A complete asymptotic expansion for operators of exponential type with $p(x) = x(1+x)^2$

Ulrich Abel¹ · Vijay Gupta²

Received: 29 May 2020 / Accepted: 6 December 2020 / Published online: 23 December 2020
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Abstract

In the year 1978, Ismail and May studied operators of exponential type and proposed some new operators which are connected with a certain characteristic function $p(x)$. Several of these operators were not separately studied by researchers due to its unusual behavior. The topic of the present paper is the local rate of approximation of a sequence of exponential type operators R_n belonging to $p(x) = x(1+x)^2$. As the main result we derive a complete asymptotic expansion for the sequence $(R_n f)(x)$ as n tends to infinity.

Keywords Ismail–May operators · Asymptotic expansion · Lagrange inversion · Exponential type operators

Mathematics Subject Classification 41A25 · 41A36

1 Introduction

Let $C_\gamma[0, \infty)$ be the class of continuous functions f on $[0, \infty)$ satisfying the growth condition $f(t) = O(e^{\gamma t})$ as $t \rightarrow \infty$, for some $\gamma > 0$.

✉ Ulrich Abel
ulrich.abel@mnd.thm.de

Vijay Gupta
vijaygupta2001@hotmail.com

¹ Fachbereich MND, Technische Hochschule Mittelhessen, Wilhelm-Leuschner-Straße 13, 61169 Friedberg, Germany

² Department of Mathematics, Netaji Subhas University of Technology, Sector 3 Dwarka, New Delhi 110078, India

In 1978, Ismail and May [10] studied operators of exponential type. Among some new operators they defined, for $f \in C_\gamma[0, \infty)$, the positive linear operators

$$(R_n f)(x) = \sum_{v=0}^{\infty} r_{n,v}(x) f\left(\frac{v}{n}\right) \quad (0 < x < +\infty) \quad (1)$$

[10, Eq. (3.14)], where

$$r_{n,v}(x) = e^{-(n+v)x/(1+x)} \frac{n(n+v)^v}{v!} \left(\frac{x}{1+x}\right)^v.$$

The corresponding fundamental functions satisfy the differential equation

$$\frac{\partial}{\partial x} r_{n,v}(x) = \frac{v-nx}{p(x)} r_{n,v}(x), \quad (2)$$

where $p(x) = x(1+x)^2$. Therefore, R_n as defined in (1) are named operators of exponential type.

The other operators due to Ismail and May [10] were studied by Totik [18], Sato [15], Ismail [11], Pop and Fărcaș [14], Altomare and Raşa [2], Gupta [7], Gupta and Rassias [9] and several others.

Using the notation

$$\omega_\beta(v, \alpha) = \frac{\alpha(\alpha + v\beta)^{v-1}}{v!} e^{-(\alpha + v\beta)} \quad (v = 0, 1, 2, \dots)$$

the operators (1) can be written in the concise form

$$(R_n f)(x) = \sum_{v=0}^{\infty} \omega_\beta(v, \alpha) f\left(\frac{v}{n}\right)$$

with the parameters $\beta = \frac{x}{1+x}$ and $\alpha = n\beta$. By using Lagrange inversion, Jain [12, Lemma 1] showed that, for $0 < \alpha < +\infty$ and $|\beta| < 1$,

$$\sum_{v=0}^{\infty} \omega_\beta(v, \alpha) = 1. \quad (3)$$

As a consequence of (3) the operators (1) preserve constant functions. As we shall see below, the operators (1) preserve linear functions. For a different proof of (3) see [19]. Note that this is a probability distribution which is called generalized Poisson distribution. It has been considered by Consul and Jain [5], [4] and several other authors.

