Kadec–Klee properties of Orlicz–Lorentz sequence spaces equipped with the Orlicz norm

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Abstract

The necessary and sufficient conditions for both the Kadec–Klee property as well as the Kadec–Klee property with respect to the coordinatewise convergence in Orlicz–Lorentz sequence spaces equipped with the Orlicz norm and generated by arbitrary Orlicz functions as well as any non-increasing weight sequences are given. Moreover, for their subspaces of elements with an order continuous norm the full characterization of the Kadec–Klee property with respect to the coordinatewise convergence is presented. Some tools useful in the proofs of the main results are also provided.

Keywords Orlicz–Lorentz sequence spaces · Subspace of order continuous elements · Orlicz norm · Kadec–Klee property · Kadec–Klee property with respect to the coordinatewise convergence

Mathematics Subject Classification 46B20 · 46B45 · 46A45 · 46A80 · 46B42

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1 Introduction

Throughout this paper $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ denote the sets of reals, nonnegative reals and natural numbers, respectively. By $S(X)$ and $B(X)$ denote the unit sphere and the unit ball of a Banach space $X = (X, \|\cdot\|)$, respectively.

The triple $(\mathbb{N}, 2^{\mathbb{N}}, m)$ stands for the counting measure space while $\ell^0 = \ell^0(m)$ denotes the space of all real sequences $x : \mathbb{N} \rightarrow \mathbb{R}$. For every $x = (x(i))_{i=1}^{\infty} \in \ell^0$, let $\text{supp}(x) := \{i \in \mathbb{N} : x(i) \neq 0\}$ and $|x|(i) := |x(i)|$ for all $i \in \mathbb{N}$, that is, $|x|$ denotes the absolute value of x .

Given any $x \in \ell^0$ we define its distribution function $\mu_x : [0, +\infty) \rightarrow \{0, \infty\} \cup \mathbb{N}$ by

$$\mu_x(\lambda) = m(\{i \in \mathbb{N} : |x(i)| > \lambda\}),$$

where $\lambda \geq 0$, m is the counting measure on $2^{\mathbb{N}}$ (see [2,21,23]) and its non-increasing rearrangement $x^* = (x^*(i))_{i=1}^{\infty}$ as

$$x^*(i) = \inf\{\lambda \geq 0 : \mu_x(\lambda) < i\}$$

for all $i \in \mathbb{N}$ (under the convention $\inf \emptyset = \infty$). We say that two sequences $x, y \in \ell^0$ are equimeasurable if $\mu_x(\lambda) = \mu_y(\lambda)$ for all $\lambda \geq 0$. It is obvious that equimeasurability of x and y gives the equality $x^* = y^*$.

In the whole paper Φ denotes an Orlicz function (see [4,25,28]), that is, $\Phi : [-\infty, \infty] \rightarrow [0, \infty]$ (our definition is extended from R into $R^e := [-\infty, \infty]$ by assuming $\Phi(-\infty) = \Phi(\infty) = \infty$) and Φ is convex, even, vanishing and continuous at zero, left continuous on $(0, \infty)$ and not identically equal to zero on $(-\infty, \infty)$. Let us denote

$$\begin{aligned} a_{\Phi} &= \sup\{u \geq 0 : \Phi(u) = 0\}, \\ b_{\Phi} &= \sup\{u \geq 0 : \Phi(u) < \infty\}. \end{aligned}$$

Note that the left continuity of Φ on $(0, \infty)$ is equivalent to the fact that $\lim_{u \rightarrow (b_{\Phi})^-} \Phi(u) = \Phi(b_{\Phi})$.

We say that an Orlicz function Φ satisfies condition δ_2 ($\Phi \in \delta_2$ for short) if there exist constants $u_0 > 0$ with $\Phi(u_0) > 0$ and $K > 0$ such that $\Phi(2u) \leq K\Phi(u)$ for any $u \in [0, u_0]$.

For any Orlicz function Φ we define its complementary (in the sense of Young) function Ψ by the formula

$$\Psi(u) = \sup_{v>0}\{|u|v - \Phi(v)\}$$

for all $u \in \mathbb{R}$.

Let $\omega : \mathbb{N} \rightarrow \mathbb{R}_+$ be a non-increasing sequence, called a weight sequence. In the whole paper we will assume that $\omega(1) > 0$.

Recall that Orlicz–Lorentz spaces, which are a natural generalization of Orlicz and Lorentz spaces, were introduced in the early nineties of the twentieth century

[18,24,27]. These investigations gave an impulse to further study of the spaces, results of which have been published among others in papers [1,3,7,10–15,19,22,33].

Let us define a modular $I_{\Phi,\omega} : \ell^0 \rightarrow [0, \infty]$ by formula

$$I_{\Phi,\omega}(x) = \sum_{i=1}^{\infty} \Phi(x^*(i))\omega(i)$$

and the Orlicz–Lorentz sequence space

$$\lambda_{\Phi,\omega} = \{x \in \ell^0 : I_{\Phi,\omega}(\beta x) < \infty \text{ for some } \beta > 0\}.$$

In most of the previous papers one studied Orlicz–Lorentz spaces equipped with the Luxemburg norm

$$\|x\|_{\Phi,\omega} = \inf \left\{ \lambda > 0 : I_{\Phi,\omega} \left(\frac{x}{\lambda} \right) \leq 1 \right\}.$$

It is easy to show that if $\sum_{i=1}^{\infty} \omega(i) < \infty$ or $a_{\varphi} > 0$, then $\lambda_{\Phi,\omega} = l^{\infty}$ with equivalent norms. Otherwise, we have $\lambda_{\Phi,\omega} \hookrightarrow c_0$.

Several years ago the first papers considering different properties in Orlicz–Lorentz spaces endowed with the Orlicz norm defined as

$$\|x\|_{\Phi,\omega}^O = \sup \left\{ \sum_{n=1}^{\infty} x^*(i) y^*(i) \omega(i) : I_{\Psi,\omega}(y) \leq 1 \right\}$$

appeared (see [15,33] and also [7]). In [8] it was shown that for arbitrary Orlicz function Φ and any non-increasing weight function ω , equality

$$\|x\|_{\Phi,\omega}^O = \|x\|_{\Phi,\omega}^A$$

holds for any $x \in \lambda_{\Phi,\omega}$, where the Amemiya norm is defined by the formula

$$\|x\|_{\Phi,\omega}^A = \inf_{k>0} \frac{1}{k} \{1 + I_{\Phi,\omega}(kx)\}. \quad (1)$$

Since $\|x\|_{\Phi,\omega} = \inf_{k>0} \{\frac{1}{k} \max(1, I_{\Phi,\omega}(kx))\}$ for any $x \in \lambda_{\Phi,\omega}$ (see [8]), the both norms, that is, the Luxemburg and the Orlicz norms are equivalent. More precisely, for any $x \in \lambda_{\Phi,\omega}$ we have $\|x\|_{\Phi,\omega} \leq \|x\|_{\Phi,\omega}^O \leq 2\|x\|_{\Phi,\omega}$. It is well known that both $(\lambda_{\Phi,\omega}, \|\cdot\|_{\Phi,\omega})$ and $(\lambda_{\Phi,\omega}, \|\cdot\|_{\Phi,\omega}^O)$ are symmetric Banach spaces (see [8,18,24]).

A natural question about the attainability by any $x \in \lambda_{\Phi,\omega} \setminus \{0\}$ of the infimum in formula (1) arises, that is, whether there is $k = k(x) > 0$ for which equality

$$\|x\|_{\Phi,\omega}^O = \|x\|_{\Phi,\omega}^A = \frac{1}{k} \{1 + I_{\Phi,\omega}(kx)\}$$

holds. In order to answer this question, let us define for any $x \in \lambda_{\Phi, \omega} \setminus \{0\}$ the following constants

$$\begin{aligned}\lambda_{\infty} &= \lambda_{\infty}(x) = \sup\{\lambda > 0: I_{\Phi, \omega}(\lambda x) < \infty\}, \\ k^* &= k^*(x) = \inf\{k > 0: I_{\Psi, \omega}(p(kx^*)) \geq 1\}, \\ k^{**} &= k^{**}(x) = \sup\{k > 0: I_{\Psi, \omega}(p(kx^*)) \leq 1\},\end{aligned}$$

where p is the right-hand side derivative of Φ . Define the function f on $(0, \infty)$,

$$f(k) = f_x(k) = \frac{1}{k} (1 + I_{\Phi, \omega}(kx)). \quad (2)$$

The function f is continuous on the interval $(0, \lambda_{\infty})$ and left-continuous at λ_{∞} provided that $\lambda_{\infty} < \infty$. We also have $0 < k^* \leq k^{**} \leq \lambda_{\infty} \leq \infty$. Moreover, as it was shown in [8], the function f is strictly decreasing on the interval $(0, k^*)$, strictly increasing on the interval $(k^{**}, \lambda_{\infty})$ if $k^{**} < \lambda_{\infty}$ as well as constant and equal to $\|x\|_{\Phi, \omega}^O$ on the interval $[k^*, k^{**}]$ if $k^* < k^{**}$. Therefore, assuming for any $x \in \lambda_{\Phi, \omega} \setminus \{0\}$ that

$$K(x) = \begin{cases} [k^*, k^{**}] & \text{if } k^{**} < \infty \\ [k^*, \infty) & \text{if } k^* < k^{**} = \infty \\ \emptyset & \text{if } k^* = \infty, \end{cases} \quad (3)$$

we get that if $k^* < \infty$, then $\|x\|_{\Phi, \omega}^O = \frac{1}{k} \{1 + I_{\Phi, \omega}(kx)\}$ for any $k \in K(x)$ and $\|x\|_{\Phi, \omega}^O = \lim_{k \rightarrow \infty} \frac{1}{k} \{1 + I_{\Phi, \omega}(kx)\}$ if $k^* = \infty$ (see [8]).

In the further part of this paper we will need to use the following

Remark 1 Note that if $b_{\Phi} < \infty$, then for any $x \in \lambda_{\Phi, \omega} \setminus \{0\}$, we have that $k^{**} \leq \lambda_{\infty} \leq \frac{b_{\Phi}}{x^*(1)}$. In the case of $b_{\Phi} = \infty$ and $\lim_{u \rightarrow \infty} \frac{\Phi(u)}{u} = \lim_{u \rightarrow \infty} \frac{1}{u} \int_0^u p(s) ds = \lim_{u \rightarrow \infty} p(u) = \infty$, we have also that $k^{**} < \infty$ for any $x \in \lambda_{\Phi, \omega} \setminus \{0\}$. Indeed, if $\lambda_{\infty} = \infty$, then there exists $k_x > 0$ such that $\Psi(p(k_x x^*(1)))\omega(1) > 1$ and $k^{**} \leq k_x$.

