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The lateral order on Riesz spaces and orthogonally additive operators

Volodymyr Mykhaylyuk[1,2] · **Marat Pliev**[3,6] · **Mikhail Popov**[4,5]

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Abstract

The paper contains a systematic study of the lateral partial order $\sqsubseteq$ in a Riesz space (the relation $x \sqsubseteq y$ means that x is a fragment of y) with applications to nonlinear analysis of Riesz spaces. We introduce and study lateral fields, lateral ideals, lateral bands and consistent subsets and show the importance of these notions to the theory of orthogonally additive operators, like ideals and bands are important for linear operators. We prove the existence of a lateral band projection, provide an elegant formula for it and prove some properties of this orthogonally additive operator. One of our main results (Theorem 7.5) asserts that, if D is a lateral field in a Riesz space E with the intersection property, X a vector space and $T_0 : D \to X$ an orthogonally additive operator, then there exists an orthogonally additive extension $T : E \to X$ of T_0. The intersection property of E means that every two-point subset of E has an infimum with respect to the lateral order. In particular, the principal projection property implies the intersection property.

Keywords Riesz spaces · Fragments · Orthogonally additive operators · Laterally continuous operators

Mathematics Subject Classification Primary 46A40; Secondary 47H30 · 47H99

1 Introduction

The main idea of the paper is to show the importance of the so called *lateral partial order* $\sqsubseteq$ on a Riesz space for analysis of Riesz spaces, especially for the study of orthogonally additive operators. By the relation $x \sqsubseteq y$ between elements of a Riesz

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Extended author information available on the last page of the article

space we mean that x is a *fragment*[1] of y, that is, $x \perp (y - x)$. To the best of our knowledge, we present here the first systematic study of the lateral order.

Among well known results on the subject, it is worth mentioning that the set $\mathfrak{F}_e$ of all fragments of an element e of a Riesz space E is a Boolean algebra with respect to the lateral order with zero 0 and unit e. A prominent role of $\mathfrak{F}_e$ was discovered in the classical Freudenthal Spectral Theorem. First the importance of lateral order was mentioned by Abramovich in 1971 [1], where the author showed that, in a Dedekind complete (respectively, order σ-complete) normed lattice every norm bounded increasing net (respectively, sequence) of positive elements has a supremum, provided so does every norm bounded laterally increasing net (respectively, sequence); see also [2] for another application of laterally increasing nets.

While the present paper was being prepared over the last couple of years, several papers were published by the authors and their co-authors which used some of the tools developed here concerning lateral order. (see [10,11,17–19,21,23]).

Organization of the paper. In Sect. 2 we give necessary information on Riesz spaces and orthogonally additive operators. In Sect. 3 we develop elementary techniques for investigation of the lateral order. Since every subset of a Riesz space is laterally bounded from below by zero, if we speak of a *laterally bounded subset* we mean that it is laterally bounded from above. We show that every laterally bounded finite set has a lateral supremum and provide with a formula for it. If a Riesz space E has the principal projection property then any finite set has a lateral infimum. Moreover, the lateral infimum xy of a two-point set $\{x, y\}$ satisfies $\mathfrak{F}_x \cap \mathfrak{F}_y = \mathfrak{F}_{xy}$. However, in an arbitrary Riesz space a two-point set need not have a lateral infimum. We do not know whether every Riesz space without the principal projection property admits such an example. We define in an obvious way and investigate the set-theoretical operations on the set $\mathfrak{F}_e$ of all fragments of an element $e \in E$. If the lateral infimum xy of a two-point set $\{x, y\}$ exists then we define the set-theoretical difference $x \setminus y$ for arbitrary elements $x, y \in E$. We also prove a number of natural properties of the defined operations for the sake of the other sections.

Section 4 is devoted to lateral convergence of nets in Riesz spaces and different types of lateral continuity. One of the main messages is to show that, as was first noted in [10], the standard notion of the lateral convergence considered in a number of papers by different authors is unnatural and too restrictive. More precisely, laterally increasing order convergent nets were defined to be laterally convergent. The new notion proposed in [10] and developed in the present paper naturally generalizes the lateral convergence from laterally increasing nets to arbitrary laterally bounded nets. Due to this idea, the main result of [10] asserts that the lateral continuity of an orthogonally additive operator T at zero implies the lateral continuity of T at any point, which is even impossible to formulate for the old notion because there is no nontrivial laterally increasing net which laterally converges to zero. Our contribution to the subject in this section consists of providing with the following examples. Example 4.15 contains a construction in $L_p[0, 1]$ with $1 \le p \le \infty$ of an orthogonally additive laterally-to-norm continuous operator which is not order-to-norm continuous. We also prove that the norm of a Dedekind σ-complete Banach lattice is a laterally-to-norm continuous

[1] A component, if one follows another, much less apposite terminology.

function if and only if it is an order-to-norm continuous function. Finally, Example 4.18 shows that the lateral continuity of an orthogonally additive operator at one (nonzero) point does not imply its lateral continuity at all other points (e.g., at zero).

In Sect. 5 we introduce and study the notion of a consistent set in a Riesz space. We say that a subset G of a Riesz space E is called *consistent* if every two-point subset of G is laterally bounded in E. The idea of the notion is to generalize laterally bounded sets to consistent sets, which is much more general. For instance, let $0 < p \leq \infty$ and let e be any element of $L_0[0, 1]$ which does not belong to $L_p[0, 1]$. Then $\mathfrak{F}_e \cap L_p[0, 1]$ is a consistent set in $L_p[0, 1]$ which is not laterally bounded in $L_p[0, 1]$. We show that consistent sets can be useful in different contexts. The main result of the section Theorem 5.4 asserts that a Riesz space E is laterally complete if and only if every consistent set $A \subset E$ has a lateral supremum.

In Sect. 6 we introduce the notions of a lateral field, a lateral ideal and a lateral band in a Riesz space. During the paper have being written, these notions were already used in some other papers by the authors and their co-authors, confirming their importance. We must say that, lateral ideals and lateral bands are so important for orthogonally additive operators, as the usual ideals and bands are important for linear operators on Riesz spaces. We show that the kernel of a positive orthogonally additive operator is a lateral ideal, however not every lateral ideal equals the kernel of some orthogonally additive operator. Theorem 6.9 gives an elegant formula for the lateral band projection which is a disjointness preserving laterally continuous orthogonally additive operator and has some additional properties. Theorem 6.10 and its corollary assert that the lateral field (ideal, band) generated by a consistent set is consistent, and that any maximal consistent subset of a Riesz space is a lateral band.

Section 7 is devoted to extensions of orthogonally additive operators. It contains a survey on the subject and a positive solution of a problem posed in [11]. We say that a Riesz space E has the intersection property if every two-point subset of E has the infimum with respect to the lateral order $\sqsubseteq$ (for instance, the principal projection property implies the intersection property). The main result of the section (Theorem 7.5) asserts that, if D is a lateral field in a Riesz space E with the intersection property, X a vector space and $T_0 : D \to X$ an orthogonally additive operator then there exists an orthogonally additive extension $T : E \to X$ of T_0.

2 Preliminaries

In this section we give some preliminary information on Riesz spaces and orthogonally additive operators. For the terminology and notation that are familiarly used in the paper, we refer the reader to [5]. All Riesz spaces considered in the paper are assumed to be Archimedean.

There are two types of order convergence of nets in Riesz spaces.

Definition 2.1 Following terminology from [12] and notation from [3], we say that a net $(x_\alpha)_{\alpha \in A}$ in a Riesz space E is

- *strongly order convergent* to a limit $x \in E$ if there is a net $(y_\alpha)_{\alpha \in A}$ in E such that $y_\alpha \downarrow 0$ and $|x_\alpha - x| \leq y_\alpha$ for some $\alpha_0 \in A$ and all $\alpha \geq \alpha_0$ (write $x_\alpha \xrightarrow{\text{s-o}} x$);

- *weakly order convergent* to a limit $x \in E$ if there is a net $(y_\beta)_{\beta \in B}$ in E such that $y_\beta \downarrow 0$ and for every $\beta \in B$ there exists $\alpha_0 \in A$ such that $|x_\alpha - x| \leq y_\beta$ for all $\alpha \geq \alpha_0$ (write $x_\alpha \xrightarrow{\text{w}-\text{o}} x$).

Obviously, every strongly order convergent net is weakly convergent net to the same limit, but the converse is not true [3]. However, the two types of order convergence are equivalent if E is Dedekind complete (because both are equivalent to the equality $\bigwedge_\alpha \bigvee_{\gamma > \alpha} |x_\gamma - x| = 0$). Another case, where the weak order convergence implies strong one, is monotonicity of $(x_\alpha)_{\alpha \in A}$. In these two cases we write $x_\alpha \xrightarrow{\text{o}} x$.

The equality $x = \bigsqcup_{i=1}^{n} x_i$ means that $x = \sum_{i=1}^{n} x_i$ and $x_i \perp x_j$ if $i \neq j$.

