p_0'}-\frac{\beta\theta}{p_1}}(u)\mathrm{b}(\rho(u))$*,* $u \in (0,1)$*.*

(b) If $\theta = 0$*, then*

$$\big(L^{(p_0,\alpha}, L^{p_1),\beta}\big)_{0,\mathrm{b},E} = L^{\mathcal{L}}_{p_0,B_0,E,1,L_{p_0}} \cap L^{\mathcal{L},\mathcal{L}}_{p_0,\mathrm{b}\circ\rho,E,\ell^{\alpha/p_0'-1}(u),L_1,1,L_{p_0}} \tag{5.7}$$

where $B_0(u) = \ell^{\frac{\alpha}{p_0'}}(u)\mathrm{b}(\rho(u))$*,* $u \in (0,1)$*.*

(c) If $\theta = 1$ *and* $\|\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty$*, then*

$$\big(L^{(p_0,\alpha}, L^{p_1),\beta}\big)_{1,\mathrm{b},E} = L^{\mathcal{R}}_{p_1,B_1,E,1,L_{p_1}} \cap L^{\mathcal{R},\mathcal{R}}_{p_1,\mathrm{b}\circ\rho,E,\ell^{-\beta/p_1}(u),L_\infty,1,L_{p_1}}$$

where $B_1(u) = \ell^{-\frac{\beta}{p_1}}(u)\mathrm{b}(\rho(u))$*,* $u \in (0,1)$*.*

Remark 5.11 Comparing (5.7) with Theorem 5.7 (b) from [24] one can deduce the identity

$$\begin{aligned} &L^{\mathcal{L}}_{p_0,B_0,E,1,L_{p_0}} \cap (L_{p_0}, L^{p_1),\beta})^{\mathcal{L}}_{0,b\circ\rho^{\#},E,\ell^{\alpha/p_0'-1}(u),L_1} \\ &= L^{\mathcal{L}}_{p_0,B_0,E,1,L_{p_0}} \cap L^{\mathcal{L},\mathcal{L}}_{p_0,\mathrm{b}\circ\rho,E,\ell^{\alpha/p_0'-1}(u),L_1,1,L_{p_0}} \end{aligned}$$

where $\rho^{\#}(u) = u\ell^{\frac{\alpha}{p_0'}}(u)$, $u \in (0,1)$.

5.2.1 Example: the case when $E = L_M$

Let L_M denote the Orlicz space on $(0,\infty)$, that is, the set of all measurable functions f such that the Luxemburg-Nakano norm

$$\|f\|_{L_M} = \inf\left\{\lambda > 0 : \int_0^\infty M\Big(\frac{|f(t)|}{\lambda}\Big)dt \le 1\right\} < \infty,$$

where M is an Orlicz function (increasing convex function on $[0,\infty)$ with $M(0) = 0$). These spaces generalize the L_q spaces, which correspond to $M(t) = t^q$.

We next define the family of *Lorentz-Orlicz* spaces, as considered by Torchinsky in [49]. When M is an Orlicz function and $\varphi : [0,1] \to [0,\infty)$ is increasing and concave, the Lorentz-Orlicz space $\Lambda(\varphi, M)$ is the set of all measurable f on $(0,1)$ such that

$$\|f\|_{\Lambda(\varphi,M)} = \inf\left\{\lambda > 0 : \int_0^1 M\Big(\frac{\varphi(t)f^{**}(t)}{\lambda}\Big)\frac{dt}{t} \le 1\right\} < \infty.$$

It is known that the above norm $\|\cdot\|_{\Lambda(\varphi,M)}$ is equivalent to the quasi-norm obtained by replacing f^{**} with f^*, provided that the upper extension index $\rho_\varphi < 1$; see, e.g. [12,Lemma 2.16]. That is,

$$\|f\|_{\Lambda(\varphi,M)} \sim \|\varphi(t)f^*(t)\|_{\widetilde{L}_M}.$$

In particular, the Lorentz-Orlicz spaces $\Lambda(\varphi, M)$ are ultrasymmetric, in the sense of (5.1); see also [5,Theorem 2].

In the special case when $\varphi(t) = t^{1/p}\mathrm{b}(t)$, with $1 < p < \infty$ and $\mathrm{b} \in SV$, then the Lorentz-Orlicz space $\Lambda(t^{1/p}\mathrm{b}(t), M)$ coincides with the Lorentz-Karamata type space L_{p,b,L_M}, namely

$$\Lambda(t^{1/p}\mathrm{b}(t), M) = L_{p,\mathrm{b},L_M}.$$

Therefore, we can summarize as follows the results stated in part (a) of Corollaries 5.5 and 5.7, in the special case that $E = L_M$ and $\mathrm{b} \equiv 1$.