Now, assume that $\lim_{u \rightarrow \infty} \frac{\Phi(u)}{u} = B < \infty$ and let $x \in \lambda_{\Phi, \omega} \setminus \{0\}$. First, we will show that if $\Psi(B) \sum_{i=1}^{m(\text{supp } x)} \omega(i) > 1$, then $k^{**} < \infty$. In order to do it, let $n_x \in \mathbb{N}$ be such that

$$\Psi(B) \sum_{i=1}^{n_x} \omega(i) > 1. \quad (4)$$

In the case when $m(\text{supp } x) < \infty$, we take $n_x = m(\text{supp } x)$ and in the case when $m(\text{supp } x) = \infty$, we can find $n_x \in \mathbb{N}$, for which inequality (4) holds. Since $\lim_{u \rightarrow \infty} p(u) = B$, there exists $u_x > 0$ for which $\Psi(p(u_x)) \sum_{i=1}^{n_x} \omega(i) > 1$. Defining $k_x = \frac{u_x}{x^*(n_x)}$, we obtain that $I_{\Psi, \omega}(p(k_x x^*)) > 1$, so $k^{**} \leq k_x$.

In the case of $\Psi(B) \sum_{i=1}^{m(\text{supp } x)} \omega(i) < 1$ or $\Psi(B) \sum_{i=1}^{m(\text{supp } x)} \omega(i) = 1$ and $p(u) < B$ for any $u > 0$, we get that $k^* = \infty$, which means $K(x) = \emptyset$.

Finally, assume that $\Psi(B) \sum_{i=1}^{m(\text{supp } x)} \omega(i) = 1$ and $p(u) = B$ starting from some $u_0 > 0$. If $m(\text{supp } x) < \infty$, then $k^* \leq \frac{u_0}{x^*(n_x)} < k^{**} = \infty$, where as above $n_x = m(\text{supp } x)$. Let now $m(\text{supp } x) = \infty$. If $i_0 = \sup\{i \in \mathbb{N}: \omega(i) > 0\} < \infty$, then $k^* \leq$

$\frac{u_0}{x^*(i_0)} < k^{**} = \infty$. However, if $\omega(i) > 0$ for any $i \in \mathbb{N}$, then $k^* \leq \frac{u_0}{x^*(\infty)} < k^{**} = \infty$, whenever $x^*(\infty) = \lim_{i \rightarrow \infty} x^*(i) > 0$, and $k^* = \infty$ otherwise.

Let $(E, \leq, \|\cdot\|_E)$ be a Banach sequence lattice. An element $x \in E$ is said to be order continuous if for any sequence (x_n) in E_+ (the positive cone of E) such that $x_n \leq |x|$ for any $n \in \mathbb{N}$ and $x_n \rightarrow 0$ coordinatewisely, there holds $\|x_n\|_E \rightarrow 0$ as $n \rightarrow \infty$. The subspace E_a of all order continuous elements in E is an order ideal in E . Space E is called order continuous if $E_a = E$.

Since the Luxemburg and the Orlicz norms are equivalent, the subspace of order continuous elements for both these norms is the same (as a subset of the elements of the space $\lambda_{\Phi, \omega}$). We will denote it as $(\lambda_{\Phi, \omega})_a$. The following theorem holds true:

Theorem 1 *The space $(\lambda_{\Phi, \omega}, \|\cdot\|_{\Phi, \omega}^O)$ is order continuous if and only if $\sum_{i=1}^{\infty} \omega(i) = \infty$ and $\Phi \in \delta_2$.*

This theorem is the consequence of the fact that the space $(\lambda_{\Phi, \omega}, \|\cdot\|_{\Phi, \omega})$ (with the Luxemburg norm) is order continuous if and only if $\sum_{i=1}^{\infty} \omega(i) = \infty$ and $\Phi \in \delta_2$ (cf. [18, Theorem 2.4]) as well as the equivalence of the Luxemburg and the Orlicz norms.

A Banach space $X = (X, \|\cdot\|)$ is said to have the Kadec–Klee property or property H (resp. the Kadec–Klee property with respect to the coordinatewise convergence or property H_c) if for any sequence $(x_n)_{n=1}^{\infty}$ in X and $x \in X$ such that $\lim_{n \rightarrow \infty} \|x_n\| = \|x\|$, we have $\|x_n - x\| \rightarrow 0$ provided $x_n \rightarrow x$ weakly as $n \rightarrow \infty$ (resp. provided $x_n(i) \rightarrow x(i)$ coordinatewisely). Note that this property was originally considered by Radon [29] and next by Riesz [30, 31], where it was proved that L^p spaces ($1 < p < \infty$) have property H , while $L^1[0, 1]$ has not. The newer results concerning the Kadec–Klee properties a reader can find for example in [3, 5, 6, 9, 10, 17, 32].

2 Auxiliary results

Remembering Remark 1, we can formulate the following

Lemma 1 *Let $\lim_{u \rightarrow \infty} \frac{\Phi(u)}{u} = B < \infty$ and $x \in \lambda_{\Phi, \omega}$. If $\Psi(B) \sum_{i=1}^{m(\text{supp } x)} \omega(i) \leq 1$ (equivalently $k^{**} = \infty$), then*

$$\|x\|_{\Phi, \omega}^O = B \sum_{i=1}^{\infty} x^*(i) \omega(i).$$

Moreover, if $\Psi(B) \sum_{i=1}^{\infty} \omega(i) > 1$, then $m(\text{supp } x) < \infty$.

Proof Note that $\Psi(B) < \infty$ and $\Psi(u) = \infty$ for any $u > B$. Hence, for any $y \in \lambda_{\Psi, \omega}$ such that $I_{\Psi, \omega}(y) \leq 1$, we get that $y(i) \leq B$ for any $i \in \mathbb{N}$. Therefore, for the same y , we obtain

$$\sum_{i=1}^{\infty} x^*(i) y^*(i) \omega(i) \leq B \sum_{i=1}^{\infty} x^*(i) \omega(i).$$

Defining

$$z = (\underbrace{B, B, \dots, B}_{m(\text{supp } x) \text{ times}}, 0, 0, \dots),$$

we get that $I_{\Psi, \omega}(z) = \sum_{i=1}^{m(\text{supp } x)} \Psi(B)\omega(i) \leq 1$ and

$$\sum_{i=1}^{m(\text{supp } x)} x^*(i)z^*(i)\omega(i) = B \sum_{i=1}^{m(\text{supp } x)} x^*(i)\omega(i) = B \sum_{i=1}^{\infty} x^*(i)\omega(i).$$

Hence it follows that

$$\|x\|_{\Phi, \omega}^O = B \sum_{i=1}^{\infty} x^*(i)\omega(i).$$

Assume now additionally that $\Psi(B) \sum_{i=1}^{\infty} \omega(i) > 1$. Then, $a_{\Psi} < b_{\Psi} = B$ and $0 < \Psi(B) < \infty$. Hence, there exists $n_0 \in \mathbb{N}$ such that $\Psi(B) \sum_{i=1}^{n_0} \omega(i) > 1$, whence we get that $m(\text{supp } x) < n_0$. $\square$

Let us note the following observation. It is known that if a sequence $(x_n)_{n=1}^{\infty}$ from c_0 is uniformly convergent to x , then $x_n^*(i) \rightarrow x^*(i)$ for all $i \in \mathbb{N}$. Consequently, the sequence $(x_n^*)_{n=1}^{\infty}$ is uniformly convergent to x^* . However, the above implication is not true in general, if the assumption about uniform convergence of $(x_n)_{n=1}^{\infty}$ to x is replaced by the assumption that $(x_n)_{n=1}^{\infty}$ is convergent to x coordinatewisely. This fact can be easily visualized by the following example. Let us take the sequence $(x_n)_{n=1}^{\infty}$ such that $x_n(n) = 1$ and $x_n(i) = 0$ for $i \neq n$, where $n \in \mathbb{N}$. Then $(x_n)_{n=1}^{\infty}$ converges to zero coordinatewisely but $x_n^*(1)$ does not converge to zero. However, Lemma 2 holds. This Lemma is a kind of generalization of Lemma 3.1 from [10]. We will present its proof below just for the completeness of this presentation as well as for the convenience of a reader.

Lemma 2 *Let Φ vanishes only at zero and $\omega(i) > 0$ for any $i \in \mathbb{N}$. Then, for any $x \in c_0$ and any sequence $(x_n)_{n=1}^{\infty}$ from c_0 such that $x_n \rightarrow x$ coordinatewisely and $I_{\Phi, \omega}(x_n) \rightarrow I_{\Phi, \omega}(x) = \alpha < \infty$ as $n \rightarrow \infty$, we obtain that $x_n^*(i) \rightarrow x^*(i)$ for any $i \in \mathbb{N}$. Consequently, $x_n^* \rightarrow x^*$ uniformly.*

Proof Assume that x and a sequence $(x_n)_{n=1}^{\infty}$ satisfy the assumptions of the lemma and that, passing to a subsequence if necessary, there exists $i_0 \in \mathbb{N}$ such that $|x_n^*(i_0) - x^*(i_0)| > \varepsilon$ for some $\varepsilon > 0$ and all $n \in \mathbb{N}$. We will consider now three cases (passing to a subsequence if necessary).

Case 1. Assume that $0 \leq x_n^*(i_0) < x^*(i_0)$ for each $n \in \mathbb{N}$. Then

$$x^*(i_0) > x_n^*(i_0) + \varepsilon \quad (5)$$

for the same n . Note that the set $A = \{i \in \mathbb{N} : |x(i)| > \frac{\varepsilon}{2}\}$ is finite because $x \in c_0$. Let us denote $m_0 = m(A)$, $y = x\chi_A$ and $y_n = x_n\chi_A$ for each $n \in \mathbb{N}$. Since

$y_n \rightarrow y$ uniformly, we get that y_n^* tends to y^* coordinatewisely (so uniformly as well). Simultaneously, $x^*(i) = y^*(i)$ for $i = 1, 2, \dots, m_0$ and $i_0 \leq m_0$. Then $x^*(i_0) = y^*(i_0)$. Consequently, we can find $n_0 \in \mathbb{N}$ such that $|y_n^*(i_0) - x^*(i_0)| = |y_n^*(i_0) - y^*(i_0)| \leq \frac{\varepsilon}{2}$ for $n \geq n_0$, whence, by (5), there holds that $y_n^*(i_0) \geq x_n^*(i_0) + \frac{\varepsilon}{2}$ for $n \geq n_0$. On the other hand, $y_n \leq x_n$ for any $n \in \mathbb{N}$, whence $y_n^* \leq x_n^*$ for any $n \in \mathbb{N}$, which arrive us to a contradiction.