The operators (1) are connected in a certain sense with a previous definition. In 1972 Jain [12] introduced the sequence of linear operators defined, for functions $f \in C[0, +\infty)$, by

$$(P_n^{[\beta]} f)(x) = \sum_{v=0}^{\infty} \omega_{\beta}(v, nx) f\left(\frac{v}{n}\right) \quad (0 < x < +\infty), \quad (4)$$

whenever the sum in (4) is convergent. It is easy to see that, for $\beta = 0$, definition (4) reduces to the classical Szász–Mirakyan operators S_n [13, 17], defined by

$$(S_n f)(x) = e^{-nx} \sum_{v=0}^{\infty} \frac{(nt)^v}{v!} f\left(\frac{v}{n}\right).$$

If e_r ($r = 0, 1, 2, \dots$) denote the monomials defined by $e_r(x) = x^r$ the first moments are given by

$$\begin{aligned} P_n^{[\beta]} e_0 &= e_0, & P_n^{[\beta]} e_1 &= \frac{1}{1-\beta} e_1, \\ P_n^{[\beta]} e_2 &= \frac{1}{(1-\beta)^2} e_2 + \frac{1}{n(1-\beta)^3} e_1 \end{aligned}$$

[12, Eqs. (2.12)–(2.14)]. Formulas for $P_n^{[\beta]} e_r$ ($r = 3, 4$) can be found in [6, Lemma 2.1]. It is obvious that a necessary condition for convergence of the operators (4) as $n \rightarrow \infty$ is that the parameter β in (4) depends on n such that $\beta = \beta_n \rightarrow 0$ as $n \rightarrow \infty$. Jain [12, Theorem 2.1] claimed that if $f \in C[0, +\infty)$ and $\beta_n \rightarrow 0$ as $n \rightarrow \infty$ then the sequence $\left((P_n^{[\beta]} f)(x)\right)_{n=1}^{\infty}$ converges uniformly to $f(x)$ on each compact subset of $[0, +\infty)$. His proof is based on Korovkin's theorem and does not consider any condition on the growth rate of the function f as $x \rightarrow +\infty$. For more details, see [1].

Both operators (1) and (4) are principally different, because in the second definition β tends to zero as $n \rightarrow \infty$. For $\beta = \beta(x)$ independent of n we have to develop a new approach.

In this paper, we study the local rate of convergence of the operators R_n as n tends to infinity. Our main result is a pointwise complete asymptotic expansion of the form

$$(R_n f)(x) \sim f(x) + \sum_{k=1}^{\infty} \frac{1}{n^k} a_k(f, x) \quad (n \rightarrow \infty),$$

for f sufficiently smooth. The latter formula means that, for all positive integers q , there holds

$$(R_n f)(x) = f(x) + \sum_{k=1}^q \frac{1}{n^k} a_k(f, x) + o(n^{-q}) \quad (n \rightarrow \infty).$$

The coefficients $a_k(f, x)$, which are independent of n , are explicitly determined in terms of Stirling numbers of the second kind.

Most of the above mentioned papers are based on the direct computation of the first moments of the Jain operators. A more systematic and deeper treatment seems to be desirable. In this note, we study the asymptotic behaviour of the sequence $((R_n f)(x))_{n=1}^\infty$ as n tends to infinity. In particular, by a result derived by using methods of complex analysis developed in [1], we give explicit expressions for the coefficients $a_k(f, x)$. In 2016, Abel and Agratini [1] presented the complete asymptotic expansion of the sequence $\left(\left(P_n^{[\beta_n]} f\right)(x)\right)_{n=1}^\infty$ as n tends to infinity.

In this paper, we apply concise expressions for the moments derived in [1] by using methods of complex analysis.

2 Main results

In what follows we put, for $| \beta | < 1$,

$$g(z) := \frac{1 + z - e^{\beta z}}{ze^{\beta z}} \quad (z \neq 0). \quad (5)$$

The function g can be defined on $\mathbb{C}$ by analytic continuation. Hence, it is an entire function having the power-series expansion

$$g(z) = \sum_{v=0}^{\infty} \left(1 - \frac{\beta}{v+1}\right) \frac{(-\beta z)^v}{v!} \quad (z \in \mathbb{C}).$$

Throughout the paper we put

$$c_{\ell,j}(\beta) = \left(\frac{\partial}{\partial z}\right)^j g^{-\ell}(z) \Big|_{z=0}. \quad (6)$$