Definition 2.2 Let E be a Riesz space and F a real vector space. A function $T : E \to F$ is said to be an *orthogonally additive operator* if $T(x + y) = T(x) + T(y)$ for every disjoint elements $x, y \in E$.

It is clear that $T(0) = 0$. The set of all orthogonally additive operators is a real vector space with respect to the natural linear operations.

Definition 2.3 Let E and F be Riesz spaces. An orthogonally additive operator $T : E \to F$ is called:

- *positive* if $Tx \geq 0$ holds in F for all $x \in E$;
- *order bounded* if T sends order bounded subsets of E to order bounded subsets of F.

An order bounded orthogonally additive operator $T : E \to F$ is called an *abstract Uryson* operator.

Observe that if $T : E \to F$ is a positive orthogonally additive operator and $x \in E$ is such that $T(x) \neq 0$ then $T(-x) \neq -T(x)$ (otherwise both $T(x) \geq 0$ and $T(-x) \geq 0$ would imply $T(x) = 0$). Thus, the positivity of orthogonally additive operators is completely different from that for linear operators, and so the only linear operator which is positive in the above sense is zero. However, there are lots of natural positive orthogonally additive operators: the positive, negative parts, modulus of an elements $x \to x^+, x \to x^-, x \to |x|$, the pth power of the norm of an element of $L_p(\mu)$-space $x \to \|x\|^p$ and many others.

A positive orthogonally additive operator need not be order bounded. Indeed, every function $T : \mathbb{R} \to \mathbb{R}$ with $T(0) = 0$ is an orthogonally additive operator, and obviously, not all of such functions are order bounded.

Another useful observation is that, if $T : E \to F$ is a positive orthogonally additive operator and $x \sqsubseteq y$ in E then $T(x) \leq T(y)$, no matter whether y is positive or not. Indeed, $x \sqsubseteq y$ implies $y = x \sqcup (y - x)$ and hence $T(y) = T(x) + T(y - x) \geq T(x)$ because $T(y - x) \geq 0$.

Orthogonally additive operators in Riesz spaces were introduced and studied in 1990 by Mazón and Segura de León (see [13,14]). Today this class of operators is an object of active investigations (see e.g. [6–8,10,11,17,19–23]).

The set of all abstract Uryson operators from E to F is denoted by $\mathcal{U}(E, F)$.

Consider the following order on $\mathcal{U}(E, F)$: $S \leq T$ whenever $(T - S) \geq 0$. Then $\mathcal{U}(E, F)$ becomes an ordered vector space.

Theorem 2.4 [13, Theorem 3.2] *Let E and F be Riesz spaces with F Dedekind complete. Then $\mathcal{U}(E, F)$ is a Dedekind complete Riesz space. Moreover, for any $S, T \in \mathcal{U}(E, F)$ and $f \in E$ the following assertions hold*

(1) $(T \vee S)(f) := \sup\{Tg_1 + Sg_2 : f = g_1 \sqcup g_2\}$.
(2) $(T \wedge S)(f) := \inf\{Tg_1 + Sg_2 : f = g_1 \sqcup g_2\}$.
(3) $T^+(f) := \sup\{Tg : g \sqsubseteq f\}$.
(4) $T^-(f) := -\inf\{Tg : g;\ g \sqsubseteq f\}$.
(5) $|Tf| \leq |T|(f)$.

3 The lateral order and set-theoretical operations on a Riesz space

Let E be a Riesz space. Consider the binary relation $\sqsubseteq$ on E defined at the very beginning of the paper. We need the following simple (most likely, known) properties of this relation which we provide with a proof for the sake of completeness.

Proposition 3.1 *Let E be a Riesz space and $x, y \in E$.*

(1) *If $x \sqsubseteq y$ then*
   (a) $x^+ \sqsubseteq y^+$ *and* $x^- \sqsubseteq y^-$;
   (b) $x^+ \leq y^+$ *and* $x^- \leq y^-$;
   (c) $x^- \perp y^+$ *and* $x^+ \perp y^-$;
   (d) $|x| \sqsubseteq |y|$.

(2) $x \sqsubseteq y$ *if and only if* $x^+ \sqsubseteq y^+$ *and* $x^- \sqsubseteq y^-$.

Proof Assume $x \sqsubseteq y$, that is, $y = x \sqcup (y - x)$. Then $y^+ = x^+ \sqcup (y - x)^+$, which implies $x^+ \leq y^+$ and $(y - x)^+ = y^+ - x^+$. Hence $y^+ = x^+ \sqcup (y^+ - x^+)$, i.e., $x^+ \sqsubseteq y^+$. Analogously, $x^- \leq y^-$ and $x^- \sqsubseteq y^-$. Thus, (a), (b) and the "only if" part of item (2) is proved.

(c) By (b), $0 \leq x^- \wedge y^+ \leq y^- \wedge y^+ = 0$. The second part of (c) is proved analogously.

(d) By (a), $x^+ \perp y^+ - x^+$, and by (c), $x^+ \perp y^-$. Moreover, $x^+ \perp x^-$. Hence $x^+ \perp |y| - |x|$. Analogously, $x^- \perp |y| - |x|$. The latter two relations yield $|x| \perp |y| - |x|$, that is, $|x| \sqsubseteq |y|$.

The "if" part of (2). Suppose $x^+ \sqsubseteq y^+$ and $x^- \sqsubseteq y^-$. Then the first relation implies $x^+ \leq y^+$. Then $0 \leq x^+ \wedge y^- \leq y^+ \wedge y^- = 0$, and hence $x^+ \perp y^-$. Taking into account $x^+ \perp (y^+ - x^+)$ and $x^+ \perp x^-$, one gets $x^+ \perp (y^+ - x^+ - y^- + x^-)$, i.e., $x^+ \perp (y - x)$. Analogously, $x^- \perp (y - x)$, and thus, $x \perp (y - x)$. □

The following statement is an easily exercise.

Proposition 3.2 *Let E be a Riesz space. Then $\sqsubseteq$ is a partial order on E.*

Definition 3.3 Let E be a Riesz space. The partial order $\sqsubseteq$ on E we call the *lateral order* on E. A subset $G \subseteq E$ is said to be *laterally bounded* in E if $G \subseteq \mathfrak{F}_x$ for some $x \in E$. We do not mention here "from above" because every subset is automatically laterally bounded from below by zero. The lateral supremum and infimum are defined as usual in a partially ordered set, using the order $\sqsubseteq$ on E.

Item (1) of the following proposition is well known for $e \geq 0$, [5, Theorem 3.15].

Proposition 3.4 *Let E be a Riesz space and $e \in E$. Then*

(1) *the set $\mathfrak{F}_e$ of all fragments of e is a Boolean algebra with zero 0, unit e with respect to the operations $x \cup y = (x^+ \vee y^+) - (x^- \vee y^-)$ and $x \cap y = (x^+ \wedge y^+) - (x^- \wedge y^-)$;*
(2) *if $e \geq 0$ then the lateral order $\sqsubseteq$ on $\mathfrak{F}_e$ coincides with the lattice order $\leq$, and hence the lateral supremum (infimum) of an arbitrary set $A \subseteq \mathfrak{F}_e$ equals its lattice supremum (infimum);*
(3) *$x \cup y$ equals the supremum, and $x \cap y$ equals the infimum of a two-point set $\{x, y\} \subseteq \mathfrak{F}_e$ with respect to the lateral order $\sqsubseteq$ both in $\mathfrak{F}_e$ and E.*

Proof (1) By [5, Theorem 3.15], $\mathfrak{F}_{e^+}$ and $\mathfrak{F}_{e^-}$ are Boolean algebras with zero 0, units e^+ and e^- respectively and operations $\vee$ and $\wedge$, that coincide with the lattice operations on E. Consider the direct sum $\mathfrak{F}_{e^+} \oplus \mathfrak{F}_{e^-}$, that is, the cartesian product $\mathfrak{F}_{e^+} \times \mathfrak{F}_{e^-}$ with zero $(0, 0)$, unit (e^+, e^-) and operations $(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \vee y_2)$ and $(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \wedge y_2)$. Obviously, $\mathfrak{F}_{e^+} \oplus \mathfrak{F}_{e^-}$ is a Boolean algebra. Then the bijection $\tau : \mathfrak{F}_{e^+} \oplus \mathfrak{F}_{e^-} \to \mathfrak{F}_e$ given by $\tau(x, y) = x - y$ for any $(x, y) \in \mathfrak{F}_{e^+} \oplus \mathfrak{F}_{e^-}$ (the facts that $\tau(x, y) \in \mathfrak{F}_e$, and that τ is one-to-one follow from Proposition 3.1), induces a Boolean algebra structure on $\mathfrak{F}_e$. It remains to observe that τ sends $(0, 0)$ to 0, (e^+, e^-) to $e^+ - e^- = e$, and the induces operations are given by the formulas given in the statement of (1).