Case 2. Assume that $0 < x^*(i_0) < x_n^*(i_0)$ for all $n \in \mathbb{N}$ and let us define $u = x^*(i_0) + \varepsilon$, $v = x^*(i_0) + \frac{\varepsilon}{2}$ and

$$\delta = \Phi(u)\omega(i_0) - \Phi(v)\omega(i_0). \quad (6)$$

Then, there exists $m_0 \in \mathbb{N}$, $m_0 \geq i_0$ such that $\sum_{i=1}^{m_0} \Phi(x^*(i))\omega(i) \geq \alpha - \frac{\delta}{4}$ and $x^*(m_0 + 1) < x^*(m_0)$. Define the set $B = \{i \in \mathbb{N} : |x(i)| \geq x^*(m_0)\}$, $z = x\chi_B$ and $z_n = x_n\chi_B$ for any $n \in \mathbb{N}$. We get that $m(B) = m_0 < \infty$, so $z^*(i) = x^*(i)$ for $i = 1, 2, \dots, m_0$ and $z_n \rightarrow z$ uniformly, whence $z_n^* \rightarrow z^*$ coordinatewisely and, consequently, $\Phi \circ z_n^* \rightarrow \Phi \circ z^*$ coordinatewisely. Therefore, there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{i=1}^{m_0} \Phi(z_n^*(i))\omega(i) \geq \alpha - \frac{\delta}{2} \quad (7)$$

for any $n \geq n_0$. Without loss of generality, we can also assume that

$$z_n^*(i_0) < z^*(i_0) + \frac{\varepsilon}{2} = x^*(i_0) + \frac{\varepsilon}{2} \quad (8)$$

for $n \geq n_0$. Since $z_n \leq x_n$, we get that $z_n^* \leq x_n^*$ and, consequently, applying (6) and (8), we get

$$\begin{aligned} I_{\Phi, \omega}(x_n) &\geq \sum_{i=1}^{i_0-1} \Phi(x_n^*(i))\omega(i) + \Phi(u)\omega(i_0) + \sum_{i=i_0+1}^{m_0} \Phi(x_n^*(i))\omega(i) \\ &= \sum_{i=1}^{i_0-1} \Phi(x_n^*(i))\omega(i) + \Phi(v)\omega(i_0) + \delta + \sum_{i=i_0+1}^{m_0} \Phi(x_n^*(i))\omega(i) \\ &\geq \sum_{i=1}^{m_0} \Phi(z_n^*(i))\omega(i) + \delta \geq \alpha + \frac{\delta}{2} \end{aligned}$$

for $n \geq n_0$, where the last inequality comes from (7), a contradiction.

Case 3. Assume that $0 = x^*(i_0) < x_n^*(i_0)$ for any $n \in \mathbb{N}$. Let $C = \{i \in \mathbb{N} : |x(i)| > 0\}$. Then $m(C) = m_0 < i_0$. Taking $u = \varepsilon$, $v = 0$ and $\delta = \sum_{i=m_0+1}^{i_0} \Phi(u)\omega(i)$, similarly as before, we will get a contradiction. $\square$

Remark 2 In Lemma 2 (also in Lemma 3) the assumptions that Φ vanishes only at zero and $\omega(i) > 0$ for any $i \in \mathbb{N}$ can not be omitted. Indeed, assume firstly that $a_\Phi > 0$ and define $x = a_\Phi e_1$ and $x_n = a_\Phi e_1 + a_\Phi e_n$ for $n \geq 2$. Then $I_{\Phi, \omega}(x) = I_{\Phi, \omega}(x_n) = 0$ for $n \geq 2$, $x_n \rightarrow x$ coordinatewisely and $x_n^*(2) = a_\Phi > 0 = x^*(2)$ for $n \geq 2$.

Now, assume that $a_\Phi = 0$ and there exists $j \in \mathbb{N}$ such that $w(j) > w(j+1) = 0$. Let $u > 0$ be such that $\Phi(u) < \infty$ and let us define $x = \sum_{i=1}^j ue_i$ and $x_n = \sum_{i=1}^j ue_i + ue_n$ for $n \geq j+1$. Then $I_{\Phi,\omega}(x) = I_{\Phi,\omega}(x_n) = \Phi(u) \sum_{i=1}^j \omega(i)$ for $n \geq j+1$, $x_n \rightarrow x$ coordinatewisely and $x_n^*(j+1) = u > 0 = x^*(j+1)$ for $n \geq j+1$.

Next lemma can be proved analogously as Lemma 3.2. from [10].

Lemma 3 *Let Φ vanishes only at zero and $\omega(i) > 0$ for any $i \in \mathbb{N}$. Then $(x_n - x)^* \rightarrow 0$ uniformly as $n \rightarrow \infty$ for any $x \in c_0$ and any sequence $(x_n)_{n=1}^\infty$ from c_0 such that $x_n \rightarrow x$ coordinatewisely and $I_{\Phi,\omega}(x_n) \rightarrow I_{\Phi,\omega}(x) = \alpha < \infty$ as $n \rightarrow \infty$.*

Remark 3 Note that in Lemma 3 assumptions $x \in c_0$ and $x_n \in c_0$ for any $\mathbb{N}$ are essential. Indeed, for any fixed $u > 0$ let us define $x(i) = u$ for $i \in \mathbb{N}$, $x_n(n) = 0$ and $x_n(i) = u$ for $n \in \mathbb{N}$ and $i \neq n$. Obviously, $x \in \ell^\infty \setminus c_0$, $x_n \in \ell^\infty \setminus c_0$ for $n \in \mathbb{N}$ and $x_n \rightarrow x$ coordinatewisely. We also have that $x^*(i) = u = x_n^*(i)$ for $i, n \in \mathbb{N}$. Therefore, $x_n^* \rightarrow x^*$ coordinatewisely. What is more, if $\Phi(u) < \infty$ and $\sum_{i=1}^\infty \omega(i) < \infty$, we get that $I_{\Phi,\omega}(x) = I_{\Phi,\omega}(x_n) < \infty$ for $n \in \mathbb{N}$. At the same time $(x_n - x)^*(1) = u$ for $n \in \mathbb{N}$.

The proof of the lemma below can be found in [8].

Lemma 4 *Let (x_n) be a sequence of elements of the space $\lambda_{\Phi,\omega}$. Then the following properties hold.*

- (1) *If $\lim_{n \rightarrow \infty} \|x_n\|_{\Phi,\omega}^O = 0$, then $\lim_{n \rightarrow \infty} I_{\Phi,\omega}(x_n) = 0$.*
- (2) *Assume that $\sum_{i=1}^\infty \omega(i) = \infty$. Then the implication that if $\lim_{n \rightarrow \infty} I_{\Phi,\omega}(x_n) = 0$, then $\lim_{n \rightarrow \infty} \|x_n\|_{\Phi,\omega}^O = 0$ holds true if and only if $\Phi \in \delta_2$.*
- (3) *If $\sum_{i=1}^\infty \omega(i) < \infty$, then the implication that if $\lim_{n \rightarrow \infty} I_{\Phi,\omega}(x_n) = 0$, then $\lim_{n \rightarrow \infty} \|x_n\|_{\Phi,\omega}^O = 0$ holds true if and only if $a_\Phi = 0$.*

3 Main results

Before we present the main results of our paper, we will recall what so far has been known about the property of the Kadec–Klee type in Orlicz, Lorentz and Orlicz–Lorentz spaces. In the case of Orlicz spaces the suitable criteria for both the Orlicz as well as the Luxemburg norms was given in [4,9,26]. Note that in the case of non-atomic measure in place of the Kadec–Klee property with respect to the coordinatewise convergence we consider the Kadec–Klee property for the local convergence in measure. Analogous results for Lorentz spaces can be found in [5]. Whereas, in the case of Orlicz–Lorentz spaces the research connected to the properties of the Kadec–Klee type were led only for the Luxemburg norm (see [3,16,17] and also [10]). In particular, we get that the Orlicz–Lorentz sequence space $\lambda_{\Phi,\omega}$ equipped with the Luxemburg norm $\|\cdot\|_{\Phi,\omega}$ has the Kadec–Klee property with respect to the coordinatewise convergence if and only if $\Phi \in \delta_2$, $\sum_{i=1}^\infty \omega(i) = \infty$ and $\Phi(b_\Phi)\omega(1) \geq 1$.

Theorem 2 *The following conditions are equivalent:*

- (1) *The function Φ satisfies condition δ_2 and $\sum_{i=1}^{\infty} \omega(i) = \infty$.*
- (2) *The Orlicz–Lorentz sequence space $(\lambda_{\Phi, \omega}, \|\cdot\|_{\Phi, \omega}^O)$ has the Kadec–Klee property with respect to the coordinatewise convergence.*
- (3) *The Orlicz–Lorentz sequence space $(\lambda_{\Phi, \omega}, \|\cdot\|_{\Phi, \omega}^O)$ has the Kadec–Klee property.*

Proof It is obvious that (2) implies (3). Since the Kadec–Klee property implies an order continuity of the space $(\lambda_{\Phi, \omega}, \|\cdot\|_{\Phi, \omega}^O)$ (see [9, Proposition 2.1]) and the last means that $\Phi \in \delta_2$ and $\sum_{i=1}^{\infty} \omega(i) = \infty$ (see Theorem 1), we obtain that (3) implies (1). Now, we will show (1) implies (2).

