



Topological concepts in partially ordered vector spaces

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Abstract

In the context of partially ordered vector spaces one encounters different sorts of order convergence and order topologies. This article investigates these notions and their relations. In particular, we study and relate the order topology presented by Floyd, Vulikh and Dobbertin, the order bound topology studied by Namioka and the concept of order convergence given in the works of Abramovich, Sirotkin, Wolk and Vulikh. We prove that the considered topologies disagree for all infinite dimensional Archimedean vector lattices that contain order units. For reflexive Banach spaces equipped with ice cream cones we show that the order topology, the order bound topology and the norm topology agree and that order convergence is equivalent to norm convergence.

Keywords Order topology · Order bound topology · Order convergence · Partially ordered vector space · Riesz space · Vector lattice · Order neighbourhood · Minkowski norm · Ice cream cone · Net catching element

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1 Introduction

In the study of partially ordered vector spaces one uses topological concepts like order convergence and order continuity as can be seen for example in [1–5]. In particular one encounters different types of order convergence, which lead to different types of order continuity as well as different types of order topology. The different types of order continuity are for example studied in [3] and [6] and we will focus in this paper on properties and relationships of different sorts of topologies that can be defined in partially ordered vector spaces. In particular we will investigate properties and relationships of the *order topology* [2,7–9] and the *order bound topology* [5,10]. In

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order to define the order topology, recall that we write $x_\alpha \downarrow 0$ whenever $(x_\alpha)_{\alpha \in A}$ is a decreasing net (i.e. $\alpha \leq \beta$ implies $x_\alpha \geq x_\beta$) which satisfies $\inf_{\alpha \in A} x_\alpha = 0$.

Definition 1.1 A subset $M \subseteq X$ is called an *order neighbourhood* of $x \in X$, if for all nets $(\check{x}_\alpha)_{\alpha \in A}$ with $\check{x}_\alpha \downarrow 0$ there is $\alpha \in A$ such that $[x - x_\alpha, x + x_\alpha] \subseteq M$. A subset O of X is called *order open* if O is an order neighbourhood of each of its points. The *order topology* $\tau_o(X)$ (or simply τ_o) is defined as the set of all order open subsets. Complements of order open sets are called *order closed*.

Following [5] we define the order bound topology for Archimedean partially ordered vector spaces that contain order units as follows.

Definition 1.2 An element $u \in X_+$ is called an *order unit*, whenever for all $x \in X$ there is $n \in \mathbb{N}$ with $x \leq nu$. For $x \in X$ we define the *Minkowski norm* as $\|x\|_u := \inf\{\lambda > 0; \pm x \leq \lambda u\}$. Recall that whenever X is Archimedean and u is an order unit, then $\|\cdot\|_u$ is indeed a norm on X . The corresponding topology $\tau_{ob}(X)$ (or simply τ_{ob}) is independent from the choice of u and called the *order bound topology*.

It is natural to ask under which assumptions these topologies coincide. In Corollary 4.8 and Theorem 4.11 we provide the following characterization.

Theorem A *If X is an Archimedean partially ordered vector space (that contains order units) the following statements are equivalent.*

- (i) $\tau_o = \tau_{ob}$.
- (ii) For all nets $(x_\alpha)_{\alpha \in A}$ with $x_\alpha \downarrow x$ there holds $x_\alpha \xrightarrow{\tau_{ob}} x$.
- (iii) X contains non-empty, order open and order bounded sets.
- (iv) There exists $x \in X_+$ such that for any net $(x_\alpha)_{\alpha \in A}$ with $x_\alpha \downarrow 0$ there exists $\alpha \in A$ with $x_\alpha \leq x$.

We will use this characterization to show that for all finite dimensional Archimedean and directed partially ordered vector spaces there holds $\tau_o = \tau_{ob}$ (Example 6.1). As τ_{ob} is a linear topology we obtain in particular that this topology is the standard topology of X , i.e. the unique linear topology on X . In Example 6.2 we will furthermore show the following.

Example For a reflexive Banach space equipped with an ice cream cone the order topology, the order bound topology and the norm topology agree and order convergence is equivalent to norm convergence.

However, for infinite dimensional Archimedean vector lattices X (with order units) we obtain $\tau_{ob} \neq \tau_o$ from the following result, which follows from Corollary 4.8 and Theorem 6.4.

Theorem B *An Archimedean vector lattice contains non-empty, order open and order bounded sets, if and only if it is finite dimensional.*

Note that there exists a close relation between the order topology τ_o and various concepts of order convergence [3,6]. It is thus natural to study the interplay of order convergence and the order bound topology. The following version of order convergence appears in [1–3,6,9].

Definition 1.3 Let $x \in X$ and let $(x_\alpha)_{\alpha \in A}$ be a net in X . We say that $(x_\alpha)_{\alpha \in A}$ *order converges* to x (and write $x_\alpha \xrightarrow{o} x$) whenever there exists a net $(\check{x}_\beta)_{\beta \in B}$ that satisfies $\check{x}_\beta \downarrow 0$ and such that for any $\beta \in B$ there exists $\alpha_\beta \in A$ with $\pm(x_\alpha - x) \leq \check{x}_\beta$ for all $\alpha \geq \alpha_\beta$. In this case we call x an *order limit* of $(x_\alpha)_{\alpha \in A}$.

We prove the following in Theorem 3.1.

Theorem C *Let X be an Archimedean partially ordered vector space (that contains order units). Every τ_{ob} -convergent net is also order convergent (to the same limit).*

The reader might have noticed that we stressed the existence of order units in our statements. We did this as the order bound topology τ_{ob} can also be defined in case X does not contain order units (and is not Archimedean). In this case τ_{ob} can be defined as the linear topology generated by the zero neighbourhood base consisting of all convex and circled sets that absorb all order intervals [5, 10]. We will see in Theorem 3.1 that for the implication discussed in Theorem C (τ_{ob} -convergence implies order convergence) the Archimedean property and the existence of order units are not only sufficient, but also necessary.

We end our introduction presenting the following remarkable parallelism between order units and elements x as considered in Theorem A(iv). In order to be more precise, we say that $x \in X_+$ is *net catching*, whenever for any net $(x_\alpha)_{\alpha \in A}$ with $x_\alpha \downarrow 0$ there exists $\alpha \in A$ that satisfies $x_\alpha \leq x$. In Corollary 3.2, Theorem 4.6 and Proposition 4.7 we show the following.

Theorem D *Let X be an Archimedean partially ordered vector space.*

- (i) *The τ_{ob} -interior of X_+ is the set of all order units. Whenever there exist order units in X there holds $\tau_o \subseteq \tau_{ob}$.*
- (ii) *The τ_o -interior of X_+ is the set of all net catching elements. Whenever there exist net catching elements in X there holds $\tau_{ob} \subseteq \tau_o$.*

In particular, whenever $\tau_o = \tau_{ob}$ an element $x \in X$ is net catching, if and only if it is an order unit.

Combining the Theorems B and D we obtain in particular that for any infinite dimensional Archimedean vector lattice that contains order units the inclusion $\tau_o \subseteq \tau_{ob}$ is strict. So far, we do not know, whether for all Archimedean partially ordered vector spaces there holds $\tau_o \subseteq \tau_{ob}$. However, in Theorem 4.15 we prove that for Archimedean vector lattices all convex order open sets are τ_{ob} -open.