The main result is a complete asymptotic expansion for the sequence of the Ismail–May operators $(R_n f)(x)$ as n tends to infinity. The coefficients are explicitly determined in terms of Stirling numbers of the second kind. To this end we introduce some notations. Let $z^{\underline{k}}$ denote the falling factorial, defined by $z^{\underline{k}} = z(z-1) \cdots (z-k+1)$, for $k \in \mathbb{N}$, and $z^{\underline{0}} = 1$. Furthermore, we use the rising factorial defined by $z^{\overline{k}} = z(z+1) \cdots (z+k-1)$, for $k \in \mathbb{N}$, and $z^{\overline{0}} = 1$. The Stirling numbers of the second kind are denoted by $\sigma(r, \ell)$. Recall that they can be defined by the relation

$$z^r = \sum_{\ell=1}^r \sigma(r, \ell) z^{\underline{\ell}} \quad (z \in \mathbb{C}, r = 1, 2, \dots). \quad (7)$$

In particular, we have $\sigma(r, \ell) = 0$, if $\ell > r$. In addition, one defines $\sigma(0, 0) = 1$ and $\sigma(r, 0) = 0$, for $r \in \mathbb{N}$.

Theorem 1 Let $q \in \mathbb{N}$ and $x > 0$. For each function $f \in C_\gamma(0, \infty)$ having a derivative of order $2q$ at x , the Ismail–May operators (1) possess the complete asymptotic expansion

$$(R_n f)(x) = f(x) + \sum_{k=1}^q \frac{1}{n^k} a_k(f, x) + o(n^{-q}) \quad (n \rightarrow \infty), \quad (8)$$

where the coefficients $a_k(f, x)$ are given by

$$a_k(f, x) = \sum_{s=k+1}^{2k} A_{k,s}(x) f^{(s)}(x) \quad (k \in \mathbb{N}) \quad (9)$$

with

$$A_{k,s}(x) = \frac{1}{s!} \sum_{\ell=0}^k \sum_{r=k+1}^s (-1)^{s-r} \binom{s}{r} \sigma(r, r-\ell) \binom{r-\ell-1}{k-\ell} \\ \times \frac{x^{s-k}}{(1+x)^{r-k}} C_{r-\ell, k-\ell} \left(\frac{x}{1+x} \right).$$

More explicitly, we have

$$(R_n f)(x) = f(x) + \frac{x(1+x)^2}{2n} f^{(2)}(x) \\ + \frac{4x(1+x)^3(1+3x)f^{(3)}(x) + 3x^2(1+x)^4 f^{(4)}(x)}{24n^2} + x(1+x)^4 \\ \times \frac{(30x^2 + 20x + 2)f^{(4)}(x) + 4x(1+x)(1+3x)f^{(5)}(x) + x^2(1+x)^2 f^{(6)}(x)}{48n^3} \\ + o(n^{-3})$$

as $n \rightarrow \infty$. As an immediate consequence, we obtain the following Voronovskaja-type result.

Corollary 1 Let $x > 0$ and $\gamma > 0$. If the function $f \in C_\gamma(0, \infty)$ has a second order derivative at x , then

$$\lim_{n \rightarrow \infty} n((R_n f)(x) - f(x)) = \frac{x(1+x)^2}{2} f''(x).$$

We emphasize the fact that operators L_n of exponential type in general satisfy such a Voronovskaja-type formula. Ismail and May [10, Proposition 2.9] proved that under certain conditions

$$\lim_{n \rightarrow \infty} n((L_n f)(x) - f(x)) = \frac{1}{2} p(x) f''(x),$$

provided that the operator L_n is associated to the function $p(x)$ via a partial differential equation of type (2). Corollary 1 reflects the fact that the Ismail–May operator R_n is associated to the polynomial $p(x) = x(1+x)^2$.

3 Auxiliary results

Because g is an entire function with $g(0) = 1 - \beta \neq 0$, we have $c_{\ell,0}(\beta) = (1 - \beta)^{-\ell}$. For $j \geq 1$, application of the chain rule yields

$$c_{\ell,j}(\beta) = \sum_{i=1}^j (-1)^i \ell^{\bar{i}} \frac{Q_{j,i}(\beta)}{(1 - \beta)^{\ell+i}}, \quad (10)$$

where $Q_{j,i}$ are certain polynomials which are independent of ℓ .

