Nuclear operators on Banach function spaces

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Abstract

Let X be a Banach space and E be a perfect Banach function space over a finite measure space $(\Omega, \Sigma, \lambda)$ such that $L^\infty \subset E \subset L^1$. Let E' denote the Köthe dual of E and $\tau(E, E')$ stand for the natural Mackey topology on E . It is shown that every nuclear operator $T : E \rightarrow X$ between the locally convex space $(E, \tau(E, E'))$ and a Banach space X is Bochner representable. In particular, we obtain that a linear operator $T : L^\infty \rightarrow X$ between the locally convex space $(L^\infty, \tau(L^\infty, L^1))$ and a Banach space X is nuclear if and only if its representing measure $m_T : \Sigma \rightarrow X$ has the Radon-Nikodym property and $|m_T|(\Omega) = \|T\|_{nuc}$ (= the nuclear norm of T). As an application, it is shown that some natural kernel operators on L^∞ are nuclear. Moreover, it is shown that every nuclear operator $T : L^\infty \rightarrow X$ admits a factorization through some Orlicz space L^φ , that is, $T = S \circ i_\infty$, where $S : L^\varphi \rightarrow X$ is a Bochner representable and compact operator and $i_\infty : L^\infty \rightarrow L^\varphi$ is the inclusion map.

Keywords Banach function spaces · Mackey topologies · Mixed topologies · Vector measures · Nuclear operators · Bochner representable operators · Kernel operators · Radon–Nikodym property · Orlicz spaces · Orlicz-Bochner spaces

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1 Introduction and preliminaries

We assume that $(X, \|\cdot\|_X)$ is a real Banach space. For terminology concerning Riesz spaces and function spaces, we refer the reader to [9, 13, 27].

We assume that $(\Omega, \Sigma, \lambda)$ is a finite measure space. Let L^0 denote the corresponding space of λ -equivalence classes of all Σ -measurable real functions on Ω . Then L^0 is a super Dedekind complete Riesz space, equipped with the topology $\mathcal{T}_o$ of con-

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vergence in measure. By $\mathcal{S}(\Sigma)$ we denote the space of all real Σ -simple functions $s = \sum_{i=1}^n c_i \mathbb{1}_{A_i}$, where the sets $A_i \in \Sigma$ are pairwise disjoint.

Let $(E, \|\cdot\|_E)$ be a Banach function space, where E is an order ideal of L^0 such that $L^\infty \subset E \subset L^1$, and $\|\cdot\|_E$ is a Riesz norm on E . By $\mathcal{T}_E$ we denote the $\|\cdot\|_E$ -norm topology on E . By E' we denote the Köthe dual of E , that is,

$$E' := \left\{ v \in L^0 : \int_{\Omega} |u(\omega) v(\omega)| d\lambda < \infty \text{ for all } u \in E \right\}.$$

The associated norm $\|\cdot\|_{E'}$ on E' is defined for $v \in E'$ by

$$\|v\|_{E'} = \sup \left\{ \int_{\Omega} |u(\omega) v(\omega)| d\lambda : u \in E, \|u\|_E \leq 1 \right\}.$$

We will assume that E is perfect, that is, $E = E''$ and $\|u\|_E = \|u\|_{E''}$. The order continuous dual $E_n^\sim$ of E separates the points of E and $E_n^\sim$ can be identified with E' through the Riesz isomorphism $E' \ni v \mapsto F_v \in E_n^\sim$, where

$$F_v(u) = \int_{\Omega} u(\omega) v(\omega) d\lambda \text{ for } u \in E \text{ and } \|F_v\| = \|v\|_{E'}$$

(see [13, Theorem 6.1.1]). The Mackey topology $\tau(E, E')$ is a locally convex-solid Hausdorff topology with the Lebesgue property (see [9, Corollary 82H]). Then $\tau(E, E') \subset \mathcal{T}_E$ and $\tau(E, E') = \mathcal{T}_E$ if the norm $\|\cdot\|_E$ is order continuous.

The most important classes of Banach function spaces are Lebesgue spaces L^p ($1 \leq p \leq \infty$) and Orlicz spaces L^φ (see [19]).

Now we present a characterization of $(\tau(E, E'), \|\cdot\|_X)$ -continuous linear operators $T : E \rightarrow X$ (see [17, Proposition 2.2]).

Proposition 1.1 *For a bounded linear operator $T : E \rightarrow X$ the following statements are equivalent:*

- (i) T is $(\tau(E, E'), \|\cdot\|_X)$ -continuous.
- (ii) $\|T(u_n)\|_X \rightarrow 0$ if $u_n(\omega) \rightarrow 0$ λ -a.e. and $|u_n(\omega)| \leq |u(\omega)|$ λ -a.e. for some $u \in E$ and all $n \in \mathbb{N}$.
- (iii) For each $u \in E$, $\|T(u \mathbb{1}_{A_n})\|_X \rightarrow 0$ whenever $\lambda(A_n) \rightarrow 0$.

For terminology and basic facts concerning vector measure, we refer the reader to [4, 6, 7, 22]. For a finitely additive measure $m : \Sigma \rightarrow X$, by $|m|(A)$ we denote the variation of m on $A \in \Sigma$. A measure $m : \Sigma \rightarrow X$ is said to be λ -continuous if $\|m(A_n)\|_X \rightarrow 0$ whenever $\lambda(A_n) \rightarrow 0$.

Let $L^1(X)$ denote the Banach space of λ -equivalence classes of all X -valued Bochner integrable functions g defined on Ω , equipped with the norm $\|g\|_1 := \int_{\Omega} \|g(\omega)\|_X d\lambda$.

Recall that a λ -continuous measure $m : \Sigma \rightarrow X$ of finite variation is said to have the Radon-Nikodym property with respect to λ if there exists a function $g \in L^1(X)$

such that $m(A) = \int_A g(\omega) d\lambda$ for all $A \in \Sigma$. Then we write $m = g\lambda$ and a function g is called the *density* of m with respect to λ .

Assume that $m : \Sigma \rightarrow X$ is a λ -continuous measure. Following [6, § 13] for $A \in \Sigma$, we put

$$|m|_{E'}(A) := \sup \left\{ \sum_{i=1}^n |c_i| \|m(A_i)\|_X \right\},$$

where the supremum is taken for all functions $s = \sum_{i=1}^n c_i \mathbb{1}_{A_i} \in \mathcal{S}(\Sigma)$ such that $A_i \subset A$ for $1 \leq i \leq n$ and $\|s\|_E \leq 1$. The set function $|m|_{E'}$ will be called a E' -variation of the measure m .

If, in particular, $E = L^\infty$, then $|m|_{L^1}(A) = |m|(A)$ for $A \in \Sigma$.