Corollary 5.12 *Let M be an Orlicz function, $1 < p_0 < p_1 < \infty$, $\alpha, \beta > 0$ and $0 < \theta < 1$. Then,*

$$\big(L^{p_0),\alpha}, L^{p_1),\beta}\big)_{\theta,1,L_M} = \Lambda(t^{1/p}\ell^{-A}(t), M)$$

and

$$\big(L^{(p_0,\alpha}, L^{(p_1,\beta}\big)_{\theta,1,L_M} = \Lambda(t^{1/p}\ell^{B}(t), M),$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $A = \frac{\alpha(1-\theta)}{p_0} + \frac{\beta\theta}{p_1}$ and $B = \frac{\alpha(1-\theta)}{p_0'} + \frac{\beta\theta}{p_1'}$.

Similar statements can be written for the interpolation spaces formed by the pairs $\big(L^{p_0),\alpha}, L^{(p_1,\beta}\big)$ and $\big(L^{(p_0,\alpha}, L^{p_1),\beta}\big)$.

5.3 Generalized Gamma spaces

Our goal in this subsection is to obtain interpolation formulae for couples formed by two Generalized Gamma spaces with double weight; see [28].

Definition 5.13 Let $1 \leq p < \infty$, $1 \leq q \leq \infty$ and w_1, w_2 two weights on $(0, 1)$ satisfying the following conditions:

(c1) There exist $K_{21} > 0$ such that $w_2(2t) \leq K_{12} w_2(t)$, for all $t \in (0, 1/2)$. The space $L^p(0, 1; w_2)$ is continuously embedded in $L^1(0, 1)$.
(c2) The function $\int_0^t w_2(s)ds$ belongs to $L^{\frac{q}{p}}(0, 1; w_1)$.

The *Generalized Gamma* space with double weights $G\Gamma(p, q, w_1, w_2)$ is the set of all measurable functions f on $(0, 1)$ such that

$$\|f\|_{G\Gamma} = \left(\int_0^1 w_1(t)\left(\int_0^t w_2(s)(f^*(s))^p\, ds\right)^{\frac{q}{p}} dt\right)^{\frac{1}{q}} < \infty.$$

These spaces are a generalization of the spaces $G\Gamma(p, q, w_1) := G\Gamma(p, q, w_1, 1)$, introduced in [29], while the spaces $G\Gamma(p, \infty, w_1, w_2)$ appeared in [31].

If we assume that $uw_1(u)$ and w_2 are slowly varying functions, we can identify the Generalized Gamma space as an $\mathcal{L}$-space in the following way

$$G\Gamma(p, q, w_1, w_2) = L^{\mathcal{L}}_{p,(uw_1(u))^{1/q}, L_q, (w_2(u))^{1/p}, L_p}.$$

Thus, we can apply the results from Sect. 4.3 to interpolate two Generalized Gamma spaces with double weights.

Corollary 5.14 *Let E be an r.i. space and* $\mathrm{b}, uw_1(u), w_2, uw_3(u), w_4 \in SV(0, 1)$. *Let* $1 < p_0 < p_1 < \infty$, $1 \leq q \leq \infty$ *and consider the function*

$$\rho(u) = u^{\frac{1}{p_0}-\frac{1}{p_1}} \frac{w_2^{\frac{1}{p_0}}(u)\|(tw_1(t))^{\frac{1}{q_0}}\|_{\widetilde{L}_{q_0}(u,1)}}{w_4^{\frac{1}{p_1}}(u)\|(tw_3(t))^{\frac{1}{q_1}}\|_{\widetilde{L}_{q_1}(u,1)}}, \quad u \in (0, 1).$$

(a) If $0 < \theta < 1$, *then*

$$\big(G\Gamma(p_0, q_0, w_1, w_2), G\Gamma(p_1, q_1, w_3, w_4)\big)_{\theta,\mathrm{b},E} = L_{p,B_\theta,E}$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *and* $B_\theta(u)$ *is equal to*

$$\left((w_2(u))^{\frac{1}{p_0}}\|(tw_1(t))^{\frac{1}{q_0}}\|_{\widetilde{L}_{q_0}(u,1)}\right)^{1-\theta}\left((w_4(u))^{\frac{1}{p_1}}\|(tw_3(t))^{\frac{1}{q_1}}\|_{\widetilde{L}_{q_1}(u,1)}\right)^{\theta}\mathrm{b}(\rho(u)),$$

for $u \in (0,1)$.

