Repeated quasi-integration on locally compact spaces

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Abstract

When X is locally compact, a quasi-integral (also called a quasi-linear functional) on $C_c(X)$ is a homogeneous, positive functional that is only assumed to be linear on singly-generated subalgebras. We study simple and almost simple quasi-integrals, i.e., quasi-integrals whose corresponding compact-finite topological measures assume exactly two values. We present equivalent conditions for a quasi-integral to be simple or almost simple. We give a criterion for repeated quasi-integration (i.e., iterated integration with respect to topological measures) to yield a quasi-linear functional. We find a criterion for a double quasi-integral to be simple or almost simple. We describe how a product of topological measures acts on open and compact sets. We show that different orders of integration in repeated quasi-integrals give the same quasi-integral if and only if the corresponding topological measures are both measures or one of the corresponding topological measures is a positive scalar multiple of a point mass.

Keywords Quasi-integral · Repeated quasi-integration · Simple and almost simple quasi-integrals · Quasi-linear functional · Topological measure

Mathematics Subject Classification 28C05 · 28A25

1 Introduction

Mathematical interpretations of quantum physics by Mackey and Kadison (see, for example, [15–17]) led to very interesting mathematical problems, including the extension problem for probability measures in von Neumann algebras. The extension problem may be regarded as a special case of the linearity problem for physical states, which is closely related to the existence of quasi-linear functionals. Aarnes [1] introduced quasi-linear functionals (that are not linear) on $C(X)$ for a compact

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Hausdorff space X and corresponding set functions, generalizing measures (initially called quasi-measures, now topological measures). He connected the two by establishing a representation theorem thereby giving an impetus to the field that has already resulted in a substantial body of work.

In [12] Entov and Polterovich first linked the theory of quasi-linear functionals to symplectic topology. They established that quasi-linear functionals can be viewed as an algebraic way of packing certain information contained in Floer theory, and in particular in spectral invariants of Hamiltonian diffeomorphisms, and proved many new results. Paper [12] has been cited over 100 times, and quasi-linear functionals and topological measures have been studied and used in many subsequent papers, as well as in a monograph ([18]).

Quasi-linear functionals are functionals that are linear on singly-generated subalgebras. Such functionals are obtained by integration of continuous functions with respect to topological measures. Topological measures lack subadditivity and other properties typical for measures but, on the other hand, many properties of measures still hold for topological measures. Likewise, quasi-linear functionals in some respects are similar to linear functionals, and in other respects they are strikingly different. Some of these differences and similarities will be demonstrated in the current paper.

The vast majority of papers devoted to quasi-linear functionals and topological measures deal with compact spaces. The author has written several papers extending the theory to the locally compact setting. The current paper, devoted to repeated integration with respect to topological measures, is an important part of this series. First, the results of this paper are necessary for future research on topics that involve repeated integration. Second, topological measures are a subclass of deficient topological measures, which also correspond via integration to certain non-linear functionals. Knowledge about repeated integration with respect to topological measures is necessary for understanding (repeated) integration with respect to deficient topological measures.

In [13] Grubb gives a nice treatment of repeated quasi-integration on a product of compact spaces. In this paper we generalize all results from [13] and some results from [2] to a locally compact setting. We also obtain new results, see Theorems 4.3, 4.4, 4.5, and Proposition 4.1. In proving results about quasi-linear functionals and topological measures in a locally compact setting one loses certain convenient features present in the compact case: quasi-integrals and topological measures are not finite in general, subalgebras do not contain constants, quasi-integrals are no longer states, etc. On another level, to tackle the problems related to repeated integration on locally compact spaces, one needs a variety of results about quasi-linear functionals and topological measures. These include a Representation Theorem showing that quasi-linear functionals are obtained by integration with respect to compact-finite topological measures, continuity of quasi-integrals with respect to the topology of uniform convergence on compacta, and others. These results are obtained in [9, 10, 19], and [8].

In this paper X is a Hausdorff, locally compact space. In Sect. 2 we give necessary definitions and facts. We define quasi-integrals and topological measures on a locally compact space and outline the correspondence between them. In Sect. 3 we study simple and almost simple quasi-integrals and topological measures, which are necessary for investigating repeated quasi-integration. We present equivalent conditions for a

quasi-integral to be simple or almost simple. In Sect. 4 we define and study repeated quasi-integrals. In particular, we give criteria for repeated quasi-integration to yield a quasi-linear functional, and for a double quasi-integral to be simple or almost simple. In Sect. 5 we give formulas that describe how a product of compact-finite topological measures acts on open and compact sets. We show that different orders of integration in repeated quasi-integrals give the same quasi-integral if and only if corresponding compact-finite topological measures are both measures or one of the corresponding topological measures is a positive scalar multiple of a point mass.

2 Preliminaries

We use continuous functions with the uniform norm; $C_c(X)$ is the set of real-valued continuous functions on X with compact support, and $C_c^+(X)$ is the set of nonnegative functions from $C_c(X)$. By $\text{supp } f$ we mean $\{x : f(x) \neq 0\}$. We denote by 1 the constant function $1(x) = 1$, by id the identity function $id(x) = x$, and by 1_E the indicator function of the set E . When we consider maps into extended real numbers, they are not identically ∞ . We denote by $\overline{E}$ the closure of a set E , and by $\sqcup$ a union of disjoint sets. $\mathcal{O}(X)$, $\mathcal{C}(X)$, and $\mathcal{K}(X)$ stand, respectively, for the collections of open, closed, and compact subsets of X .

We are interested in repeated quasi-integrals, i.e. repeated quasi-integration with respect to topological measures on locally compact spaces. Since $X \times Y$ is locally compact iff X and Y are locally compact, we assume that X and Y are locally compact.

We will use the following definitions and facts.

Definition 2.1 A topological measure on a locally compact space X is a set function $\mu : \mathcal{C}(X) \cup \mathcal{O}(X) \rightarrow [0, \infty]$ satisfying the following conditions:

- (TM1) If $A, B, A \sqcup B \in \mathcal{K}(X) \cup \mathcal{O}(X)$ then $\mu(A \sqcup B) = \mu(A) + \mu(B)$.
- (TM2) If $U \in \mathcal{O}(X)$ then $\mu(U) = \sup\{\mu(K) : K \in \mathcal{K}(X), K \subseteq U\}$.
- (TM3) If $F \in \mathcal{C}(X)$ then $\mu(F) = \inf\{\mu(U) : U \in \mathcal{O}(X), F \subseteq U\}$.

A topological measure μ on X is called compact-finite if $\mu(K) < \infty$ for $K \in \mathcal{K}(X)$; μ is finite if $\mu(X) < \infty$.

The following theorem is [8, Theorem 4.9].

Theorem 2.1 Let μ be a topological measure on a locally compact space X . The following are equivalent:

- (a) If C, K are compact subsets of X , then $\mu(C \cup K) \leq \mu(C) + \mu(K)$.
- (b) If U, V are open subsets of X , then $\mu(U \cup V) \leq \mu(U) + \mu(V)$.
- (c) μ admits a unique extension to an inner regular on open sets, outer regular Borel measure m on the Borel σ -algebra of subsets of X . m is a Radon measure iff μ is compact-finite. If μ is finite then m is a finite outer regular and inner closed regular Borel measure.