(2) Assume $e \geq 0$ and $x, y \in \mathfrak{F}_e$. By Proposition 3.1, $x, y \geq 0$.

Let $x \sqsubseteq y$. By (1)(b) of Proposition 3.1, we get $x \leq y$.

Let $x \leq y$. Then $0 \leq x \wedge (e - y) \leq x \wedge (e - x) = 0$, and hence $x \perp (e - y)$. Since $x \perp (e - x)$ and the disjoint complement is a linear subspace [5, Theorem 3.3], we obtain $x \perp (y - x)$, and hence $x \sqsubseteq y$.

(3) follows from (2) and Proposition 3.1. □

Let G be a nonempty subset of E. Extending the notation of Proposition 3.4, the $\sqsubseteq$-supremum and the $\sqsubseteq$-infimum of G in E we denote by $\bigcup G$ and $\bigcap G$ respectively. We also write $\bigcup_{k=1}^{m} x_k$ or $x_1 \cup \ldots \cup x_n$ for $\bigcup\{x_1, \ldots, x_m\}$, and similarly for infima using the symbols $\cap$ and $\bigcap$. This notation is natural due to the following reasons. First, if we deal with a function lattice then the graph of the lateral supremum equals the union of the graphs, in their nonzero parts. Analogously, the graph of the lateral infimum equals the intersection of the graphs, in their nonzero parts. Second reason is that, by Proposition 3.4 (1), $\mathfrak{F}_e$ is a Boolean algebra with respect to the lateral order. Then by Stone's representation theorem [9, Theorem 7.11], $\mathfrak{F}_e$ is Boolean isomorphic to the algebra of subsets of some set. Such a Boolean isomorphism sends the lateral supremum to the union, and the lateral infimum to the intersection. To distinguish the lateral supremum from the set-theoretical union, the reader just has to check whether the symbol concerns elements or sets of elements of a Riesz space. Another thing which differs the lateral supremum from the set-theoretical union is the bold style for lateral suprema and infima.

By 3.2, using the well known equality $x + y = x \vee y + x \wedge y$ [5, Theorem 1.2], we obtain the following consequence.

Corollary 3.5 *Let E be a Riesz space, $e \in E$ and $x, y \sqsubseteq e$. Then $x + y = x \cup y + x \cap y$. If, moreover, $x \perp y$ then $x + y = x \cup y$.*

As a direct consequence of item (2) of Proposition 3.1 one has the following statement.

Corollary 3.6 *Let E be a Riesz space and $G \subseteq E$. Set $G^+ = \{g^+ : g \in G\}$ and $G^- = \{g^- : g \in G\}$.*

(1) *G is laterally bounded if and only if both G^+ and G^- are laterally bounded.*
(2) *$\bigcup G$ exists if and only if both $\bigcup G^+$ and $\bigcup G^-$ exist. Moreover, $\bigcup G = \bigcup G^+ - \bigcup G^-$ in the case of the existence.*
(3) *$\bigcap G$ exists if and only if both $\bigcap G^+$ and $\bigcap G^-$ exist. Moreover, $\bigcap G = \bigcap G^+ - \bigcap G^-$ in the case of the existence.*

Using Proposition 3.2 (1), one can easily prove by induction the following statement.

Corollary 3.7 *Every laterally bounded finite subset $\{x_1, \dots, x_n\}$ of a Riesz space E has a $\sqsubseteq$-supremum $\bigcup\{x_1, \dots, x_n\} = (x_1^+ \vee \dots \vee x_n^+) - (x_1^- \vee \dots \vee x_n^-)$.*

The next statement reduces the lateral supremum and infimum to the usual ones for laterally bounded sets.

Corollary 3.8 *Let G be a laterally bounded subset of a Riesz space E. Set $G^+ = \{f^+ : f \in G\}$ and $G^- = \{f^- : f \in G\}$. Then $\bigcup G$ exists if and only if the usual suprema $\sup G^+$ and $\sup G^-$ exist. In this case $\bigcup G = \sup G^+ - \sup G^-$. If, moreover, $G \subseteq E^+$ then $\bigcup G$ exists if and only if the usual suprema $\sup G$ exists. In this case $\bigcup G = \sup G$. Analogous assertions are true for the infimum of a laterally bounded subset G.*

Proof By (ii) of Corollary 3.6, it is enough to consider the case where $G \subseteq E^+$. But in this case the statement follows from Proposition 3.2 (2). □

Corollary 3.9 *Let G be a laterally bounded subset of a Dedekind complete Riesz space E. Then $\bigcup G = \sup G^+ - \sup G^-$ and $\bigcap G = \inf G^+ - \inf G^-$ exist, where $G^+ = \{f^+ : f \in G\}$ and $G^- = \{f^- : f \in G\}$.*

A somewhat different approach is needed for investigation of lateral infima.

Remark 3.10 For simplicity of notation, we set the convention to write xy instead of $x \cap y$ for elements x, y of a Riesz space E. Next, to avoid lots of parentheses, for example, in the expression $(x \cap y) - z$ we get a deal to operate the lateral infimum xy first. So, we write $xy - z$ instead.

As already mentioned above, every subset of a Riesz space is laterally bounded from below by zero. However, a two-point set need not have $\sqsubseteq$-infimum, as the following example shows.

Example 3.11 There exists a Riesz space E and a two-point set $\{x, y\}$ in E having no lateral infimum xy.

Proof Let E be the Riesz space of all functions $x : [0, 2] \to \mathbb{R}$ which are continuous at the point 1 with the point-wise order: $x \le y$ if and only if $x(t) \le y(t)$ for all $t \in [0, 2]$. As the two-point set $\{x, y\}$ we take any two continuous at each point $t \in [0, 2]$ functions having the following properties.

(1) $x(t) = y(t) = 1$ for all $t \in [0, 1]$;
(2) $0 < x(t) < 1$ for all $t \in (1, 2]$;
(3) $y(t) > 1$ for all $t \in (1, 2]$.

Now we prove that xy does not exist. Assume on the contrary the existence of $z = xy$. We show that $z(t) = 1$ for all $t \in [0, 1)$ and $z(t) = 0$ for all $t \in (1, 2]$, which contradicts the continuity of z at the point 1. We will use the following property of E (which is a common property of every Riesz space of functions): if $u \sqsubseteq v$ in E then for each $t \in \mathbb{R}$ either $u(t) = 0$ or $u(t) = v(t)$.

Fix any point $t \in [0, 1)$. Choose $\alpha \in \mathbb{R}$ with $0 \le t < \alpha < 1$. Since $\mathbf{1}_{[0,\alpha]} \sqsubseteq x$ and $\mathbf{1}_{[0,\alpha]} \sqsubseteq y$, we get that $\mathbf{1}_{[0,\alpha]} \sqsubseteq z$, hence $z(t) = 1$.

Fix any point $t \in (1, 2]$. Since $z \sqsubseteq x$, either $z(t) = 0$ or $z(t) = x(t)$. Analogously, since $z \sqsubseteq y$, either $z(t) = 0$ or $z(t) = y(t)$. Finally, since $x(t) \ne y(t)$, we obtain $z(t) = 0$. ◻

Nevertheless, such examples exist only in Riesz spaces without the principal projection property. Let E be a Riesz space with the principal projection property. By P_v, $v \in E$ we denote the order projection of E onto the band generated by v.

Lemma 3.12 *Let E be a Riesz space with the principal projection property and $x, y \in E$. Then the following assertions are equivalent:*

(1) *$x \sqsubseteq y$;*
(2) *$P_x y = x$.*

Proof 1) $\Rightarrow$ 2). We have that $y = (y - x) \sqcup x$ and $P_x y = P_x((y - x) \sqcup x) = P_x x = x$.