(b) If $\theta = 0$ *then*

$$\begin{aligned}&\big(G\Gamma(p_0,q_0,w_1,w_2),G\Gamma(p_1,q_1,w_3,w_4)\big)_{0,\mathrm{b},E}\\&\quad= L^{\mathcal{L}}_{p_0,B_0,E,(w_2(u))^{1/p_0},L_{p_0}} \cap L^{\mathcal{L},\mathcal{L}}_{p_0,\mathrm{b}\circ\rho,E,(uw_1(u))^{1/q_0},L_{q_0},(w_2(u))^{1/p_0},L_{p_0}}\end{aligned}$$

where $B_0(u) = \|(tw_1(t))^{\frac{1}{q_0}}\|_{\widetilde{L}_{q_0}(u,1)}\mathrm{b}(\rho(u))$, $u \in (0,1)$.

(c) If $\theta = 1$ *and* $\|\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty$, *then*

$$\begin{aligned}&\big(G\Gamma(p_0,q_0,w_1,w_2),G\Gamma(p_1,q_1,w_3,w_4)\big)_{1,\mathrm{b},E}\\&\quad= L^{\mathcal{L},\mathcal{R}}_{p_1,\mathrm{b}\circ\rho,E,(uw_3(u))^{1/q_1},L_{q_1},(w_4(u))^{1/p_1},L_{p_1}}.\end{aligned}$$

5.4 *A* and *B*-type spaces

Finally, we consider the A and B-type spaces studied by Pustylnik in [44].

Definition 5.15 Given $1 < p < \infty$, $\alpha < 1$ and E an r.i. space. The space $A_{p,\alpha,E}$ is the set of all measurable functions f on $(0,1)$ such that

$$\|f\|_{A_{p,\alpha,E}} = \left\|\ell^{\alpha-1}(t)\int_t^1 s^{\frac{1}{p}} f^{**}(s)\,\frac{ds}{s}\right\|_{\widetilde{E}(0,1)} < \infty$$

assumed that the function $(1+u)^{\alpha-1}$ belongs to E (i.e. $\|\ell^{\alpha-1}(t)\|_{\widetilde{E}(0,1)} < \infty$ see [44]). The space $B_{p,\alpha,E}$ is the set of all measurable functions f on $(0,1)$ such that

$$\|f\|_{B_{p,\alpha,E}} = \left\|\sup_{0<s<t} s^{\frac{1}{p}}\ell^{\alpha-1}(s) f^{**}(s)\right\|_{\widetilde{E}(0,1)} < \infty.$$

The spaces of B-type when $\alpha = 0$ first appeared in [12]. General versions of these spaces were studied in [46]. The main feature of the A and B-type spaces is that they appear in a natural way as optimal domain and range spaces, respectively, in the corresponding weak type interpolation theorems [44, 47]. The norms $\|\cdot\|_{A_{p,\alpha,E}}$ and $\|\cdot\|_{B_{p,\alpha,E}}$ are equivalent with the corresponding quasi-norms obtained via replacing f^{**} by f^*.

The A and B-type spaces can be seen as $\mathcal{R}$ and $\mathcal{L}$-spaces, respectively. Indeed,

$$A_{p,\alpha,E} = L^{\mathcal{R}}_{p,\ell^{\alpha-1}(t),E,1,L_1} \quad B_{p,\alpha,E} = L^{\mathcal{L}}_{p,1,E,\ell^{\alpha-1}(t),L_\infty}.$$

Then, we can apply the results from Sect. 4 to obtain the following interpolation formulae.

Corollary 5.16 *Let* E, E_0, E_1 *be r.i. spaces,* $\mathrm{b} \in SV(0,1)$, $1 < p_0 < p_1 < \infty$, $\alpha, \beta < 1$ *and assume that* $(1+u)^{\alpha-1}$, $(1+u)^{\beta-1}$, *belongs to* E_0, E_1, *respectively.*

Consider the function

$$\rho(u) = u^{\frac{1}{p_0}-\frac{1}{p_1}} \frac{\|\ell^{\alpha-1}(t)\|_{\widetilde{E}_0(0,u)}}{\|\ell^{\beta-1}(t)\|_{\widetilde{E}_1(0,u)}}, \qquad u \in (0,1).$$