Remark 2.1 Let X be a locally compact noncompact space. For $f \in C_c(X)$ we have $0 \in f(X)$. By a singly generated subalgebra of $C_c(X)$ generated by f we mean the smallest closed subalgebra of $C_c(X)$ containing f ; it has the form

$$B(f) = \{\phi \circ f : \phi(0) = 0, \phi \in C(f(X))\}.$$

When X is compact, by a singly generated subalgebra of $C(X)$ generated by f we mean the smallest closed subalgebra of $C(X)$ containing f and 1; it has the form:

$$A(f) = \{\phi \circ f : \phi \in C(f(X))\}.$$

(Cf. [9, Lemma 1.4]).

Definition 2.2 Let X be locally compact. A quasi-integral (or a quasi-linear functional) on $C_c(X)$ is a map $\rho : C_c(X) \rightarrow \mathbb{R}$ such that:

- (QI1) ρ is homogeneous, i.e. $\rho(af) = a\rho(f)$ for $a \in \mathbb{R}$.
- (QI2) For each $f \in C_c(X)$ we have: $\rho(g + h) = \rho(g) + \rho(h)$ for g, h in the singly generated subalgebra $B(f)$.
- (QI3) ρ is positive, i.e. $f \geq 0 \implies \rho(f) \geq 0$.

When X is compact, we call ρ a quasi-state if $\rho(1) = 1$.

In this paper we are interested in quasi-integrals on $C_c(X)$ and compact-finite topological measures for the reason given in the next remark.

Remark 2.2 There is an order-preserving isomorphism between compact-finite topological measures on X and quasi-integrals on $C_c(X)$, and μ is a measure iff the corresponding functional is linear. See [9, Theorem 3.9] for this result and [19, Theorem 3.9] for the first version of the representation theorem. We outline the correspondence.

- (I) Suppose μ is a compact-finite topological measure on a locally compact space X , $f \in C_c(X)$. Then there exists a finite measure m_f on $\mathbb{R}$ with $\text{supp } m_f \subseteq f(X)$ such that

$$m_f(W) = \mu(f^{-1}(W \setminus \{0\})) \text{ for every open set } W \in \mathbb{R},$$

thus,

$$m_f(W) = \mu(f^{-1}(W)) \text{ for every open set } W \in \mathbb{R} \setminus \{0\}.$$

If μ is finite then

$$m_f(W) = \mu(f^{-1}(W)) \text{ for every open set } W \in \mathbb{R}. \quad (1)$$

The measure m_f is the Stieltjes measure associated with the function $F(t) = \mu(f^{-1}((t, \infty) \setminus \{0\}))$ (respectively, $F(t) = \mu(f^{-1}((t, \infty)))$ if μ is finite.) (See [9, Lemma 2.6].)

Define a quasi-integral $\rho = \rho_\mu$ on $C_c(X)$ by:

$$\rho_\mu(f) = \int_{\mathbb{R}} id \, dm_f. \quad (2)$$

We also write $\rho_\mu(f) = \int_X f \, d\mu$. If μ is a measure then $\rho_\mu(f) = \int_X f \, d\mu$ in the usual sense. On singly generated subalgebras ρ_μ acts as follows: for every $\phi \in C([a, b])$ (with $\phi(0) = 0$ if X is locally compact but not compact)

$$\rho_\mu(\phi \circ f) = \int_{\mathbb{R}} \phi \, d\mu_f = \int_{[a,b]} \phi \, d\mu_f, \quad (3)$$

where $[a, b]$ is any interval containing $f(X)$. (See [9, Proposition 2.11, Theorem 2.12].)

(II) Let ρ be a quasi-integral on $C_c(X)$. The corresponding compact-finite topological measure $\mu = \mu_\rho$ is given as follows:

If U is open, $\mu_\rho(U) = \sup\{\rho(f) : f \in C_c(X), 0 \leq f \leq 1, \text{supp } f \subseteq U\}$,

if F is closed, $\mu_\rho(F) = \inf\{\mu_\rho(U) : F \subseteq U, U \in \mathcal{O}(X)\}$.

If K is compact, $\mu_\rho(K) = \inf\{\rho(g) : g \in C_c(X), g \geq 1_K\} = \inf\{\rho(g) : g \in C_c(X), 0 \leq g \leq 1, g = 1 \text{ on } K\}$. (See [9, Definition 3.1, Lemma 3.2].)

Remark 2.3 Suppose X is locally compact.

(1) Let ρ be a quasi-integral. If $f, g \geq 0$, $f \cdot g = 0$ then f, g belong to the same singly generated subalgebra. If $f \cdot g = 0$ then $\rho(f + g) = \rho(f) + \rho(g)$; hence, $\rho(f) = \rho(f^+) - \rho(f^-)$. Also, $\rho(0) = 0$. (See [9, Lemma 2.1, Lemma 2.2].)

(2) It is easy to see that if $c \geq 0$ and μ is a topological measure then $\rho_{c\mu} = c\rho_\mu$.

(3) If compact $C \subseteq U$, U is open then there is an open set V with compact closure such that $C \subseteq V \subseteq \bar{V} \subseteq U$ (see, for example, [11, Ch. XI, 6.2]).

The following is a part of [9, Theorem 4.5].

Theorem 2.2 Suppose X is locally compact and ρ is a quasi-integral on $C_c(X)$. If $f, g \in C_c(X)$, $f, g \geq 0$, $\text{supp } f, \text{supp } g \subseteq K$, where K is compact, then

$$|\rho(f) - \rho(g)| \leq \|f - g\| \mu(K),$$

where μ is the compact-finite topological measure corresponding to ρ . In particular, for any $f \in C_c(X)$

$$|\rho(f)| \leq \|f\| \mu(\text{supp } f).$$

If $f, g \in C_c(X)$, $\text{supp } f, \text{supp } g \subseteq K$, where K is compact, then

$$|\rho(f) - \rho(g)| \leq 2 \|f - g\| \mu(K).$$

Remark 2.4 If we replace condition (TM1) in Definition 2.1 by finite additivity on compact sets only, we obtain the definition of a deficient topological measure. Deficient topological measures correspond to certain nonlinear functionals that generalize quasi-linear functionals. See [8, 14, 20, 21], and [6] for more information.

We would like to conclude this section with some examples.

Definition 2.3 A set A is bounded if $\overline{A}$ is compact. If X is locally compact, non-compact, a set A is solid if A is connected, and $X \setminus A$ has only unbounded connected components. If X is compact, a set A is solid if A and $X \setminus A$ are connected.

Many examples of topological measures that are not measures are obtained in the following way. Define a so-called solid-set function on bounded open solid and compact solid sets in a locally compact, connected, locally connected, Hausdorff space. A solid set function extends to a unique topological measure. See [3, Definition 2.3, Theorem 5.1], [10, Definition 6.1, Theorem 10.7].

