2-Local derivations of real AW*-algebras are derivation

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Abstract

2-Local derivations on real matrix algebras over unital semi-prime Banach algebras are considered. Using the real analogue of the result that any 2-local derivation on the algebra $M_{2^n}(A)$ ($n \geq 2$) is a derivation, it is shown that any 2-local derivation on real AW*-algebra for which the enveloping algebra is (complex) AW*-algebra, is a derivation, where A is a unital semi-prime Banach algebra with the inner derivation property.

Keywords Matrix algebra · Real AW*-algebra · Derivation · Inner derivation property · 2-local derivation

Mathematics Subject Classification 46L10, 46L37, 46L57, 47B47, 47C15

1 Introduction

Given an algebra $\mathcal{A}$, a linear operator $D : \mathcal{A} \rightarrow \mathcal{A}$ is called a *derivation*, if $D(xy) = D(x)y + xD(y)$, for all $x, y \in \mathcal{A}$. Each element $a \in \mathcal{A}$ implements a derivation D_a on $\mathcal{A}$ defined as $D_a(x) = [a, x] = ax - xa$, $x \in \mathcal{A}$. Such derivations are said to be *inner derivations*. A map $\Delta : \mathcal{A} \rightarrow \mathcal{A}$ (not linear in general) is called a *2-local derivation*, if for every $x, y \in \mathcal{A}$, there exists a derivation $D_{x,y} : \mathcal{A} \rightarrow \mathcal{A}$ such that $\Delta(x) = D_{x,y}(x)$ and $\Delta(y) = D_{x,y}(y)$.

In the paper [1] P.Semrl introduced the notion of 2-local derivations and described 2-local derivations on the algebra $B(H)$ of all bounded linear operators on the infinite-

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dimensional separable (complex) Hilbert space H . A similar description for the finite-dimensional case appeared later in [2]. In the papers [3–5] and [6] the authors extended the Semrl's result for arbitrary finite, semi-finite and purely infinite von Neumann algebras, respectively. The real analogue of Semrl's result is received in the paper [7], i.e. it is described 2-local derivations on the real W^* -algebra $B(H)$ of all bounded linear operators on the infinite-dimensional separable real Hilbert space H .

In the paper [8] the authors investigated 2-local derivations on matrix algebras over unital semi-prime Banach algebras. For a unital semi-prime Banach algebra A with the inner derivation property it is proved that any 2-local derivation on the algebra $M_{2^n}(A)$, $n \geq 2$, is a derivation. They apply this result to AW^* -algebras and it is showed that any 2-local derivation on an arbitrary (complex) AW^* -algebra is a derivation. In the present paper step by step we will prove analogous result for real AW^* -algebras.

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2 Preliminaries

Let $B(H)$ be the algebra of all bounded linear operators on a complex Hilbert space H . A weakly closed $*$ -subalgebra M containing the identity operator $\mathbf{1}$ in $B(H)$ is called a W^* -algebra. A real $*$ -subalgebra $R \subset B(H)$ is called a *real W^* -algebra* if it is closed in the weak operator topology, $\mathbf{1} \in R$ and $R \cap iR = \{0\}$. A real W^* -algebra R is called a *real factor* if its center $Z(R)$ consists of the elements $\{\lambda \mathbf{1}, \lambda \in \mathbb{R}\}$. We say that a real W^* -algebra R is of the type I_{fin} , I_∞ , Π_1 , Π_∞ , or III_λ , ($0 \leq \lambda \leq 1$) if the enveloping W^* -algebra $R + iR$ has the corresponding type in the ordinary classification of W^* -algebras.

We say that an algebra A has the *inner derivation property* if every derivation on A is inner. Recall that an algebra A is said to be *semi-prime* if $aAa = 0$ implies that $a = 0$.

In work [8] (Theorem 2.1) it is proved, that if A is a unital semi-prime (complex) Banach algebra with the inner derivation property and $M_{2^n}(A)$ is the algebra of $2^n \times 2^n$ -matrices over A , then any 2-local derivation on $M_{2^n}(A)$ is a derivation.

The proof of this theorem without changes to pass for real Banach algebras. Therefore we shall formulate this result in a real case.

Theorem 1 *Let A be a unital semi-prime real Banach algebra with the inner derivation property and let $M_{2^n}(A)$ is the real algebra of $2^n \times 2^n$ -matrices over A . Then any 2-local derivation on $M_{2^n}(A)$ is a derivation.*

We apply Theorem 1 to the description of 2-local derivations on real AW^* -algebras.