Lemma 1 *Let $\alpha \in \mathbb{R}$ and $|\beta| < 1$. Then the series*

$$\sum_{v=0}^{\infty} \omega_{\beta}(v, \alpha) z^v$$

converges for $|z| < e^{\beta-1}/\beta$.

Note that $e^{\beta-1}/\beta > 1$, for $0 < \beta < 1$.

Proof Since

$$\begin{aligned} \frac{\omega_{\beta}(v+1, \alpha) z^{v+1}}{\omega_{\beta}(v, \alpha) z^v} &= \frac{(\alpha + (v+1)\beta)^v}{(v+1)(\alpha + v\beta)^{v-1}} e^{-\beta} z = \frac{(\alpha + v\beta)(\alpha + (v+1)\beta)^v}{(v+1)(\alpha + v\beta)^v} e^{-\beta} z \\ &= \frac{\alpha + v\beta}{v+1} \left(1 + \frac{1}{\alpha/\beta + v}\right)^v e^{-\beta} z \rightarrow \beta e^{1-\beta} z \quad (v \rightarrow \infty) \end{aligned}$$

the ratio test implies the convergence of the series $\sum_{v=0}^{\infty} \omega_{\beta}(v, \alpha) z^v$ if $|\beta e^{1-\beta} z| < 1$. $\square$

In [1], the following formula was proved.

Lemma 2 *Let $0 < \alpha < +\infty$, $\beta \in [0, 1)$ and $\ell \in \mathbb{N}$. Then*

$$\sum_{v=0}^{\infty} v^{\ell} \omega_{\beta}(v, \alpha) = \sum_{j=0}^{\ell-1} \binom{\ell-1}{j} c_{\ell,j}(\beta) \alpha^{\ell-j}.$$

Lemma 3 *The moments of the Ismail–May operators (1) have the representation*

$$R_n e_0 = e_0,$$

$$(R_n e_r)(x) = \sum_{k=0}^{r-1} \frac{1}{n^k} \left(\frac{x}{1+x} \right)^{r-k} \sum_{\ell=0}^k \sigma(r, r-\ell) \binom{r-\ell-1}{k-\ell} c_{r-\ell, k-\ell} \left(\frac{x}{1+x} \right) \\ (r = 1, 2, \dots),$$

where $\sigma(r, r-\ell)$ denote the Stirling numbers of the second kind and $c_{\ell, j}(\beta)$ is as defined in (6).

In particular, we have $R_n e_1 = e_1$. Consequently, the Ismail–May operators preserve linear functions. Furthermore,

$$(R_n e_2)(x) = x^2 + \frac{x(1+x)^2}{n}$$

(cf. [8, Lemma 1]).

Proof of Lemma 3 For $r = 0$, the assertion is a direct consequence of (3). Now let $r \geq 1$. Using relation (7), i.e.,

$$v^r = \sum_{\ell=1}^r \sigma(r, \ell) v^\ell$$

we obtain

$$(R_n e_r)(x) = n^{-r} \sum_{\ell=1}^r \sigma(r, \ell) \sum_{v=0}^{\infty} \omega_{x/(1+x)} \left(v, n \frac{x}{1+x} \right) v^\ell$$

and, by Lemma 2, the desired formula follows after a short calculation. $\square$

For each positive real number x define the function ψ_x by $\psi_x(t) = t - x$. We study the central moments $(R_n \psi_x^s)(x)$ ($s = 0, 1, 2, \dots$).

Lemma 4 *The central moments of the operators (1) have the representation*

$$(R_n \psi_x^s)(x) = \sum_{k=1}^{s-1} \frac{x^{s-k}}{n^k} \sum_{\ell=0}^k \sum_{r=k+1}^s (-1)^{s-r} \binom{s}{r} \sigma(r, r-\ell) \binom{r-\ell-1}{k-\ell} \\ \times \left(\frac{1}{1+x} \right)^{r-k} c_{r-\ell, k-\ell} \left(\frac{x}{1+x} \right).$$