Example 2.1 Suppose that λ is the Lebesgue measure on $X = \mathbb{R}^2$, the set P consists of two points $p_1 = (0, 0)$ and $p_2 = (2, 0)$. For each bounded open solid or compact solid set A let $\nu(A) = 0$ if $A \cap P = \emptyset$, $\nu(A) = \lambda(A)$ if A contains one point from P , and $\nu(A) = 2\lambda(A)$ if A contains both points from P . Then ν is a solid-set function (see [10, Example 15.5]), and ν extends to a unique topological measure on X . Let K_i be the closed ball of radius 1 centered at p_i for $i = 1, 2$. Then K_1, K_2 and $C = K_1 \cup K_2$ are compact solid sets, $\nu(K_1) = \nu(K_2) = \pi$, $\nu(C) = 4\pi$. Since ν is not subadditive, it can not be a measure. The quasi-linear functional corresponding to ν is not linear.

Example 2.2 Let $X = \mathbb{R}^2$ or a square, n be a natural number, and let P be a set of distinct $2n + 1$ points. For each bounded open solid or compact solid set A let $\nu(A) = i/n$ if A contains $2i$ or $2i + 1$ points from P . The set function ν defined in this way is a solid-set function, and it extends to a unique topological measure on X that assumes values $0, 1/n, \dots, 1$. See [2, Example 2.1], [4, Examples 4.14, 4.15]), and [10, Example 15.9]. The resulting topological measure is not a measure. For instance, when X is the square and $n = 1$, it is easy to represent $X = A_1 \cup A_2 \cup A_3$, where each A_i is a compact solid set containing one point from P . Then $\nu(A_i) = 0$ for $i = 1, 2, 3$, while $\nu(X) = 1$. Since ν is not subadditive, it is not a measure, and the quasi-linear functional ρ corresponding to ν is not linear. In [9, Example 4.13] we take $n = 2$ and show that there are $f, g \geq 0$ such that $\rho(f + g) \neq \rho(f) + \rho(g)$. An interesting property of quasi-linear functionals is demonstrated by ρ . If X is locally compact, non-compact, $n = 1$, for the functional ρ we consider a new functional ρ_g defined by $\rho_g(f) = \rho(gf)$, where $g \geq 0$. The new functional ρ_g corresponds to a set function (in, fact, a deficient topological measure) obtained by integrating g over closed and open sets with respect to a topological measure ν . We can choose $g \geq 0$ so that ρ_g is no longer linear on singly generated subalgebras, but only linear on singly generated cones. See [7, Example 35, Theorem 43] for detail.

For more examples of topological measures and quasi-integrals on locally compact spaces see [5] and the last sections of [10] and [9].

Remark 2.5 Our examples 2.1 and 2.2, as well as examples in many papers starting from [1], show that there are topological measures that are not measures, and that there are quasi-linear functionals that are not linear. There are also examples of deficient topological measures that are not topological measures, see, for instance, [21] and [8]. We can say that, in general, the collection of all regular Borel measures and all Radon measures is properly contained in the collection of all topological measures, which, in turn, is properly contained in the collection of all deficient topological measures (see also [8, Remark 4.3].) We can state the same about corresponding functionals.

3 Almost simple quasi-integrals

A nontrivial topological measure assumes at least two values. Topological measures that assume exactly two values are important for proving results about repeated quasi-integration.

Definition 3.1 Let X be locally compact. A topological measure is called simple if it only assumes values 0 and 1. A topological measure is almost simple if it is a positive scalar multiple of a simple topological measure. A quasi-integral is simple (almost simple) if the corresponding topological measure is simple (almost simple).

Lemma 3.1 Suppose X is locally compact and μ is a compact-finite topological measure that assumes more than two values. Let ρ be the corresponding quasi-integral. Then there are functions $f_1, f_2 \in C_c^+(X)$ such that $f_1 f_2 = 0$, $\rho(f_1) = \rho(f_2) = 1$. Functions f_1, f_2 belong to the same singly generated subalgebra.

Proof Choose compact C_1 such that $0 < \mu(C_1) < \mu(X)$. If $\mu(X) < \infty$ by inner regularity of μ on $X \setminus C_1$ choose compact $C_2 \subseteq X \setminus C_1$ such that $0 < \mu(C_2) < \mu(X)$. If $\mu(X) = \infty$, i.e. $\mu(X \setminus C_1) = \infty$, choose a compact $C_2 \subseteq X \setminus C_1$ with $\mu(C_2) > n$. Let U_1, U_2 be open disjoint sets containing C_1, C_2 . For $\epsilon > 0$ let $f_i \in C_c(X)$ be such that $f_i = 1$ on C_i , $\text{supp } f_i \subseteq U_i$ and $0 < \mu(C_i) \leq \rho(f_i) \leq \mu(C_i) + \epsilon$ for $i = 1, 2$. Then $f_1 f_2 = 0$ and by calibrating f_i we may assume that $\rho(f_i) = 1$ for $i = 1, 2$. (This proof is adapted from part of an argument in [13, Theorem 1].) The last statement follows from Remark 2.3. $\square$

The next theorem extends results for a simple quasi-state on $C(X)$ where X is compact, given in [2, Sect. 2].

Theorem 3.1 Let X be locally compact. The following are equivalent for a quasi-integral ρ on $C_c(X)$:

- (i) ρ is simple.
- (ii) m_f is a point mass at $y = \rho(f) \in f(X)$.
- (iii) $\rho(\phi \circ f) = \phi(\rho(f))$ for any $\phi \in C([a, b])$ (with $\phi(0) = 0$ if X is locally compact but not compact), where $f(X) \subseteq [a, b]$.
- (iv) ρ is multiplicative on each singly generated subalgebra, i.e. for each $f \in C_c(X)$ we have: $\rho(gh) = \rho(g)\rho(h)$ for g, h in the singly generated subalgebra $B(f)$.
- (v) $\rho(f^2) = (\rho(f))^2$ for each $f \in C_c(X)$.

Proof (i) $\implies$ (ii). If ρ is simple, i.e. the corresponding topological measure μ is simple, then the measure m_f in part (I) of Remark 2.2 is a point mass. From formula (2) we see that m_f is a point mass at $y = \rho(f)$. To show that $y \in f(X)$, suppose the opposite, and choose an open set W such that $y \in W$, $W \cap f(X) = \emptyset$. Then $m_f(W) = 1$, while $\mu(f^{-1}(W)) = \mu(\emptyset) = 0$, which contradicts formula (1).

(ii) $\implies$ (iii). Since m_f is a point mass at $y = \rho(f) \in f(X)$, by formula (3)

$$\rho(\phi \circ f) = \int_{[a, b]} \phi \, dm_f = \phi(y) = \phi(\rho(f)).$$

(iii) $\implies$ (iv). Let $\phi \circ f, \psi \circ f \in B(f)$. Then $\rho((\phi \circ f)(\psi \circ f)) = \rho((\phi \psi) \circ f) = (\phi \psi)(\rho(f)) = \phi(\rho(f)) \psi(\rho(f)) = \rho(\phi \circ f) \rho(\psi \circ f)$.

(iv) $\implies$ (v). Obvious.