3 2-Local derivations on real AW^* -algebras

Firstly, we shall remind some definitions and the facts from the theory of complex and real AW^* -algebras. Let A be a real or complex $*$ -algebra and let S be a nonempty subset of A . Put

$$R(S) = \{x \in A \mid sx = 0 \text{ for all } s \in S\}$$

and call $R(S)$ the *right-annihilator* of S . Similarly

$$L(S) = \{x \in A \mid xs = 0 \text{ for all } s \in S\}$$

denotes the *left-annihilator* of S . Following [12] we introduce the following notions.

Definition 1 A $*$ -algebra A is called a *Baer $*$ -algebra* if for any nonempty $S \subset A$, $R(S) = gA$ for an appropriate projection g .

Since $L(S) = (R(S^*))^* = (hA)^* = Ah$ the definition is symmetric and can be given in terms of the left-annihilator and a suitable projection h . Here $S^* = \{s^* \mid s \in S\}$. In the particular case, where we consider only one point sets $S = \{x\}$, $x \in A$, we obtain the more general definition of a *Rickart $*$ -algebra*. It is known that a Rickart $*$ -algebra is a Baer $*$ -algebra if and only if its projections form a complete lattice or every orthogonal family of projections has a supremum (i.e. a least upper bound).

Let A be a Banach $*$ -algebra over the field $\mathbb{C}$. The algebra A is called a *C*-algebra*, if $\|aa^*\| = \|a\|^2$ for any $a \in A$. A real Banach $*$ -algebra R is called a *real C*-algebra*, if $\|aa^*\| = \|a\|^2$ and an element $\mathbf{1} + aa^*$ is invertible for any $a \in R$. It is easy to see that R is a real C*-algebra if and only if a norm on R can be extended onto the complexification $A = R + iR$ of the algebra R so that algebra A is a C*-algebra.

Definition 2 A (complex) C*-algebra M which is a Baer $*$ -algebra is called an *AW*-algebra*.

This definition is equivalent to the definition given by Kaplansky [9], namely a C*-algebra is an AW*-algebra if and only if it satisfies the following conditions:

- (A) In the partially ordered set of projections, any set of orthogonal projections has a supremum;
- (B) Any maximal abelian $*$ -subalgebra is generated by its projections.

Every W*-algebra is, of course, an AW*-algebra, however, the converse is not true as it was shown by Dixmier [10]. Given an AW*-algebra M , its *center* is $Z_M = \{x \in M \mid xy = yx \text{ for all } y \in M\}$. An AW*-algebra is called an *AW*-factor*, if its center consists of complex multiples of the identity $\mathbf{1}$, i.e. $Z_M = \{\lambda\mathbf{1} \mid \lambda \in \mathbb{C}\}$.

Now following Kaplansky [11, Appendix III] we introduce the main subject of the paper.

Definition 3 A real C*-algebra which is a Baer $*$ -ring is called a *real AW*-algebra*.

It is clear that any real AW*-algebra contains an identity $\mathbf{1}$, and we say that a real AW*-algebra A is a *real AW*-factor* if its center consists of real multiples of $\mathbf{1}$.

Remark 1 1. Unlike the complex case in the real case it is not possible to give a definition in terms of conditions (A) and (B) above, because in maximal abelian $*$ -subalgebras skew-hermitian elements can not be generated by projection.

2. A slightly more general notion of real AW*-algebras was given also by Berberian [12, p. 26, Exercise 14A]. He defined a *real AW*-algebra* as a Banach *-algebra over the field of real numbers such that $\|x^*x\| = \|x\|^2$ for all $x \in A$ and such that A is a Baer *-ring. In this case the field $\mathbb{C}$ of complex numbers with the identical involution $x^* = x$ becomes a real AW*-algebra, but it is not a real C*-algebra, because it is not a *symmetric* *-algebra, which means $\mathbf{1} + x^*x$ is invertible for any $x \in A$.

Any real W*-algebra (real W*-factor) is a real AW*-algebra (resp. a real AW*-factor). But the converse is not true. Any complex AW*-algebra is a real AW*-algebra. Complex AW*-factors are not real AW*-factors, because their centers are complex multiples of $\mathbf{1}$.

For real C*-algebras and W*-algebras we know that their complexification are C*- and W*-algebras respectively. But in AW*-algebras case we have: there is a real AW*-algebra R such that the complex C*-algebra $R + iR$ is not an AW*-algebra (see [13, Proposition 4.2.3]).

Now, we shall prove the main result of paper.

Theorem 2 *Let R be an arbitrary real AW*-algebra and suppose that its complexification $M = R + iR$ is a (complex) AW*-algebra. Then any 2-local derivation Δ on R is a derivation.*

Proof Let us first note that any (complex) AW*-algebra is semi-prime, and it is clear that a real algebra A is semi-prime if and only if its complexification $A + iA$ is semi-prime. Therefore, any real AW*-algebra is also semi-prime. It is also known [14, Theorem 2] that AW*-algebra has the inner derivation property.