2) $\Rightarrow$ 1). $y = P_x y \sqcup (y - P_x y) = x \sqcup (y - x)$ and therefore $(y - x) \perp x$. ◻

Theorem 3.13 *Every finite set G in a Riesz space E with the principal projection property has a lateral infimum $z = \bigcap G$ in E. Moreover, $\mathfrak{F}_z = \bigcap_{x \in G} \mathfrak{F}_x$. In particular, for any two elements $x, y \in E$ one has $\mathfrak{F}_x \cap \mathfrak{F}_y = \mathfrak{F}_{xy}$.*

Proof Obviously, it is enough to prove the theorem for a two-point set $G = \{x, y\}$. Set $z = x - P_{x-y}x$ and observe that $z = y - P_{x-y}y$. Indeed,

$$P_{x-y}x - P_{x-y}y = P_{x-y}(x - y) = x - y.$$

Moreover

$$P_{x-y}z = P_{x-y}(x - P_{x-y}x) = P_{x-y}x - P_{x-y}x = 0$$

and hence $P_{x-y} \perp P_z$. Show that both relations $z \sqsubseteq x$ and $z \sqsubseteq y$ hold. Indeed,

$$P_z x = P_z(z + P_{x-y}x) = P_z z + P_z \circ P_{x-y}x = z;$$
$$P_z y = P_z(z + P_{x-y}y) = P_z z + P_z \circ P_{x-y}y = z.$$

Now take $t \in E$ such that $t \sqsubseteq x$ and $t \sqsubseteq y$. Then $t \perp (x - t)$ and $t \perp (y - t)$. Hence $t \perp \big((x - t) - (y - t)\big)$ that is, $t \perp (x - y)$. and $P_t \perp P_{x-y}$. Now we may write

$$P_t z = P_t(x - P_{x-y}x) = P_t x - P_t \circ P_{x-y}x = P_t x \overset{\text{Lemma } 3.12}{=} t.$$

Thus by the Lemma 3.12 we have $t \sqsubseteq z$ and therefore both statements $z = xy$ and $\mathfrak{F}_x \cap \mathfrak{F}_y = \mathfrak{F}_z$ are proved. □

Definition 3.14 We say that a Riesz space E has the *intersection property* if every two-point subset $\{x, y\} \subset E$ has a lateral infimum xy.

By Theorem 3.13, the following holds.

Corollary 3.15 *The principal projection property of a Riesz space E implies the intersection property of E.*

Problem 3.16 *Does there exist a Riesz space with the intersection property failing the principal projection property?*

We consider some more set-theoretical operations on the Boolean algebra $\mathfrak{F}_e$ of fragments of an element e.

Definition 3.17 Let E be a Riesz space and $e \in E$. We define the *set-theoretical difference* $\setminus$ and the *symmetric difference* Δ on $\mathfrak{F}_e$ by setting $x \setminus y = x\,(e - y)$ and $x \Delta y = (x \setminus y) \cup (y \setminus x)$ for any $x, y \in \mathfrak{F}_e$ respectively.

The above operations one can define in another way using the following statement.

Proposition 3.18 *Let E be a Riesz space and $e \in E$. Then for any $x, y \in \mathfrak{F}_e$ one has*

(1) $x = xy \sqcup (x \setminus y)$;
(2) $x \setminus y = x - xy$;
(3) $x \Delta y = (x \setminus y) \sqcup (y \setminus x)$;
(4) $x \cup y = x \sqcup (y \setminus x)$.

Proof (1) The equality $A = (A \cap B) \cup (A \setminus B)$ for subsets of any set X is obvious. Hence by Proposition 3.2 and Stone's theorem we obtain $x = xy \cup (x \setminus y)$. Analogously, the equality $(A \cap B) \cap (A \setminus B) = \emptyset$ for sets yields $xy(x \setminus y) = 0$. In view of Corollary 3.5, one has $xy + (x \setminus y) = xy \cup (x \setminus y) = x$.

(2) follows from (1).
(3) The equality for sets $(A\backslash B) \cap (B\backslash A) = \emptyset$ implies $(x\backslash y) \perp (y\backslash x)$. Then use Corollary 3.5.
(4) The assertion follows from the obvious equalities for sets. □

Using the lateral infimum, we define the set-theoretical difference for not necessarily laterally bounded elements.

Definition 3.19 Let E be a Riesz space. For any $x, y \in E$ such that xy exists we define the set-theoretical difference by setting

$$x\backslash y \overset{\text{def}}{=} x\backslash xy. \tag{3.1}$$

Since the two-point set $\{x, xy\}$ is laterally bounded by x, the operation is well defined. Observe that, in case of existence, one has

$$x\backslash y = x - xy. \tag{3.2}$$

Indeed, since $x, xy \in \mathfrak{F}_x$, by (3.1) and (2) of Proposition 3.18

$$x\backslash y = x\backslash xy = x - x(xy) = x - xy.$$

If E has the intersection property (in particular, if E has the principal projection property) then $x\backslash y$ is well defined for any $x, y \in E$.

Proposition 3.20 *Let E be a Riesz space and $x, y, u, v \in E$ with $x \sqsubseteq y$ and $u \sqsubseteq v$. Then the following assertions hold.*

(i) *If xu, yv exist then $xu \sqsubseteq yv$.*
(ii) *If $y \cup v$ exists then $x \cup u$ exists and $x \cup u \sqsubseteq y \cup v$.*
(iii) *If xv, yu exist then $x\backslash v \sqsubseteq y\backslash u$.*

The proof is obvious.

We need the following statement for the proof of the main result of Sect. 7.

Proposition 3.21 *Let E be a Riesz space and $x, y, z \in E$. Then the following assertions hold under the assumption of existence of operations results.*

(i) *If $x \sqsubseteq y$ then $y = x \sqcup (y\backslash x)$.*
(ii) $x = xy \sqcup (x\backslash y)$.
(iii) $(x\backslash y)\, y = 0$.
(iv) $(x\backslash y)\backslash y = x\backslash y$.
(v) $x\,(y \sqcup z) = xy \sqcup xz$ *(here it is enough to assume the existence of the left hand side).*
(vi) $x\,(y\backslash z) = xy\backslash xz = xy\backslash z$.
(vii) $(x\backslash z)(y\backslash z) = xy\backslash z$.
(viii) $(x\backslash z)\backslash(y\backslash z) = (x\backslash y)\backslash z$.

Remark that all items of Proposition 3.21 are evident for elements of a Boolean algebra. So, no proof is needed for laterally bounded sets of elements.

Proof (i) follows from the observation that all the terms belong to $\mathfrak{F}_y$ which is a Boolean algebra.

(ii) immediately follows from (i).

(iii) Set $u = (x\backslash y)\, y$. Then $u \sqsubseteq xy$ and $u \sqsubseteq x\backslash y$. By (ii) $u = 0$.

(iv) By (iii), $(x\backslash y)\backslash\big((x\backslash y)\, y\big) = (x\backslash y)\backslash 0 = x\backslash y$.

(v) Set $u = x\,(y \sqcup z)$. Since $xy \sqsubseteq u$ and $xz \sqsubseteq u$, we have $xy \sqcup xz \sqsubseteq u$. To prove the other side inequality, set $v = u\backslash(xy \sqcup xz)$. By (i) of Proposition 3.20 and (i) of Proposition 3.21, $vxy \sqsubseteq v\,(xy \sqcup xz) = 0$. Since $v \sqsubseteq u \sqsubseteq x$, one has $v = vx$ and hence $vy = vxy = 0$. Analogously, $vz = 0$. Observe that $v, y, z \sqsubseteq y \sqcup z$. Hence, as for elements of the Boolean algebra $\mathfrak{F}_{y\sqcup z}$, we obtain $v = vy \sqcup vz = 0$, that is, $u = xy \sqcup xz$.

(vi) Set $u = x\,(y\backslash z) = x\,(y\backslash yz)$. Then $u \sqsubseteq xy$ implies $u \perp xy - u$ and $yz \sqsubseteq y$ implies $yz \perp y - yz$ and hence

$$|u| \wedge |xyz| \leq |y - yz| \wedge |yz| = 0$$

by (i). Thus, $u \perp xy - u - xyz$, that is, $u \sqsubseteq xy - xyz = xy\backslash z$.
To prove the other side inequality $xy\backslash xyz \sqsubseteq y\backslash yz$ we set $x_1 = xy$ and $y_1 = yz$. Then the desired inequality is $x_1\backslash x_1 y_1 \sqsubseteq y\backslash y_1$. Taking into account that all the terms of the later inequality are laterally bounded by y, we obtain the inequality from the corresponding inclusion for sets.

(vii) Using twice (v) and then (iv) we get

$$(x\backslash z)(y\backslash z) = (x\backslash z)\, y\backslash z = (xy\backslash z)\backslash z = xy\backslash z.$$

(viii) By the definitions and (vii),

$$\begin{aligned}(x\backslash z)\backslash(y\backslash z) &= (x\backslash xz)\backslash(x\backslash z)(y\backslash z)\\ &= (x\backslash xz)\backslash(xy\backslash xyz).\end{aligned}$$

Substituting $xy = x_1$ and $xz = x_2$ we continue the chain of equalities

$$= (x\backslash x_2)\backslash(x_1\backslash x_2).$$

Now all the terms are elements of the Boolean algebra P_x. Hence we continue as follows

$$= (x\backslash x_1)\backslash x_2 = (x\backslash xy)\backslash xz = (x\backslash y)\backslash(x\backslash y)\, xz.$$

By (vi), $(x\backslash y)\, x = xx\backslash y = x\backslash y$. Hence we continue

$$= (x\backslash y) - (x\backslash y)\, z = (x\backslash y)\backslash z.$$

□

4 Lateral convergence and lateral continuity

Some version of laterally (i.e., horizontally) convergence of nets and corresponding continuity of maps acting between Riesz spaces was considered in [13,20] and other papers. A net (x_α) in a Riesz space E in the mentioned above papers is said to be laterally convergent to an element $x \in E$ if $x_\alpha \sqsubseteq x_\beta$ as $\alpha < \beta$ and $x_\alpha \xrightarrow{\text{o}} x$ (by (1) of Proposition 3.1, a laterally increasing net equals a difference of two order increasing nets, and hence, the strong and weak order convergence coincide for a laterally increasing net).