(v) $\implies$ (i). Suppose to the contrary that ρ is not simple, i.e. the corresponding compact-finite topological measure μ is not simple. Then there is a compact set $K \subseteq X$ with $a = \mu(K) \in \mathbb{R}$, $a \neq 0, 1$. By part 2 of Remark 2.3 we may assume that $a < 1$. Let $\epsilon > 0$ be such that $a > (a + \epsilon)^2$. By part (II) of Remark 2.2 choose $g \in C_c(X)$, $g \geq 1_K$ with $\rho(g) < a + \epsilon$. Since $g^2 \geq 1_K$, we have $\rho(g^2) \geq \mu(K) = a > (a + \epsilon)^2 > (\rho(g))^2$, which gives a contradiction. $\square$

Remark 3.1 If μ is almost simple but not simple, write $\mu = c\mu'$, where μ' is simple and $c > 0$. Then $\rho = c\rho'$, where quasi-integrals ρ and ρ' correspond to μ and μ' , and ρ is no longer multiplicative on singly generated subalgebras.

The following theorem follows immediately from Theorem 3.1 and Lemma 3.1.

Theorem 3.2 *Let X be locally compact. The following are equivalent for a quasi-integral ρ on $C_c(X)$:*

- (i) ρ is almost simple.
- (ii) If $fg = 0$ then $\rho(f)\rho(g) = 0$.
- (iii) If $fg = 0$, $f, g \geq 0$ then $\rho(f)\rho(g) = 0$.
- (iv) If $fg = 0$, $f, g \leq 0$ then $\rho(f)\rho(g) = 0$.

When X is compact the equivalence of (ii) and (iv) in Theorem 3.2 with the condition “ ρ is simple” is given by [22, Theorem 3.10].

4 Repeated quasi-integrals

Let μ be a compact-finite topological measure on X with corresponding quasi-integral ρ , and ν be a compact-finite topological measure on Y with corresponding quasi-integral η .

Definition 4.1 For a set A in $X \times Y$ let $A_y = \{x : (x, y) \in A\}$, and let $A_x = \{y : (x, y) \in A\}$.

Remark 4.1 If the set A is closed/compact/open then so is the set A_y . Note also that $(A \setminus B)_y = A_y \setminus B_y$.

Let $f \in C_c(X \times Y)$. Define continuous functions f_y on X and f_x on Y by $f_y(x) = f(x, y) = f_x(y)$. Compact $C = \pi_1(\text{supp } f)$, where $\pi_1 : X \times Y \rightarrow X$ is the canonical projection, contains $\text{supp } f_y$ for any y . We have $f_y \in C_c(X)$, and $f_x \in C_c(Y)$.

Definition 4.2 Define real-valued functions $T_\rho(f)$ on Y and $S_\eta(f)$ on X by:

$$T_\rho(f)(y) = \rho(f_y), \quad S_\eta(f)(x) = \eta(f_x).$$

Proposition 4.1 Suppose $X \times Y$ is locally compact and $f \in C_c(X \times Y)$. If ρ is a quasi-integral on $C_c(X)$ then $T_\rho(f) \in C_c(Y)$ and $\|T_\rho(f)\| \leq \|f\| \mu(\pi_1(\text{supp } f))$. Similarly, if η is a quasi-integral on $C_c(Y)$ then $S_\eta(f) \in C_c(X)$ and $\|S_\eta(f)\| \leq \|f\| \nu(\pi_2(\text{supp } f))$. Here $\pi_1 : X \times Y \rightarrow X$ and $\pi_2 : X \times Y \rightarrow Y$ are canonical projections.

Proof Let $y \in Y$. We shall show that $T_\rho(f)$ is a continuous function at y . Compact $C = \pi_1(\text{supp } f)$ contains $\text{supp } f_y$. Let $\epsilon > 0$. For each $x \in C$ let U_x be a neighborhood of x and $V_{x,y}$ be a neighborhood of y such that $|f(x, y) - f(x', y')| < \epsilon$ whenever $(x', y') \in U_x \times V_{x,y}$. Open sets U_x cover C , so let $U_{x_1}, \dots, U_{x_n}$ be a finite subcover, and let $V_y = \bigcap_{i=1}^n V_{x_i,y}$. Take $y' \in V_y$. Then for each $x \in C$ there is i such that $x \in U_{x_i}$, so $(x, y') \in U_{x_i} \times V_y \subseteq U_{x_i} \times V_{x_i,y}$, and then $|f_y(x) - f_{y'}(x)| = |f(x, y) - f(x, y')| < \epsilon$. For any $x \notin C$ we have $f_y(x) = f_{y'}(x) = 0$. Therefore, $\|f_y - f_{y'}\| < \epsilon$ (for any $y' \in V_y$). Since $\text{supp } f_y, \text{supp } f_{y'} \subseteq C$, by Theorem 2.2 for any $y' \in V_y$ we have:

$$|T_\rho(f)(y) - T_\rho(f)(y')| = |\rho(f_y) - \rho(f_{y'})| \leq 2 \|f_y - f_{y'}\| \mu(C) < 2\epsilon\mu(C),$$

and the continuity of $T_\rho(f)$ at y follows.

By Theorem 2.2 $|T_\rho(f)(y)| = |\rho(f_y)| \leq \|f_y\| \mu(C) \leq \|f\| \mu(C)$ for any y . Thus,

$$\|T_\rho(f)\| \leq \|f\| \mu(C) = \|f\| \mu(\pi_1(\text{supp } f)).$$

Since X and Y are locally compact, for $x \in X$ let $U(x)$ be a neighborhood of x such that $\overline{U(x)}$ is compact in X , and let $V(y)$ be a neighborhood of $y \in Y$ such that $\overline{V(y)}$ is compact in Y . Open sets $U(x) \times V(y)$ cover $\text{supp } f$, so let $U_1 \times V_1, \dots, U_n \times V_n$ be a finite subcover of $\text{supp } f$. Let $G = \overline{V_1} \cup \dots \cup \overline{V_n}$, a compact in Y . For each $x \in X$ and each $y \notin G$ we have $(x, y) \notin \text{supp } f$, so $f(x, y) = 0$. This means that $f_y = 0$ for each $y \notin G$. Then $T_\rho(f)(y) = \rho(f_y) = 0$ for each $y \notin G$. Hence, $T_\rho(f) \in C_c(Y)$. $\square$

We are now ready to define repeated quasi-integrals.

Definition 4.3 Using Definition 4.2 and Proposition 4.1 we define the following functionals on $C_c(X \times Y)$:

$$\begin{aligned} (\eta \times \rho)(f) &= \eta(T_\rho(f)) = \int_Y T_\rho(f) d\nu, \\ (\rho \times \eta)(f) &= \rho(S_\eta(f)) = \int_X S_\eta(f) d\mu. \end{aligned}$$

Remark 4.2 Functionals $\eta \times \rho$ and $\rho \times \eta$ are real-valued since η and ρ are quasi-integrals.