Case 1. At the beginning let us assume that $b_{\Phi} < \infty$ or $b_{\Phi} = \infty$ and $\lim_{u \rightarrow \infty} \frac{\Phi(u)}{u} = \infty$. Let $x \in S(\lambda_{\Phi, \omega}, \|\cdot\|_{\Phi, \omega}^O)$, $x_n \in S(\lambda_{\Phi, \omega}, \|\cdot\|_{\Phi, \omega}^O)$ for $n \in \mathbb{N}$ and $x_n \rightarrow x$ coordinatewisely. By the assumptions of Φ (see Remark 1), for any $n \in \mathbb{N}$ we can find $k_n > 1$ such that $1 = \|x_n\|_{\Phi, \omega}^O = \frac{1}{k_n} (1 + I_{\Phi, \omega}(k_n x_n))$. First, we will show the boundedness of the sequence $(k_n)_{n=1}^{\infty}$. Define

$$m_1 = \max\{i \in \mathbb{N} : x^*(i) = x^*(1)\}. \quad (9)$$

By $\sum_{i=1}^{\infty} \omega(i) = \infty$, we have $m_1 < \infty$ and $x^*(m_1 + 1) < x^*(1)$. By virtue of the rearrangement definition, there exist natural numbers $i_1, i_2, \dots, i_{m_1}$ such that

$$|x(i_1)| = |x(i_2)| = \dots = |x(i_{m_1})| = x^*(1)$$

and $|x(i)| \leq x^*(m_1 + 1)$ for $i \in \mathbb{N} \setminus \{i_1, i_2, \dots, i_{m_1}\}$. Since $x_n \rightarrow x$ coordinatewisely, there exists $n_0 \in \mathbb{N}$ such that

$$x_n(i_j) \geq b := \frac{x^*(1) + x^*(m_1 + 1)}{2}$$

for $j = 1, 2, \dots, m_1$ and $n \geq n_0$. Hence $x_n^*(i) \geq b$ for $i = 1, 2, \dots, m_1$ and $n \geq n_0$. Therefore, if $b_{\Phi} < \infty$, then we have $k_n \leq b_{\Phi}/b$ for any $n \geq n_0$. Let now $b_{\Phi} = \infty$ and $\lim_{u \rightarrow \infty} \frac{\Phi(u)}{u} = \infty$. If $\lim_{n \rightarrow \infty} k_n = \infty$ (passing to a subsequence if necessary), then by the assumption that $\lim_{u \rightarrow \infty} \frac{\Phi(u)}{u} = \infty$, we obtain

$$1 = \frac{1}{k_n} (1 + I_{\Phi, \omega}(k_n x_n)) \geq \frac{1}{k_n} \sum_{i=1}^{m_1} \Phi(k_n x_n^*(i)) \omega(i) \geq b \frac{\Phi(b k_n)}{b k_n} \sum_{i=1}^{m_1} \omega(i) > 1$$

beginning from some natural $n_1 \geq n_0$, a contradiction. So $(k_n)_{n=1}^{\infty}$ is bounded.

Now, we will show that any cluster point k_0 of the sequence $(k_n)_{n=1}^{\infty}$ belongs to $K(x)$, that is,

$$\frac{1}{k_0} (1 + I_{\Phi, \omega}(k_0 x)) = \|x\|_{\Phi, \omega}^O = 1. \quad (10)$$

From the properties of the Orlicz norm

$$\frac{1}{k_0} (1 + I_{\Phi, \omega}(k_0 x)) \geq 1. \quad (11)$$

In order to get the opposite inequality, let us define the sequence (m_l) in the following way: m_1 is given by formula (9) and

$$m_l = \max\{i \in \mathbb{N}: x^*(i) = x^*(m_{l-1} + 1)\}$$

for $l = 2, 3, 4, \dots$. Note that sequence (m_l) is increasing. If $m(\text{supp } x) = \infty$, then $\lim_{l \rightarrow \infty} m_l = \infty$. In the case of $m(\text{supp } x) = p < \infty$, we can find $l_x \leq p$ such that $m_{l_x} = p$. For any l the set $N_l = \{i \in \mathbb{N}: |x(i)| \geq x^*(m_l)\}$ is finite, more precisely, $m(N_l) = m_l$. Let $\lim_{s \rightarrow \infty} k_{n(s)} = k_0$. Since $k_{n(s)}x_{n(s)} \rightarrow k_0x$ coordinatewisely, then $k_{n(s)}x_{n(s)}\chi_{N_l} \rightarrow k_0x\chi_{N_l}$ uniformly for any fixed l , whence $k_{n(s)}x_{n(s)}^*\chi_{N_l} \rightarrow k_0x^*\chi_{N_l}$ coordinatewisely for any fixed l and, consequently,

$$\sum_{i=1}^{m_l} \Phi(k_{n(s)}x_{n(s)}^*(i))\omega(i) \rightarrow \sum_{i=1}^{m_l} \Phi(k_0x^*(i))\omega(i)$$

as $s \rightarrow \infty$. Hence, for any fixed l , there holds

$$1 \geq \lim_{s \rightarrow \infty} \frac{1}{k_{n(s)}} \left(1 + \sum_{i=1}^{m_l} \Phi(k_{n(s)}x_{n(s)}^*(i))\omega(i) \right) = \frac{1}{k_0} \left(1 + \sum_{i=1}^{m_l} \Phi(k_0x^*(i))\omega(i) \right).$$

From the arbitrariness of $l \in \mathbb{N}$, we get

$$\frac{1}{k_0} \left(1 + \sum_{i=1}^{\infty} \Phi(k_0x^*(i))\omega(i) \right) \leq 1.$$

This inequality together with inequality (11) give us equality (10).

Let now $(x_{n(s)})$ be an arbitrary subsequence of the sequence (x_n) . Without loss of generality, passing to a subsequence if necessary, we can assume that sequence $(k_{n(s)})$ is convergent to some k_0 . Thus $k_{n(s)}x_{n(s)} \rightarrow k_0x$ coordinatewisely and $k_0 \in K(x)$, whence

$$\frac{1}{k_0} (1 + I_{\Phi, \omega}(k_0x)) = 1 = \frac{1}{k_{n(s)}} (1 + I_{\Phi, \omega}(k_{n(s)}x_{n(s)})) \quad \text{for } s \in \mathbb{N}$$

and, consequently,

$$\lim_{s \rightarrow \infty} I_{\Phi, \omega}(k_{n(s)}x_{n(s)}) = I_{\Phi, \omega}(k_0x).$$

By virtue of Lemmas 2 and 3, we get

$$k_{n(s)}x_{n(s)}^*(i) \rightarrow k_0x^*(i) \tag{12}$$

for any $i \in \mathbb{N}$ and

$$(k_{n(s)}x_{n(s)} - k_0x)^* \rightarrow 0 \tag{13}$$

uniformly as $s \rightarrow \infty$.

Since ℓ^1 has the Kadec–Klee property with respect to the coordinatewise convergence, we obtain

$$\sum_{i=1}^{\infty} \left| \Phi(k_{n(s)}x_{n(s)}^*(i)) - \Phi(k_0x^*(i)) \right| \omega(i) = \|(\Phi(k_{n(s)}x_{n(s)}^*) - \Phi(k_0x^*))\omega\|_{\ell^1} \rightarrow 0$$

as $s \rightarrow \infty$. By virtue of Lemma 2, p. 97 from [20], there exist $y \in \ell_+^1$ (the positive cone of ℓ^1) and a subsequence $(k_{n(s(t))}x_{n(s(t))}^*)$ of the sequence $(k_{n(s)}x_{n(s)}^*)$ such that

$$|\Phi(k_{n(s(t))}x_{n(s(t))}^*) - \Phi(k_0x^*)|\omega \leq y. \quad (14)$$

If $k_0x^*(1) < u_0$ (the constant from condition δ_2), then $k_{n(s(t))}x_{n(s(t))}^*(1) \leq u_0$, starting from some $t_0 \in \mathbb{N}$. Consequently, applying the rearrangement properties and the condition δ_2 , we get for $i = 2j$ and $t \geq t_0$ that

$$\begin{aligned} & \Phi((k_{n(s(t))}x_{n(s(t))} - k_0x)^*(2j))\omega(2j) \\ & \leq \Phi(k_{n(s(t))}x_{n(s(t))}^*(j) + k_0x^*(j))\omega(2j) \\ & \leq \Phi\left(2 \cdot \frac{k_{n(s(t))}x_{n(s(t))}^*(j) + k_0x^*(j)}{2}\right)\omega(j) \\ & \leq \frac{K}{2} \left(\Phi(k_{n(s(t))}x_{n(s(t))}^*(j)) + \Phi(k_0x^*(j))\right)\omega(j) \\ & \leq \frac{K}{2} (y(j) + 2\Phi(k_0x^*(j))\omega(j)). \end{aligned}$$

Similarly, for $i = 2j + 1$ and $t \geq t_0$, we have

$$\begin{aligned} & \Phi((k_{n(s(t))}x_{n(s(t))} - k_0x)^*(2j + 1))\omega(2j + 1) \\ & \leq \Phi((k_{n(s(t))}x_{n(s(t))} - k_0x)^*(2j))\omega(2j) \\ & \leq \frac{K}{2} (y(j) + 2\Phi(k_0x^*(j))\omega(j)). \end{aligned}$$

Moreover, for $i = 1$ and $t \geq t_0$, we get

$$\begin{aligned} \Phi((k_{n(s(t))}x_{n(s(t))} - k_0x)^*(1))\omega(1) & \leq \Phi(k_{n(s(t))}x_{n(s(t))}^*(1) + k_0x^*(j))\omega(1) \\ & \leq \frac{K}{2} (y(1) + 2\Phi(k_0x^*(1))\omega(1)). \end{aligned}$$

Let us define the sequence $z = (z(i))_{i=1}^{\infty}$ by the formula

$$z(i) = \begin{cases} \frac{K}{2} (y(1) + 2\Phi(k_0x^*(1))\omega(1)) & \text{for } i = 1, \\ \frac{K}{2} (y(j) + 2\Phi(k_0x^*(j))\omega(j)) & \text{if } i = 2j \text{ or } i = 2j + 1, j \in \mathbb{N}. \end{cases}$$

Then $z \in \ell^1$ and $\Phi \circ (k_{n(s(t))}x_{n(s(t))} - k_0x)^* \cdot \omega \leq z$ for $t \geq t_0$. Moreover, by (13) we have $\Phi \circ (k_{n(s(t))}x_{n(s(t))} - k_0x)^* \cdot \omega \rightarrow 0$ uniformly as $t \rightarrow \infty$. Hence, by the Lebesgue dominated convergence theorem, $I_{\Phi, \omega}(k_{n(s(t))}x_{n(s(t))} - k_0x) \rightarrow 0$, whence, by Lemma 4, $\|k_{n(s(t))}x_{n(s(t))} - k_0x\|_{\Phi, \omega}^O \rightarrow 0$. Since $k_{n(s(t))} \rightarrow k_0$, we obtain $\|x_{n(s(t))} - x\|_{\Phi, \omega}^O \rightarrow 0$.