However, the assumption of lateral increasing on the net in the above definition is too restrictive and unjustified. It is very natural to replace this restriction with the condition of lateral boundedness of the net. This idea was realized in a paper by Gumenchuk [10]. We modify Gumenchuk's idea by considering strong and weak order convergence.

Lateral convergence

We introduce several types of lateral convergence of a net to be laterally bounded and order convergent in the corresponding sense. Since attaching of additional terms at the beginning of a net cannot spoil its convergence, we define a slightly weaker notion of lateral boundedness.

Definition 4.1 A net $(x_\alpha)_{\alpha\in A}$ in a Riesz space E is said to be *eventually laterally bounded* if there is $\alpha_0 \in A$ such that the net $(x_\alpha)_{\alpha\geq\alpha_0}$ is laterally bounded.

Definition 4.2 Let E be a Riesz space and $x \in E$. An eventually laterally bounded net (x_α) in E is said to be

- *strongly laterally convergent* to x (write $x_\alpha \xrightarrow{\text{s}-\ell} x$) if $x_\alpha \xrightarrow{\text{s}-\text{o}} x$;
- *weakly laterally convergent* to x (write $x_\alpha \xrightarrow{\text{w}-\ell} x$) if $x_\alpha \xrightarrow{\text{w}-\text{o}} x$.

Observe that the same example showing that the weak order convergence of a net does not imply its strong order convergence, which is referred as Fremlin's example in the literature (see e.g. [25, p. 141]), works for the lateral convergence. We provide this example for the sake of completeness.

Example 4.3 There exists a Riesz space E and a net in E which is weakly laterally convergent, but is not strongly laterally convergent.

Indeed, consider the one-point compactification K of an uncountable discrete space, and let $E = C(K)$. If $(x_n)_{n=1}^{\infty}$ denotes the characteristic functions of a sequence of distinct singletons in K, the sequence (x_n), being laterally bounded from above by the constant function 1, is weakly laterally convergent but not strongly laterally convergent to zero.

Surely, if either E is Dedekind complete or the net is laterally monotone then the relations $x_\alpha \xrightarrow{\text{s}-\ell} x$ and $x_\alpha \xrightarrow{\text{w}-\ell} x$ are equivalent, and we write $x_\alpha \xrightarrow{\ell} x$.

To distinguish the laterally convergent nets in the sense of Definition 4.2 from the laterally convergent nets considered in papers [13,20], we offer another name for the latter notion.

Definition 4.4 A net (x_α) in a Riesz space E is said to be *up-laterally convergent* to $x \in E$ (write $x_\alpha \xrightarrow{\ell\uparrow} x$) if $x_\alpha \sqsubseteq x_\beta \sqsubseteq x$ as $\alpha < \beta$ and $x_\alpha \xrightarrow{\mathrm{o}} x$.

A subset A of a Riesz space E is said to be *strongly order closed* (respectively, *weakly order closed*) if for every net (x_α) in A and every $x \in E$ the condition $x_\alpha \xrightarrow{\mathrm{s-o}} x$ (respectively, $x_\alpha \xrightarrow{\mathrm{w-o}} x$) implies that $x \in A$. Likewise, we say that $A \subseteq E$ is *strongly laterally closed* (respectively, *weakly laterally closed*) if for every net (x_α) in A and every $x \in E$ the condition $x_\alpha \xrightarrow{\mathrm{s-}\ell} x$ (respectively, $x_\alpha \xrightarrow{\mathrm{w-}\ell} x$) implies that $x \in A$. Obviously, a strongly (respectively, weakly) order closed subset is strongly (respectively, weakly) laterally closed, however, the set $\{n^{-1}x_0 : n \in \mathbb{N}\}$ where $x_0 \in E\setminus\{0\}$, is both strongly and weakly laterally closed but is neither strongly nor weakly order closed. Among other obvious implications we point our the following ones: a weakly order closed subset is strongly order closed, and a weakly laterally closed subset is strongly laterally closed.

It is an easy technical exercise to show that the set $\mathfrak{F}_e$ is weakly order closed and hence, weakly laterally closed (see [10] for a short proof that $\mathfrak{F}_e$ is strongly order closed, which also can be applied to prove that $\mathfrak{F}_e$ is weakly order closed). Hence, we obtain the following useful consequence.

Proposition 4.5 *Let E be a Riesz space, $e \in E$ and $x_\alpha \xrightarrow{\mathrm{w-}\ell} x$ or $x_\alpha \xrightarrow{\mathrm{s-}\ell} x$, where $x \in E$ and $x_\alpha \sqsubseteq e$ for all $\alpha \geq \alpha_0$. Then $x \sqsubseteq e$.*

It is natural that, for a laterally convergent net in a Dedekind complete Riesz space, a majorizing net (which appears in the definition of the order convergence) could be found among the fragments of the modulus of the upper lateral bound.

Proposition 4.6 [10] *Let E be a Dedekind complete Riesz space, $e \in E$, $x_\alpha \xrightarrow{\ell} x$, where $x \in E$ and $x_\alpha \sqsubseteq e$ for all $\alpha \geq \alpha_0$. Then there is a net (v_α) with the same index set such that $v_\alpha \sqsubseteq |e|$ and $|x_\alpha - x| \sqsubseteq v_\alpha$ for all $\alpha \geq \alpha_0$ and $v_\alpha \downarrow 0$.*

Another useful fact from [10] describes relationships between the lateral convergence of a net and of its positive and negative parts.

Lemma 4.7 [10] *Let E be a Riesz space, (x_α) a net in E and $x \in E$. Then the following assertions are equivalent.*

(i) $x_\alpha \xrightarrow{\mathrm{s-}\ell} x$;
(ii) $x_\alpha^+ \xrightarrow{\mathrm{s-}\ell} x^+$, $x_\alpha^- \xrightarrow{\mathrm{s-}\ell} x^-$ *and (x_α) is eventually laterally bounded;*
(iii) *The set $\{x\} \cup \{x_\alpha : \alpha \geq \alpha_0\}$ is laterally bounded for some index α_0 and $x_\alpha \Delta x \xrightarrow{\mathrm{s-}\ell} 0$.*

Moreover, each of (i)–(iii) *implies $|x_\alpha| \xrightarrow{\mathrm{s-}\ell} |x|$.*

The same is true for weak lateral convergence.

Actually, the proof in [10] of Lemma 4.7 concerns strong lateral convergence. However, the same proof is actual for weak lateral convergence.

There is an example in [10] showing that the eventually laterally boundedness assumption in (ii) cannot be removed.

It is well known that the order convergence of nets in the classical Banach lattices L_p with $1 \le p \le \infty$ does not coincide with the convergence in any topology on L_p. We remark that the lateral convergence of nets in L_p, as well as the up-lateral convergence, is much farther from topological convergence, because the sum of two up-lateral convergent nets need not be laterally convergent. Consider, for example, sequences $x_n = \mathbf{1}_{\left[\frac{1}{n},1\right]}$ and $y_n = \mathbf{1}_{\left[0,1-\frac{1}{n}\right]}$. Then $x_n \xrightarrow{\ell\uparrow} \mathbf{1}_{[0,1]}$ and $y_n \xrightarrow{\ell\uparrow} \mathbf{1}_{[0,1]}$, however the sequence $x_n + y_n = \mathbf{1}_{\left[0,\frac{1}{n}\right)} + 2 \cdot \mathbf{1}_{\left[\frac{1}{n},1-\frac{1}{n}\right]} + \mathbf{1}_{\left(1-\frac{1}{n},1\right]}$ is not eventually laterally bounded, and hence, is not laterally convergent.

Lateral boundedness

We show that an orthogonally additive operator is lateral bounded if and only if it preserves disjointness.

Definition 4.8 An orthogonally additive operator $T : E \to F$ between Riesz spaces E and F is said to be *laterally bounded* if T sends laterally bounded subsets of E to laterally bounded subsets of F.

In other words, T is laterally bounded provided for every $e \in E$ there exists $f \in F$ such that $T(\mathfrak{F}_e) \subseteq \mathfrak{F}_f$.

Theorem 4.9 *For an orthogonally additive operator $T : E \to F$ between Riesz spaces E and F the following assertions are equivalent:*

(1) *T is laterally bounded;*
(2) *T preserves disjointness;*
(3) $(\forall e \in E)$ $T(\mathfrak{F}_e) \subseteq \mathfrak{F}_{T(e)}$;
(4) *T is lateral order preserving, that is, for every $x, y \in E$ the relation $x \sqsubseteq y$ implies $T(x) \sqsubseteq T(y)$.*

For the proof, we need the following lemma.