Let $L^0(X)$ stand for the linear space of λ -equivalence classes of all strongly Σ -measurable functions $g : \Omega \rightarrow X$. Let

$$E(X) = \left\{ g \in L^0(X) : \|g(\cdot)\|_X \in E \right\}.$$

Then $E(X)$ equipped with the norm $\|g\|_{E(X)} := \|\cdot\|_E$ is a Banach space, called a Köthe-Bochner space (see [14]).

Definition 1.1 A bounded linear operator $T : E \rightarrow X$ is said to be *Bochner representable*, if there exists $g \in E'(X)$ such that

$$T(u) = \int_{\Omega} u(\omega) g(\omega) d\lambda \quad \text{for } u \in E.$$

The concept of nuclear operators between Banach spaces is due to Ruston [21]. Grothendieck carried over the concept of nuclear operators to locally convex spaces [10, 11] (see also [26, p. 289], [18], [23, Chap. 3, § 7], [4, Chap. 6], [5, 22]).

Following [23, Chap. 3, § 7] (see also [2, Chap. 4], [12, 17.3, p. 379]), we have

Definition 1.2 A linear operator $T : E \rightarrow X$ is said to be $\tau(E, E')$ -nuclear if there exist a sequence (v_n) in E' such that the family $\{F_{v_n} : n \in \mathbb{N}\}$ is $\tau(E, E')$ -equicontinuous, a bounded sequence (x_n) in X and a sequence $(\alpha_n) \in \ell^1$ such that

$$T(u) = \sum_{n=1}^{\infty} \alpha_n \left(\int_{\Omega} u v_n d\lambda \right) x_n \quad \text{for } u \in E. \quad (1.1)$$

Let

$$\|T\|_{nuc} := \inf \left\{ \sum_{n=1}^{\infty} |\alpha_n| \|v_n\|_{E'} \|x_n\|_X \right\},$$

where the infimum is taken over all sequences (v_n) in E' , (x_n) in X and $(\alpha_n) \in \ell^1$ such that T admits a representation (1.1).

It is known that a $\tau(E, E')$ -nuclear operator $T : E \rightarrow X$ is $(\tau(E, E'), \|\cdot\|_X)$ -continuous and $\tau(E, E')$ -compact, that is, $T(V)$ is relatively norm compact in X for some $\tau(E, E')$ -neighborhood V of 0 in E (see [23, Chap. 3, § 7, Corollary 1], [12, Corollary 4, p. 379]).

In this paper we study $\tau(E, E')$ -nuclear operators $T : E \rightarrow X$. In Section 2 it is shown that every $\tau(E, E')$ -nuclear operator $T : E \rightarrow X$ is Bochner representable (see Theorem 2.3 below). In particular, we obtain that a linear operator $T : L^\infty \rightarrow X$ is $\tau(L^\infty, L^1)$ -nuclear if and only if its representing measure $m_T : \Sigma \rightarrow X$ has the Radon-Nikodym property and $|m_T|(\Omega) = \|T\|_{nuc}$ (see Theorem 2.5 below). As an application, we obtain that some natural kernel operators on L^∞ are $\tau(L^\infty, L^1)$ -nuclear (see Proposition 2.9 below). In Section 3 it is shown that every $\tau(L^\infty, L^1)$ -nuclear operator $T : L^\infty \rightarrow X$ admits a factorization through some Orlicz space L^φ , that is, $T = S \circ i_\infty$, where $S : L^\varphi \rightarrow X$ is a Bochner representable, compact operator and $i_\infty : L^\infty \rightarrow L^\varphi$ denotes the inclusion map (see Corollary 3.5).

2 Nuclear operators on Banach function spaces

Assume that $T : E \rightarrow X$ is a linear operator. Then the measure $m_T : \Sigma \rightarrow X$ defined by

$$m_T(A) := T(\mathbb{1}_A) \text{ for } A \in \Sigma$$

is called a *representing measure* of T .

If, in particular, T is $(\tau(E, E'), \|\cdot\|_X)$ -continuous, then using Proposition 1.1 we obtain that m_T is countably additive. Since $m_T(A) = 0$ if $\lambda(A) = 0$, by the Pettis theorem m_T is λ -continuous, that is, $m_T \ll \lambda$.

The following lemma will be useful.

Lemma 2.1 *Let $T : E \rightarrow X$ be a $(\tau(E, E'), \|\cdot\|_X)$ -continuous linear operator. If $|m_T|_{E'}(\Omega) < \infty$ and m_T has the Radon-Nikodym property with respect to λ with a density $g \in L^1(X)$, then $g \in E'(X)$ and*

$$\|\mathbb{1}_A g\|_{E'(X)} = |m_T|_{E'}(A) \text{ for all } A \in \Sigma,$$

and

$$T(u) = \int_{\Omega} u(\omega) g(\omega) d\lambda \text{ for all } u \in E.$$

Proof First we shall show that for $A \in \Sigma$,

$$|m_T|_{E'}(A) = \sup \left\{ \int_A |s(\omega)| \|g(\omega)\|_X d\lambda : s \in \mathcal{S}(\Sigma), \|s\|_E \leq 1 \right\}. \quad (2.1)$$

Note that $|m_T|(A) = \int_A \|g(\omega)\|_X d\lambda$. For $s = \sum_{i=1}^k c_i \mathbb{1}_{A_i} \in \mathcal{S}(\Sigma)$, we have

$$\sum_{i=1}^k c_i m_T(A \cap A_i) = \sum_{i=1}^k c_i \int_{A \cap A_i} g(\omega) d\lambda = \int_A s(\omega) g(\omega) d\lambda.$$

We now show that for $s = \sum_{i=1}^k c_i \mathbb{1}_{A_i} \in \mathcal{S}(\Sigma)$ and $\|s\|_E \leq 1$, we have

$$\sum_{i=1}^k |c_i| \int_{A \cap A_i} \|g(\omega)\|_X d\lambda = \sum_{i=1}^k |c_i| |m_T|(A \cap A_i) \leq |m_T|_{E'}(A).$$

Indeed, let $\varepsilon > 0$ be given. Then for each $1 \leq i \leq k$, there exists a Σ -partition $(A_{i,j})_{j=1}^{j_i}$ of $A \cap A_i$ such that

$$|m_T|(A \cap A_i) \leq \sum_{j=1}^{j_i} \|m_T(A_{ij})\|_X + \frac{\varepsilon}{k|c_i|}.$$

Hence

$$\sum_{i=1}^k |c_i| |m_T|(A \cap A_i) \leq \sum_{i=1}^k \left(\sum_{j=1}^{j_i} \|c_i m_T(A_{ij})\|_X \right) + \varepsilon \leq |m_T|_{E'}(A) + \varepsilon,$$

because $\sum_{i=1}^k \left(\sum_{j=1}^{j_i} c_i \mathbb{1}_{A_{ij}} \right) = \sum_{i=1}^k c_i \mathbb{1}_{A_i}$.