Lemma 4.1 Suppose η and ρ are quasi-integrals. Let $\rho = c\rho'$, where $c > 0$. Then $\eta \times \rho$ is a quasi-integral iff $\eta \times \rho'$ is a quasi-integral, and $\eta \times \rho = c(\eta \times \rho')$. Similarly, if $\eta = k\eta'$, $k > 0$ then $\eta \times \rho$ is a quasi-integral iff $\eta' \times \rho$ is a quasi-integral, and $\eta \times \rho = k(\eta' \times \rho)$.

Proof We have $T_\rho(f)(y) = cT_{\rho'}(f)(y)$ for every y , i.e. $T_\rho(f) = cT_{\rho'}(f)$. Since η is homogeneous, we see that $\eta \times \rho = c(\eta \times \rho')$. Since the set of quasi-integrals is a positive cone, $\eta \times \rho$ is a quasi-integral iff $\eta \times \rho'$ is. $\square$

We would like to know whether $\eta \times \rho = \rho \times \eta$. We shall see later that, unlike the case of linear functionals, this is not usually the case.

For $g \in C_c(X)$ and $h \in C_c(Y)$ let $(g \otimes h)(x, y) = g(x)h(y)$. Then $(g \otimes h) \in C_c(X \times Y)$. From Definition 4.3 we obtain

- Proposition 4.2** (1) $T_\rho(g \otimes h) = \rho(g)h$.
 (2) $(\eta \times \rho)(g \otimes h) = (\rho \times \eta)(g \otimes h) = \rho(g)\eta(h)$.

Remark 4.3 (1) If μ is the compact-finite topological measure corresponding to quasi-integral ρ , ν is the compact-finite topological measure corresponding to η , let $\nu \times \mu$ and $\mu \times \nu$ be the compact-finite topological measure corresponding to quasi-integrals $\eta \times \rho$ and $\rho \times \eta$, respectively. Using part (2) of Proposition 4.2 and part (II) of Remark 2.2 it is easy to see that

$$\mu(X)\nu(Y) \leq (\nu \times \mu)(X \times Y), \quad \mu(X)\nu(Y) \leq (\mu \times \nu)(X \times Y).$$

By Definition 4.3, Theorem 2.2, and Proposition 4.1 the opposite inequalities also hold, so

$$(\nu \times \mu)(X \times Y) = \mu(X)\nu(Y), \quad (\mu \times \nu)(X \times Y) = \mu(X)\nu(Y).$$

(2) It is easy to see that if η and ρ are homogeneous, then so are $\eta \times \rho$ and $\rho \times \eta$; and that if η and ρ are positive, then so are $\eta \times \rho$ and $\rho \times \eta$. Yet, $\rho \times \eta$ and $\eta \times \rho$ are not always quasi-integrals. The criterion for $\eta \times \rho$ to be a quasi-integral is given in Theorem 4.1.

- Theorem 4.1** (1) If the compact-finite topological measure corresponding to ρ assumes more than two values and $\eta \times \rho$ is a quasi-integral, then η is linear.
 (2) If η is linear or ρ is simple then $\eta \times \rho$ is a quasi-integral.
 (3) Suppose η and ρ are quasi-integrals. $\eta \times \rho$ is a quasi-integral iff η is linear or ρ is almost simple.

Proof 1. Our proof follows part of [13, Theorem 1]. It is given for completeness and because intermediate results from the proof are needed elsewhere in the paper. Suppose that $\eta \times \rho$ is a quasi-integral and the compact-finite topological measure μ corresponding to ρ assumes more than two values. By Lemma 3.1 choose $f_1, f_2 \in C_c(X)$ such that $\rho(f_1) = \rho(f_2) = 1$ and $f_1 f_2 = 0$. Take any $g, h \in C_c(Y)$. For any $y \in Y$, $(g(y)f_1)(h(y)f_2) = 0$, so using Remark 2.3 we have:

$$\begin{aligned} T_\rho(f_1 \otimes g + f_2 \otimes h)(y) &= \rho((f_1 \otimes g + f_2 \otimes h)_y) = \rho(g(y)f_1 + h(y)f_2) \\ &= \rho(g(y)f_1) + \rho(h(y)f_2) = g(y)\rho(f_1) + h(y)\rho(f_2), \end{aligned}$$

i.e.

$$T_\rho(f_1 \otimes g + f_2 \otimes h) = \rho(f_1)g + \rho(f_2)h = g + h. \quad (4)$$

Since $\eta \times \rho$ is a quasi-integral and $(f_1 \otimes g)(f_2 \otimes h) = 0$, using Proposition 4.2 and Remark 2.3 we see that

$$\begin{aligned} \eta(g) + \eta(h) &= \rho(f_1)\eta(g) + \rho(f_2)\eta(h) \\ &= (\eta \times \rho)(f_1 \otimes g) + (\eta \times \rho)(f_2 \otimes h) \\ &= (\eta \times \rho)(f_1 \otimes g + f_2 \otimes h) \\ &= \eta(T_\rho(f_1 \otimes g + f_2 \otimes h)) = \eta(g + h). \end{aligned}$$

Thus, η is linear on $C_c(Y)$.

2. To show that $\eta \times \rho$ is a quasi-integral, by the second part of Remark 4.3 we are left to check condition (QI2) of Definition 2.2. Let $\phi, \psi \in C([a, b])$ where $[a, b]$ contains the ranges of f and $T_\rho(f)$. Recall that $f_y \in C_c(X)$ and note that $(\phi \circ f)_y = \phi \circ f_y$. In particular, $\phi \circ f_y$ and $\psi \circ f_y$ are in the same subalgebra generated by f_y . One may apply the argument from [13, Theorem 1] to prove the statement.
3. ($\Rightarrow$). Suppose $\eta \times \rho$ is a quasi-integral. If ρ is not almost simple then μ assumes more than two values, and by part (1) η must be linear.
($\Leftarrow$). If η is linear then by part (2) $\eta \times \rho$ is a quasi-integral. Suppose $\rho = c\rho'$, where ρ' is simple, $c > 0$. By part (2) $\eta \times \rho'$ is a quasi-integral, and then by Lemma 4.1 so is $\eta \times \rho$.

□

Corollary 4.1 *Suppose $\mu(X) = 1$. The functional $\eta \times \rho$ is a quasi-integral iff η is linear or ρ is simple.*

Remark 4.4 We may phrase the results of Theorem 4.1 in terms of topological measures. If μ is a compact-finite topological measure on X (with corresponding quasi-integral ρ) and ν is a compact-finite topological measure on Y (with corresponding quasi-integral η) we may define a compact-finite product topological measure $\nu \times \mu$ on $X \times Y$ (corresponding to quasi-integral $\eta \times \rho$) if either ν is a measure or μ is an almost simple topological measure. Similar results hold for topological measure $\mu \times \nu$ corresponding to quasi-integral $\rho \times \eta$.