The paper is organized as follows. In Sect. 2 we fix some notation. In Sect. 3 we will consider the interplay of order convergence and the Minkowski norm and present a proof of Theorem C. Sect. 4 is devoted to the interplay of order units, net catching elements, the order topology and the order bound topology. In particular, we will prove the Theorems A and D and present examples. This section ends with a brief investigation of convex order open sets. In order to present more interesting examples, such as the statement about ice cream cones of reflexive Banach spaces, we will provide a sufficient condition for the existence of net catching elements in Sect. 5. This technique is used in Sect. 6. In this section we furthermore prove Theorem B.

2 Preliminaries

2.1 Nets and subnets

A mapping $x: A \rightarrow M$ is called a *net*, whenever A is a non-empty set equipped with a reflexive and transitive relation \leq that is *directed*, i.e. for $\alpha, \alpha' \in A$ there exists $\alpha'' \in A$ with $\alpha \leq \alpha''$ and $\alpha' \leq \alpha''$. In this case we denote $(x_\alpha)_{\alpha \in A}$ with $x_\alpha = x(\alpha)$. We say that another net $(y_\beta)_{\beta \in B}$ is a *subnet* of $(x_\alpha)_{\alpha \in A}$, whenever there exists a function $h: B \rightarrow A$, such that (i) for any $\beta \leq \beta'$ there holds $\alpha \leq \alpha'$, (ii) for any $\alpha \in A$ exists $\beta \in B$ with $\alpha \leq h(\beta)$, and (iii) for all $\beta \in B$ we have $y_\beta = x_{h(\beta)}$. See [11] for further details.

2.2 Partially ordered vector spaces

A *partially ordered vector space* is a real vector space X equipped with a reflexive, transitive and antisymmetric binary relation \leq such that for every $\lambda \in \mathbb{R}$ with $0 \leq \lambda$ and $x, y, z \in X$ with $x \leq y$ one has that $x + z \leq y + z$ and $\lambda x \leq \lambda y$. We denote $X_+ := \{x \in X; 0 \leq x\}$ for the *cone* and call the elements of X_+ *positive*. For $x, y \in X$ we write $x < y$ if $x \leq y$ and $x \neq y$. For $U, V \subseteq X$ we write $U \leq V$ if for every $x \in U$ and $y \in V$ we have $x \leq y$. For $x \in X$, we abbreviate $V \leq \{x\}$ by $V \leq x$, and $x \leq V$ is defined similarly. For $x, y \in X$ the *order interval* is given by $[x, y] := \{z \in X; x \leq z \leq y\}$. For a subset of X , the notions *bounded above*, *bounded below*, *order bounded*, *upper (or lower) bound* and *infimum (or supremum)* are defined as usual.

Consider a partially ordered vector space X . X is called *directed* if for every $x, y \in X$ there exists $z \in X$ with $x, y \leq z$. Note that X is directed, if and only if X_+ is *generating*, i.e. $X = X_+ - X_+$. We say that X is *Archimedean* if for every $x, y \in X$ with $nx \leq y$ for all $n \in \mathbb{N}$ one has that $x \leq 0$. We call X a *vector lattice* if for every non-empty finite subset of X the infimum and the supremum exist in X . X is called *Dedekind complete* whenever every non-empty set that is bounded from above has a supremum, and every non-empty set that is bounded from below has an infimum. Note that every Dedekind complete vector lattice is Archimedean. An element $u \in X_+$ is called an *order unit*, if for all $x \in X$ there is $n \in \mathbb{N}$ such that $x \leq nu$. An element $x \in X_+$ is called *net catching*, if for all nets $(x_\alpha)_{\alpha \in A}$ with $x_\alpha \downarrow 0$ there exists $\alpha \in A$ such that $x_\alpha \leq x$. We call $M \subseteq X$ *order dense* in X if for every $y \in X$ one has

$$\sup\{x \in M; x \leq y\} = y = \inf\{x \in M; y \leq x\}.$$

A non-empty convex subset B of $X_+ \setminus \{0\}$ is called a *base (of X_+)*, if for each $x \in X_+ \setminus \{0\}$ there are unique $b \in B$ and $\lambda \in \mathbb{R}_+$ such that $x = \lambda b$. Consider subsets $M, N \subseteq X$. We say that M is *circled* whenever for all $\lambda \in [-1, 1]$ we have $\lambda M \subseteq M$. Furthermore, we say that M *absorbs* N , if there is $\mu > 0$ with $\lambda N \subseteq M$ for all $\lambda \in [-\mu, \mu]$.

2.3 Embedding theorems

Let X and Y be partially ordered vector spaces. A linear mapping $f: X \rightarrow Y$ is said to be *positive* whenever all $x \in X_+$ satisfy $f(x) \geq 0$. We call a linear map $f: X \rightarrow Y$ *order reflecting*, whenever $f(x) \geq 0$ implies $x \geq 0$ for all $x \in X$. Note that every order reflecting mapping is injective. A *linear order embedding* is a positive and order reflecting map. A *linear order dense embedding* is a linear order embedding $f: X \rightarrow Y$ for which the image $f(X)$ is order dense in Y . A *linear order isomorphism* is a surjective order embedding. For a reference of the following embedding theorem see [2, Theorem IV.11.1] or [12, Chapter V.3].

Theorem 2.1 *If X is an Archimedean and directed partially ordered vector space, then there exists a Dedekind complete vector lattice X^δ and a linear order dense embedding $\iota: X \rightarrow X^\delta$. In this context $\iota: X \rightarrow X^\delta$ is called a Dedekind completion of X .*

Recall that a topological space is called *extremely disconnected* whenever the closure of any open set is open. For reference of the following see [13, Theorem 45.5] or [2, Theorem V.2.2 and Theorem V.4.1].

Theorem 2.2 *If X is a Dedekind complete vector lattice that contains order units, then there exists an extremely disconnected compact Hausdorff space Ω and a linear order isomorphism $\iota: X \rightarrow C(\Omega)$. Here $C(\Omega)$ is the vector lattice of all continuous function $f: \Omega \rightarrow \mathbb{R}$ equipped with the pointwise order.*

2.4 Order convergence and the order topology

We next summarize some well known results that can be found in varying degrees of generality in [1–3,6,9] concerning the order topology and the order convergence as defined in the introduction.

Theorem 2.3 *Let X be a partially ordered vector space.*

- (i) *A subset $M \subseteq X$ is order closed, if and only if the order limits of nets in M are contained in M .*
- (ii) *Any net $(x_\alpha)_{\alpha \in A}$ that satisfies $x_\alpha \downarrow x$ order converges to x .*
- (iii) *Any net that order converges to $x \in X$ also converges to x w.r.t. the order topology τ_o .*
- (iv) *X_+ is order closed.*
- (vi) *Whenever τ_o is a linear topology, then X is Archimedean and directed.*
- (v) *$\tau_o(\mathbb{R})$ is the standard topology of \mathbb{R} . Order convergence is equivalent to the standard notion of convergence of \mathbb{R} .*

Proof The statement (vi) follows from [6, Lemma 8.1] and (v) can easily be seen as the standard topology of the Dedekind complete lattice \mathbb{R} is generated by the set of all order intervals. See [6, Theorem 3.14, Proposition 3.6, and Corollary 3.18] for the rest of the statements. □

If X and Y are partially ordered vector spaces we call a positive map $f: X \rightarrow Y$ *order continuous*, whenever any net $(x_\alpha)_{\alpha \in A}$ with $x_\alpha \downarrow 0$ satisfies $f(x_\alpha) \downarrow 0$. From [6, Theorem 4.4 and Corollary 4.5] we quote the following results.

- Theorem 2.4** (i) Every linear order dense embedding between partially ordered vector spaces is order continuous.
- (ii) For any positive map $f: X \rightarrow Y$ between partially ordered vector spaces the following statements are equivalent.
- f is order continuous.
 - f is continuous with respect to $\tau_o(X)$ and $\tau_o(Y)$.
 - For any order convergent net $(x_\alpha)_{\alpha \in A}$ in X with order limit $x \in X$ the net $(f(x_\alpha))_{\alpha \in A}$ is order convergent with order limit $f(x)$.