Then we have

$$\begin{aligned} \sum_{i=1}^k |c_i| \|m_T(A \cap A_i)\|_X &\leq \sum_{i=1}^k |c_i| |m_T|(A \cap A_i) \\ &= \sum_{i=1}^k |c_i| \int_{A \cap A_i} \|g(\omega)\|_X d\lambda = \int_A \left(\sum_{i=1}^k |c_i| \mathbb{1}_{A_i}(\omega) \right) \|g(\omega)\|_X d\lambda \\ &= \int_A |s(\omega)| \|g(\omega)\|_X d\lambda \leq |m_T|_{E'}(A). \end{aligned}$$

Taking supremum on the left side, we get

$$|m_T|_{E'}(A) = \sup \left\{ \int_A |s(\omega)| \|g(\omega)\|_X d\lambda : s \in \mathcal{S}(\Sigma), \|s\|_E \leq 1 \right\}.$$

We shall now show that $g \in E'(X)$, that is, $\|g(\cdot)\|_X \in E'$.

Indeed, let $u \in E$. Then there exists a sequence (s_n) in $\mathcal{S}(\Sigma)$ such that $0 \leq s_n(\omega) \uparrow |u(\omega)|$ λ -a.e. (see [13, Corollary I.6]). Choose $c > 0$ such that $\|cu\|_E \leq 1$. Then by the Fatou lemma,

$$\int_{\Omega} c |u(\omega)| \|g(\omega)\|_X d\lambda \leq \sup_n \int_{\Omega} c s_n(\omega) \|g(\omega)\|_X d\lambda \leq |m_T|_{E'}(\Omega),$$

and this means that $\|g(\cdot)\|_X \in E'$, that is, $g \in E'(X)$. Moreover, for $A \in \Sigma$ and $u \in E$ with $\|u\|_E \leq 1$, using (2.1) we get

$$\int_{\Omega} |u(\omega)| \|\mathbb{1}_A(\omega) g(\omega)\|_X d\lambda \leq \sup_n \int_{\Omega} s_n(\omega) \|\mathbb{1}_A(\omega) g(\omega)\|_X d\lambda \leq |m_T|_{E'}(A)$$

and it follows that $\|\mathbb{1}_A g\|_{E'(X)} \leq |m_T|_{E'}(A)$. In view of (2.1) we get $|m_T|_{E'}(A) \leq \|\mathbb{1}_A g\|_{E'(X)}$. Hence $|m_T|_{E'}(A) = \|\mathbb{1}_A g\|_{E'(X)}$.

Note that for $s = \sum_{i=1}^k c_i \mathbb{1}_{A_i} \in \mathcal{S}(\Sigma)$, we have

$$T(s) = \sum_{i=1}^k c_i m_T(A_i) = \int_{\Omega} s(\omega) g(\omega) d\lambda.$$

Let $u \in E$ be given. Then there exists a sequence (s_n) in $\mathcal{S}(\Sigma)$ such that $|s_n(\omega) - u(\omega)| \rightarrow 0$ λ -a.e. and $|s_n(\omega)| \leq |u(\omega)|$ λ -a.e. for all $n \in \mathbb{N}$. Then $|s_n(\omega) - u(\omega)| \|g(\omega)\|_X \leq 2|u(\omega)| \|g(\omega)\|_X$ λ -a.e., where $u \|g(\cdot)\|_X \in L^1$. Hence by the Lebesgue dominated convergence theorem,

$$\left\| \int_{\Omega} s_n(\omega) g(\omega) d\lambda - \int_{\Omega} u(\omega) g(\omega) d\lambda \right\|_X \leq \int_{\Omega} |s_n(\omega) - u(\omega)| \|g(\omega)\|_X d\lambda \rightarrow 0.$$

On the other hand, in view of Proposition 1.1 we have

$$\|T(s_n) - T(u)\|_X = \|T(s_n - u)\|_X \rightarrow 0.$$

$$\text{Hence } T(u) = \int_{\Omega} u(\omega) g(\omega) d\lambda. \quad \square$$

As a consequence of Lemma 2.1, we have

Proposition 2.2 Assume that $T : E \rightarrow X$ is a Bochner representable operator, that is, there exists $g \in E'(X)$ such that

$$T(u) = \int_{\Omega} u(\omega) g(\omega) d\lambda \quad \text{for all } u \in E.$$

Then the following statements hold:

- (i) T is $(\tau(E, E'), \|\cdot\|_X)$ -continuous.
- (ii) For every $A \in \Sigma$, $|m_T|_{E'}(A) = \|\mathbb{1}_A g\|_{E'(X)}$.

Proof (i) Assume that $u_n(\omega) \rightarrow 0$ λ -a.e. and $|u_n(\omega)| \leq |u(\omega)|$ λ -a.e. for some $u \in E$ and all $n \in \mathbb{N}$. Since for $n \in \mathbb{N}$,

$$\|T(u_n)\|_X \leq \int_{\Omega} |u_n(\omega)| \|g(\omega)\|_X d\lambda,$$

where $u \|g(\cdot)\|_X \in L^1$, by the Lebesgue dominated convergence theorem, we get $\|T(u_n)\|_X \rightarrow 0$. Hence in view of Proposition 1.1 T is $(\tau(E, E'), \|\cdot\|_X)$ -continuous.

(ii) Assume that $s = \sum_{i=1}^k c_i \mathbb{1}_{A_i} \in \mathcal{S}(\Sigma)$ and $\|s\|_E \leq 1$. Then

$$\begin{aligned} \sum_{i=1}^k |c_i| \|m_T(A_i)\|_X &= \sum_{i=1}^k |c_i| \int_{A_i} \|g(\omega)\|_X d\lambda = \int_{\Omega} |s(\omega)| \|g(\omega)\|_X d\lambda \\ &\leq \|s\|_E \|g\|_{E'(X)} \leq \|g\|_{E'(X)}. \end{aligned}$$

Hence $|m_T|_{E'(\Omega)} \leq \|g\|_{E'(X)}$. Using (i) and Lemma 2.1, we get $|m_T|_{E'}(A) = \|\mathbb{1}_A g\|_{E'(X)}$ for all $A \in \Sigma$. $\square$

The following result shows a relationship between $\tau(E, E')$ -nuclear operators and Bochner representable operators $T : E \rightarrow X$.