Let now $k_0x^*(1) \geq u_0$. Since $x \in c_0$, there exists $j_0 \in \mathbb{N}$ such that $k_0x^*(j_0) < u_0$. Hence, by (12) and (13), we can find $t_0 \in N$ such that for any $t \geq t_0$, we have $(k_{n(s(t))}x_{n(s(t))} - k_0x)^*(1) \leq u_0$ and $k_{n(s(t))}x_{n(s(t))}^*(j_0) \leq u_0$. Defining $z = (z(i))_{i=1}^\infty$ by the formula

$$z(i) = \begin{cases} \Phi(u_0)\omega(i) & \text{for } i = 1, \dots, 2j_0 - 1, \\ \frac{K}{2} (y(j) + 2\Phi(k_0x^*(j))\omega(j)) & \text{if } i = 2j \text{ or } i = 2j + 1, j \geq j_0, \end{cases}$$

we get again that $z \in \ell^1$ and $\Phi \circ (k_{n(s(t))}x_{n(s(t))} - k_0x)^* \cdot \omega \leq z$ for $t \geq t_0$. Proceeding analogously as above we obtain $\|x_{n(s(t))} - x\|_{\Phi, \omega}^O \rightarrow 0$.

Finally, applying the double extract convergence theorem, we get $\|x_n - x\|_{\Phi, \omega}^O \rightarrow 0$.

Case 2. Assume that $\lim_{u \rightarrow \infty} \frac{\Phi(u)}{u} = B < \infty$ and let $x \in S(\lambda_{\Phi, \omega}, \|\cdot\|_{\Phi, \omega}^O)$, $x_n \in S(\lambda_{\Phi, \omega}, \|\cdot\|_{\Phi, \omega}^O)$ for $n \in \mathbb{N}$ and $x_n \rightarrow x$ coordinatewisely.

First, assume additionally that $\Psi(B) \sum_{i=1}^{m(\text{supp } x)} \omega(i) > 1$ or $\Psi(B) \sum_{i=1}^{m(\text{supp } x)} \omega(i) = 1$ and $p(u) = B$ starting with some u_0 . If $\Psi(B) \sum_{i=1}^{m(\text{supp } x)} \omega(i) > 1$, then there exist $j_x \in \mathbb{N}$, $j_x \leq m(\text{supp } x)$ and $u_x > 0$ such that $\Psi(p(u_x)) \sum_{i=1}^{j_x} \omega(i) > 1$. Without loss of generality we can assume that $x^*(j_x) > x^*(j_x + 1)$, whence $m(N_x) = j_x$, where $N_x = \{i \in \mathbb{N} : |x(i)| \geq x^*(j_x)\}$. Since (x_n) is coordinatewisely convergent to x , there exists $n_0 \in \mathbb{N}$ such that $|x_n(i)| \geq \frac{x^*(j_x) + x^*(j_x + 1)}{2}$ for $i \in N_x$ and $n \geq n_0$. Hence, $I_{\Psi, \omega}(p(k_1x_n^*)) > 1$ for $k_1 = \frac{2u_x}{x^*(j_x) + x^*(j_x + 1)}$ and $n \geq n_0$ and, consequently, $k_n^{**} := k^{**}(x_n) \leq k_1$ for the same n . Next, proceeding analogously as in Case 1, we obtain that $x_n \rightarrow x$ in norm.

We proceed analogously in the case when $\Psi(B) \sum_{i=1}^{m(\text{supp } x)} \omega(i) = 1$ and there exists $u_0 > 0$ such that $p(u) = B$ for all $u \geq u_0$. Indeed, note that in this case $\Psi(B) > 0$ and $m(\text{supp } x) < \infty$ by virtue of the assumption that $\sum_{i=1}^\infty \omega(i) = \infty$. Taking this time $j_x = m(\text{supp } x)$ and using again the fact that $x_n \rightarrow x$ coordinatewisely, we can find $n_0 \in \mathbb{N}$ such that $|x_n(i)| \geq \frac{1}{2}x^*(j_x)$ for any $i \in \text{supp } x$ and $n \geq n_0$. Hence, taking $k_1 = \frac{2u_0}{x^*(j_x)}$, we get that $I_{\Psi, \omega}(p(k_1x_n^*)) \geq \Psi(p(u_0)) \sum_{i=1}^{m(\text{supp } x)} \omega(i) = 1$ for all $n \geq n_0$, which means that for any $n \geq n_0$ there is $k_n \leq k_1$ such that $\|x_n\|_{\Phi, \omega}^O = \frac{1}{k_n}(1 + I_{\Phi, \omega}(k_nx_n))$. Next, proceeding again in the same way as in Case 1, we can finish this part of the proof.

Finally, assume that $\Psi(B) \sum_{i=1}^{m(\text{supp } x)} \omega(i) < 1$ or $\Psi(B) \sum_{i=1}^{m(\text{supp } x)} \omega(i) = 1$ and $p(u) < B$ for any $u > 0$, which means that $K(x) = \emptyset$, or equivalently,

$$\frac{1}{k}(1 + I_{\Phi, \omega}(kx)) > 1 \quad (15)$$

for any $k > 0$ (see Remark 1). Let $(x_{n(s)})$ be an arbitrary subsequence of the sequence (x_n) . We will consider two subcases.

Subcase 1. Assume, passing to a subsequence if necessary, that there exists $s_0 \in \mathbb{N}$ such that $\Psi(B) \sum_{i=1}^{m(\supp x_{n(s)})} \omega(i) \leq 1$ for any $s \geq s_0$. By virtue of Lemma 1, we obtain

$$B \sum_{i=1}^{\infty} x^*(i) \omega(i) = 1 = B \sum_{i=1}^{\infty} x_{n(s)}^*(i) \omega(i)$$

for all $s \geq s_0$. Hence, by Lemma 2 (for $\Phi_1(u) = |u|$, we have $\lim_{s \rightarrow \infty} I_{\Phi_1, \omega}(x_{n(s)}) = I_{\Phi_1, \omega}(x) = \frac{1}{B}$), we get $x_{n(s)}^*(i) \rightarrow x^*(i)$ for any $i \in \mathbb{N}$. Applying again the fact that ℓ^1 has the Kadec–Klee property with respect to coordinatewise convergence, we obtain

$$\sum_{i=1}^{\infty} |x_{n(s)}^*(i) - x^*(i)| \omega(i) = \|(x_{n(s)}^* - x^*) \omega\|_{\ell^1} \rightarrow 0$$

as $s \rightarrow \infty$. Hence, by virtue of Lemma 4, p. 97 from [20], there exist $y \in \ell_+^1$ and a subsequence $(x_{n(s(t))}^*)$ of the sequence $(x_{n(s)}^*)$ such that

$$|x_{n(s(t))}^* - x^*| \omega \leq y.$$

Hence, similarly to Case 1, we get

$$\begin{aligned} (x_{n(s(t))} - x)^*(2j) \omega(2j) &\leq (x_{n(s(t))}^*(j) + x^*(j)) \omega(j) \\ &\leq y(j) + 2x^*(j) \omega(j). \end{aligned}$$

Consequently, defining

$$z(i) = \begin{cases} y(1) + 2x^*(1) \omega(1) & \text{for } i = 1 \\ y(j) + 2x^*(j) \omega(j) & \text{for } i = 2j \text{ or } i = 2j + 1, j \in \mathbb{N}, \end{cases}$$

we have that $z = (z(i))_{i=1}^{\infty} \in \ell^1$ and $(x_{n(s(t))} - x)^* \omega \leq z$. Since, by virtue of Lemma 3, $(x_{n(s(t))} - x)^* \rightarrow 0$ uniformly as $t \rightarrow \infty$, applying again the Lebesgue dominated convergence theorem, we get that $\lim_{t \rightarrow \infty} \sum_{i=1}^{\infty} (x_{n(s(t))} - x)^*(i) \omega(i) = 0$. Hence, it is enough to show that, starting with some t_0 , there holds

$$\|x_{n(s(t))} - x\|_{\Phi, \omega}^O = B \sum_{i=1}^{\infty} (x_{n(s(t))} - x)^*(i) \omega(i), \quad (16)$$

whence we finally get

$$\lim_{t \rightarrow \infty} \|x_{n(s(t))} - x\|_{\Phi, \omega}^O = 0.$$

Obviously, formula (16) is satisfied if $\Psi(B) = 0$. Now, assume that $\Psi(B) > 0$. Since $\sum_{i=1}^{\infty} \omega(i) = \infty$, we get that $m(\text{supp } x) < \infty$. Hence, and by the assumption that x_n is coordinatewise convergent to x , we obtain that $\text{supp } x \subset \text{supp } x_{n(s(t))}$ starting with some t_0 and, without loss of generality, we can assume that $s(t_0) \geq s_0$. Consequently, $\Psi(B) \sum_{i=1}^{m(\text{supp}(x_{n(s(t))} - x))} \omega(i) \leq 1$ for all $t \geq t_0$, whence from Lemma 1 formula (16) arrives for the same t .

Subcase 2. Assume, starting with some $s_0 \in \mathbb{N}$, that $\Psi(B) \sum_{i=1}^{m(\text{supp}(x_{n(s)}))} \omega(i) > 1$ holds. Then $\Psi(B) > 0$. Since $\sum_{i=1}^{\infty} \omega(i) = \infty$, we get that $i_x := m(\text{supp } x) < \infty$. Let

$$b_{n(s)} = \sup\{|x_{n(s)}(i)| : i \in \mathbb{N} \setminus \text{supp } x\}.$$

Since $x_{n(s)} \in c_0$, then for any $s \in \mathbb{N}$ there exists i_s such that $b_{n(s)} = |x_{n(s)}(i_s)|$. We will show indirectly that $\lim_{s \rightarrow \infty} b_{n(s)} = 0$. In order to do it suppose, passing to a subsequence if necessary, that there is $\delta > 0$ such that $b_{n(s)} \geq \delta$ starting with some $s_1 \geq s_0$. Without loss of generality we can assume that $\delta \leq \frac{1}{2}x^*(i_x)$. Moreover, since $x_{n(s)} \rightarrow x$ coordinatewisely, we can assume that $|x_{n(s)}(i)| \geq \frac{1}{2}x^*(i_x)$ for any $i \in \text{supp } x$ and $s \geq s_1$.