Lemma 4.10 *For any elements x, y of a Riesz space E the following are equivalent:*

(i) $x \perp y$;
(ii) *the set $\{x, y, x + y\}$ is laterally bounded.*

Proof of Lemma 4.10 (i) $\Rightarrow$ (ii). Observe that if (i) is true then the set $\{x, y, x + y\}$ is laterally bounded by $x + y$ from above.

(ii) $\Rightarrow$ (i). Let $e \in E$ be such that $\{x, y, x+y\} \subseteq \mathfrak{F}_e$. Using (1) of Proposition 3.18, represent $x + y$ as a disjoint sum $x + y = (x \backslash y) \sqcup 2xy \sqcup (y \backslash x)$. Then every summand of the disjoint sum is a fragment of $x + y$, and hence, a fragment of e. Thus, xy and $2xy$ are fragments of e. Hence, $xy \perp (e - xy)$ and $2xy \perp (e - 2xy)$. By [25, p. 64], $2xy \perp \big((e - xy) - (e - 2xy)\big) = xy$. This yields $xy = 0$, and hence, $x \perp y$. □

Proof of Theorem 4.9 (1) $\Rightarrow$ (2). Assume $x, y \in E$ and $x \perp y$. By Lemma 4.10, the set $\{x, y, x+y\}$ is laterally bounded in E. By (1), the set

$$\{T(x), T(y), T(x+y)\} = \{T(x), T(y), T(x) + T(y)\}$$

is laterally bounded in F. Again, by Lemma 4.10, $T(x) \perp T(y)$.

(2) $\Rightarrow$ (3). Fix any $e \in E$ and $x \in \mathfrak{F}_e$. Since $x \perp (e - x)$, by (2) one has $T(x) \perp T(e-x)$. By the orthogonal additivity of T, we obtain $T(e) = T(x) + T(e-x)$ and hence $T(x) \perp \big(T(e) - T(x)\big)$, that is, $Tx \in \mathfrak{F}_{T(e)}$.

(3) $\Leftrightarrow$ (4) and (3) $\Rightarrow$ (1) are obvious. □

Lateral continuity

One of the motivations to consider the lateral convergence is to generalize theorems in a natural way. For instance, Theorem 5.1 from [15] (see also [24, Theorem 10.17]) asserting that every AM-compact order-to-norm continuous linear operator from an atomless Dedekind complete Riesz space to a Banach space is narrow, was then generalized in [20, Theorem 3.2] simultaneously in three directions: every C-compact up-laterally-to-norm continuous orthogonally additive operator acting from an atomless Dedekind complete Riesz space to a Banach space is narrow. One of the directions is due to the fact that the class of up-laterally-to-norm continuous maps is wider than the class of order-to-norm continuous maps, because the up-lateral convergence implies the order convergence.

Definition 4.11 Let E, F be Riesz spaces. A function $f : E \to F$ is said to be

- *strongly laterally continuous* at a point $x \in E$ provided for every net (x_α) in E, if $x_\alpha \overset{\mathrm{s}-\ell}{\longrightarrow} x$ then $f(x_\alpha) \overset{\mathrm{s}-\ell}{\longrightarrow} f(x)$;
- *weakly laterally continuous* at a point $x \in E$ provided for every net (x_α) in E, if $x_\alpha \overset{\mathrm{w}-\ell}{\longrightarrow} x$ then $f(x_\alpha) \overset{\mathrm{w}-\ell}{\longrightarrow} f(x)$;
- *up-laterally continuous* at a point $x \in E$ provided for every net (x_α) in E, if $x_\alpha \overset{\ell\uparrow}{\longrightarrow} x$ then $f(x_\alpha) \overset{\ell\uparrow}{\longrightarrow} f(x)$.

A function $f : E \to F$ is said to be *strongly laterally* (or *weakly laterally*, or *up-laterally*) *continuous* provided f is so at every point $x \in E$.

Remark 4.12 The above three notions of lateral continuity are incomparable.

(a) Notice that for any Riesz spaces E and F every function $f : E \to F$ is up-laterally continuous at zero. But there are Riesz spaces E, F and a linear operator $T : E \to F$ which is not strongly and weakly laterally continuous at zero. Indeed, let Y be an uncountable discrete space, $X \subseteq Y$ be a countable subspace of Y, $L = Y \sqcup \{\infty\}$ be the one-point compactification of Y, $K = X \sqcup \{\infty\}$, $E = C(K)$ and $F = C(L)$. Consider the operator $T : E \to F$ defined by

$$T(f)(y) = \begin{cases} f(y), & y \in K; \\ f(\infty), & y \in L \setminus K. \end{cases}$$

Then T is not strongly and weakly laterally continuous at zero. Thus, ***the up-lateral continuity does not imply the strong lateral continuity and does not imply the weak lateral continuity.***

(b) Let $E = F = c_0$. We consider the linear operator $T : E \to F$ defined by

$$T(\xi_1, \xi_2, \xi_3, \xi_4, \dots) = \left(\frac{\xi_1+\xi_2}{2}, \frac{\xi_3+\xi_4}{2}, \dots\right).$$

It is easy to see that T is both strongly and weakly laterally continuous. However, T is not up-laterally continuous at the point $x_0 = (1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \dots)$. Thus, ***the strong lateral continuity and the weak lateral continuity do not imply the up-lateral continuity.***

(c) Let $E = c$ be the Banach lattice of all converging sequences and $F = C(K)$, where K is the one-point compactification of an uncountable discrete space X, $(x_n)_{n=1}^{\infty}$ be a sequence of distinct points $x_n \in X$ and $y_n = \mathbf{1}_{\{x_n\}} \in F$ for every $n \in \mathbb{N}$. We consider the linear operator $T : E \to F$ defined by

$$T(u) = \sum_{n\in\mathbb{N}} \xi_n(y_n - y_{n+1}) = \xi_1 y_1 + \sum_{n\geq 2}(\xi_{n+1} - \xi_n) y_n,$$

where $u = (\xi_n)_{n\in\mathbb{N}} \in E$. Since $\lim\limits_{n\to\infty}(\xi_{n+1} - \xi_n) = 0$ for every $u = (\xi_n)_{n\in\mathbb{N}} \in E$, T is well defined.

Show that T is weakly laterally continuous at 0. Denote by $\mathcal{A}$ the system of all finite sets

$$A \subseteq X\setminus\{x_k : k \in \mathbb{N}\}.$$

Consider the following order $\leq$ on the set $B = \mathbb{N} \times \mathcal{A}$:

$$(n_1, A_1) \leq (n_2, A_2) \ \equiv \ n_1 \leq n_2 \text{ and } A_1 \subseteq A_2.$$

For every $\beta = (n, A) \in B$ we put

$$y_\beta = \max\left\{\frac{1}{n}\cdot\mathbf{1}, \mathbf{1} - \mathbf{1}_{\{x_k:1\leq k\leq n\}\cup A}\right\}.$$

It is easy to see that $y_\beta \downarrow 0$. Moreover, for every net $(u_\alpha)_\alpha$ in E such that $u_\alpha \overset{\text{w}-\ell}{\longrightarrow} 0$ and for every $\beta \in B$ there exists α_0 such that $|T(u_\alpha)| \leq y_\beta$ for all $\alpha \geq \alpha_0$. Consequently, T is weakly laterally continuous at 0.

On the other hand, we consider the sequence $(u_n)_{n\in\mathbb{N}}$ of elements $u_n = \mathbf{1} - \mathbf{1}_{\{k:1\leq k\leq n-1\}} \in E$. Then $(u_n)_{n\in\mathbb{N}}$ strongly laterally converges to 0. But $T(u_n) = y_n$ for every $n \in \mathbb{N}$ and $(T(u_n))_{n\in\mathbb{N}}$ does not strongly laterally converge to 0. Thus, T is not strongly laterally continuous at 0. So, the ***weak lateral continuity does not imply the strong lateral continuity.***

(d) Let $E = C(K)$, where K is the one-point compactification of an uncountable discrete space, and $F = l_\infty$. It is easy to construct function (not necessarily orthogonally additive) $f : E \to F$ such that f is strongly laterally continuous and is not weakly laterally continuous at 0. So, the ***strong lateral continuity does not imply the weak lateral continuity.***

In case of Dedekind complete E and F we use the term "*laterally continuous*" for both versions "strongly laterally continuous" and "weakly laterally continuous", because these versions are equivalent.

We say that a net (x_α) is *eventually constant* if $x_\alpha = x_{\alpha_0}$ for some α_0 and all $\alpha \geq \alpha_0$. Observe that the only eventually constant nets are laterally convergent in the Riesz space $\mathbb{R}$ with the usual order. So, every map $f : \mathbb{R} \to F$ for every Riesz space F is both strongly laterally continuous and weakly laterally continuous. On the other hand, the only laterally continuous functions from an atomless Riesz space E to $\mathbb{R}$ are constant functions. Another drawback of the notion of lateral continuity for linear and orthogonally additive operators is that the sum of two laterally continuous operators need not be laterally continuous.