Theorem 2.3 *Assume that $T : E \rightarrow X$ is a $\tau(E, E')$ -nuclear operator. Then T is Bochner representable and $|m_T|_{E'(\Omega)} \leq \|T\|_{nuc}$.*

Proof Let $\varepsilon > 0$ be given. Then there exists a bounded sequence (v_n) in E' , a bounded sequence (x_n) in X and a sequence $(\alpha_n) \in \ell^1$ such that

$$T(u) = \sum_{n=1}^{\infty} \alpha_n \left(\int_{\Omega} u v_n d\lambda \right) x_n \quad \text{for } u \in E$$

and

$$\sum_{n=1}^{\infty} |\alpha_n| \|v_n\|_{E'} \|x_n\|_X \leq \|T\|_{nuc} + \varepsilon. \quad (2.2)$$

Hence we have

$$m_T(A) = \sum_{n=1}^{\infty} \alpha_n \left(\int_A v_n d\lambda \right) x_n. \quad (2.3)$$

We shall now show that $|m_T|_{E'(\Omega)} \leq \|T\|_{nuc}$. Indeed, let $s = \sum_{i=1}^k c_i \mathbb{1}_{A_i} \in \mathcal{S}(\Sigma)$ with $\|s\|_E \leq 1$. Then using (2.2) we get

$$\begin{aligned} \sum_{i=1}^k |c_i| \|m_T(A_i)\|_X &= \sum_{i=1}^k |c_i| \sum_{n=1}^{\infty} \alpha_n \left(\int_{A_i} v_n(\omega) d\lambda \right) \|x_n\|_X \\ &\leq \sum_{n=1}^{\infty} |\alpha_n| \left(\sum_{i=1}^k |c_i| \int_{A_i} |v_n(\omega)| d\lambda \right) \|x_n\|_X \\ &= \sum_{n=1}^{\infty} |\alpha_n| \|x_n\|_X \int_{\Omega} |s(\omega)| |v_n(\omega)| d\lambda \end{aligned}$$

$$\leq \sum_{n=1}^{\infty} |\alpha_n| \|x_n\|_X \|v_n\|_{E'} \leq \|T\|_{nuc} + \varepsilon.$$

Hence we get

$$|m_T|_{E'}(\Omega) \leq \|T\|_{nuc}. \quad (2.4)$$

For $n \in \mathbb{N}$, let $g_n := \sum_{i=1}^n \alpha_i v_i \otimes x_i$. Then for $n, k \in \mathbb{N}$ with $n > k$, we have

$$\begin{aligned} \int_{\Omega} \|g_n(\omega) - g_k(\omega)\|_X d\lambda &= \int_{\Omega} \left\| \sum_{i=k+1}^n \alpha_i v_i(\omega) x_i \right\|_X d\lambda \\ &\leq \int_{\Omega} \left(\sum_{i=k+1}^n |\alpha_i| |v_i(\omega)| \|x_i\|_X \right) d\lambda = \sum_{i=k+1}^n |\alpha_i| \|x_i\|_X \int_{\Omega} |v_i| d\lambda \\ &\leq \sum_{i=k+1}^n |\alpha_i| \|x_i\|_X \|\mathbb{1}_{\Omega}\|_E \|v_i\|_{E'} \leq \sup_{j \in \mathbb{N}} \|x_j\|_X \sup_{j \in \mathbb{N}} \|v_j\|_{E'} \|\mathbb{1}_{\Omega}\|_E \sum_{i=k+1}^n |\alpha_i|. \end{aligned}$$

This follows that (g_n) is a Cauchy sequence in the Banach space $L^1(X)$, so there exists $g \in L^1(X)$ such that $\|g_n - g\|_1 \rightarrow 0$. One can easily show that

$$\left(\int_A v d\lambda \right) x = \int_A v(\omega) x d\lambda \quad \text{for } v \in E', x \in X, A \in \Sigma.$$

Hence for $A \in \Sigma$, we have

$$\begin{aligned} \left\| \sum_{i=1}^n \alpha_i \left(\int_A v_i d\lambda \right) x_i - \int_A g d\lambda \right\|_X &= \left\| \int_A \sum_{i=1}^n \alpha_i v_i(\omega) x_i d\lambda - \int_A g(\omega) d\lambda \right\|_X \\ &\leq \int_A \left\| \sum_{i=1}^n \alpha_i v_i(\omega) x_i - g(\omega) \right\|_X d\lambda = \int_A \|g_n(\omega) - g(\omega)\|_X d\lambda \rightarrow 0. \end{aligned}$$

Then in view of (2.3) for $A \in \Sigma$, we get

$$m_T(A) = \int_A g(\omega) d\lambda.$$

Using Lemma 2.1 and (2.4) we see that $g \in E'(X)$ with $\|g\|_{E'(X)} = |m_T|_{E'}(\Omega) \leq \|T\|_{nuc}$ and $T(u) = \int_{\Omega} u(\omega) g(\omega) d\lambda$ for all $u \in E$. $\square$

Now we shall study $\tau(L^{\infty}, L^1)$ -nuclear operators $T : L^{\infty} \rightarrow X$.

Making use of the Dunford–Pettis theorem (see [3, Theorem, p. 93]) we have

Theorem 2.4 *For a subset H of L^1 the following statements are equivalent:*

- (i) H is relatively weakly compact.
- (ii) $\sup_{v \in H} \|v\|_1 < \infty$ and H is uniformly integrable.
- (iii) $\{F_v : v \in H\}$ is $\tau(L^{\infty}, L^1)$ -equicontinuous.

Note that in view of Theorem 2.4 and Definition 1.2, a linear operator $T : L^\infty \rightarrow X$ is $\tau(L^\infty, L^1)$ -nuclear if there exist a bounded uniformly integrable sequence (v_n) in L^1 , a bounded sequence (x_n) in X and a sequence $(\alpha_n) \in \ell^1$ such that

$$T(u) = \sum_{n=1}^{\infty} \alpha_n \left(\int_{\Omega} u v_n d\lambda \right) x_n \quad \text{for all } u \in L^\infty \quad (2.5)$$

and then

$$\|T\|_{nuc} = \inf \left\{ \sum_{n=1}^{\infty} |\alpha_n| \|v_n\|_1 \|x_n\|_X \right\},$$

where the infimum is taken over all sequences (v_n) in L^1 , (x_n) in X and $(\alpha_n) \in \ell^1$ such that T admits a representation (2.5).

Now we can state our main result.

Theorem 2.5 *For a linear operator $T : L^\infty \rightarrow X$ the following statements are equivalent:*

- (i) m_T has the Radon-Nikodym property with respect to λ .
- (ii) T is Bochner representable.
- (iii) T is a $\tau(L^\infty, L^1)$ -nuclear operator.

In this case $|m_T|(\Omega) = \|T\|_{nuc}$.