It is easy to shown that any real AW*-algebra has also the inner derivation property, i.e. every derivation of real AW*-algebra is an inner. Indeed, let R be a real AW*-algebra and let $D : R \rightarrow R$ be a derivation. D can be extended by the linearity to a derivation on $M = R + iR$ as $\overline{D}(x + iy) = D(x) + iD(y)$. Since $\overline{D}$ is an inner there is an element $z = a + ib$ ($a, b \in R$) such that $\overline{D}(x + iy) = [z, x + iy]$ for all $x, y \in R$. Hence $D(x) = \overline{D}(x) = [z, x] = [a + ib, x] = [a, x] + i[b, x]$. Therefore from $D(x) \in R$ we have $b = 0$, i.e. $z = a \in R$. Thus $D(x) = [a, x]$.

Now, let z be a central projection in R . Then z is a central projection in M . It is known that $\overline{D}(z) = 0$, and therefore $D(z) = 0$. Then it is easy to see that $\Delta(z) = 0$ for any 2-local derivation Δ on R . For $x \in R$ we consider the elements x and zx . Then there is a derivation D on R such that $\Delta(zx) = D(zx)$ and $\Delta(x) = D(x)$. Then we have

$$\Delta(zx) = D(zx) = D(z)x + zD(x) = zD(x) = z\Delta(x).$$

It means that every 2-local derivation Δ maps zR into zR for each central projection $z \in R$. Thus we may consider the restriction of Δ onto eR . By [13, Proposition 4.4.3] an arbitrary real AW*-algebra can be decomposed along a central projection into the direct sum of an abelian real AW*-algebra, and real AW*-algebras of type I_n , $n \geq 2$, type I_∞ , type II and type III. We will consider these cases separately.

Let R be an abelian real AW^* -algebra. It is well-known that any derivation on an abelian (complex) W^* -algebra $R + iR$ is identically zero. Therefore, the derivation $\overline{D}(x + iy) = D(x) + iD(y)$ on $R + iR$ is identically zero, where D is a derivation on R . Hence D is identically zero, i.e. any 2-local derivation on an abelian AW^* -algebra is also identically zero.

If R is a real AW^* -algebra of type I_n , $n \geq 2$, with the center $Z(R)$, then it is isomorphic to the algebra $M_n(Z(R))$. By Lemma 2.3 [8] (as it is already told above, that the proof of theorem 2.1 and Lemmas from [8] without changes to pass for real Banach algebras) there exists a derivation D on $R \equiv M_n(Z(R))$ such that $\Delta \equiv D$. So, Δ is a derivation.

Let the real AW^* -algebra R have one of the types I_∞ , II or III. Then using the methods developed in [13, §§4.3-4.7] and similarly following the scheme of the proof of Lemmas 4.5 and 4.12 in [9], the algebra R can be represented as a sum of mutually equivalent orthogonal projections e_1, e_2, e_3, e_4 from R . Then the map $x \rightarrow \sum_{i,j=1}^4 e_i x e_j$ defines an isomorphism between the algebra R and the matrix algebra $M_4(Q)$, where $Q = e_{1,1} R e_{1,1}$. It is easy to see that Q and $Q + iQ$ are real and complex C^* -algebras, respectively. Since $M = R + iR$ is AW^* -algebra by [12, Proposition 8 (iii), 23p.] C^* -algebra eMe is also AW^* -algebra, where e is an arbitrary projection in M . Then by [13, Proposition 4.3.1] a real C^* -algebra $Q = e_{1,1} R e_{1,1}$ is a real AW^* -algebra and its complexification $Q + iQ = e_{1,1} M e_{1,1}$ is also a (complex) AW^* -algebra. Therefore Q is a unital semi-prime real Banach algebra with the inner derivation property. Hence Theorem 1 implies that any 2-local derivation on R is a derivation. $\square$

Remark 2 Everywhere in the work we considered a real AW^* -algebra with a (complex) AW^* -algebra of its complexification. Moreover, in the definition of real C^* -algebra the condition of convertibility of an element $\mathbf{1} + xx^*$ (for all x) is required. It is equivalent to that a norm on real C^* -algebra can be extended onto its complexification so that it is a (complex) C^* -algebra. But in the [12, Exercise 14A] in definition of real C^* -algebra convertibility of $\mathbf{1} + xx^*$ is not required. In this connection we shall formulate following questions.

Question let R be a real Baer * -ring. Suppose that

- (i) R is a real Banach * -algebra with $\|xx^*\| = \|x\|^2$, for any $x \in R$, or
- (ii) R is a real AW^* -algebra (not necessary its complexification is a (complex) AW^* -algebra). Then
 - (1) is any derivation of R is inner?
 - (2) is any 2-local derivation on R is derivation?

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