2.5 The order bound topology

The following definition generalizes the definition of the order bound topology presented in the introduction. See [5, Sect. 2.8] for details.

Definition 2.5 Let $\mathcal{B}_{ob}(X)$ (or simply \mathcal{B}_{ob}) be the set of all subsets $V \subseteq X$ such that V is convex, circled and absorbs all order intervals. Note that \mathcal{B}_{ob} is a base at zero for a linear topology $\tau_{ob}(X)$ (or simply τ_{ob}). This topology is called the *order bound topology*.

The order bound topology τ_{ob} is introduced and studied in [10]. For further details we recommend [5], where τ_{ob} is called 'order topology'. The following statements can be found in [5, Sect. 2.8].

Theorem 2.6 Let X be an Archimedean partially ordered vector space and let u be an order unit in X .

- The norm $\|\cdot\|_u$ generates τ_{ob} .
- The respective closed unit ball satisfies $\overline{B_1^{\|\cdot\|_u}}(0) = [-u, u]$.

3 Order convergence and the order bound topology

As presented above the link between the order convergence and the order topology was investigated in various works like for example [6,9]. We thus focus next on the link between the order bound topology and the order convergence.

Theorem 3.1 Let X be a partially ordered vector space. The following statements are equivalent.

- Any τ_{ob} -convergent net is order convergent (to the same limit).
- X is Archimedean and contains order units.

From Theorem 2.3 we recall that order convergence implies convergence with respect to the order topology τ_o . We thus obtain the following.

Corollary 3.2 If X is an Archimedean partially ordered vector space that contains order units, then the order bound topology is finer than the order topology, i.e. $\tau_o \subseteq \tau_{ob}$.

In order to show Theorem 3.1 we will need the following lemma.

Lemma 3.3 *Let X be an Archimedean partially ordered vector space that contains order units. Then the set of order units U equipped with the reversed order from X is directed and satisfies $\inf U = 0$ with respect to the order of X .*

Proof Let $u, v \in U$. There is $n \in \mathbb{N}$ such that $v \leq nu$. As $(1/n)v$ is also an order unit which satisfies $(1/n)v \leq \{u, v\}$ we obtain U to be directed with respect to the reversed order.

Clearly $0 \leq U$ and we consider $z \in X$ with $z \leq U$. For $u \in U$ and $n \in \mathbb{N}$ we know that also $(1/n)u$ is an order unit. Thus there holds $nz \leq u$ for all $n \in \mathbb{N}$ and we obtain $z \leq 0$ as u is an order unit. This shows $\inf U = 0$. □

Proof of Theorem 3.1 To show that (i) implies (ii) we show first that X is Archimedean and consider $x, y \in X$ with $nx \leq y$ for all $n \in \mathbb{N}$. Any $V \in \mathcal{B}_{ob}$ absorbs $[y, y]$ there exists $\mu > 0$ with $\lambda[y, y] \subseteq V$ for all $\lambda \in [-\mu, \mu]$. For $n \in \mathbb{N}$ with $n \geq 1/\mu$ we observe $(1/n)y \in (1/n)[y, y] \subseteq V$. Hence $((1/n)y)_{n \in \mathbb{N}}$ converges to 0 w.r.t. τ_{ob} and (i) implies that this net order converges to 0. Thus there exists a net $(\check{y}_\beta)_{\beta \in B}$ with $\check{y}_\beta \downarrow 0$ and such that for $\beta \in B$ there exists $n \in \mathbb{N}$ with $x \leq (1/n)y \leq \check{y}_\beta$. Hence $x \leq 0$ and we obtain X to be Archimedean.

To show that X contains order units let $A := \{(x, V); V \in \mathcal{B}_{ob}, x \in V\}$ be ordered by $(x, V) \leq (y, U)$ whenever $V \supseteq U$. Then A is easily seen to be directed and we define $x_{(x, V)} := x$ for $(x, V) \in A$ in order to obtain a net $(x_\alpha)_{\alpha \in A}$. As \mathcal{B}_{ob} is a base at zero for τ_{ob} we obtain $x_\alpha \xrightarrow{\tau_{ob}} 0$ and (i) implies $x_\alpha \xrightarrow{o} 0$. Thus there exists a net $(\check{x}_\beta)_{\beta \in B}$ with $\check{x}_\beta \downarrow 0$ and such that for any $\beta \in B$ there exists $\alpha_\beta \in A$ such that $\pm x_\alpha \leq \check{x}_\beta$ for all $\alpha \geq \alpha_\beta$. We will now show that \check{x}_β is an order unit for any $\beta \in B$.

Let $\beta \in B$ and consider $(y, V) = \alpha_\beta \in A$ as above. For $v \in V$ we have $(v, V) \geq (x, V)$ and hence $\pm v = \pm x_{(v, V)} \leq \check{x}_\beta$. We observe $V \leq \check{x}_\beta$. For $x \in X$ we consider the order interval $[x, x]$ and obtain from the definition of \mathcal{B}_{ob} that V absorbs $[x, x]$. Thus there exists $\mu > 0$ with $\lambda x \leq V$ for all $\lambda \in [-\mu, \mu]$. For $n \in \mathbb{N}$ with $n \geq 1/\mu$ we obtain $x \leq nV \leq n\check{x}_\beta$.

To show that (ii) implies (i) we consider a net $(x_\alpha)_{\alpha \in A}$ with $x_\alpha \xrightarrow{\tau_{ob}} x \in X$ but assume w.l.o.g. that $x = 0$. Let B be the set of all order units, equipped with the reversed order from X . Define $\check{x}_\beta := \beta$ for $\beta \in B$. By Lemma 3.3 the net $(\check{x}_\beta)_{\beta \in B}$ satisfies $\check{x}_\beta \downarrow 0$. For $\beta \in B$ we know that the norm $\|\cdot\|_\beta$ generates the order bound topology and thus $\|x_\alpha\|_\beta \rightarrow 0$. Thus there is $\alpha_\beta \in A$ such that $\inf\{\lambda > 0; \pm x_\alpha \leq \lambda\beta\} = \|x_\alpha\|_\beta \leq \frac{1}{2}$ for $\alpha \geq \alpha_\beta$. We obtain $\pm x_\alpha \leq \beta = \check{x}_\beta$ for every $\alpha \geq \alpha_\beta$. This shows $x_\alpha \xrightarrow{o} 0$. □

Remark 3.4 Note that the net $(\check{x}_\beta)_{\beta \in B}$ constructed in the second part of the proof is independent from $(x_\alpha)_{\alpha \in A}$.

4 The interplay of the order topology and the order bound topology

4.1 About order units and net catching elements

We start the investigation of the interplay of the order topology and the order bound topology by relating net catching elements with order units.

Lemma 4.1 *Let X be a partially ordered vector space.*

- (i) *If X contains net catching elements, then every order unit in X is net catching.*
(ii) *If X is Archimedean and directed, then every net catching element of X is an order unit.*

Proof To show (i) let u be an order unit of X and consider a net catching element $x \in X$. There is $n \in \mathbb{N}$ such that $x \leq nu$. It is straight forward to show that nu and hence u is net catching.

To show (ii) assume $x \in X$ to be net catching and consider $y \in X$. As X is directed there exists $z \in X_+$ such that $z \geq y$. The Archimedean property of X implies $(1/n)z \downarrow 0$. Thus since x is net catching there is $n \in \mathbb{N}$ such that $(1/n)z \leq x$, i.e. $y \leq z \leq nx$.