Proof (i) $\Rightarrow$ (ii) Assume that (i) holds with the density $g \in L^1(X)$. Note that for $s \in \mathcal{S}(\Sigma)$, we have

$$\int_{\Omega} s dm_T = \int_{\Omega} s(\omega) g(\omega) d\lambda.$$

Let $u \in L^\infty$. Choose a sequence (s_n) in $\mathcal{S}(\Sigma)$ such that $\|u - s_n\|_\infty \rightarrow 0$. Then we have

$$T(u) = \int_{\Omega} u dm_T = \lim \int_{\Omega} s_n dm_T = \lim \int_{\Omega} s_n(\omega) g(\omega) d\lambda = \int_{\Omega} u(\omega) g(\omega) d\lambda.$$

(ii) $\Rightarrow$ (i) This is obvious.

(ii) $\Rightarrow$ (iii) Assume that (ii) holds, that is, there exists $g \in L^1(X)$ such that

$$T(u) = \int_{\Omega} u(\omega) g(\omega) d\lambda \quad \text{for all } u \in L^\infty.$$

Let $L^1 \hat{\otimes} X$ denote the projective tensor product of L^1 and X , equipped with the norm π defined for $w \in L^1 \hat{\otimes} X$, by

$$\pi(w) := \inf \left\{ \sum_{n=1}^{\infty} |\alpha_n| \|v_n\|_1 \|x_n\|_X \right\},$$

where the infimum is taken over all sequences (v_n) in L^1 , (x_n) in X with $\lim \|v_n\|_1 = 0 = \lim \|x_n\|_X$ and $(\alpha_n) \in \ell^1$ such that $w = \sum_{n=1}^{\infty} \alpha_n v_n \otimes x_n$ (see [22, Proposition 2.8, pp. 21–22]). It is known that $L^1 \hat{\otimes} X$ is isometrically isomorphic to $L^1(X)$ through the isometry J , where

$$J(v \otimes x) := v(\cdot) x \text{ for } v \in L^1, x \in X$$

(see [4, Example 10, p. 228], [22, Example 2.19, p. 29], [5, Theorem 1.1.10, p. 14]). Let $\varepsilon > 0$ be given. Then there exist sequence (v_n) in L^1 and (x_n) in X with $\lim_n \|v_n\|_1 = 0 = \lim \|x_n\|_X$ and $(\alpha_n) \in \ell^1$ such that

$$J^{-1}(g) = \sum_{n=1}^{\infty} \alpha_n v_n \otimes x_n \text{ in } (L^1 \hat{\otimes} X, \pi)$$

and

$$\sum_{n=1}^{\infty} |\alpha_n| \|v_n\|_1 \|x_n\|_X \leq \pi(J^{-1}(g)) + \varepsilon = \|g\|_1 + \varepsilon. \quad (2.6)$$

Hence

$$g = J\left(\sum_{n=1}^{\infty} \alpha_n v_n \otimes x_n\right) = \sum_{n=1}^{\infty} \alpha_n v_n(\cdot) x_n \text{ in } (L^1(X), \|\cdot\|_1).$$

Note that

$$\begin{aligned} & \left\| \int_{\Omega} u(\omega) g(\omega) d\lambda - \sum_{i=1}^n \alpha_i \left(\int_{\Omega} u(\omega) v_i(\omega) d\lambda \right) x_i \right\|_X \\ & \leq \int_{\Omega} |u(\omega)| \left\| g(\omega) - \sum_{i=1}^n \alpha_i v_i(\omega) x_i \right\|_X d\lambda \\ & \leq \|u\|_{\infty} \int_{\Omega} \left\| g(\omega) - \sum_{i=1}^n \alpha_i v_i(\omega) x_i \right\|_X d\lambda = \|u\|_{\infty} \cdot \left\| g - \sum_{i=1}^n \alpha_i v_i \otimes x_i \right\|_1 \end{aligned}$$

Hence

$$T(u) = \sum_{n=1}^{\infty} \alpha_n \left(\int_{\Omega} u v_n d\lambda \right) x_n \text{ for } u \in L^{\infty}.$$

Since $\lim \|v_n\|_1 = 0$, the set $\{v_n : n \in \mathbb{N}\}$ is uniformly integrable and it follows that T is $\tau(L^{\infty}, L^1)$ -nuclear. In view of (2.6) we get

$$\|T\|_{nuc} \leq \|g\|_1 = |m_T|(\Omega). \quad (2.7)$$

(iii) $\Rightarrow$ (ii) Assume that (iii) holds. Then by Theorem 2.3 T is Bochner representable and

$$|m_T|(\Omega) = \|g\|_1 \leq \|T\|_{nuc}. \quad (2.8)$$

Thus (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) hold and using (2.7) and (2.8), we get $|m_T|(\Omega) = \|T\|_{nuc}$. $\square$

Let $ba_\lambda(\Sigma)$ denote the Banach space of all bounded finitely additive real measures μ on Σ such that $\mu(A) = 0$ if $\lambda(A) = 0$, equipped with norm $\|\mu\| := |\mu|(\Omega)$. The Banach dual $(L^\infty)^*$ of L^∞ can be identified with $ba_\lambda(\Sigma)$ through the integration mapping $ba_\lambda(\Sigma) \ni \mu \mapsto F_\mu \in (L^\infty)^*$, where $F_\mu(u) = \int_\Omega u d\mu$ for all $u \in L^\infty$ and $\|\mu\|(\Omega) = \|F_\mu\|$.

Recall (see [26, p. 279]) that a linear operator $T : L^\infty \rightarrow X$ is said to be *nuclear* if there exist a bounded sequence (μ_n) in $ba_\lambda(\Sigma)$, a bounded sequence (x_n) in X and a sequence $(\alpha_n) \in \ell^1$ such that

$$T(u) = \sum_{n=1}^{\infty} \alpha_n \left(\int_{\Omega} u d\mu_n \right) x_n \quad \text{for all } u \in L^\infty.$$

It is well known that a bounded linear operator $T : L^\infty \rightarrow X$ is $(\tau(L^\infty, L^1), \|\cdot\|_X)$ -continuous if and only if its representing measure $m_T : \Sigma \rightarrow X$ is λ -continuous (see [17, Proposition 3.1]). Hence due to Swartz (see [24, Theorem 1 and Theorem 3]) we have

Theorem 2.6 *Assume that $T : L^\infty \rightarrow X$ is a $(\tau(L^\infty, L^1), \|\cdot\|_X)$ -continuous linear operator. Then the following statements are equivalent:*

- (i) T is nuclear.
- (ii) m_T has the Radon-Nikodym property with respect to λ .