Proof of Lemma 4 For $s = 0$, the assertion is a direct consequence of Lemma 3. Now let $s \geq 1$. By application of the binomial formula, we obtain

$$(R_n \psi_x^s)(x) = \sum_{r=0}^s \binom{s}{r} (-x)^{s-r} (R_n e_r)(x)$$

$$\begin{aligned}
&= (-x)^s + \sum_{r=1}^s \binom{s}{r} (-x)^{s-r} \sum_{k=0}^{r-1} \frac{1}{n^k} \left(\frac{x}{1+x} \right)^{r-k} \\
&\quad \times \sum_{\ell=0}^k \sigma(r, r-\ell) \binom{r-\ell-1}{k-\ell} c_{r-\ell, k-\ell} \left(\frac{x}{1+x} \right).
\end{aligned}$$

Extracting the terms which are independent of n (i.e., the case $k = 0$), we see that the constant term

$$\begin{aligned}
&(-x)^s + \sum_{r=1}^s \binom{s}{r} (-x)^{s-r} \left(\frac{x}{1+x} \right)^r c_{r,0} \left(\frac{x}{1+x} \right) \\
&= (-x)^s + (-x)^s \sum_{r=1}^s (-1)^r \binom{s}{r} = 0
\end{aligned}$$

vanishes, where we used that $c_{r,0}(\beta) = \lim_{z \rightarrow 0} g^{-r}(z) = (1-\beta)^{-r}$ (see Eq. (6)). This is the desired formula. $\square$

As was recently proved by Gupta and Agrawal [8, Lemma 1], the central moments $(R_n \psi_x^s)(x)$ of the operators (1) decay with order $O(n^{-\lfloor (s+1)/2 \rfloor})$ as $n \rightarrow \infty$. We give an independent proof of this fact.

Lemma 5 *The central moments of the operators (1) satisfy*

$$(R_n \psi_x^s)(x) = O\left(n^{-\lfloor (s+1)/2 \rfloor}\right) \quad (n \rightarrow \infty).$$

Proof The assertion is trivial for $s = 0$. Now let $s \geq 1$. By Lemma 4, we have the representation

$$(R_n \psi_x^s)(x) = \sum_{k=1}^{s-1} \frac{x^{s-k}}{n^k} \sum_{\ell=0}^k M_{s,k,\ell}(x),$$

where

$$\begin{aligned}
M_{s,k,\ell}(x) &= \sum_{r=k+1}^s (-1)^{s-r} \binom{s}{r} \sigma(r, r-\ell) \binom{r-\ell-1}{k-\ell} \left(\frac{1}{1+x} \right)^{r-k} \\
&\quad \times c_{r-\ell, k-\ell} \left(\frac{x}{1+x} \right).
\end{aligned}$$

We have to show that $M_{s,k,\ell}(x) = 0$, if $2 \leq 2k < s$, for $\ell = 0, \dots, k$. First we consider the case $\ell = 0$. Because $\sigma(r, r) = 1$, we have

$$M_{s,k,0}(x) = \sum_{r=k+1}^s (-1)^{s-r} \binom{s}{r} \binom{r-1}{k} \left(\frac{1}{1+x} \right)^{r-k} c_{r,k} \left(\frac{x}{1+x} \right).$$

Noting that $k \geq 1$, we have, by (10),

$$c_{r,k}(\beta) = \sum_{i=1}^k (-1)^i r^{\bar{i}} \frac{Q_{k,i}(\beta)}{(1-\beta)^{r+i}}.$$

Hence,

$$\begin{aligned} M_{s,k,0}(x) &= \sum_{i=1}^k (-1)^i Q_{k,i} \left(\frac{x}{1+x} \right) \sum_{r=k+1}^s (-1)^{s-r} \binom{s}{r} r^{\bar{i}} \binom{r-1}{k} \left(\frac{1}{1+x} \right)^{r-k} (1+x)^{r+i} \\ &= \sum_{i=1}^k (-1)^i Q_{k,i} \left(\frac{x}{1+x} \right) (1+x)^{k+i} \sum_{r=k+1}^s (-1)^{s-r} \binom{s}{r} (r+1)^{\bar{i}-1} r \binom{r-1}{k}. \end{aligned}$$