Remark 4.2 Note that an element $x \in X$ is net catching, if and only if it is net catching with respect to the order restricted to the *directed part* $X_+ - X_+$. As non-directed partially ordered vector spaces do not contain order units it is natural to assume directedness in (ii).

Example 4.3 Let l_∞ be the Dedekind complete vector lattice of all bounded and real sequences, equipped with the pointwise order. Then $e := (1)_{k \in \mathbb{N}}$ is an order unit of l_∞ that is not net catching. Thus Lemma 4.1(i) implies that l_∞ has no net catching elements.

Proof For $n, k \in \mathbb{N}$ let $e_k^{(n)} := 0$, if $k < n$ and $e_k^{(n)} := 2$ otherwise. This defines a sequence $(e^{(n)})_{n \in \mathbb{N}}$ in l_∞ such that $e^{(n)} \downarrow 0$. Since for no $n \in \mathbb{N}$ we have $e^{(n)} \leq e$ we obtain e to be not net catching. \square

There also exist net catching elements, which are not order units.

Example 4.4 Equip l_∞ with the lexicographic order, i.e. with $(x_n)_{n \in \mathbb{N}} \leq (y_n)_{n \in \mathbb{N}}$ whenever $(x_n)_{n \in \mathbb{N}} = (y_n)_{n \in \mathbb{N}}$ or whenever there exists $n \in \mathbb{N}$ such that $x_n < y_n$ and $x_m = y_m$ for all $m < n$. l_∞ equipped with \leq is a (non-Archimedean) vector lattice that contains order units and furthermore net catching elements which are not order units.

Proof Note that the lexicographic order is *total*, i.e. that there holds $x \leq y$ or $y \leq x$ for all $x, y \in l_\infty$. One easily obtains that $(u_n)_{n \in \mathbb{N}} \in l_\infty$ is an order unit with respect to this ordering, if and only if $u_1 > 0$. Consider $x = (x_n)_{n \in \mathbb{N}} \in l_\infty$ with $x_1 = 0$ and $x_n := 1$ for $n \geq 2$, which is obviously no order unit. To show that x is net catching let $(x_\alpha)_{\alpha \in A}$ be a net in l_∞ with $x_\alpha \downarrow 0$. As $0 < x$ and as \leq is total we observe that there exists $\alpha \in A$ such that $x_\alpha \leq x$. \square

4.2 A parallelism between order units and net catching elements

We next investigate the interplay of order units with the order bound topology, and of net catching elements with the order topology respectively.

Lemma 4.5 *Let X be a partially ordered vector space and $x \in X$.*

- (i) x is an order unit, if and only if $[-x, x] \in \mathcal{B}_{ob}$.
- (ii) x is net catching, if and only if $[-x, x]$ is an order neighbourhood of 0.

Proof '(i)' Let us assume that x is an order unit. Since $[-x, x]$ is clearly convex and circled it remains to show that $[-x, x]$ absorbs order intervals. Consider an order interval $[y, z] \subseteq X$. As x is an order unit there is $n \in \mathbb{N}$ with $-y, z \leq nx$ and we obtain $(1/n)[y, z] \subseteq [-x, x]$. As $[-x, x]$ is circled this shows that $[-x, x]$ absorbs $[y, z]$.

To show the converse assume $[-x, x] \in \mathcal{B}_{ob}$ and let $y \in X$. As $[-x, x]$ absorbs $[y, y]$ there is $\mu > 0$ with $\lambda[y, y] \subseteq [-x, x]$ for all $\lambda \in [-\mu, \mu]$. For $n \in \mathbb{N}$ with $n \geq 1/\mu$ we obtain $y \leq nx$, which proves x to be an order unit.

'(ii)' Assume that x is net catching. For a net $(x_\alpha)_{\alpha \in A}$ with $x_\alpha \downarrow 0$ there exists $\alpha \in A$ with $x_\alpha \leq x$. Thus $[-x_\alpha, x_\alpha] \subseteq [-x, x]$ and we obtain $[-x, x]$ to be an order neighbourhood. Similarly one shows that whenever $[-x, x]$ is an order neighbourhood, then x is net catching. □

In analogy with Corollary 3.2 we next show the following.

Theorem 4.6 *If there are net catching elements in X , then the order topology is finer than the order bound topology, i.e. $\tau_{ob} \subseteq \tau_o$.*

Proof We show that every $V \in \mathcal{B}_{ob}$ is an order neighbourhood of 0. Let x be a net catching element. Since V absorbs all order intervals there is $\lambda > 0$ such that $[-\lambda x, \lambda x] \subseteq V$. Note that λx is net catching. Thus Lemma 4.5 shows $[-\lambda x, \lambda x] \subseteq V$ to be an order neighbourhood of 0. □

Proposition 4.7 *Let X be a partially ordered vector space.*

- (i) The τ_{ob} -interior of X_+ is the set of all order units of X .
- (ii) The τ_o -interior of X_+ is the set of all net catching elements of X .

Proof '(i)' Let u be contained in the τ_{ob} -interior of X_+ and let $x \in X$. There exists $V \in \mathcal{B}_{ob}$ such that $V + u \subseteq X_+$. As V absorbs all order intervals we argue as above to find $n \in \mathbb{N}$ with $[-x, -x] \subseteq nV$ and observe $nu - x \leq nV + nu \subseteq nX_+ = X_+$, i.e. $x \leq nu$.

To show the converse we will show that the set U of all order units is order open. Indeed, whenever $u \in U$, then $(1/2)u \in U$. Thus by Lemma 4.5 we obtain $[(-1/2)u, (1/2)u] \in \mathcal{B}_{ob}$. As $u + [(-1/2)u, (1/2)u] = [(1/2)u, (3/2)u]$ consists only of order units we obtain that U is a τ_{ob} -neighbourhood of u .

'(ii)' Let x be contained in the τ_o -interior $X_+^{o\tau_o}$ of X_+ . To show that x is net catching let $(x_\alpha)_{\alpha \in A}$ be a net in X_+ with $x_\alpha \downarrow 0$. Since $X_+^{o\tau_o}$ is order open it is an order neighbourhood of x . Thus there is $\alpha \in A$ such that $[x - x_\alpha, x + x_\alpha] \subseteq X_+^{o\tau_o} \subseteq X_+$ and hence $x_\alpha \leq x$. The converse can be shown similarly to the corresponding implication of (i).

Corollary 4.8 *Let X be a partially ordered vector space.*

- (i) X contains order units, if and only if there exist non-empty τ_{ob} -open subsets of X that are order bounded.

(ii) X contains net catching elements, if and only if there exist non-empty order open subsets of X that are order bounded.

Proof If the τ_{ob} -interior $X_+^{o(\tau_{ob})}$ is non-empty, then $((2x) - X_+^{o(\tau_{ob})}) \cap X_+^{o(\tau_{ob})}$ is a non-empty, τ_{ob} -open and order bounded set for any $x \in X_+^{o(\tau_{ob})}$. Conversely, if O is a non-empty, τ_{ob} -open and order bounded set, then there exist $x, y \in X$ with $-x \leq O \ni y$ and $x + y \in O + x \subseteq X_+$ is an interior point of X_+ with respect to τ_{ob} . This shows (i) and (ii) can be obtained by considering τ_o instead of τ_{ob} . \square

4.3 Examples

Example 4.9 Recall from Example 4.3 that l_∞ equipped with the pointwise order is an Archimedean vector lattice that contains order units but no net catching elements. From Corollary 3.2 we know that $\tau_o \subseteq \tau_{ob}$. This inclusion is strict, as we know from Proposition 4.7 that the interior of X_+ with respect to τ_o is empty while the interior of X_+ with respect to τ_{ob} contains elements.