Combining Theorem 2.5 and Theorem 2.6, we get

Corollary 2.7 *For a linear operator $T : L^\infty \rightarrow X$ the following statements are equivalent:*

- (i) T is $\tau(L^\infty, L^1)$ -nuclear.
- (ii) T is $(\tau(L^\infty, L^1), \|\cdot\|_X)$ -continuous and nuclear.

As an application of Theorem 2.5 we show that some natural kernel operators on L^∞ are $\tau(L^\infty, L^1)$ -nuclear.

Assume that K_1 and K_2 are compact Hausdorff spaces and $k(\cdot, \cdot) \in C(K_2 \times K_1)$. Let $\mathcal{B}_0$ be the σ -algebra of Borel sets in K_1 and $\lambda : \mathcal{B}_0 \rightarrow [0, \infty)$ be a countably additive measure.

We will need the following lemma.

Lemma 2.8 *For every $u \in L^\infty(\lambda)$, the mapping $\Psi_u : K_1 \ni s \mapsto u(s) k(\cdot, s) \in C(K_2)$ is continuous.*

Proof Let $s_0 \in K_1$ and $\varepsilon > 0$ be given. Then for every $t \in K_2$ there exist a neighborhood V_t of t and a neighborhood W_t of s_0 such that

$$|k(z, s) - k(t, s_0)| \leq \frac{\varepsilon}{\|u\|_\infty} \quad \text{for all } z \in V_t, s \in W_t.$$

Hence there exist $t_1, \dots, t_n \in K_2$ such that $K_2 = \bigcup_{i=1}^n V_{t_i}$. Let us put $W := \bigcap_{i=1}^n W_{t_i}$. For $t \in K_2$, choose i_0 with $1 \leq i_0 \leq n$ such that $t \in V_{t_{i_0}}$. Then for $s \in W$, we have $|k(t, s) - k(t, s_0)| \leq \frac{\varepsilon}{\|u\|_\infty}$. Hence

$$\|\Psi_u(s) - \Psi_u(s_0)\|_\infty = \sup_{t \in K_2} |k(t, s) - k(t, s_0)| \|u\|_\infty \leq \varepsilon.$$

This means that Ψ_u is continuous. $\square$

Note that the Banach space $C(K_1, C(K_2))$ can be embedded in the Banach space $L^1(C(K_2))$ such that with each function from $C(K_1, C(K_2))$ is associated its λ -equivalence class in $L^1(C(K_2))$.

In view of Lemma 2.8 we can define a kernel operator $T_k : L^\infty(\lambda) \rightarrow C(K_2)$ by

$$T_k(u) := \int_{K_1} u(s) k(\cdot, s) d\lambda(s) \quad \text{for all } u \in L^\infty(\lambda).$$

For $t \in K_2$, let $\delta_t(w) := w(t)$ for all $w \in C(K_2)$. Then $\delta_t \in C(K_2)^*$ and according to the Hille's theorem (see [DU, Theorem 6, p.47]), we have

$$T_k(u)(t) = \delta_t(T_k(u)) = \int_{K_1} u(s) k(t, s) d\lambda(s) \quad \text{for all } u \in L^\infty(\lambda), t \in K_2.$$

Then

$$m_{T_k}(A) = T_k(\mathbb{1}_A) = \int_A k(\cdot, s) d\lambda(s) \quad \text{for all } A \in \mathcal{B}_O,$$

where the mapping $K_1 \ni s \mapsto k(\cdot, s) \in C(K_2)$ belongs to $L^1(C(K_2))$.

Hence for $A \in \mathcal{B}_O$,

$$|m_{T_k}|(A) = \int_A \|k(\cdot, s)\|_\infty d\lambda(s) = \int_A \sup_{t \in K_2} |k(t, s)| d\lambda(s).$$

Hence as a consequence of Theorem 2.5 we have

Proposition 2.9 *The kernel operator $T_k : L^\infty(\lambda) \rightarrow C(K_2)$ is $\tau(L^\infty(\lambda), L^1(\lambda))$ -nuclear and*

$$\|T_k\|_{nuc} = \int_{K_1} \sup_{t \in K_2} |k(t, s)| d\lambda(s).$$

3 Application of the theory of Orlicz spaces to vector measures

First we recall terminology and basic facts concerning Orlicz spaces and Orlicz-Bochner spaces (see [19] for more details). By a *Young function* we mean here a convex continuous mapping $\varphi : [0, \infty) \rightarrow [0, \infty)$ that vanishes only at 0 and $\varphi(t)/t \rightarrow 0$ as $t \rightarrow 0$ and $\varphi(t)/t \rightarrow \infty$ as $t \rightarrow \infty$. By φ^* we denote the complementary function of φ in the sense of Young, that is, $\varphi^*(t) = \sup\{ts - \varphi(s) : s \geq 0\}$ for $t \geq 0$. Note that $\varphi^{**} = \varphi$.

The corresponding *Orlicz space* L^φ is an ideal of L^0 defined by

$$L^\varphi := \left\{ u \in L^0 : \int_\Omega \varphi(\alpha |u(\omega)|) d\lambda < \infty \text{ for some } \alpha > 0 \right\},$$

and equipped with the topology $\mathcal{T}_\varphi$, defined by two equivalent norms:

$$\begin{aligned} \|u\|_\varphi &:= \inf \left\{ \alpha > 0 : \int_\Omega \varphi(|u(\omega)| / \alpha) d\lambda \leq 1 \right\}, \\ \|u\|_\varphi^0 &:= \sup \left\{ \int_\Omega |u(\omega) v(\omega)| d\lambda : v \in L^{\varphi^*}, \int_\Omega \varphi^*(|v(\omega)|) d\lambda \leq 1 \right\}, \end{aligned}$$

called the Luxemburg norm and the Orlicz norm. Then we have:

$\|u\|_\varphi \leq 1$ if and only if $\int_\Omega \varphi(|u(\omega)|) d\lambda \leq 1$,

$\|u_n\|_\varphi \rightarrow 0$ if and only if $\int_\Omega \varphi(\alpha |u_n(\omega)|) d\lambda \rightarrow 0$ for every $\alpha > 0$,

$$\begin{aligned} E^\varphi &:= \left\{ u \in L^\varphi : \int_\Omega \varphi(\alpha |u(\omega)|) d\lambda < \infty \text{ for all } \alpha > 0 \right\} \\ &= \left\{ u \in L^\varphi : \|\mathbb{1}_{A_n} u\|_\varphi \rightarrow 0 \text{ if } \lambda(A_n) \rightarrow 0 \right\}, \end{aligned}$$

$$(L^\varphi)' = L^{\varphi^*}.$$

The *Orlicz-Bochner space* $L^\varphi(X)$ is defined by

$$\begin{aligned} L^\varphi(X) &:= \left\{ g \in L^0(X) : \int_\Omega \varphi(\alpha \|g(\omega)\|_X) d\lambda < \infty \text{ for some } \alpha > 0 \right\} \\ &= \{g \in L^0(X) : \|g(\cdot)\|_X \in L^\varphi\}. \end{aligned}$$

For $g \in L^\varphi(X)$, let

$$\|g\|_\varphi := \|\|g(\cdot)\|_X\|_\varphi \quad \text{and} \quad \|g\|_\varphi^0 := \|\|g(\cdot)\|_X\|_\varphi^0.$$

Then

$$E^\varphi(X) := \left\{ g \in L^0(X) : \|g(\cdot)\|_X \in E^\varphi \right\}.$$

Assume that φ is Young function. If $m : \Sigma \rightarrow X$ is a λ -continuous measure, we write $|m|_{\varphi^*}$ instead of $|m|_{(L^\varphi)'}$.