The inner sum is equal to

$$\begin{aligned} &\frac{1}{k!} \sum_{r=k+1}^s (-1)^{s-r} \binom{s}{r} r^{k+1} P_{k-1}(r) \\ &= \frac{s^{k+1}}{k!} \sum_{r=0}^{s-k-1} (-1)^{s-k-1-r} \binom{s-k-1}{r} P_{k-1}(r+k+1), \end{aligned}$$

where P_{k-1} is a certain polynomial of degree at most $k-1$. The latter sum vanishes if $k-1 < s-k-1$ or equivalently, if $2k < s$. Now we turn to the case $\ell \geq 1$. We take advantage of the representation [3, pp. 226–227] of the Stirling numbers of the second kind

$$\sigma(r, r-\ell) = \sum_{\alpha=1}^{\ell} \binom{r}{\ell+\alpha} \sigma_2(\ell+\alpha, \alpha) \quad (1 \leq \ell \leq r)$$

in terms of the associated Stirling numbers of the second kind $\sigma_2(\mu, \nu)$ (see [3, Pages 221–222]). Therefore,

$$\begin{aligned} M_{s,k,\ell}(x) &= \sum_{\alpha=1}^{\ell} \sigma_2(\ell+\alpha, \alpha) \sum_{r=k+1}^s (-1)^{s-r} \binom{s}{r} \binom{r}{\ell+\alpha} \binom{r-\ell-1}{k-\ell} \left(\frac{1}{1+x} \right)^{r-k} \\ &\quad \times c_{r-\ell, k-\ell} \left(\frac{x}{1+x} \right). \end{aligned}$$

Note that

$$\binom{r}{\ell+\alpha} \binom{r-\ell-1}{k-\ell} = \binom{r}{k+1} P_{\alpha,k,\ell}(r),$$

where $P_{\alpha,k,\ell}$ is a certain polynomial of degree at most $\alpha - 1$ (whose coefficients may depend on α, k, ℓ). By (10), we have

$$c_{r-\ell,k-\ell}(\beta) = \sum_{i=0}^{k-\ell} (-1)^i (r-\ell)^{\bar{i}} \frac{Q_{k-\ell,i}(\beta)}{(1-\beta)^{r-\ell+i}}.$$

Finally, we obtain

$$\begin{aligned} M_{s,k,\ell}(x) &= \sum_{\alpha=1}^{\ell} \sigma_2(\ell + \alpha, \alpha) \sum_{r=k+1}^s (-1)^{s-r} \binom{s}{r} \binom{r}{k+1} P_{\alpha,k,\ell}(r) \left(\frac{1}{1+x}\right)^{r-k} \\ &\quad \times \sum_{i=0}^{k-\ell} (-1)^i (r-\ell)^{\bar{i}} Q_{k-\ell,i}\left(\frac{x}{1+x}\right) (1+x)^{r-\ell+i} \\ &= \sum_{\alpha=1}^{\ell} \sigma_2(\ell + \alpha, \alpha) \sum_{i=0}^{k-\ell} (-1)^i Q_{k-\ell,i}\left(\frac{x}{1+x}\right) (1+x)^{k-\ell+i} \\ &\quad \times \sum_{r=k+1}^s (-1)^{s-r} \binom{s}{r} \binom{r}{k+1} P_{\alpha,k,\ell}(r) (r-\ell)^{\bar{i}}. \end{aligned}$$

Making use of the binomial identity $\binom{s}{r} \binom{r}{k+1} = \binom{s}{k+1} \binom{s-k-1}{r-k-1}$ shows that the inner sum is equal to

$$\begin{aligned} &\binom{s}{k+1} \sum_{r=k+1}^s (-1)^{s-r} \binom{s-k-1}{r-k-1} P_{\alpha,k,\ell}(r) (r-\ell)^{\bar{i}} \\ &= \binom{s}{k+1} \sum_{r=0}^{s-k-1} (-1)^{s-k-1-r} \binom{s-k-1}{r} P_{\alpha,k,\ell}(r+k+1) (r+k+1-\ell)^{\bar{i}} \\ &= 0, \end{aligned}$$

if $k-1 < s-k-1$, because $P_{\alpha,k,\ell}(r+k+1)(r+k+1-\ell)^{\bar{i}}$ is a polynomial in the variable r of degree at most $\alpha-1+i \leq \ell-1+k-\ell = k-1$. Hence, $M_{s,k,\ell}(x) = 0$ if $2k < s$. This completes the proof. $\square$

Proposition 1 (*Localization theorem*) Let $\gamma, \delta > 0$ and fix $x > 0$. If $f \in C_{\gamma}[0, \infty)$ vanishes on the interval $(x-\delta, x+\delta) \cap (0, \infty)$ then

$$(R_n f)(x) = O(n^{-m}) \quad (n \rightarrow \infty),$$

for arbitrary large $m > 0$.