Example 4.10 Recall from Example 4.4 that l_∞ equipped with the lexicographic order is a vector lattice that contains net catching elements that are not order units. From Theorem 4.6 we obtain $\tau_{ob} \subseteq \tau_o$. This inclusion is strict as the interior of X_+ with respect to τ_o is not the same as the interior of X_+ with respect to τ_{ob} .

4.4 Characterizations of $\tau_o = \tau_{ob}$

From our investigations we conclude the following characterization.

Theorem 4.11 *Let X be an Archimedean partially ordered vector space that contains order units. Then the following statements are equivalent.*

- (i) $\tau_o = \tau_{ob}$.
- (ii) For all nets $(x_\alpha)_{\alpha \in A}$ with $x_\alpha \downarrow x$ there holds $x_\alpha \xrightarrow{\tau_{ob}} x$.
- (iii) X contains net catching elements.

Proof Recall from Theorem 2.3 that $x_\alpha \downarrow x$ implies $x_\alpha \xrightarrow{\tau_o} x$. Thus (i) implies (ii).

To show that (ii) implies (iii) consider an order unit $u \in X_+$. Let $(x_\alpha)_{\alpha \in A}$ be a net in X with $x_\alpha \downarrow 0$. From (ii) we know $x_\alpha \rightarrow 0$ with respect to τ_{ob} . Now recall from Lemma 4.5 that $[-u, u] \in \mathcal{B}_{ob}$. Thus there is $\alpha \in A$ with $x_\alpha \in [-u, u]$ and we observe $x_\alpha \leq u$.

To show that (iii) implies (i) recall from Corollary 4.8 that the existence of order units implies $\tau_o \subseteq \tau_{ob}$. If X contains net catching elements we furthermore know from Theorem 4.6 $\tau_o \supseteq \tau_{ob}$ and obtain (i). \square

Having the parallelism between order units and net catching elements in mind it is natural to ask, how $\tau_o = \tau_{ob}$ can be characterized under the assumption of the existence of net catching elements.

Theorem 4.12 *Let X be a partially ordered vector space that contains net catching elements. Then the following statements are equivalent.*

- (i) $\tau_o = \tau_{ob}$.
- (ii) τ_o is a linear topology.
- (iii) X is Archimedean and directed.
- (iv) X is Archimedean and contains order units.

In particular, if one of these statements is valid, then each net catching element $x \in X$ is an order unit of X and $\| \cdot \|_x$ is a norm that generates the order topology τ_o .

Proof As the order bound topology is a linear topology we obtain that (i) implies (ii). From Theorem 2.3 we furthermore obtain that (ii) implies (iii). If X is Archimedean and directed, then any net catching element is an order unit by Lemma 4.1. Thus (iii) implies (iv). From Theorem 4.11 we furthermore obtain that (iv) implies (i). \square

Recall that order convergence implies convergence with respect to τ_o . From Theorem 4.12 and Theorem 3.1 we thus obtain the following corollary.

Corollary 4.13 *If X is Archimedean and directed and contains net catching elements, then the following statements are equivalent for any net $(x_\alpha)_{\alpha \in A}$ in X .*

$$(i)x_\alpha \xrightarrow{o} 0. \quad (ii)x_\alpha \xrightarrow{\tau_o} 0. \quad (iii)x_\alpha \xrightarrow{\tau_{ob}} 0.$$

Remark 4.14 In [9, Corollary 4.11] a different proof for the equivalence of (i) and (ii) in Corollary 4.13 is given. Note that the existence of net catching elements is not stated explicitly in [9] but used in the proof. Indeed, it is used that the set of all non-empty, order bounded and order open sets is not-empty, a property equivalent to the existence of net catching elements as shown in Corollary 4.8. See [6, Example 8.3] for an example of an Archimedean vector lattice for which τ_o -convergence does not imply order convergence.

4.5 About convex order open sets

So far we do not now, whether for Archimedean and directed partially ordered vector spaces (without order units) any order open set is τ_{ob} -open. For the following partial result in this direction recall that a partially ordered vector space X is said to satisfy the (Riesz) decomposition property whenever for $x, u, v \in X_+$ with $x \leq u + v$ there exist $u', v' \in X_+$ with $x = u' + v', u' \leq u$ and $v' \leq v$. Any vector lattice satisfies the decomposition property. See [5] for further details on this notion.

Theorem 4.15 *If X is an Archimedean and directed partially ordered vector space that satisfies the decomposition property, then any convex order open set is τ_{ob} -open.*

The proof of Theorem 4.15 is split into several lemmas about the local structure of order neighbourhoods. For the rest of this section we assume that X is a partially ordered vector space.

Lemma 4.16 *Any order neighbourhood of 0 contains a circled order neighbourhood of 0.*

Proof Let $U \subseteq X$ be an order neighbourhood of 0. Define $I(U) := \{x \in X_+; [-x, x] \subseteq U\}$. The set $V := \bigcup_{x \in I(U)} [-x, x]$ is easily seen to be circled and to be contained in U . To show that V is an order neighbourhood of 0 let $(x_\alpha)_{\alpha \in A}$ be a net in X with $x_\alpha \downarrow 0$. As U is an order neighbourhood of 0 there is $\alpha \in A$ with $[-x_\alpha, x_\alpha] \subseteq U$. Thus $x_\alpha \in I(U)$ and we obtain $[-x_\alpha, x_\alpha] \subseteq V$. \square

Remark 4.17 The construction of the set V is given in [9] and it is shown there that V is an order neighbourhood of 0.

Lemma 4.18 *If X satisfies the decomposition property, then any convex order neighbourhood of 0 contains a convex and circled order neighbourhood of 0.*

Proof Let U be a convex order neighbourhood of 0 and define V as in the proof of Lemma 4.16. It remains to show that V is convex and we consider $x, y \in V$ and $\lambda \in (0, 1)$. There are $x', y' \in X_+$ with $x \in [-x', x'] \subseteq U$ and $y \in [-y', y'] \subseteq U$. Denoting $z' := \lambda x' + (1 - \lambda)y'$ we observe $\lambda x + (1 - \lambda)y \in [-z', z']$.

To show $z' \in I(U)$ we consider $\tilde{z} \in [-z', z']$. There holds $0 \leq \tilde{z} + z' \leq 2z' = 2\lambda x' + 2(1 - \lambda)y'$ and from the decomposition property we observe that there exist $u, v \in X_+$ with $\tilde{z} + z' = u + v, u \leq 2\lambda x'$ and $v \leq 2(1 - \lambda)y'$. We compute

$$\tilde{x} := \frac{1}{\lambda}(u - \lambda x') \in \frac{1}{\lambda}[-\lambda x', \lambda x'] = [-x', x'] \subseteq U$$

and similarly $\tilde{y} := (1/(1 - \lambda))(v - (1 - \lambda)y') \in U$. As U is convex we observe

$$\tilde{z} = u + v - z' = u + v - (\lambda x' + (1 - \lambda)y') = \lambda \tilde{x} + (1 - \lambda)\tilde{y} \in U.$$

This shows $[-z', z'] \subseteq U$, i.e. $z' \in I(U)$. We conclude $\lambda x + (1 - \lambda)y \in [-z', z'] \subseteq V$. \square

Lemma 4.19 *If X is Archimedean and directed, then any order neighbourhood of 0 absorbs all order intervals.*

Proof Let U be an order neighbourhood of 0. By Lemma 4.16 U contains a circled order neighbourhood V of 0. Let $[x, y]$ be a non-empty order interval in X . Since X is directed there is $w \in X$ with $w \geq -x$ and $w \geq y$. From the Archimedean property we observe $(1/n)w \downarrow 0$. Now recall that V is an order neighbourhood of 0. We thus obtain the existence of $n \in \mathbb{N}$ with $(1/n)[x, y] \subseteq [-(1/n)w, (1/n)w] \subseteq V$. For $\lambda \in [-1/n, 1/n]$ we use that V is circled to see $\lambda[x, y] \subseteq V \subseteq U$, which proves U to absorb $[x, y]$. \square

Proof of Theorem 4.15 Let O be a convex and order open set and consider $x \in O$. Then $O - x$ is a convex order neighbourhood of 0. From Lemma 4.18 we know that $O - x$ contains a convex and circled order neighbourhood V of 0. By Lemma 4.19 we know that V absorbs all order intervals and hence $V \in \mathcal{B}_{ob}$. This proves O to be τ_{ob} -open. \square

We finish this section with an Example of an Archimedean vector lattice that contains a circled order neighbourhood that does not contain convex sets that absorb order intervals.