Theorem 4.2 *If ρ and η are simple quasi-integrals, then so is $\eta \times \rho$.*

Proof Assume that ρ and η are simple. Let $f \in C_c(X \times Y)$ and $\phi \in C([a, b])$ (with $\phi(0) = 0$ if X is locally compact but not compact), where $[a, b]$ contains the ranges of f and $T_\rho(f)$. Since ρ is simple, by part (iii) of Theorem 3.1 we have $T_\rho(\phi \circ f)(y) = \phi \circ T_\rho(f)(y)$ for all y . Since η is simple, we have

$$(\eta \times \rho)(\phi \circ f) = \eta(T_\rho(\phi \circ f)) = \eta(\phi \circ T_\rho(f)) = \phi(\eta \circ T_\rho(f)) = \phi((\eta \times \rho)f).$$

By Theorem 3.1 $\eta \times \rho$ is simple. (Cf. [13, Corollary 1].)

□

Theorem 4.3 Suppose ρ , η , and $\eta \times \rho$ are quasi-integrals. $\eta \times \rho$ is almost simple iff ρ and η are almost simple.

Proof If ρ and η are almost simple, write $\rho = c\rho'$, $\eta = k\eta'$ where $c, k > 0$ and ρ', η' are simple quasi-integrals. By Lemma 4.1 and Theorem 4.2 $\eta \times \rho = ck(\eta' \times \rho')$ is almost simple.

Now assume that $\eta \times \rho$ is almost simple. Write $\eta \times \rho = c\xi$ where $c > 0$ and ξ is a simple quasi-integral. Suppose that neither of ρ , η is almost simple. By Lemma 3.1 choose $h_1, h_2 \in C_c^+(Y)$ such that $\eta(h_1) = \eta(h_2) = 1$, $h_1h_2 = 0$, and $g_1, g_2 \in C_c^+(X)$ such that $\rho(g_1) = \rho(g_2) = 1$, $g_1g_2 = 0$.

Since $(g_1 \otimes h_1) \cdot (g_2 \otimes h_2) = 0$, by Remark 2.3 functions $g_1 \otimes h_1, g_2 \otimes h_2$ belong to the same singly generated subalgebra. Since $\eta \times \rho$ is simple, it is multiplicative on this subalgebra and then using Proposition 4.2 we have:

$$\begin{aligned} (\eta \times \rho)(g_1g_2 \otimes h_1h_2) &= (\eta \times \rho)((g_1 \otimes h_1) \cdot (g_2 \otimes h_2)) \\ &= ((\eta \times \rho)(g_1 \otimes h_1)) \cdot ((\eta \times \rho)(g_2 \otimes h_2)) \\ &= \rho(g_1)\eta(h_1)\rho(g_2)\eta(h_2) = 1. \end{aligned}$$

Then

$$\xi(g_1g_2 \otimes h_1h_2) = \frac{1}{c}.$$

But by Proposition 4.2 also $(\eta \times \rho)(g_1g_2 \otimes h_1h_2) = \rho(g_1g_2)\eta(h_1h_2) = 0$, so

$$\xi(g_1g_2 \otimes h_1h_2) = 0.$$

The contradiction shows that at least one of ρ , η must be almost simple. If ρ is almost simple and η is not almost simple, we pick $g \in C_c^+(X)$ such that $\rho(g) = 1$. With h_1, h_2 as above, and g instead of g_1, g_2 , the argument above shows that we again obtain $\xi(g_1g_2 \otimes h_1h_2) = 1/c$ and $\xi(g_1g_2 \otimes h_1h_2) = 0$. Thus, the case " ρ is almost simple, and η is not almost simple" is impossible. Similarly, the case " η is almost simple and ρ is not almost simple" is impossible. Therefore, both ρ and η are almost simple. $\square$

We extend Theorem 4.2 to:

Theorem 4.4 Suppose ρ , η , and $\eta \times \rho$ are quasi-integrals. If any two of them are simple, then so is the third one.

Proof Suppose ρ and $\eta \times \rho$ are simple. By Theorem 4.3 η is almost simple, and we write $\eta = k\eta'$ where $k > 0$ and η' is simple. Then $\eta \times \rho = k(\eta' \times \rho)$, and with Theorem 4.2 both $\eta \times \rho$ and $\eta' \times \rho$ are simple. Then $k = 1$ (use part (v) of Theorem 3.1), so η is simple. The case " η and $\eta \times \rho$ are simple" is similar. $\square$

Remark 4.5 Taking simple quasi-integrals ρ and η , for $c > 0$ we have $\eta \times \rho = (c\rho) \times (c^{-1}\eta)$. Thus, the fact that $\eta \times \rho$ is simple does not imply that ρ and η are simple. However, we have the following:

Theorem 4.5 Suppose ρ , η , $\eta \times \rho$ are quasi-integrals. Then $\eta \times \rho$ is simple iff $\eta \times \rho = \eta' \times \rho'$ for some simple quasi-integrals ρ' and η' .

Proof If ρ and η are simple, then $\eta \times \rho$ is simple by Theorem 4.2. Now assume that $\eta \times \rho$ is simple. By Theorem 4.3 ρ and η are almost simple. Write $\rho = c\rho'$, $\eta = k\eta'$, where $c, k > 0$ and ρ', η' are simple. Then $\eta \times \rho = ck(\eta' \times \rho')$, both $\eta \times \rho$ and $\eta' \times \rho'$ are simple, so $ck = 1$. $\square$

5 Products of topological measures and Fubini's theorem

The next two theorems describe how $\nu \times \mu$ acts on sets.

Theorem 5.1 Let μ be a compact-finite topological measure on X and ν a finite measure on Y . Then for U open in $X \times Y$ we have:

$$(\nu \times \mu)(U) = \int_Y \mu(U_y) d\nu(y).$$

If μ is finite, for a compact set K in $X \times Y$ we also have

$$(\nu \times \mu)(K) = \int_Y \mu(K_y) d\nu(y).$$

Proof We shall show that for any open $U \subseteq X \times Y$ we have $(\nu \times \mu)(U) = \int_Y \mu(U_y) d\nu(y)$. The argument follows that in [13, Theorem 2].

Observe that the function $y \rightarrow \mu(U_y)$ is lower semicontinuous, hence, ν -measurable. [Indeed, suppose $\mu(U_y) > \alpha$, and choose compact $K \subseteq \mu(U_y)$ such that $\mu(K) > \alpha$. Since $K \times \{y\} \subseteq U$, there is a neighborhood V of y such that $K \times V \subseteq U$. Then for $y \in V$ we have $K \subseteq U_y$, so $\mu(U_y) > \alpha$.]