If $\Psi(B) \sum_{i=1}^{i_x+1} \omega(i) \leq 1$, then by virtue of Lemma 1, for any $s \geq s_1$ there holds

$$\begin{aligned} 1 &= \|x_{n(s)}\|_{\Phi, \omega}^O \geq \|x_{n(s)}\chi_{\text{supp } x} + \delta e_{i_s}\|_{\Phi, \omega}^O \\ &= B \sum_{i=1}^{i_x} (x_{n(s)}\chi_{\text{supp } x})^*(i)\omega(i) + B\delta\omega(i_x + 1). \end{aligned} \quad (17)$$

Since $\lim_{s \rightarrow \infty} (x_{n(s)}\chi_{\text{supp } x})^*(i) = x^*(i)$ for any $i \in \text{supp } x$, we have

$$\lim_{s \rightarrow \infty} B \sum_{i=1}^{i_x} (x_{n(s)}\chi_{\text{supp } x})^*(i)\omega(i) = B \sum_{i=1}^{i_x} x^*(i)\omega(i) = 1,$$

whence we conclude that there is $s_2 \geq s_1$ such that

$$B \sum_{i=1}^{i_x} (x_{n(s)}\chi_{\text{supp } x})^*(i)\omega(i) \geq 1 - \frac{1}{2}B\delta\omega(i_x + 1)$$

for any $s \geq s_2$. Hence,

$$B \sum_{i=1}^{i_x} (x_{n(s)}\chi_{\text{supp } x})^*(i)\omega(i) + B\delta\omega(i_x + 1) > 1$$

for any $s \geq s_2$, a contradiction to (17).

Now, assume that $\Psi(B) \sum_{i=1}^{i_x+1} \omega(i) > 1$. Then, there exists $u_x > 0$ such that $\Psi(p(u_x)) \sum_{i=1}^{i_x+1} \omega(i) > 1$. Defining $k_1 = \frac{u_x}{\delta}$, we have that $I_{\Psi}(p(k_1(x_{n(s)}\chi_{\text{supp } x} +$

$\delta e_{i_s}^*) > 1$ for all $s \geq s_1$. Therefore, for each $s \geq s_1$ there is $k_s \in [1, k_1]$ such that

$$\begin{aligned} 1 &= \|x_{n(s)}\|_{\Phi, \omega}^O \geq \|x_{n(s)}\chi_{\text{supp } x} + \delta e_{i_s}\|_{\Phi, \omega}^O \\ &= \frac{1}{k_s} \left\{ 1 + \sum_{i=1}^{i_x} \Phi(k_s(x_{n(s)}\chi_{\text{supp } x})^*(i))\omega(i) + \Phi(k_s\delta)\omega(i_x + 1) \right\} \\ &\geq \frac{1}{k_s} \left\{ 1 + \sum_{i=1}^{i_x} \Phi(k_s(x_{n(s)}\chi_{\text{supp } x})^*(i))\omega(i) \right\} + \Phi(\delta)\omega(i_x + 1), \end{aligned} \quad (18)$$

where the inequality $\Phi(\delta) \leq \frac{\Phi(k_s\delta)}{k_s}$ is a consequence of the fact that function $\frac{\Phi(u)}{u}$ is non-decreasing.

On the other hand, by (15),

$$\begin{aligned} &\lim_{s \rightarrow \infty} \frac{1}{k_1} \left\{ 1 + \sum_{i=1}^{i_x} \Phi(k_1(x_{n(s)}\chi_{\text{supp } x})^*(i))\omega(i) \right\} \\ &= \frac{1}{k_1} \left\{ 1 + \sum_{i=1}^{i_x} \Phi(k_1(x^*(i))\omega(i)) \right\} > 1. \end{aligned}$$

Hence, by the fact that function $f_{x_{n(s)}\chi_{\text{supp } x}}$ (see formula (2)) is strictly decreasing, there is $s_2 \geq s_1$ such that for all $s \geq s_2$, we have

$$\begin{aligned} &\frac{1}{k_s} \left\{ 1 + \sum_{i=1}^{i_x} \Phi(k_s(x_{n(s)}\chi_{\text{supp } x})^*(i))\omega(i) \right\} \\ &\geq \frac{1}{k_1} \left\{ 1 + \sum_{i=1}^{i_x} \Phi(k_1(x_{n(s)}\chi_{\text{supp } x})^*(i))\omega(i) \right\} > 1, \end{aligned}$$

a contradiction to (18).

Therefore, $\lim_{s \rightarrow \infty} b_{n(s)} = 0$ holds. Let $s_1 \geq s_0$ be such that $b_{n(s)} < \frac{1}{2}x^*(i_x)$ for $s \geq s_1$ and $|x_{n(s)}(i)| > \frac{1}{2}x^*(i_x)$ for any $i \in \text{supp } x$ and $s \geq s_1$. Hence, $x_{n(s)}^*(i) = (x_{n(s)}\chi_{\text{supp } x})^*(i)$ for $i = 1, 2, \dots, i_x$ and $x_{n(s)}^*(i) = (x_{n(s)}\chi_{\mathbb{N} \setminus \text{supp } x})^*(i - i_x)$ for $i > i_x$. At the same time $x_{n(s)}^*(i) \rightarrow x^*(i)$ for $i = 1, 2, \dots, i_x$. Now, we will show that $\lim_{s \rightarrow \infty} k_{n(s)} = \infty$ where

$$1 = \|x_{n(s)}\|_{\Phi, \omega}^O = \frac{1}{k_{n(s)}} \left\{ 1 + I_{\Phi, \omega}(k_{n(s)}x_{n(s)}) \right\}.$$

Indeed, in the opposite case, there exists a subsequence $(k_{n(s(t))})$ such that $\lim_{t \rightarrow \infty} k_{n(s(t))} = k < \infty$. Then, by inequality (15), we get

$$1 = \lim_{t \rightarrow \infty} \|x_{n(s(t))}\|_{\Phi, \omega}^O = \lim_{t \rightarrow \infty} \frac{1}{k_{n(s(t))}} \left\{ 1 + \sum_{i=1}^{\infty} \Phi(k_{n(s(t))}x_{n(s(t))}^*(i))\omega(i) \right\}$$

$$\begin{aligned}
&\geq \lim_{t \rightarrow \infty} \frac{1}{k_{n(s(t))}} \left\{ 1 + \sum_{i=1}^{i_x} \Phi(k_{n(s(t))}(x_{n(s(t))} \chi_{\text{supp } x})^*(i)) \omega(i) \right\} \\
&= \frac{1}{k} \left\{ 1 + \sum_{i=1}^{i_x} \Phi(kx^*(i)) \omega(i) \right\} > \|x\|_{\Phi, \omega}^O = 1,
\end{aligned}$$

a contradiction. We also have for any $s \geq s_1$ that

$$\begin{aligned}
1 = \|x_{n(s)}\|_{\Phi, \omega}^O &= \frac{1}{k_{n(s)}} \left\{ 1 + \sum_{i=1}^{\infty} \Phi(k_{n(s)} x_{n(s)}^*(i)) \omega(i) \right\} \\
&= \frac{1}{k_{n(s)}} \left\{ 1 + \sum_{i=1}^{i_x} \Phi(k_{n(s)}(x_{n(s)} \chi_{\text{supp } x})^*(i)) \omega(i) \right. \\
&\quad \left. + \sum_{i=i_x+1}^{\infty} \Phi(k_{n(s)}(x_{n(s)} \chi_{\mathbb{N} \setminus \text{supp } x})^*(i - i_x)) \omega(i) \right\}.
\end{aligned}$$

Since

$$\begin{aligned}
&\lim_{s \rightarrow \infty} \frac{1}{k_{n(s)}} \sum_{i=1}^{i_x} \Phi(k_{n(s)}(x_{n(s)} \chi_{\text{supp } x})^*(i)) \omega(i) \\
&= \lim_{s \rightarrow \infty} \sum_{i=1}^{i_x} \frac{\Phi(k_{n(s)}(x_{n(s)} \chi_{\text{supp } x})^*(i))}{k_{n(s)}(x_{n(s)} \chi_{\text{supp } x})^*(i)} (x_{n(s)} \chi_{\text{supp } x})^*(i) \omega(i) \\
&= B \sum_{i=1}^{i_x} x^*(i) \omega(i) = 1,
\end{aligned}$$

we obtain

$$\lim_{s \rightarrow \infty} \frac{1}{k_{n(s)}} \left\{ 1 + \sum_{i=i_x+1}^{\infty} \Phi(k_{n(s)}(x_{n(s)} \chi_{\mathbb{N} \setminus \text{supp } x})^*(i - i_x)) \omega(i) \right\} = 0. \quad (19)$$

Defining

$$\bar{x}_{n(s)} = b_{n(s)} \chi_{\text{supp } x} + x_{n(s)} \chi_{\mathbb{N} \setminus \text{supp } x}$$

for $s \geq s_1$, we have

$$|x - x_{n(s)}| \leq |(x - x_{n(s)}) \chi_{\text{supp } x}| + |\bar{x}_{n(s)}|.$$

Consequently,

$$\|x - x_{n(s)}\|_{\Phi, \omega}^O \leq \|(x - x_{n(s)}) \chi_{\text{supp } x}\|_{\Phi, \omega}^O + \|\bar{x}_{n(s)}\|_{\Phi, \omega}^O$$

$$\leq \|(x - x_{n(s)})\chi_{\text{supp } x}\|_{\Phi, \omega}^O + \|\bar{x}_{n(s)}\|_{\Phi, \omega}^O$$

for the same s . Note that for any $s \geq s_1$ there holds

$$\begin{aligned} \|\bar{x}_{n(s)}\|_{\Phi, \omega}^O &\leq \frac{1}{k_{n(s)}} \{1 + I_{\Phi, \omega}(k_{n(s)}\bar{x}_{n(s)})\} \\ &= \frac{1}{k_{n(s)}} \left\{ 1 + \sum_{i=1}^{i_x} \Phi(k_{n(s)}b_{n(s)})\omega(i) + \sum_{i=i_x+1}^{\infty} \Phi(k_{n(s)}(x_{n(s)}\chi_{\mathbb{N} \setminus \text{supp } x})^*(i - i_x))\omega(i) \right\} \\ &= \frac{1}{k_{n(s)}} \sum_{i=1}^{i_x} \Phi(k_{n(s)}b_{n(s)})\omega(i) + \frac{1}{k_{n(s)}} \left\{ 1 + \sum_{i=i_x+1}^{\infty} \Phi(k_{n(s)}(x_{n(s)}\chi_{\mathbb{N} \setminus \text{supp } x})^*(i - i_x))\omega(i) \right\}. \end{aligned}$$

Hence, by virtue of

$$\begin{aligned} \lim_{s \rightarrow \infty} \frac{1}{k_{n(s)}} \sum_{i=1}^{i_x} \Phi(k_{n(s)}b_{n(s)})\omega(i) &= \lim_{s \rightarrow \infty} \frac{\Phi(k_{n(s)}b_{n(s)})}{k_{n(s)}} \sum_{i=1}^{i_x} \omega(i) \\ &= \lim_{s \rightarrow \infty} Bb_{n(s)} \sum_{i=1}^{i_x} \omega(i) = 0 \end{aligned}$$

and equality (19), we get that $\lim_{s \rightarrow \infty} \|\bar{x}_{n(s)}\|_{\Phi, \omega}^O = 0$. Meanwhile, $\lim_{s \rightarrow \infty} \|(x - x_{n(s)})\chi_{\text{supp } x}\|_{\Phi, \omega}^O = 0$, whence we finally obtain that $\lim_{s \rightarrow \infty} \|x - x_{n(s)}\|_{\Phi, \omega}^O = 0$.