Example 4.20 Let l_0 be the space of all sequences $x = (x_n)_{n \in \mathbb{N}}$ in \mathbb{R} with finite support $\text{supp}(x) = \{n \in \mathbb{N}; x_n \neq 0\}$. l_0 is an Archimedean vector lattice equipped with the pointwise order. Let M be the set of all x , for which there exists $F \subseteq \mathbb{N}$ finite with $\text{supp}(x) \subseteq F$ and $|x_n| \leq 1/|F|$ for $n \in F$. Then M is a circled order neighbourhood of 0 that does not contain a convex set which absorbs all order intervals. In particular, we obtain from Lemma 4.19 that M does not contain a convex order neighbourhood.

Proof Clearly M is circled. To show that M is an order neighbourhood let $(x^{(\alpha)})_{\alpha \in A}$ be a net in l_0 with $x^{(\alpha)} \downarrow 0$. Considering $\alpha_0 \in A$ we obtain the existence of $N \in \mathbb{N}$ with $\text{supp}(x^{(\alpha)}) \subseteq \text{supp}(x^{(\alpha_0)}) \subseteq \{1, \dots, N\}$ for all $\alpha \geq \alpha_0$. Let $\alpha \geq \alpha_0$ such that $x_k^{(\alpha)} \leq 1/N$ for $k = 1, \dots, N$ to obtain $[-x^{(\alpha)}, x^{(\alpha)}] \subseteq M$.

Assume now that there exists a convex subset $C \subseteq M$ that absorbs all order intervals and let $x \in C$ and $N \in \mathbb{N}$. Choose $F \subseteq \mathbb{N}$ disjoint from $\text{supp}(x)$ with $|F| = N$. Let $y \in l_0$ be defined by $y_n := 1$ for $n \in F$ and $y_n = 0$ for $n \notin F$. As C absorbs order intervals there exists $\lambda > 0$ such that $\lambda y \in C$. Since C is convex we obtain $z := 1/2x + 1/2y \in C \subseteq M$. Thus for $n \in \text{supp}(x)$ we compute $0 \leq (1/2)|x_n| = |(1/2)x_n + (1/2)y_n| \leq 1/|\text{supp}(z)| \leq 1/N$. As $N \in \mathbb{N}$ was arbitrary this shows $x = 0$ and we have shown $C = \{0\}$, a contradiction as C is supposed to absorb order intervals. □

5 A sufficient condition for net catching elements

In order to be able to present more examples of partially ordered vector spaces that contain net catching elements we will prove the following in this section.

Theorem 5.1 *Let X be a partially ordered vector space and B be a base of X_+ . If there exists a linear topology τ on X such that B is τ -compact and such that X_+ is τ -closed, then all upper bounds of B are net catching.*

To prove this theorem it will be convenient to consider the following correspondence between strictly positive linear functionals and bases. Recall that a linear functional $f : X \rightarrow \mathbb{R}$ is called *strictly positive*, whenever we have $0 < f(x)$ for all non-zero $x \in X_+$. For a strictly positive linear functional f the set $B_f := \{x \in X_+; f(x) = 1\}$ is a base of X_+ . Furthermore, for any base B of X_+ there exists a unique strictly positive linear functional $f : X \rightarrow \mathbb{R}$ such that $B = B_f$. Thus there is a one-to-one correspondence between strictly positive linear functionals and bases of X_+ . For reference see [5]. We show next that order continuity translates into being order closed in this correspondence.

Proposition 5.2 *Let $f : X \rightarrow \mathbb{R}$ be a strictly positive linear functional. Then B_f is order closed if and only if f is order continuous.*

Proof Assume that f is order continuous and note that f is positive. From Sect. 2.4 we thus know that f is continuous with respect to $\tau_o(X)$ and the standard topology of \mathbb{R} . As X_+ is always order closed we observe that $B = X_+ \cap f^{-1}(\{1\})$ is order closed.

To show the converse assume B_f to be order closed and let $(x_\alpha)_{\alpha \in A}$ be a net in X with $x_\alpha \downarrow 0$. Clearly $(f(x_\alpha))_{\alpha \in A}$ is a net in \mathbb{R}_+ such that $f(x_\alpha) \downarrow$. To get a

contradiction assume that there does not hold $f(x_\alpha) \downarrow 0$. Then there exists $\lambda > 0$ with $f(x_\alpha) \geq \lambda$ for all $\alpha \in A$. We define $y_\alpha := (1/f(x_\alpha))x_\alpha$ for all $\alpha \in A$ to obtain a net $(y_\alpha)_{\alpha \in A}$ in B_f . For $\alpha \in A$ we observe

$\pm y_\alpha \leq (1/\lambda)x_\alpha$, which proves $y_\alpha \xrightarrow{o} 0$. Since B_f is order closed we obtain that $0 \in B_f$, i.e. $f(0) = 1$, a contradiction. \square

The following proposition gives a sufficient condition on X to have an order closed base.

Lemma 5.3 *Let B be a base of X_+ . If there exists a linear topology τ on X such that B is τ -compact and such that X_+ is τ -closed, then B is order closed.*

Proof Let $f: X \rightarrow \mathbb{R}$ be the unique strictly positive linear functional on X such that $B = \{x \in X_+; f(x) = 1\}$. By Proposition 5.2 it suffices to show that f is order continuous. Let $(x_\alpha)_{\alpha \in A}$ be a net in X such that $x_\alpha \downarrow 0$. Since f is positive we obtain $(f(x_\alpha))_{\alpha \in A}$ to be a net in \mathbb{R}_+ that satisfies $f(x_\alpha) \downarrow$. To obtain a contradiction we assume that $f(x_\alpha) \downarrow 0$ is not satisfied. As above we obtain that $0 < \lambda := \inf_{\alpha \in A} f(x_\alpha)$ and define $y_\alpha := (1/f(x_\alpha))x_\alpha$ in order to obtain a net $(y_\alpha)_{\alpha \in A}$ in the τ -compact set B . Thus there exists a subnet $(z_\gamma)_{\gamma \in C}$ of $(y_\alpha)_{\alpha \in A}$ that converges with respect to τ to some $z \in B$. Let us fix a map $h: C \rightarrow A$ as in the definition of a subnet.