Example 4.13 Let $0 \leq p \leq \infty$. Then there exist linear operators $S, T : L_p[0, 1] \to L_p[0, 1]$ that are up-laterally continuous and laterally continuous, however the sum $S + T$ is not both up-laterally continuous and laterally continuous.

Proof Let S be the identity operator and set $(Tx)(t) = x(1 - t)$ for all $x \in L_p[0, 1]$ and $t \in [0, 1]$. Then obviously S and T are up-laterally continuous and laterally continuous. To show that the sum $S + T$ is not, consider the sequence $x_n = \mathbf{1}_{\left[\frac{1}{n}, 1\right]}$. Then we have the convergence in both senses $x_n \xrightarrow{\ell\uparrow} \mathbf{1}_{[0,1]}$ and $x_n \xrightarrow{\ell} \mathbf{1}_{[0,1]}$, however the sequence

$$(S + T)\, x_n = \mathbf{1}_{\left[0, \frac{1}{n}\right)} + 2 \cdot \mathbf{1}_{\left[\frac{1}{n}, 1-\frac{1}{n}\right]} + \mathbf{1}_{\left(1-\frac{1}{n}, 1\right]}$$

is not eventually laterally bounded and hence fails to be convergent both up-laterally and laterally. □

This is why it is important to investigate mixed versions of continuity like the following ones.

Definition 4.14 Let E be a Riesz space and X a normed space. A function $f : E \to X$ is said to be *s-order-to-norm continuous* (or w-order-to-norm continuous) at a point $x \in E$ if for any net (x_α) in E with $x_\alpha \xrightarrow{\text{s-o}} x$ (respectively, $x_\alpha \xrightarrow{\text{w-o}} x$) one has $\|f(x_\alpha) - f(x)\| \to 0$. The same function f is called *s-order-to-norm continuous* (or *w-order-to-norm continuous*), if it is so at each point $x \in E$.

In a similar way one can define *s-laterally-to-norm*, *w-laterally-to-norm, up-laterally-to-norm* continuous functions from a Riesz space to a normed space. Moreover, let E, F be Riesz spaces and (φ, ψ) be any pair of distinct types of convergence from the following ones: s-order, w-order, s-lateral, w-lateral, up-lateral. A function $f : E \to F$ is said to be φ-to-ψ continuous at a point $x \in E$, if for every φ-convergent to x net (x_α) in E one has $x_\alpha \xrightarrow{\psi} x$.

Since the lateral convergence implies the order convergence, the order-to-norm continuity of a function $f : E \to X$ implies the laterally-to norm continuity of f. The following example shows that the converse assertion is not true even for linear operators.

Example 4.15 Let $1 \le p \le \infty$. Then there is a laterally-to-norm continuous orthogonally additive operator $T : L_p \to \mathbb{R}$ which is not order-to-norm continuous.

Proof Given any $x \in L_p$, we set

$$T(x) = \sum_{a \in \mathbb{R}} a \cdot \mu\big(x^{-1}(\{a\})\big).$$

To see that T is well defined, we set $A_x = \big\{a \in \mathbb{R}\backslash\{0\} : \mu\big(x^{-1}(\{a\})\big) \neq 0\big\}$ and

$$B_x = \bigcup_{a \in A_x} x^{-1}(\{a\}).$$

Note that the set A_x is at most countable, B_x is a measurable subset of [0, 1] and

$$T(x) = \int_{B_x} x \, d\mu.$$

We will use the following simple observation: $B_x \subseteq \operatorname{supp} x$.

Show that T is an orthogonally additive operator. Assume $x, y \in L_p$ and $x \perp y$, that is, $\operatorname{supp} x \cap \operatorname{supp} y = \emptyset$. Then $B_x \cap B_y = \emptyset$, $B_{x+y} = B_x \cup B_y$ and hence

$$\begin{aligned} T(x+y) &= \int_{B_{x+y}} (x+y)\, d\mu = \int_{B_x} (x+y)\, d\mu + \int_{B_y} (x+y)\, d\mu \\ &= \int_{B_x} x\, d\mu + \int_{B_y} y\, d\mu = T(x) + T(y). \end{aligned}$$

Prove the laterally-to-norm continuity of T. Assume (x_α) is a net is L_p and $x_\alpha \xrightarrow{\ell} x$. Choose $e \in L_p$ and an index α_0 so that $x, x_\alpha \sqsubseteq e$ for all $\alpha \ge \alpha_0$ (see Proposition 4.5). Fix any $\varepsilon > 0$ and choose $\delta > 0$ so that for any measurable set $A \subseteq [0, 1]$ the inequality $\mu(A) < \delta$ implies $\int_A |e| \, d\mu < \varepsilon/2$. Since $x_\alpha \xrightarrow{\mathrm{o}} x$, by [4, Lemma 8.17] $x_\alpha \to x$ a.e. on [0, 1], and hence, $x_\alpha \xrightarrow{\mu} x$. Then choose $\alpha_1 \ge \alpha_0$ so that $\mu(C_\alpha) < \delta$, where $C_\alpha = \{t \in [0, 1] : |x_\alpha(t) - x(t)| \ge \varepsilon/2\}$ for all $\alpha \ge \alpha_1$. Then for every $\alpha \ge \alpha_1$ one has

$$\begin{aligned} |T(x_\alpha) - T(x)| &= \left| \int_{B_{x_\alpha}} x_\alpha \, d\mu - \int_{B_x} x\, d\mu \right| \\ &= \left| \int_{B_{x_\alpha}} e\, d\mu - \int_{B_x} e\, d\mu \right| = \int_{B_{x_\alpha} \triangle B_x} |e| \, d\mu. \end{aligned} \tag{4.1}$$

Since $\mu\big((B_{x_\alpha} \triangle B_x) \cap C_\alpha\big) \le \mu(C_\alpha) < \delta$, we get

$$\int_{(B_{x_\alpha} \triangle B_x) \cap C_\alpha} |e| \, d\mu < \frac{\varepsilon}{2} \tag{4.2}$$

by the choice of δ. Now observe:

- if $t \in [0, 1] \setminus C_\alpha$ then $|x_\alpha(t) - x(t)| < \varepsilon/2$;
- if $t \in B_{x_\alpha} \triangle B_x$ then one of the values $x_\alpha(t)$, $x(t)$ equals $e(t)$ and the other one equals zero.

Hence, if $t \in (B_{x_\alpha} \triangle B_x) \setminus C_\alpha$ then $|e(t)| < \varepsilon/2$. Therefore

$$\int_{(B_{x_\alpha} \triangle B_x) \setminus C_\alpha} |e| \, d\mu \le \frac{\varepsilon}{2}. \tag{4.3}$$

Combining (4.1) with (4.2) and (4.2), we obtain $|T(x_\alpha) - T(x)| < \varepsilon$. The laterally-to-norm continuity of T is proved.

To show that T is not order-to-norm continuous, set $x_n(t) = 1 + \frac{t}{n}$ and $\mathbf{1}(t) = 1$ for all $t \in [0, 1]$. Then $T(x_n) = 0$, $x_n \xrightarrow{\mathrm{o}} \mathbf{1}$ and $T(\mathbf{1}) = 1$. □

Proposition 4.16 *Let $(E, \|\cdot\|)$ be a Dedekind σ-complete Banach lattice. Then the following statements are equivalent:*

(1) $\|\cdot\|$ *is a laterally-to-norm continuous function;*
(2) $\|\cdot\|$ *is an order-to-norm continuous function.*

Before the proof we remark that a Banach lattice E is Dedekind σ-complete if and only if E has the principal projection property [16, p. 18], which actually is used in the proof below.

Proof Implication (2) $\Rightarrow$ (1) is obvious. We show (1) $\Rightarrow$ (2). Take a sequence (x_n) in E^+ with $x_n \downarrow 0$. It is enough to prove that $\inf\limits_n \|x_n\| = 0$. Fix $\varepsilon > 0$. For every $n \in \mathbb{N}$ let π_n denote the order projection of E onto the band in E generated by $(x_n - \varepsilon x_1)^+$. Since $x_n \downarrow 0$, it follows that $\pi_n \downarrow 0$, where the infimum is taken in the Boolean algebra $\mathfrak{B}(E)$ of all order projections on E. For $n \ge n_0$ we get

$$\begin{aligned} x_n - \varepsilon x_1 \le (x_n - \varepsilon x_1)^+ &= \pi_n((x_n - \varepsilon x_1)^+) \\ &= \pi_n(x_n - \varepsilon x_1) = \pi_n x_n - \varepsilon \pi_n x_1 \le \pi_n x_1. \end{aligned}$$

Hence, $0 \le x_n \le \varepsilon x_1 + \pi_n x_1$. Thus,

$$0 \le \|x_n\| \le \varepsilon \|x_1\| + \|\pi_n x_1\|$$

and

$$0 \le \inf_n \|x_n\| \le \varepsilon \|x_1\| + \inf_n \|\pi_n x_1\|.$$

Observe that $(\pi_n x_1)_{n\in\mathbb{N}}$ laterally converges to 0. Since $\|\cdot\|$ is a laterally-to-norm continuous function, we have that $\inf\limits_n \|\pi_n x_1\| = 0$ and

$$0 \le \inf_n \|x_n\| \le \varepsilon \|x_1\|.$$

Since $\varepsilon > 0$ is arbitrary, we deduce that $\inf\limits_n \|x_n\| = 0$. □

It is very natural that the continuity of an orthogonally additive operator is equivalent to its continuity at zero.