Summarizing both subcases and applying the double extract convergence theorem, we conclude that $\lim_{n \rightarrow \infty} \|x - x_n\|_{\Phi, \omega}^O = 0$, which finishes the proof. $\square$

Finally, let us present a criterion for the subspace of order continuous elements $(\lambda_{\Phi, \omega})_a$ of the space $(\lambda_{\Phi, \omega}, \|\cdot\|_{\Phi, \omega}^O)$ to possess the Kadec–Klee property with respect to the coordinatewise convergence.

Theorem 3 (i) If $\sum_{i=1}^{\infty} \omega(i) = \infty$, then $((\lambda_{\Phi, \omega})_a, \|\cdot\|_{\Phi, \omega}^O)$ has the Kadec–Klee property with respect to the coordinatewise convergence if and only if Φ satisfies the δ_2 -condition.

(ii) If $\sum_{i=1}^{\infty} \omega(i) < \infty$, then $((\lambda_{\Phi, \omega})_a, \|\cdot\|_{\Phi, \omega}^O)$ has the Kadec–Klee property with respect to the coordinatewise convergence if and only if $\omega(i) > 0$ for any $i \in \mathbb{N}$ and, $a_{\Phi} = 0$ if there exists $j \in \mathbb{N}$ such that $K(x_j) \neq \emptyset$ where $x_j = \underbrace{(1, 1, \dots, 1)}_{j \text{ times}}, 0, 0, \dots$.

Proof (i) If $\sum_{i=1}^{\infty} \omega(i) = \infty$ and $\Phi \in \delta_2$, then by virtue of Theorem 1, we obtain that $(\lambda_{\Phi, \omega})_a = \lambda_{\Phi, \omega}$, whence by Theorem 2, we get that $((\lambda_{\Phi, \omega})_a, \|\cdot\|_{\Phi, \omega}^O)$ has the Kadec–Klee property with respect to the coordinatewise convergence.

Now, we will show the necessity of the δ_2 -condition. Assume that $\Phi \notin \delta_2$. Let $j \in \mathbb{N}$ be such that $K(x_j) \neq \emptyset$ where $x_j = \underbrace{(1, 1, \dots, 1)}_{j \text{ times}}, 0, 0, \dots$. If $b_{\Phi} < \infty$ or $b_{\Phi} = \infty$

and $\lim_{u \rightarrow \infty} \frac{\Phi(u)}{u} = \infty$, then we have $j = 1$ (see Remark 1). While, in the case when $B = \lim_{u \rightarrow \infty} \frac{\Phi(u)}{u} < \infty$, we get $\Psi(B) > 0$ (note here that the case when $\Psi(B) = 0$

is equivalent to the equality $\Phi(u) = Bu$ for any $u \geq 0$ and this function satisfies the δ_2 -condition). Since $\sum_{i=1}^{\infty} \omega(i) = \infty$, we can find $j \in \mathbb{N}$ such that $\Psi(B) \sum_{i=1}^j \omega(i) > 1$ and, in consequence, $K(x_j) \neq \emptyset$ (see Remark 1). Let $h \in K(x_j)$.

First, assume that $a_\Phi > 0$. For $n \geq j+1$ let us define

$$x_{j,n} = \sum_{i=1}^j e_i + \min\left(1, \frac{a_\Phi}{h}\right) e_n.$$

Note that x_j and $x_{j,n}$ for $n \geq j+1$ belong to $(\lambda_{\Phi,\omega})_a$. Since $0 \leq x_j \leq x_{j,n}$ for $n \geq j+1$, we have $\|x_j\|_{\Phi,\omega}^O \leq \|x_{j,n}\|_{\Phi,\omega}^O$ for the same n . On the other hand

$$\|x_j\|_{\Phi,\omega}^O = \frac{1}{h} (1 + I_{\Phi,\omega}(hx_j)) = \frac{1}{h} (1 + I_{\Phi,\omega}(hx_{j,n})) \geq \|x_{j,n}\|_{\Phi,\omega}^O$$

for $n \geq j+1$ and, finally, $\|x_j\|_{\Phi,\omega}^O = \|x_{j,n}\|_{\Phi,\omega}^O$ for the same n . Obviously, $x_{j,n} \rightarrow x_j$ coordinatewisely. Simultaneously, $\|x_{j,n} - x_j\|_{\Phi,\omega}^O = \|\min\{1, \frac{a_\Phi}{h}\} e_n\|_{\Phi,\omega}^O > 0$, so we conclude that $((\lambda_{\Phi,\omega})_a, \|\cdot\|_{\Phi,\omega}^O)$ does not possess the Kadec–Klee property with respect to the coordinatewise convergence.

Assume right now that $a_\Phi = 0$. Since $\Phi \notin \delta_2$, we find a decreasing to zero sequence $(u_n)_{n=1}^\infty$ such that $\frac{u_1}{h} < 1$, $\Phi(2u_n) > 2^{n+1}\Phi(u_n)$ and $\Phi(u_n)\omega(1) \leq \frac{1}{2^{n+1}}$. Let j_n , where $n \in \mathbb{N}$, be such that

$$\frac{1}{2^{n+1}} < \Phi(u_n) \sum_{i=1}^{j_n} \omega(i) \leq \frac{1}{2^n}.$$

Define

$$x_{j,n} = \sum_{i=1}^j e_i + \sum_{i=j+1}^{j+j_n} \frac{u_n}{h} e_i$$

for $n \in \mathbb{N}$. Obviously, x_j and $x_{j,n}$ for $n \in \mathbb{N}$ belong to $(\lambda_{\Phi,\omega})_a$ and $x_{j,n} \rightarrow x_j$ coordinatewisely. Applying the orthogonal subadditivity of the modular, we get

$$\begin{aligned} \|x_{j,n}\|_{\Phi,\omega}^O &\leq \frac{1}{h} (1 + I_{\Phi,\omega}(hx_{j,n})) \leq \frac{1}{h} (1 + I_{\Phi,\omega}(hx_j)) + \frac{1}{h} \sum_{i=1}^{j_n} \Phi(u_n)\omega(i) \\ &\leq \|x_j\|_{\Phi,\omega}^O + \frac{1}{h2^n} \end{aligned}$$

for $n \in \mathbb{N}$, whence and by the fact that $\|x_j\|_{\Phi, \omega}^O \leq \|x_{j,n}\|_{\Phi, \omega}^O$ for $n \in \mathbb{N}$, we consequently obtain that $\|x_{j,n}\|_{\Phi, \omega}^O \rightarrow \|x_j\|_{\Phi, \omega}^O$. Simultaneously,

$$I_{\Phi, \omega}(2h(x_{j,n} - x_j)) = \sum_{i=1}^{j_n} \Phi(2u_n)\omega(i) > \sum_{i=1}^{j_n} 2^{n+1} \Phi(u_n)\omega(i) > 1$$

for $n \in \mathbb{N}$, whence $\|h(x_{j,n} - x_j)\|_{\Phi, \omega} \geq \frac{1}{2}$ for the same n , where $\|\cdot\|_{\Phi, \omega}$ is the Luxemburg norm. Therefore, $\|x_{j,n} - x_j\|_{\Phi, \omega}^O \geq \|x_{j,n} - x_j\|_{\Phi, \omega} \geq \frac{1}{2h}$ for $n \in \mathbb{N}$ and we conclude again that $(\lambda_{\Phi, \omega})_a, \|\cdot\|_{\Phi, \omega}^O$ does not possess the Kadec–Klee property with respect to the coordinatewise convergence.

(ii) *Sufficiency.* Assume that, if $\sum_{i=1}^{\infty} \omega(i) < \infty$, then, as it was shown in [12, Theorem 4.2], we have that $(\lambda_{\Phi, \omega})_a = c_0$. Generally, the proof of sufficiency is similar to the proof of Theorem 2 (implication (1) $\Rightarrow$ (2)). Therefore, we will focus only on the essential differences.

First, let us note that if $\omega(i) > 0$ for any $i \in \mathbb{N}$, then condition $K(x_j) = \emptyset$ for any $j \in \mathbb{N}$, where $x_j = (\underbrace{1, 1, \dots, 1}_{j \text{ times}}, 0, 0, \dots)$, is equivalent to conditions $\lim_{u \rightarrow \infty} \frac{\Phi(u)}{u} =$

$B < \infty$ and $\Psi(B) \sum_{i=1}^{\infty} \omega(i) \leq 1$ (see Remark 1). According to the conditions of the theorem, inequality $a_{\Phi} > 0$ can be true in that case only. But then, for all elements $x \in (\lambda_{\Phi, \omega})_a = c_0$, by virtue of Lemma 1, we get that $\|x\|_{\Phi, \omega}^O = B \sum_{i=1}^{\infty} x^*(i)\omega(i)$ and the proof can be led similarly as in Case 2, Subcase 1 (see below).

Case 1. Using the assumption $(\lambda_{\Phi, \omega})_a = c_0$ in place of assumption $\sum_{i=1}^{\infty} \omega(i) = \infty$, we get that $k_{n(s)}x_{n(s)}^*(i) \rightarrow k_0x^*(i)$ for any $i \in \mathbb{N}$. Hence, by virtue of Lemma 3, we obtain that $(k_{n(s)}x_{n(s)} - k_0x)^*$ tends to zero uniformly as $s \rightarrow \infty$. Therefore, there exists s_0 such that $(k_{n(s)}x_{n(s)} - k_0x)^*(1) < \frac{1}{2} \min\{1, b_{\Phi}\}$ for $s \geq s_0$. Thus, for any $i \in \mathbb{N}$ and $s \geq s_0$, we have

$$\Phi((k_{n(s)}x_{n(s)} - k_0x)^*(i)) \leq \Phi((k_{n(s)}x_{n(s)} - k_0x)^*(1)) \leq \Phi\left(\frac{1}{2} \min\{1, b_{\Phi}\}\right).$$

Defining $z = (z(i))_{i=1}^{\infty}$ by formula $z(i) = \Phi\left(\frac{1}{2} \min\{1, b_{\Phi}\}\right)\omega(i)$, we obtain that $\Phi \circ (k_{n(s)}x_{n(s)} - k_0x)^* \leq z$ and $z \in \ell^1$, whence by virtue of the Lebesgue dominated convergence theorem we get again that $I_{\Phi, \omega}(k_{n(s)}x_{n(s)} - k_0x) \rightarrow 0$ as $s \rightarrow \infty$. Hence, by virtue of Lemma 4, we finally get that $\|k_{n(s)}x_{n(s)} - k_0x\|_{\Phi, \omega}^O \rightarrow 0$ as $s \rightarrow \infty$.