Then $(f(x_{h(\gamma)}))_{\gamma \in C}$ is a subnet of $(f(x_\alpha))_{\alpha \in A}$ we observe $f(x_{h(\gamma)}) \downarrow \lambda$. Thus $(f(x_{h(\gamma)}))_{\gamma \in C}$ converges to λ with respect to the standard topology of \mathbb{R} . As τ is a linear topology we obtain

$$x_{h(\gamma)} = f(x_{h(\gamma)})y_{h(\gamma)} = f(x_{h(\gamma)})z_\gamma \xrightarrow{\tau} \lambda z.$$

Furthermore, we have $x_{h(\gamma)} \downarrow$ and that X_+ is τ -closed. A standard argument [5, Lemma 2.3] therefore yields $x_{h(\gamma)} \downarrow \lambda z$. Thus $x_{h(\gamma)} \downarrow 0$ allows to observe $\lambda z = 0$. Since $z \in B$ this implies $\lambda = 0$, a contradiction. \square

We obtain Theorem 5.1 by combining the following Lemma with Lemma 5.3 above.

Lemma 5.4 *If B is an order closed base of X_+ , then all upper bounds of B are net catching.*

Proof Let $x \in X$ be such that $B \leq x$. To show that x is net catching consider a net $(x_\alpha)_{\alpha \in A}$ with $x_\alpha \downarrow 0$. Furthermore, consider a strictly positive functional $f: X \rightarrow \mathbb{R}$ such that $B = \{x \in X_+; f(x) = 1\}$. From Proposition 5.2 we know that f is order continuous and thus satisfies $f(x_\alpha) \downarrow 0$. In particular, there is $\alpha \in A$ with $f(x_\alpha) \leq 1$. Whenever $f(x_\alpha) = 0$, then the strict positivity of f implies $x_\alpha = 0 \leq x$. Whenever $f(x_\alpha) > 0$, then $(1/f(x_\alpha))x_\alpha \in B$ and we obtain $x_\alpha \leq (1/f(x_\alpha))x_\alpha \leq x$. \square

6 Examples

6.1 Finite dimensional spaces

Example 6.1 For a finite dimensional, Archimedean and directed partially ordered vector space X the topology $\tau_o = \tau_{ob}$ is the standard topology of X , i.e. the unique

linear topology on X . A net is order convergent, if and only if it is convergent with respect to the standard topology. The interior of X_+ with respect to the discussed topologies is non-empty and consists of the net catching elements which are precisely the order units.

Proof Let us denote τ for the standard topology of X . As X is Archimedean we know that X_+ is τ -closed and that X_+ has a τ -closed base. See [5, Chapter 3] for reference.

From X being directed we furthermore obtain that X contains order units [5, Lemma 2.5 and Lemma 3.2]. By Theorem 4.11 it thus suffices to show that X contains net catching elements in order to obtain $\tau_o = \tau_{ob}$. Note that τ_{ob} is a linear topology and thus the standard topology of X . The statements about the order convergence and the interior of the cone follow from Proposition 4.7 and Corollary 4.13.

It remains to show that X contains net catching elements. Let u be an order unit of X and consider a norm $\|\cdot\|$ on X such that the closed unit ball $\overline{B}_1(0)$ is the convex hull of a finite set $F \subseteq X$. Since B is compact with respect to $\|\cdot\|$ there is $\kappa > 0$ such that $B \subseteq \overline{B}_\kappa(0)$. Since u is an order unit there is $n \in \mathbb{N}$ such that $\{\kappa x; x \in F\} \leq nu$ and we observe $B \subseteq \overline{B}_\kappa(0) \leq nu$. From Theorem 5.1 we obtain that nu is net catching. \square

6.2 Reflexive Banach spaces with ice cream cones

Example 6.2 Let $(X, \|\cdot\|)$ be a reflexive Banach space. Let f be a linear functional on X such that $\sup\{|f(x)|/\|x\|; x \in X\} = 1$ and $\varepsilon \in (0, 1)$. The *ice cream cone* with parameters f and ε is the cone

$$K_{f,\varepsilon} := \{x \in X; f(x) \geq \varepsilon\|x\|\}.$$

A reflexive Banach space X equipped with an ice cream cone becomes an Archimedean and directed partially ordered vector space and $K_{f,\varepsilon}$ contains $\|\cdot\|$ -interior points [5, Theorem 2.52 and Lemma 2.3]. We prove next that in this case $\tau_o = \tau_{ob}$ is the norm topology of X . A net is order convergent, if and only if it is $\|\cdot\|$ -convergent. A point in X is a $\|\cdot\|$ -interior point of the cone, iff it is an order unit, iff it is net catching.

Proof Note that the statements about order convergence and the interior of the cone follow from the first statement, Proposition 4.7 and Corollary 4.13. We show first that X (equipped with $K_{f,\varepsilon}$) contains net catching elements. Recall from [5, Theorem 2.52] that f is strictly positive and that $K_{f,\varepsilon}$ is $\|\cdot\|$ -closed.

In order to apply Theorem 5.1 we consider the cone base $B := \{x \in K_{f,\varepsilon}; f(x) = 1\}$ and the weak topology

τ of X . As X is a reflexive Banach space we know from the Banach-Alaoglu theorem that the closed ball $\overline{B}_\varepsilon^{\|\cdot\|}(0)$ is τ -compact. From $\sup\{|f(x)|/\|x\|; x \in X\} = 1$ we obtain that f is continuous with respect to $\|\cdot\|$. Thus f is continuous with respect to τ and $B = \overline{B}_\varepsilon^{\|\cdot\|}(0) \cap f^{-1}(\{1\})$ is τ -compact. Note furthermore that $K_{f,\varepsilon}$ is τ -closed as a $\|\cdot\|$ -closed and convex set. To show that B has upper bounds we consider a $\|\cdot\|$ -interior point x of $K_{f,\varepsilon}$. There exists $\kappa > 0$ such that $\overline{B}_\kappa^{\|\cdot\|}(x) \subseteq K_{f,\varepsilon}$. We define

$y := (1/\kappa)x$. For $b \in B = \overline{B}_\varepsilon^{\|\cdot\|}(0) \cap f^{-1}(\{1\})$ we compute

$$\|x - \kappa(y - b)\| = \|\kappa b\| = \kappa \|b\| \leq \kappa$$

and obtain $\kappa(y - b) \in K_{f,\varepsilon}$. Since $K_{f,\varepsilon}$ is a cone we obtain $y - b \in K_{f,\varepsilon}$, i.e. $b \leq y$. Thus y is an upper bound of B and Theorem 5.1 yields that y is a net catching element. In particular, we observe that $x = \kappa y$ is net catching.

Recall that X is Archimedean and directed. Thus by Theorem 4.12 there holds $\tau_o = \tau_{ob}$ and x is an order unit in X . It remains to show that $\|\cdot\|$ and $\|\cdot\|_x$ are equivalent norms, which we will do by showing

$$\overline{B}_\kappa^{\|\cdot\|}(0) \subseteq \overline{B}_1^{\|\cdot\|_x}(0) \subseteq \overline{B}_\lambda^{\|\cdot\|}(0),$$

where we denote $\lambda := (2/\varepsilon)f(x) + \|x\|$. For $z \in \overline{B}_\kappa^{\|\cdot\|}(0)$ we obtain $x + z \in \overline{B}_\kappa^{\|\cdot\|}(x) \subseteq K_{f,\varepsilon}$, i.e. $-x \leq z$ and similarly $z \leq x$. This shows $\overline{B}_\kappa^{\|\cdot\|}(0) \subseteq [-x, x] = \overline{B}_1^{\|\cdot\|_x}(0)$. To show the second inclusion we consider $z \in \overline{B}_1^{\|\cdot\|_x}(0) = [-x, x]$. As f is positive we know $f(z) \leq f(x)$. From $z + x \in K_{f,\varepsilon}$ we thus observe

$$\varepsilon\|z + x\| \leq f(z + x) = f(z) + f(x) \leq 2f(x).$$

Computing $\|z\| \leq \|z + x\| + \|x\| \leq \lambda$ we obtain $z \in \overline{B}_\lambda^{\|\cdot\|}(0)$. □

6.3 Spaces of continuous functions on extremely disconnected compact hausdorff spaces

Example 6.3 Let Ω be an extremely disconnected, compact and infinite Hausdorff space. Consider the Dedekind complete vector lattice $C(\Omega)$ of all continuous maps $f : \Omega \rightarrow \mathbb{R}$ equipped with the pointwise order. Any element of $C(\Omega)$ with values in $(0, \infty)$ is an order unit as any $f \in C(\Omega)$ is bounded. Nevertheless, $C(\Omega)$ contains no net catching elements. Thus by Corollary 3.2 and Theorem 4.11 the inclusion $\tau_o(C(\Omega)) \subseteq \tau_{ob}(C(\Omega))$ is valid but strict.