Let $f \in C_c(X \times Y)$, $\text{supp } f \subseteq U$. Then $f_y \in C_c(X)$ and $\text{supp } f_y \subseteq U_y$ for each $y \in Y$. Thus $\rho(f_y) \leq \mu(U_y)$ and $(\eta \times \rho)(f) = \int_Y \rho(f_y) d\nu(y) \leq \int_Y \mu(U_y) d\nu(y)$. By part (II) of Remark 2.2 $(\nu \times \mu)(U) = \sup\{(\eta \times \rho)(f) : f \in C_c(X \times Y), \text{supp } f \subseteq U\}$, and we have

$$(\nu \times \mu)(U) \leq \int_Y \mu(U_y) d\nu(y).$$

Now suppose $g \in C_c(Y)$ is such that $0 \leq g(y) \leq \mu(U_y)$ for $y \in Y$. Let $\epsilon > 0$. For each y choose a compact $K(y) \subseteq U_y$ such that $\mu(K(y)) > g(y) - \epsilon$. Since $K(y) \times \{y\} \subseteq U$, there is a neighborhood (with compact closure) $V(y)$ of y such that $K \times \overline{V(y)} \subseteq U$ and $|g(y) - g(z)| < \epsilon$ for $z \in \overline{V(y)}$. Choose $V(y_1), \dots, V(y_n)$ that cover $\text{supp } g$, and let $E = \bigcup_{i=1}^n K(y_i) \times \overline{V(y_i)}$. Then compact $E \subseteq U$ and we choose $f \in C_c(X \times Y)$ such that $1_E \leq f$, $\text{supp } f \subseteq U$. If $y \in \text{supp } g$, say, $y \in V(y_i)$, then $K(y_i) \subseteq E_y$ and $1_{K(y_i)} \leq f_y$. Then by part (II) of Remark 2.2

$g(y) - 2\epsilon < g(y_i) - \epsilon \leq \mu(K(y_i)) \leq \rho(f_y)$, and so

$$\int_Y g \, dv - 2\epsilon v(Y) \leq \int_Y \rho(f_y) \, dv(y) = (\eta \times \rho)(f) \leq (v \times \mu)(U).$$

Thus, $\int_Y g \, dv \leq (v \times \mu)(U)$. Since the function $y \rightarrow \mu(U_y)$ is lower semicontinuous,

$$\int_Y \mu(U_y) \, dv(y) = \sup \left\{ \int_Y g \, dv : g \in C_c(Y), 0 \leq g(y) \leq \mu(U_y) \right\} \leq (v \times \mu)(U).$$

We have $(v \times \mu)(U) = \int_Y \mu(U_y) \, dv(y)$.

If μ is finite, then by Remark 4.3 so is $v \times \mu$ and $(v \times \mu)(X \times Y) = \mu(X)v(Y)$. If $K \subseteq X \times Y$ is compact, by Definition 2.1 we have:

$$\begin{aligned} (v \times \mu)(K) &= (v \times \mu)(X \times Y) - (v \times \mu)((X \times Y) \setminus K) = \\ &= v(Y)\mu(X) - \int_Y \mu((X \times Y) \setminus K)_y \, dv(y) \\ &= v(Y)\mu(X) - \int_Y (\mu(X) - \mu(K_y)) \, dv(y) = \int_Y \mu(K_y) \, dv(y). \end{aligned}$$

□

Theorem 5.2 *Let μ be a simple topological measure on X and v be a compact-finite topological measure on Y . Then for U open in $X \times Y$ we have:*

$$(v \times \mu)(U) = v(\{y : \mu(U_y) = 1\}).$$

If v is finite, for K compact in $X \times Y$ we also have:

$$(v \times \mu)(K) = v(\{y : \mu(K_y) = 1\}).$$

Proof For A in $X \times Y$ let $B(A) = \{y : \mu(A_y) = 1\}$. Let U be open in $X \times Y$. It is not hard to show (for example, by applying the argument from [13, Claim 1 of Theorem 3]) that $B(U)$ is also open. Since μ is simple, for any compact $D \subseteq X \times Y$ we have $(B(D))^c = B(D^c)$, and $B(D)$ is closed.

Suppose that $K \subseteq B(U)$ is compact. For each $y \in K$, $\mu(U_y) = 1$, so there is a compact set $C(y) \subseteq U_y$ with $\mu(C(y)) = 1$. Then $C(y) \times \{y\} \subseteq U$, so there are open sets $V(y)$, $W(y)$ such that $C(y) \subseteq V(y) \subseteq X$, $y \in W(y) \subseteq Y$, and $V(y) \times \overline{W(y)} \subseteq U$. Finitely many $W(y_1), \dots, W(y_n)$ cover K . Set $D = \bigcup_{i=1}^n C(y_i) \times \overline{W(y_i)}$. Then D is compact and $D \subseteq U$. Choose $f \in C_c(X \times Y)$ such that $1_D \leq f \leq 1$, $\text{supp } f \subseteq U$. Then for $y \in K$, say, $y \in W(y_i)$, we have $f_y = 1$ on $C(y_i)$, so $1 = \mu(C(y_i)) \leq \rho(f_y) = T_\rho(f)(y)$. Thus $T_\rho(f) = 1$ on K . Then by part (II) of Remark 2.2

$$v(K) \leq \eta(T_\rho(f)) = (\eta \times \rho)(f) \leq (v \times \mu)(U).$$

Taking the supremum over $K \subseteq B(U)$ shows that $v(B(U)) \leq (v \times \mu)(U)$.

We shall show that $(\nu \times \mu)(U) \leq \nu(B(U))$. Since $(\nu \times \mu)(U) = \sup\{(\eta \times \rho)(f) : f \in C_c(X \times Y), \text{supp } f \subseteq U, 0 \leq f \leq 1\}$, we shall check that $(\eta \times \rho)(f) \leq \nu(B(U))$ for such f . Let $C = \text{supp } f$. Pick an open set V with compact closure such that $C \subseteq V \subseteq \overline{V} \subseteq U$. Note that $f_y \in C_c(X)$, $\text{supp } f_y \subseteq V_y$, $0 \leq f_y \leq 1$. Then

$$\begin{aligned} \{y : T_\rho(f)(y) > 0\} &= \{y : \rho(f_y) > 0\} \subseteq \{y : \mu(V_y) > 0\} \\ &= \{y : \mu(V_y) = 1\} = B(V) \subseteq B(\overline{V}) \subseteq B(U). \end{aligned}$$

Since $B(\overline{V})$ is closed, we have $\text{supp } T_\rho(f) \subseteq B(\overline{V}) \subseteq B(U)$. By Proposition 4.1 $\|T_\rho(f)\| \leq 1$. By part (II) of Remark 2.2 $(\eta \times \rho)(f) = \eta(T_\rho(f)) \leq \nu(B(U))$.

Now we have: $(\nu \times \mu)(U) = \nu(\{y : \mu(U_y) = 1\})$ for any open set $U \subseteq X \times Y$. When ν is finite, the formula for compact K can be proved as in Theorem 5.1. $\square$

Corollary 5.1 *Suppose $\mu = c\mu'$ is an almost simple topological measure on X , where μ' is simple and $c > 0$, and ν is a compact-finite topological measure on Y . Then for A open in $X \times Y$ we have:*

$$(\nu \times \mu)(A) = c \nu(\{y : \mu(A_y) = c\}).$$

When ν is finite, the same also holds for compact sets in $X \times Y$.