Lemma 3.1 Assume that $m : \Sigma \rightarrow X$ is a λ -continuous measure and φ is Young function. Then for $A \in \Sigma$, we have

$$|m|(A) \leq \left(\varphi^{-1} \left(\frac{1}{\lambda(A)} \right) \right)^{-1} |m|_{\varphi^*}(A).$$

Proof Let $A \in \Sigma$ and $\varepsilon > 0$ be given. Then there is a Σ -partition $(A_i)_{i=1}^n$ of A such that

$$|m|(A) \leq \sum_{i=1}^n \|m(A_i)\|_X + \left(\varphi^{-1} \left(\frac{1}{\lambda(A)} \right) \right)^{-1} \varepsilon.$$

It is known that $\|\mathbb{1}_A\|_{\varphi} = \left(\varphi^{-1} \left(\frac{1}{\lambda(A)} \right) \right)^{-1}$ (see [RR, Chap. 3.4, Corollary 7, p. 79]).

Hence

$$\left\| \sum_{i=1}^n \varphi^{-1} \left(\frac{1}{\lambda(A)} \right) \mathbb{1}_{A_i} \right\|_{\varphi} = \left\| \varphi^{-1} \left(\frac{1}{\lambda(A)} \right) \mathbb{1}_A \right\|_{\varphi} = 1$$

and

$$\varphi^{-1} \left(\frac{1}{\lambda(A)} \right) |m|(A) \leq \sum_{i=1}^n \varphi^{-1} \left(\frac{1}{\lambda(A)} \right) \|m(A_i)\|_X + \varepsilon.$$

It follows that $\varphi^{-1} \left(\frac{1}{\lambda(A)} \right) |m|(A) \leq |m|_{\varphi^*}(A)$, so

$$|m|(A) \leq \varphi^{-1} \left(\frac{1}{\lambda(A)} \right)^{-1} |m|_{\varphi^*}(A).$$

□

By $\gamma[\mathcal{T}_{\varphi}, \mathcal{T}_0|_{L^{\varphi}}]$ (in brief, γ_{φ}) we denote the natural *mixed topology* on L^{φ} (in the sense of Wiweger), that is, γ_{φ} is the finest linear topology that agrees with $\mathcal{T}_0$ on $\mathcal{T}_{\varphi}$ -bounded sets in L^{φ} (see [1, 25] for more details). Then γ_{φ} is a locally convex-solid Hausdorff topology and γ_{φ} and $\mathcal{T}_{\varphi}$ have the same bounded sets. This means that $(L^{\varphi}, \gamma_{\varphi})$ is a generalized DF-space (see [20]) and it follows that $(L^{\varphi}, \gamma_{\varphi})$ is quasi-normable (see [20, p. 422]). Hence as a consequence of the Grothendieck's classical result (see [20, p. 429]), we have

Proposition 3.2 For a linear operator $S : L^{\varphi} \rightarrow X$ the following statements are equivalent:

- (i) S is $(\gamma_{\varphi}, \|\cdot\|_X)$ -continuous and compact.
- (ii) S is γ_{φ} -compact, that is, there exists a γ_{φ} -neighborhood V of 0 in L^{φ} such that $T(V)$ is relatively norm compact in X .

We say that a Young function φ *increases essentially more rapidly* than another ψ (in symbols, $\psi \ll \varphi$) if for arbitrary $c > 0$, $\psi(ct)/\varphi(t) \rightarrow 0$ as $t \rightarrow 0$ and $t \rightarrow \infty$. Note that $L^\varphi \subset E^\psi$ if $\psi \ll \varphi$.

The following result will be useful (see [16, Theorem 2.1]).

Theorem 3.3 *Let φ be a Young function. Then*

- (i) γ_φ is generated by the family of norms $\{\|\cdot\|_\psi|_{L^\varphi} : \psi \ll \varphi\}$.
- (ii) $(L^\varphi, \gamma_\varphi)^* = \{F_v : v \in E^{\varphi^*}\}$, where $F_v(u) = \int_\Omega u(\omega) v(\omega) d\lambda$ for all $u \in L^\varphi$.

According to [15, Corollary 1.6] we have the following identity:

$$L^1 = \bigcup_{\varphi} E^\varphi, \text{ where } \varphi \text{ runs over the family of all Young functions.} \quad (3.1)$$

Now we can state the main result in this section.

Theorem 3.4 *Assume that a measure $m : \Sigma \rightarrow X$ has the Radon-Nikodym property with the density $g \in L^1(X)$. Then there exists a Young function φ such that $g \in E^{\varphi^*}(X)$ and the following statements hold:*

- (i) *The operator $S_g : L^\varphi \rightarrow X$ defined by*

$$S_g(u) := \int_\Omega u(\omega) g(\omega) d\lambda \text{ for all } u \in L^\varphi$$

is $(\gamma_\varphi, \|\cdot\|_X)$ -continuous and compact.

- (ii) S_g is γ_φ -compact, that is, there exists a Young function ψ with $\psi \ll \varphi$ such that $\{\int_\Omega u(\omega) g(\omega) d\lambda : u \in L^\varphi, \|u\|_\psi \leq 1\}$ is a relatively norm compact subset of X .
- (iii) $|m|_{\varphi^*}(\Omega) < \infty$ and $|m|_{\varphi^*}(A) = \|\mathbb{1}_A g\|_{\varphi^*}^0$ for every $A \in \Sigma$.
- (iv) $|m|_{\varphi^*}$ is λ -continuous, that is, $|m|_{\varphi^*}(A_n) \rightarrow 0$ if $\lambda(A_n) \rightarrow 0$.