Case 2. Having regard to the above considerations, the proof when $\Psi(B) \cdot \sum_{i=1}^{m(\text{supp}(x))} \omega(i) > 1$ is similar to that of Theorem 2. Also in the case when $\Psi(B) \sum_{i=1}^{m(\text{supp}(x))} \omega(i) = 1$, $p(u) = B$ starting with some u_0 and $m(\text{supp } x) < \infty$, we can repeat reasoning from the proof of Theorem 2.

The case when $(\Psi(B) \sum_{i=1}^{m(\text{supp}(x))} \omega(i) = 1, p(u) = B$ starting with some u_0 and $m(\text{supp } x) = \infty)$ we consider jointly with cases $\Psi(B) \sum_{i=1}^{m(\text{supp}(x))} \omega(i) < 1$ and $(\Psi(B) \sum_{i=1}^{m(\text{supp}(x))} \omega(i) = 1$ and $p(u) < B$ for all $u > 0)$. Similarly as in Theorem 2, we need to consider two subcases.

In *Subcase 1*, defining the majorant z , we apply inequality $(x_{n(s)} - x)^*(i) \leq (x_{n(s)} - x)^*(1)$ for $i \in \mathbb{N}$ and the fact that $(x_{n(s)} - x)^*(1) \leq 1$ starting with some s_0 (by virtue of Lemma 3, $(x_{n(s)} - x)^* \rightarrow 0$ uniformly as $s \rightarrow \infty$). Therefore, defining $z = (z(i))_{i=1}^\infty$ by formula $z(i) = \omega(i)$, we get that $z \in \ell^1$ and $(x_{n(s)} - x)^* \omega \leq z$ for any $s \geq s_0$. Next, formula (16) is proved in the same way as in the proof of Theorem 2. Let us only note that if $\Psi(B) > 0$ and $m(\text{supp } x) = \infty$, then $\Psi(B) \sum_{i=1}^\infty \omega(i) \leq 1$, which allow us to apply Lemma 1.

Coming to the proof of *Subcase 2*, note that from condition $\Psi(B) \sum_{i=1}^{m(\text{supp}(x_{n(s)}))} \omega(i) > 1$, we conclude that $\Psi(B) > 0$. Hence and from condition $\Psi(B) \sum_{i=1}^{m(\text{supp}(x))} \omega(i) \leq 1$, we get that $m(\text{supp}(x)) < \infty$. Further, the proof can be led and finished similarly as this for Theorem 2.

Necessity. First, suppose that there is $j \in \mathbb{N}$ such that $\omega(j) > \omega(j+1) = 0$. Define

$$x_j = \sum_{i=1}^j e_i \quad \text{and} \quad x_{j,n} = \sum_{i=1}^j e_i + e_n$$

for $n \geq j+1$. Then, x_j and $x_{j,n}$ for $n \geq j+1$ belong to $(\lambda_{\Phi,\omega})_a$, $x_{j,n} \rightarrow x_j$ coordinatewisely and

$$\|x_j\|_{\Phi,\omega}^O = \inf_{k>0} \frac{1}{k} (1 + I_{\Phi,\omega}(kx_j)) = \inf_{k>0} \frac{1}{k} (1 + I_{\Phi,\omega}(kx_{j,n})) = \|x_{j,n}\|_{\Phi,\omega}^O,$$

but $\|x_{j,n} - x_j\|_{\Phi,\omega}^O = \|e_n\|_{\Phi,\omega}^O > 0$, which means that $((\lambda_{\Phi,\omega})_a, \|\cdot\|_{\Phi,\omega}^O)$ does not have the Kadec–Klee property with respect to the coordinatewise convergence.

Finally, suppose that $\omega(i) > 0$ for all $i \in \mathbb{N}$, there exists $j \in \mathbb{N}$ such that $K(x_j) \neq \emptyset$, where $x_j = \sum_{i=1}^j e_i$, and $a_\Phi > 0$. Let $h \in K(x_j)$. Then, defining

$$x_{j,n} = \sum_{i=1}^j e_i + \min\left(1, \frac{a_\Phi}{h}\right) e_n$$

for $n \geq j+1$, in the same way as in the proof of the necessity of Case (i), we obtain again that $((\lambda_{\Phi,\omega})_a, \|\cdot\|_{\Phi,\omega}^O)$ does not have the Kadec–Klee property with respect to the coordinatewise convergence.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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References

1. Astashkin, S.V., Sukochev, F.A., Wong, C.P.: Distributionally concave symmetric spaces and uniqueness of symmetric structure. *Adv. Math.* **232**, 399–431 (2013)
2. Bennett, C., Sharpley, R.: *Interpolation of Operators*, Pure and Applied Mathematics. Academic Press Inc, Boston, MA (1988)
3. Cerdà, J., Hudzik, H., Kamińska, A., Mastyło, M.: Geometric properties of symmetric spaces with applications to Orlicz–Lorentz spaces. *Positivity* **2**, 311–337 (1998)
4. Chen, S.T.: Geometry of Orlicz spaces. *Dissertationes Math. (Rozprawy Mat.)* **356**, 204 (1996)
5. Chilin, V.I., Dodds, P.G., Sedaev, A.A., Sukochev, F.A.: Characterisations of Kadec–Klee properties in symmetric spaces of measurable functions. *Trans. Am. Math. Soc.* **348**, 4895–4918 (1996)
6. Ciesielski, M., Kolwicz, P., Pluciennik, R.: Local approach to Kadec–Klee properties in symmetric function spaces. *J. Math. Anal. Appl.* **426**, 700–726 (2015)
7. Cui, Y., Foralewski, P., Hudzik, H.: M -constants in Orlicz–Lorentz function spaces. *Math. Nachr.* **292**, 2556–2573 (2019)
8. Cui, Y., Foralewski, P., Kończak, J.: Orlicz–Lorentz sequence spaces equipped with the Orlicz norm. submitted
9. Dominguez, T., Hudzik, H., López, G., Mastyło, M., Sims, B.: Complete characterizations of Kadec–Klee properties in Orlicz spaces. *Houston J. Math.* **29**, 1027–1044 (2003)
10. Foralewski, P.: On some geometric properties of generalized Orlicz–Lorentz sequence spaces. *Indag. Math. (N.S.)* **24**, 346–372 (2013)
11. Foralewski, P., Hudzik, H., Kolwicz, P.: Non-squareness properties of Orlicz–Lorentz sequence spaces. *J. Funct. Anal.* **264**, 605–629 (2013)
12. Foralewski, P., Hudzik, H., Szymaszkiewicz, L.: On some geometric and topological properties of generalized Orlicz–Lorentz sequence spaces. *Math. Nachr.* **281**, 181–198 (2008)
13. Foralewski, P., Kończak, J.: Local uniform non-squareness of Orlicz–Lorentz function spaces. *Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM* **113**, 3425–3443 (2019)
14. Gong, W.Z., Shi, Z.R.: Points of monotonicity in Orlicz–Lorentz function spaces. *Nonlinear Anal.* **73**, 1300–1317 (2010)
15. Gong, W.Z., Zhang, D.X.: Monotonicity in Orlicz–Lorentz sequence spaces equipped with the Orlicz norm. *Acta Math. Sci. Ser. B (Engl. Ed.)* **36**, 1577–1589 (2016)
16. Hudzik, H., Kamińska, A., Mastyło, M.: On geometric properties of Orlicz–Lorentz spaces. *Can. Math. Bull.* **40**, 316–329 (1997)
17. Hudzik, H., Mastyło, M.: Strongly extreme points in Köthe–Bochner spaces. *Rocky Mt. J. Math.* **23**, 899–909 (1993)
18. Kamińska, A.: Some remarks on Orlicz–Lorentz spaces. *Math. Nachr.* **147**, 29–38 (1990)
19. Kamińska, A., Leśnik, K., Raynaud, Y.: Dual spaces to Orlicz–Lorentz spaces. *Studia Math.* **222**, 229–261 (2014)
20. Kantorovich, L.V., Akilov, G.P.: *Functional Analysis*, 2nd edn. Pergamon Press, Oxford (1982). English translation from the Russian edition
21. Kreĭn, S.G., Petunĭn, Yu. I., Semĕnov, E.M.: *Interpolation of linear operators*, Translations of Mathematical Monographs, vol. 54. American Mathematical Society, Providence (1982)
22. Levis, F.E., Cuenya, H.H.: Gateaux differentiability in Orlicz–Lorentz spaces and applications. *Math. Nachr.* **280**, 1282–1296 (2007)
23. Lindenstrauss, J., Tzafriri, L.: *Classical Banach spaces. II, Function spaces*, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, vol. 97, Springer, Berlin (1979)
24. Maligranda, L.: Indices and interpolation. *Dissertationes Math. (Rozprawy Mat.)* **234**, 49 (1985)
25. Maligranda, L.: Orlicz spaces and interpolation, *Seminários de Matemática [Seminars in Mathematics]*, 5. Universidade Estadual de Campinas, Departamento de Matemática, Campinas (1989)

26. Medzhitov, A.M., Sukochev, F.A.: The property (H) in Orlicz spaces. *Bull. Pol. Acad. Sci. Math.* **40**, 5–11 (1992)
27. Montgomery-Smith, S.J.: Comparison of Orlicz–Lorentz spaces. *Studia Math.* **103**, 161–189 (1992)
28. Musielak, J.: Orlicz Spaces and Modular Spaces. *Lecture Notes in Mathematics*, vol. 1034. Springer, Berlin (1983)
29. Radon, J.: Theorie und Anwendungen der absolut additiven Mengenfunktionen. *Sitz. Akad. Wiss. Wien* **122**, 1295–1438 (1913)
30. Riesz, F.: Sur la convergence en moyenne I. *Acta Sci. Math.* **4**, 58–64 (1928)
31. Riesz, F.: Sur la convergence en moyenne II. *Acta Sci. Math.* **4**, 182–185 (1929)
32. Saint Raymond, J.: Kadec–Klee property and fixed points. *J. Funct. Anal.* **266**, 5429–5438 (2014)
33. Wang, J., Ning, Z.: Rotundity and uniform rotundity of Orlicz–Lorentz sequence spaces equipped with the Orlicz norm. *Math. Nachr.* **284**, 2297–2311 (2011)

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