(a) *If* $0 < \theta < 1$, *then*

$$\left(A_{p_0,\alpha,E_0}, A_{p_1,\beta,E_1}\right)_{\theta,\mathrm{b},E} = L_{p,B_\theta,E},$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *and*

$$B_\theta(u) = \|\ell^{\alpha-1}(t)\|^{1-\theta}_{\widetilde{E}_0(0,u)} \|\ell^{\beta-1}(t)\|^{\theta}_{\widetilde{E}_1(0,u)} \mathrm{b}(\rho(u)), \quad u \in (0,1).$$

(b) *If* $\theta = 0$, *then*

$$\left(A_{p_0,\alpha,E_0}, A_{p_1,\beta,E_1}\right)_{0,\mathrm{b},E} = L^{\mathcal{R},\mathcal{L}}_{p_0,\mathrm{b}\circ\rho,E,\ell^{\alpha-1}(u),E_0,1,L_1}.$$

(c) *If* $\theta = 1$ *and* $\|\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty$, *then*

$$\left(A_{p_0,\alpha,E_0}, A_{p_1,\beta,E_1}\right)_{1,\mathrm{b},E} = L^{\mathcal{R}}_{p_1,B_1,E,1,L_1} \cap L^{\mathcal{R},\mathcal{R}}_{p_1,\mathrm{b}\circ\rho,E,\ell^{\beta-1}(u),E_1,1,L_1}$$

where $B_1(u) = \|\ell^{\beta-1}(t)\|_{\widetilde{E}_1(0,u)} \mathrm{b}(\rho(u))$, $u \in (0,1)$.

Corollary 5.17 *Let* E, E_0, E_1 *be r.i. spaces,* $\mathrm{b} \in SV(0,1)$, $1 < p_0 < p_1 < \infty$ *and* $\alpha, \beta < 1$. *Consider the function* $\rho(u) = u^{\frac{1}{p_0}-\frac{1}{p_1}} \frac{\ell^{\alpha-1}(u)\varphi_{E_0}(\ell(u))}{\ell^{\beta-1}(u)\varphi_{E_1}(\ell(u))}$, $u \in (0,1)$.

(a) *If* $0 < \theta < 1$, *then*

$$\left(B_{p_0,\alpha,E_0}, B_{p_1,\beta,E_1}\right)_{\theta,\mathrm{b},E} = L_{p,B_\theta,E},$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *and*

$$B_\theta(u) = \left(\ell^{\alpha-1}(u)\varphi_{E_0}(\ell(u))\right)^{1-\theta} \left(\ell^{\beta-1}(u)\varphi_{E_1}(\ell(u))\right)^{\theta} \mathrm{b}(\rho(u)), \quad u \in (0,1).$$

(b) *If* $\theta = 0$, *then*

$$\left(B_{p_0,\alpha,E_0}, B_{p_1,\beta,E_1}\right)_{0,\mathrm{b},E} = L^{\mathcal{L}}_{p_0,B_0,E,\ell^{\alpha-1}(u),L_\infty} \cap L^{\mathcal{L},\mathcal{L}}_{p_0,\mathrm{b}\circ\rho,E,1,E_0,\ell^{\alpha-1}(u),L_\infty}$$

where $B_0(u) = \varphi_{E_0}(\ell(u))\mathrm{b}(\rho(u))$, $u \in (0,1)$.

(c) *If* $\theta = 1$ *and* $\|\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty$, *then*

$$\left(B_{p_0,\alpha,E_0}, B_{p_1,\beta,E_1}\right)_{1,\mathrm{b},E} = L^{\mathcal{L},\mathcal{R}}_{p_1,\mathrm{b}\circ\rho,E,1,E_1,\ell^{\beta-1}(u),L_\infty}.$$

Corollary 5.18 *Let $E,\ E_0,\ E_1$ be r.i. spaces,* $\mathrm{b} \in SV(0,1)$, $1 < p_0 < p_1 < \infty$ *and* $\alpha, \beta < 1$. *Consider the function* $\rho(u) = u^{\frac{1}{p_0}-\frac{1}{p_1}} \frac{\|\ell^{\alpha-1}(t)\|_{\widetilde{E}_0(0,u)}}{\ell^{\beta-1}(u)\varphi_{E_1}(\ell(u))}$, $u \in (0,1)$.