Proof In view of (3.1) there exists a Young function ψ such that $g \in E^\psi(X)$, that is, $\|g(\cdot)\|_X \in E^\psi$. Hence $|u| \|g(\cdot)\|_X \in L^1$ for every $u \in L^{\psi^*}$. Let us put $\varphi = \psi^*$. Then $\varphi^* = \psi^{**} = \psi$ and $(L^\varphi)' = L^{\varphi^*} = L^\psi$ and $L^{\psi^*} = L^\varphi$, and so $g \in E^{\varphi^*}(X)$. Then for $u \in L^\varphi$, we have

$$\|S_g(u)\|_X \leq \int_\Omega |u(\omega)| \|g(\omega)\|_X d\lambda = F_{\|g(\cdot)\|_X}(|u|),$$

where $\|g(\cdot)\|_X \in E^{\varphi^*}$. The inequality shows that S_g is $(|\sigma|(L^\varphi, E^{\varphi^*}), \|\cdot\|_X)$ -continuous, where $|\sigma|$ denotes the absolute weak topology. Since $(|\sigma|(L^\varphi, E^{\varphi^*}))$ is the coarsest locally convex-solid topology on L^φ with dual E^{φ^*} , and by Theorem 3.3(ii) γ_φ is such a topology, it follows that S_g is $(\gamma_\varphi, \|\cdot\|_X)$ -continuous.

To show that S_g is compact, choose a sequence (f_n) of X -valued simple functions on Ω such that $\|f_n(\omega) - g(\omega)\|_X \rightarrow 0$ and $\|f_n(\omega)\|_X \leq \|g(\omega)\|_X$ λ -a.e. and for all $n \in \mathbb{N}$ (see [7, Theorem 6, p. 4]). Hence for every $\alpha > 0$, we have that $\varphi^*(\alpha \|f_n(\omega) - g(\omega)\|_X) \rightarrow 0$ λ -a.e. and $\varphi^*(\alpha \|f_n(\omega) - g(\omega)\|_X) \leq \varphi^*(2\alpha \|g(\omega)\|_X)$ λ -a.e. and for all

$n \in \mathbb{N}$, where $\int_{\Omega} \varphi^*(2\alpha \|g(\omega)\|_X) d\lambda < \infty$. By the Lebesgue dominated convergence theorem, $\int_{\Omega} \varphi^*(\alpha \|f_n(\omega) - g(\omega)\|_X) d\lambda \rightarrow 0$ and this means that $\|f_n - g\|_{\varphi^*}^0 \rightarrow 0$.

For each $n \in \mathbb{N}$, let $S_n : L^{\varphi} \rightarrow X$ be a linear operator defined by

$$S_n(u) := \int_{\Omega} u(\omega) f_n(\omega) d\lambda \quad \text{for all } u \in L^{\varphi}.$$

Note that the range of each S_n is contained in the span of finite set of values of f_n . Therefore S_n is compact and by the Hölder's inequality for each $u \in L^{\varphi}$, we have

$$\begin{aligned} \|(S_n - S_g)(u)\|_X &= \left\| \int_{\Omega} u(\omega)(f_n(\omega) - g(\omega)) d\lambda \right\|_X \\ &\leq \int_{\Omega} |u(\omega)| \|f_n(\omega) - g(\omega)\|_X d\lambda \leq \|u\|_{\varphi} \|f_n - g\|_{\varphi^*}^0. \end{aligned}$$

It follows that $\|S_n - S_g\| \rightarrow 0$, so S_g is a compact operator.

(ii) In view of (i) and Proposition 3.2 S_g is γ_{φ} -compact. Using Theorem 3.3 we obtain that (ii) holds.

(iii) Let $s = \sum_{i=1}^k c_i \mathbb{1}_{A_i} \in \mathcal{S}(\Sigma)$ and $\|s\|_{\varphi} \leq 1$. Then

$$\sum_{i=1}^k |c_i| \|m(A_i)\|_X = \sum_{i=1}^k |c_i| \int_{A_i} \|g(\omega)\|_X d\lambda = \int_{\Omega} |s(\omega)| \|g(\omega)\|_X d\lambda \leq \|g\|_{\varphi^*}^0$$

and hence $|m|_{\varphi^*}(\Omega) \leq \|g\|_{\varphi^*}^0$. Note that $m = m_{S_g}$. Hence using Lemma 2.1 we have that $|m_{\varphi^*}|(A) = \|\mathbb{1}_A g\|_{\varphi^*}^0$ for $A \in \Sigma$.

(iv) This follows from (iii) because $g \in E^{\varphi^*}(X)$. $\square$

Let $i_{\infty} : L^{\infty} \rightarrow L^{\varphi}$ denotes the inclusion map. Note that i_{∞} is $(\sigma(L^{\infty}, L^1), \sigma(L^{\varphi}, L^{\varphi^*}))$ -continuous, and it follows that i_{∞} is $(\tau(L^{\infty}, L^1), \tau(L^{\varphi}, L^{\varphi^*}))$ -continuous (see [8, Theorem 8.6.1]). Since $\gamma_{\varphi} \subset \tau(L^{\varphi}, L^{\varphi^*})$, we obtain that i_{∞} is $(\tau(L^{\infty}, L^1), \gamma_{\varphi})$ -continuous.

As a consequence of Theorem 3.4 and Theorem 2.5, we can show that every $\tau(L^{\infty}, L^1)$ -nuclear operator $T : L^{\infty} \rightarrow X$ admits a factorization through some Orlicz space L^{φ} .

Corollary 3.5 Assume that $T : L^{\infty} \rightarrow X$ is a $\tau(L^{\infty}, L^1)$ -nuclear operator. Then there exists a Young function φ such that $T = S \circ i_{\infty}$, where

- (i) $S : L^{\varphi} \rightarrow X$ is a Bochner representable and γ_{φ} -compact linear operator.
- (ii) $|m_S|_{\varphi^*}(A_n) \rightarrow 0$ if $\lambda(A_n) \rightarrow 0$.

Proof (i) In view of Theorem 2.5 the representing measure $m_T : \Sigma \rightarrow X$ has the Radon-Nikodym property with respect to λ , that is, $m_T = g\lambda$, where $g \in L^1(X)$. Hence according to Theorem 3.4 there exists a Young function φ such that $g \in E^{\varphi^*}(X)$ and an operator $S : L^\varphi \rightarrow X$ defined by

$$S(u) := \int_{\Omega} u(\omega) g(\omega) d\lambda \quad \text{for all } u \in L^\varphi,$$

is γ_φ -compact. Note that for $u \in L^\infty$, $T(u) = \int_{\Omega} u dm_T = \int_{\Omega} u(\omega) g(\omega) d\lambda = S(u)$.

(ii) Since $m_S(A) = \int_A g(\omega) d\lambda$ for all $A \in \Sigma$, using Theorem 3.4 we have that $|m_S|_{\varphi^*}(A_n) \rightarrow 0$ if $\lambda(A_n) \rightarrow 0$.

□

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