Proof The statement about the order units is standard and it remains to show that $C(\Omega)$ does not contain net catching elements. By Lemma 4.1 this can be achieved by showing that the order unit $u \in C(\Omega)$ defined by $u(\omega) := 1/2$ for all $\omega \in \Omega$ is not net catching.

As Ω is infinite and compact there exists $\delta \in \Omega$ such that $\{\delta\}$ is not open. Let A be the set of all open and closed subsets $M \subseteq \Omega$ that contain δ . We order A by reversed inclusion and obtain a directed set. For $M \in A$ we denote the characteristic function of M by χ_M and obtain $(\chi_M)_{M \in A}$ to be a decreasing net in $C(\Omega)_+$. As for no $M \in A$ we have $\chi_M \leq u$ it remains to show that there holds $\chi_M \downarrow 0$.

For this it suffices to show that any lower bound x of $\{\chi_M; M \in A\}$ satisfies $x \leq 0$. Consider $\omega \in \Omega \setminus \{\delta\}$. Since Ω is Hausdorff there are disjoint and open sets $U, O \subseteq \Omega$ with $\delta \in U$ and $\omega \in O$. Let M be the closure of U and note that M is open by the extreme disconnectedness of Ω . From $\delta \in M$ we observe that we have $M \in A$. As

$\omega \notin M$ we compute $x(\omega) \leq \chi_M(\omega) = 0$. If $x(\delta) > 0$, then the continuity of x implies $\{\delta\} = x^{-1}((0, \infty))$ to be open, a contradiction. We observe $x \leq 0$, which proves $\chi_M \downarrow 0$. □

6.4 Archimedean vector lattices

In Example 6.3 we have seen that an infinite dimensional Archimedean vector lattice might not contain net catching elements. We show next that no infinite dimensional Archimedean vector lattice contains net catching elements.

Theorem 6.4 *An Archimedean vector lattice contains net catching elements, if and only if it is finite dimensional.*

Recall from Corollary 3.2 that whenever an Archimedean vector lattice contains order units, then $\tau_o \subseteq \tau_{ob}$. As a direct consequence of Proposition 4.7 and Theorem 6.4 we thus observe the following.

Corollary 6.5 *For an Archimedean vector lattice that contains order units the inclusion $\tau_o \subseteq \tau_{ob}$ is valid but strict.*

Remark 6.6 Note that Example 4.10 provides an infinite dimensional vector lattice that contains net catching elements and for which the inclusion $\tau_{ob} \subseteq \tau_o$ is strict. Clearly this vector lattice is not Archimedean. In Example 6.2 we have seen that there are infinite dimensional Archimedean partially ordered vector spaces that contain net catching elements and satisfy $\tau_o = \tau_{ob}$. Clearly such partially ordered vector spaces are no vector lattices.

We will prove Theorem 6.4 by embedding X into a space of the form discussed in Example 6.3. We thus investigate next how net catching elements behave under embedding.

Lemma 6.7 *Let X be an Archimedean vector lattice and Y be a partially ordered vector space. Let $\iota: X \rightarrow Y$ be a linear order dense embedding. An element $x \in X$ is net catching in X , if and only if $\iota(x)$ is net catching in Y .*

Proof Assume that x is net catching w.r.t. X and consider a net $(y_\alpha)_{\alpha \in A}$ with $y_\alpha \downarrow 0$. Let B be the set of all $\beta \in X_+$ for which there exists $\alpha \in A$ with $\iota(\beta) \geq y_\alpha$. Using that ι preserves infima it is straight forward to show that for $\beta, \beta' \in B$ also the infimum $\beta \wedge \beta'$ is contained in B . Thus B equipped with the reversed order of X is directed. We define $x_\beta := \beta$ for $\beta \in B$ to obtain a net $(x_\beta)_{\beta \in B}$ in X that satisfies $x_\beta \downarrow$.

We next show that this net satisfies $x_\beta \downarrow 0$. Obviously, we have $0 \leq x_\beta$ for all $\beta \in B$. Let $z \in X$ with $z \leq x_\beta$ for all $\beta \in B$ and consider $\alpha \in A$. For $y \in \iota(X)$ with $y \geq y_\alpha$ there exists $\beta \in X$ with $\iota(\beta) = y$ and we observe $\beta \in B$. Thus $\iota(z) \leq \iota(y_\beta) = \iota(\beta) = y$. As $\iota(X)$ is order dense in Y we obtain

$$\iota(z) \leq \inf\{y \in \iota(X); y \geq y_\alpha\} = y_\alpha.$$

From the arbitrary choice of $\alpha \in A$ and $y_\alpha \downarrow 0$ we get $\iota(z) \leq 0$. As ι is order reflecting we observe $z \leq 0$, which proves $x_\beta \downarrow 0$.

Now recall that x is net catching. We therefore obtain the existence of $\beta \in B$ with $\beta = x_\beta \leq x$. From the definition of B we thus get the existence of $\alpha \in A$ with $x_\alpha \leq \iota(\beta) \leq \iota(x)$ and observe $\iota(x)$ to be net catching in Y .

To show the converse assume that $\iota(x)$ is net catching and consider a net $(x_\alpha)_{\alpha \in A}$ in X with $x_\alpha \downarrow 0$. As any linear order dense embedding is order continuous we obtain $\iota(x_\alpha) \downarrow 0$. Thus there exists $\alpha \in A$ with $\iota(x_\alpha) \leq \iota(x)$. Since ι is order reflecting we obtain x to be net catching. \square

Remark 6.8 Note that there exist infinite dimensional Archimedean partially ordered vector spaces X with net catching elements (Example 6.2). Considering the Dedekind completion X^δ and a linear order dense embedding $\iota: X \rightarrow X^\delta$ we obtain from Theorem 6.4 that $\iota(x)$ is not net catching for all net catching elements $x \in X$.

Proof of Theorem 6.4 Let us assume that X is finite dimensional. Then X contains net catching elements as presented in Example 6.1.

To show the converse assume that X is an Archimedean vector lattice that contains net catching elements. As X is Archimedean and directed we obtain from Lemma 4.1(ii) that X contains an order unit u . Let $\iota: X \rightarrow X^\delta$ be a Dedekind completion of X . Then X^δ contains the order unit $\iota(u)$ and by Theorem 2.2 we assume w.l.o.g. that X^δ is of the form $C(\Omega)$ for a compact and extremely disconnected Hausdorff space Ω . As ι is a linear order dense embedding we obtain from Lemma 6.7 that $C(\Omega)$ contains net catching elements and our considerations in Example 6.3 imply Ω to be finite. Thus $C(\Omega)$ is finite dimensional. As any linear order embedding is injective we obtain that X is finite dimensional as well. \square

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