Proof Using Lemma 4.1 and Theorem 5.2 we have: $(\nu \times \mu)(A) = c(\nu \times \mu')(A) = c\nu(\{y : \mu'(A_y) = 1\}) = c\nu(\{y : \mu(A_y) = c\})$. $\square$

Corollary 5.2 *Suppose μ is a compact-finite topological measure on X , ν is a finite topological measure on Y , and $\nu \times \mu$ is a compact-finite topological measure. If $A \subseteq X$ and $B \subseteq Y$ are both open then $(\nu \times \mu)(A \times B) = \mu(A)\nu(B)$. When μ is finite, the same also holds when $A \subseteq X$ and $B \subseteq Y$ are compact sets.*

Proof By Theorem 4.1 either μ is almost simple or ν is a finite measure. The corollary now follows from Theorem 5.1 and Corollary 5.1, and generalizes Remark 4.3. $\square$

Lemma 5.1 *If μ and ν are almost simple but not measures then $\eta \times \rho \neq \rho \times \eta$.*

Proof We first prove (as in [13, Corollary 3]) that if μ and ν are simple but not measures then $\eta \times \rho \neq \rho \times \eta$. If μ and ν are not measures, by Theorem 2.1 they are not subadditive, and we may find open sets $U, V \subseteq X$ with $\mu(U) = \mu(V) = 0$, $\mu(U \cup V) = 1$, and compact sets $C, K \subseteq Y$ with $\nu(C) = \nu(K) = 0$, $\nu(C \cup K) = 1$. Taking complements of C and K we get open sets $W, E \subseteq Y$ such that $\nu(W) = \nu(E) = 1$, $\nu(W \cap E) = 0$. Let $A = (U \times W) \cup (V \times E)$. We shall show that $(\nu \times \mu)(A) = 0$ using Theorem 5.2. If $y \in E \cap W$ then $A_y = U \cup V$, so $\mu(A_y) = 1$. For the cases $y \in W \setminus E$, $y \in E \setminus W$, $y \notin E \cup W$ we have $\mu(A_y) = 0$. Then $(\nu \times \mu)(A) = \nu(E \cap W) = 0$. A similar argument shows that $(\mu \times \nu)(A) = 1$. Since $(\nu \times \mu)(A) \neq (\mu \times \nu)(A)$, we have $\eta \times \rho \neq \rho \times \eta$.

If μ and ν are almost simple but not measures, write $\mu = c\mu'$, $\nu = k\nu'$, where $c, k > 0$ and μ', ν' are simple, but not measures. With simple quasi-integrals ρ', η' corresponding to μ', ν' we have $\eta \times \rho = ck(\eta' \times \rho') \neq ck(\rho' \times \eta') = \rho \times \eta$. $\square$

Lemma 5.2 *Suppose one of compact-finite topological measures μ, ν is a positive scalar multiple of a point mass. Then $(\nu \times \mu) = (\mu \times \nu)$.*

Proof If we prove the statement in the case when one of μ, ν is a point mass, the lemma easily follows. So suppose $\mu = \delta_{x_0}$. It is enough to show that $(\nu \times \mu) = (\mu \times \nu)$ for open sets. Let $U \subseteq X \times Y$ be open. Then $\mu(U_y) = 1$ iff $(x_0, y) \in U$, and by Theorem 5.2 we have:

$$(\nu \times \mu)(U) = \nu(\{y : \mu(U_y) = 1\}) = \nu(\{y : (x_0, y) \in U\}) = \nu(U_{x_0}).$$

By Theorem 5.1 we have:

$$(\mu \times \nu)(U) = \int_X \nu(U_x) d\mu(x) = \nu(U_{x_0}).$$

Thus, $(\nu \times \mu) = (\mu \times \nu)$. $\square$

Now we are ready to answer the question of when a version of Fubini's theorem holds for repeated quasi-integrals, in other words, when $\eta \times \rho = \rho \times \eta$.

Theorem 5.3 *Let ρ be a quasi-integral with corresponding compact-finite topological measure μ on Y and let η be a quasi-integral with corresponding compact-finite topological measure ν on X . Then $\eta \times \rho = \rho \times \eta$ if and only if μ, ν are both measures or one of μ, ν is a positive scalar multiple of a point mass.*

Proof ($\Leftarrow$) If both μ, ν are measures, then $\eta \times \rho = \rho \times \eta$ by Fubini's theorem. If one of μ, ν is a positive scalar multiple of a point mass, then $\eta \times \rho = \rho \times \eta$ by Lemma 5.2.

($\Rightarrow$) First we shall show that $\eta \times \rho$ and $\rho \times \eta$ must be quasi-integrals. Suppose $\eta \times \rho$ is not a quasi-integral, that is, ρ is not almost simple and η is not linear by Theorem 4.1. Let $f_1, f_2 \in C_c(X)$ be functions given by Lemma 3.1, so $f_1 f_2 = 0$, $\rho(f_1) = \rho(f_2) = 1$. Also choose $g, h \in C_c(Y)$ such that $\eta(g + h) \neq \eta(g) + \eta(h)$. Let $f = f_1 \otimes g + f_2 \otimes h$. As in formula (4), we have $T_\rho(f) = g + h$. Then

$$(\eta \times \rho)(f) = \eta(T_\rho(f)) = \eta(g + h).$$

Now we shall look at $(\rho \times \eta)(f) = \rho(S_\eta(f))$. For each x , $S_\eta(f)(x) = \eta(f_x) = \eta(f_1(x)g + f_2(x)h)$. Since $(f_1(x)g)(f_2(x)h) = 0$, we have:

$$\eta(f_1(x)g + f_2(x)h) = \eta(f_1(x)g) + \eta(f_2(x)h) = f_1(x)\eta(g) + f_2(x)\eta(h).$$

Thus, $S_\eta(f) = f_1\eta(g) + f_2\eta(h)$. Since $f_1\eta(g) \cdot f_2\eta(h) = 0$, we have

$$(\rho \times \eta)(f) = \rho(S_\eta(f)) = \eta(g)\rho(f_1) + \eta(h)\rho(f_2) = \eta(g) + \eta(h).$$

and we see that $\eta \times \rho \neq \rho \times \eta$. Thus, $\eta \times \rho$ must be a quasi-integral, and so must $\rho \times \eta$.

Both $\eta \times \rho$ and $\rho \times \eta$ are quasi-integrals. From Theorem 4.1 we see that this happens only when (a) both μ and ν are measures, or (b) at least one of μ or ν is a positive scalar multiple of a point mass, or (c) both μ and ν are almost simple, but not measures. The first two cases produce $\eta \times \rho = \rho \times \eta$, by Fubini's Theorem or Lemma 5.2. In the last case (c), by Lemma 5.1 $\eta \times \rho \neq \rho \times \eta$. This finishes the proof. $\square$

Remark 5.1 As in the compact case (see [13, p. 2166]), we have the following interesting phenomenon: if μ and ν are almost simple, but not measures, then $\nu \times \mu$ and $\mu \times \nu$ are different, even though they agree on rectangles. This holds even when $X = Y$ and $\mu = \nu$. Linear combinations $d(\mu \times \nu) + (m - d)(\nu \times \mu)$, where $m = \mu(X)^2$ and $0 \leq d \leq m$, give uncountably many topological measures that agree on rectangles, but are distinct. This is impossible for measures on product spaces, as they are determined by values on rectangles.

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Declarations

Conflict of interest The author declares no conflict of interest.

Data availability The article has no associated data.

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