(a) If $0 < \theta < 1$, *then*

$$\left(A_{p_0,\alpha,E_0}, B_{p_1,\beta,E_1}\right)_{\theta,\mathrm{b},E} = L_{p,B_\theta,E},$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *and*

$$B_\theta(u) = \|\ell^{\alpha-1}(t)\|^{1-\theta}_{\widetilde{E}_0(0,u)} \left(\ell^{\beta-1}(u)\varphi_{E_1}(\ell(u))\right)^\theta \mathrm{b}(\rho(u)), \quad u \in (0,1).$$

(b) If $\theta = 0$, *then*

$$\left(A_{p_0,\alpha,E_0}, B_{p_1,\beta,E_1}\right)_{0,\mathrm{b},E} = L^{\mathcal{R},\mathcal{L}}_{p_0,\mathrm{b}\circ\rho,E,\ell^{\alpha-1}(u),E_0,1,L_1}.$$

(c) If $\theta = 1$ *and* $\|\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty$, *then*

$$\left(A_{p_0,\alpha,E_0}, B_{p_1,\beta,E_1}\right)_{1,\mathrm{b},E} = L^{\mathcal{L},\mathcal{R}}_{p_1,\mathrm{b}\circ\rho,E,1,E_1,\ell^{\beta-1}(u),L_\infty}.$$

Our last result completes [24,Corollary 5.19].

Corollary 5.19 *Let $E,\ E_0,\ E_1$ be r.i. spaces,* $\mathrm{b} \in SV(0,1)$, $1 < p_0 < p_1 < \infty$ *and* $\alpha, \beta < 1$. *Consider the function* $\rho(u) = u^{\frac{1}{p_0}-\frac{1}{p_1}} \frac{\ell^{\alpha-1}(u)\varphi_{E_0}(\ell(u))}{\|\ell^{\beta-1}(t)\|_{\widetilde{E}_1(0,u)}}$, $\quad u \in (0,1)$.

(a) If $0 < \theta < 1$, *then*

$$\left(B_{p_0,\alpha,E_0}, A_{p_1,\beta,E_1}\right)_{\theta,\mathrm{b},E} = L_{p,B_\theta,E},$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *and*

$$B_\theta(u) = \left(\ell^{\alpha-1}(u)\varphi_{E_0}(\ell(u))\right)^{1-\theta} \|\ell^{\beta-1}(t)\|^{\theta}_{\widetilde{E}_0(0,u)} \mathrm{b}(\rho(u)), \quad u \in (0,1).$$

(b) If $\theta = 0$, *then*

$$\left(B_{p_0,\alpha,E_0}, A_{p_1,\beta,E_1}\right)_{0,\mathrm{b},E} = L^{\mathcal{L}}_{p_0,B_0,E,\ell^{\alpha-1}(u),L_\infty} \cap L^{\mathcal{L},\mathcal{L}}_{p_0,\mathrm{b}\circ\rho,E,1,E_0,\ell^{\alpha-1}(u),L_\infty} \tag{5.8}$$

where $B_0(u) = \varphi_{E_0}(\ell(u))\mathrm{b}(\rho(u))$, $u \in (0,1)$.

(c) If $\theta = 1$ *and* $\|\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty$, *then*

$$\left(B_{p_0,\alpha,E_0}, A_{p_1,\beta,E_1}\right)_{1,\mathrm{b},E} = L^{\mathcal{R}}_{p_1,B_1,E,1,L_1} \cap L^{\mathcal{R},\mathcal{R}}_{p_1,\mathrm{b}\circ\rho,E,\ell^{\beta-1}(u),E_1,1,L_1},$$

where $B_1(u) = \|\ell^{\beta-1}(t)\|_{\widetilde{E}_1(0,u)} \mathrm{b}(\rho(u))$, $u \in (0,1)$.

Remark 5.20 Comparing (5.8) with Theorem 5.19 (b) from [24] one can deduce that

$$L^{\mathcal{L}}_{p_0,B_0,E,\ell^{\alpha-1}(u),L_\infty} \cap \left(L_{p_0,\ell^{\alpha-1}(u),L_\infty}, A_{p_1,\beta,E_1}\right)^{\mathcal{L}}_{0,\mathrm{b}(u\varphi_{E_0}(\ell(u))),E,1,E_0}$$
$$= L^{\mathcal{L}}_{p_0,B_0,E,\ell^{\alpha-1}(u),L_\infty} \cap L^{\mathcal{L},\mathcal{L}}_{p_0,\mathrm{b}\circ\rho,E,1,E_0,\ell^{\alpha-1}(u),L_\infty}.$$

Additional remark. During the revision period, after the first draft of this paper was submitted, we had knowledge of the work by Leo Doktorski [15]. His work, which cites various results from our paper, contains limiting reiteration formulae for $\mathcal{L}$ and $\mathcal{R}$ spaces in the case when $E = L_q$, which also cover the quasi-Banach range $0 < q < \infty$. We thank Leo Doktorski for sharing his work with us, and for quoting our results in his paper.

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