Proof Let $m \in \mathbb{N}$. For a certain positive constant C , we have

$$|(R_n f)(x)| \leq C \sum_{\substack{v \geq 0 \\ |\frac{v}{n} - x| \geq \delta}} r_{n,v}(x) \exp\left(\gamma \frac{v}{n}\right) \leq C \delta^{-m} \sum_{v=0}^{\infty} r_{n,v}(x) \left|\frac{v}{n} - x\right|^m \exp\left(\gamma \frac{v}{n}\right).$$

Application of the Schwarz inequality yields

$$|(R_n f)(x)| \leq C \delta^{-m} \sqrt{\sum_{v=0}^{\infty} r_{n,v}(x) \exp\left(2\gamma \frac{v}{n}\right)} \sqrt{\sum_{v=0}^{\infty} r_{n,v}(x) \left(\frac{v}{n} - x\right)^{2m}}.$$

The first root is finite, by Lemma 1, if n is sufficiently large. The second root is equal to

$$\sqrt{(R_n \psi_x^{2m})(x)} = O(n^{-m}) \quad (n \rightarrow \infty),$$

by Lemma 5. This completes the proof. $\square$

To derive the complete asymptotic expansion for the operators (1), we use a general approximation theorem for positive linear operators. It is a modification of a result due to Sikkema [16, Theorem 1].

Lemma 6 Let $q \in \mathbb{N}$ be even and fix $x > 0$. Furthermore, let $L_n : C(0, \infty) \rightarrow C(0, \infty)$ be a sequence of positive linear operators which are applicable to polynomials. Suppose that

$$(L_n \psi_x^s)(x) = O(n^{-\lfloor (s+1)/2 \rfloor}) \quad (n \rightarrow \infty) \quad \text{for } s = 0, 1, \dots, q+2. \quad (11)$$

Then, for each bounded function f on $(0, \infty)$, possessing a derivative of order q at x , there holds

$$(L_n f)(x) = \sum_{s=0}^q \frac{f^{(s)}(x)}{s!} (L_n \psi_x^s)(x) + o(n^{-q}) \quad (n \rightarrow \infty). \quad (12)$$

Furthermore, if $f^{(q+2)}(x)$ exists, the term $o(n^{-q})$ in Eq. (12) can be replaced by $O(n^{-(q+1)})$.

Proof of Theorem 1 Fix $x > 0$. In view of the localization property (Prop. 1) we can assume that the function f is bounded on $(0, \infty)$. By Lemma 4, Lemma 5 and Lemma 6 we obtain

$$(R_n f)(x) = f(x) + \sum_{s=2}^{2q} \frac{f^{(s)}(x)}{s!} \sum_{k=\lfloor (s+1)/2 \rfloor}^{s-1} \frac{x^{s-k}}{n^k} \sum_{\ell=0}^k \sum_{r=k+1}^s (-1)^{s-r} \binom{s}{r}$$

$$\begin{aligned} & \times \sigma(r, r - \ell) \binom{r - \ell - 1}{k - \ell} \left(\frac{1}{1 + x} \right)^{r - k} c_{r - \ell, k - \ell} \left(\frac{x}{1 + x} \right) + o(n^{-q}) \\ & = f(x) + \sum_{k=1}^q \frac{1}{n^k} \sum_{s=k+1}^{2k} A_{k,s}(x) f^{(s)}(x) + o(n^{-q}) \end{aligned}$$

as $n \rightarrow \infty$, where $A_{k,s}(x)$ are certain functions independent of n and f . $\square$

Acknowledgements The authors are grateful to the anonymous referee for valuable suggestions leading to a better exposition of the manuscript.

Funding Open Access funding enabled and organized by Projekt DEAL.

Data Availability Statement The authors declare that the data supporting the findings of this study are available within the article.

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