Theorem 4.17 [10] *Let E be a Riesz space, F a Riesz space (normed space), $T : E \to F$ an orthogonally additive operator. Let T be s-laterally or s-laterally-to-order (s-laterally-to-norm, respectively) continuous at zero. Then T is s-laterally or s-laterally-to-order (s-laterally-to-norm, respectively) continuous.*

The following example shows that the lateral continuity of an orthogonally additive operator at one nonzero point does not imply its lateral continuity at all other points (e.g., at zero).

Example 4.18 There exists a non-Dedekind σ-complete Riesz space E and an orthogonally additive operator $T : E \to \mathbb{R}$ which is laterally-to-norm continuous in all senses (up, s- and w-) at one point and is not laterally-to-norm continuous in all senses at another point. Hence, the same operator is laterally-to-order continuous in all senses at one point and is not laterally-to-order continuous in all senses at another point.

Proof Let $E = l_0^\infty$ be the Riesz space of all real eventually constant sequences, that is,

$$l_0^\infty = \big\{(x_i)_{i\in\mathbb{N}} : \ (\exists k \in \mathbb{N})(\forall i \ge k)(x_i = x_k)\big\}$$

equipped with the coordinate-wise order. The sequence $x_k = \left(\frac{1}{2}, \frac{2}{3}, \dots, \frac{k}{k+1}, 0, 0, \dots\right)$ is order bounded from above by $\mathbf{1} = (1, \dots, 1, \dots)$ and has no supremum in E, so E is not Dedekind σ-complete.

Observe that $\mathfrak{F}_{\mathbf{1}} = A_1 \sqcup A_2$, where

$$A_1 = \{(x_i)_{i\in\mathbb{N}} : \ \exists\, l \in \mathbb{N} \text{ such that } \forall i > l :\ x_i = 1 \text{ and } x_i \in \{0, 1\};\ 1 \le i \le l\}$$

and

$$A_2 = \{(x_i)_{i\in\mathbb{N}} : \ \exists\, m \in \mathbb{N} \text{ such that } \forall i > m :\ x_i = 0 \text{ and } x_i \in \{0, 1\};\ 1 \le i \le m\}.$$

Define a map $\varphi : \mathfrak{F}_{\mathbf{1}} \to \mathbb{R}$ by setting

$$\varphi(u) = \begin{cases} 1 & \text{if } u \in A_1 \\ 0 & \text{if } u \in A_2. \end{cases}$$

Then $\varphi : \mathfrak{F}_{\mathbf{1}} \to \mathbb{R}$ is an orthogonally additive map from $\mathfrak{F}_{\mathbf{1}}$ to $\mathbb{R}$. Now set $\mathbf{2} = (2, \ldots, 2, \ldots)$. Define a map $\psi : \mathfrak{F}_{\mathbf{2}} \to \mathbb{R}$ by setting $\psi(v) = \sum_{i=1}^{\infty} \frac{v_i}{2^i}$, $v = (v_i) \in \mathfrak{F}_{\mathbf{2}}$. Observe that ψ is an orthogonally additive operator as well.

Show that ℓ_0^∞ has the intersection property. For any $\alpha, \beta \in \mathbb{R}$ we set $\alpha * \beta = \alpha$ if $\alpha = \beta$ and $\alpha * \beta = 0$ if $\alpha \neq \beta$. Then for each $x = (\xi_1, \xi_2, \ldots)$, $y = (\eta_1, \eta_2, \ldots) \in \ell_0^\infty$ the element $w = (\xi_1 * \eta_1, \xi_2 * \eta_2, \ldots)$ is the lateral infimum of $\{x, y\}$. Indeed, $w \sqsubseteq x$ and $w \sqsubseteq y$ by construction. Assume $z = (\gamma_1, \gamma_2, \ldots) \in \ell_0^\infty$ be any element with $z \sqsubseteq x$ and $z \sqsubseteq y$. Then for every $n \in \mathbb{N}$, if $\gamma_n \neq 0$ then $\gamma_n = \alpha_n$ and $\gamma_n = \beta_n$, and hence $\alpha_n * \beta_n = \gamma_n$. Therefore, $z \sqsubseteq w$ and the equality $w = xy$ is proved. So, ℓ_0^∞ has the intersection property and the lateral infimum xy is well defined for all $x, y \in \ell_0^\infty$ (another way to show that ℓ_0^∞ has the intersection property is to prove that ℓ_0^∞ possesses the principal projection property and then use Corollary 3.15).

Observe that for each element $e \in \ell_0^\infty$ the function $T_e : \ell_0^\infty \to \ell_0^\infty$ of taking the lateral infimum $T_e(x) = xe$ is a disjointness preserving operator. Now we define a map $T : \ell_0^\infty \to \mathbb{R}$ by setting

$$T(e) = \varphi(e\mathbf{1}) + \psi(e\mathbf{2})$$

for all $e \in \ell_0^\infty$. Remark that T is well defined because $e\mathbf{1} \in \mathfrak{F}_{\mathbf{1}}$ and $e\mathbf{2} \in \mathfrak{F}_{\mathbf{2}}$. Being of a composition of a disjointness preserving operator by an orthogonally additive operator, each summand of the above formula defines an orthogonally additive operator. Thus, so is the sum T.

Consider the sequence of elements $x_k \in \ell_0^\infty$ where $x_k = (u_{k,i})_{i \in \mathbb{N}}$ are such that $u_{k,1} = u_{k,2} = \cdots = u_{k,k} = 1$ and $\forall i > k\colon\ u_{k,i} = 0$. It is clear that $x_k \in \mathfrak{F}_{\mathbf{1}}$ for all $k \in \mathbb{N}$. Observe that $x_k \xrightarrow{\ell\uparrow} \mathbf{1}$ and hence, $x_k \xrightarrow{\text{s}-\ell} \mathbf{1}$ and $x_k \xrightarrow{\text{w}-\ell} \mathbf{1}$. On the other hand, $x_k\mathbf{2} = 0$ and hence $\psi(x_k) = 0$ for all $k \in \mathbb{N}$. Thus,

$$T(x_k) = \varphi(x_k) = 0 \longrightarrow 0 \neq 1 = \varphi(\mathbf{1}) = T(\mathbf{1}),$$

and so, T is neither up-laterally-to-norm, nor s-laterally-to-norm continuous, nor w-laterally-to-norm continuous at $\mathbf{1}$. To show that T is s-laterally-to-norm, w-laterally-to-norm and up-laterally-to-norm continuous at $\mathbf{2}$, consider any s-, w- or up-laterally convergent net (v_α) to $\mathbf{2}$. Say, (v_α) is eventually laterally bounded by $e \in \ell_0^\infty$. Let α_0 be an index such that $v_\alpha \sqsubseteq e$ for all $\alpha \geq \alpha_0$. By Proposition 4.5, $\mathbf{2} \sqsubseteq e$, which obviously implies that $e = \mathbf{2}$. Observe that for any type of lateral convergence of (v_α) one has $\psi(\mathbf{2} - v_\alpha) \longrightarrow 0$. Then for all $\alpha \geq \alpha_0$ one has

$$\mathbf{2} = v_\alpha \sqcup (\mathbf{2} - v_\alpha) \text{ and } T(\mathbf{2}) - T(v_\alpha) = T(\mathbf{2} - v_\alpha) = \psi(\mathbf{2} - v_\alpha) \longrightarrow 0,$$

so T has the desired properties. □

5 Consistent sets

The notion of a consistent subset is new. It may be convenient in different contexts. Consistent sets naturally generalize laterally bounded subsets of a Riesz space and give principally new tools for investigation of general Riesz spaces.

Definition 5.1 A subset G of a Riesz space E is called *consistent* if every two-point subset of G is laterally bounded in E.

Observe that every disjoint set is consistent. Simple examples show that a consistent set in a Riesz space need not be laterally bounded. For instance, the set $G = \{\mathbf{1}_{[0,n]} : n \in \mathbb{N}\}$, being a lateral chain, is consistent but is not laterally bounded in $L_1(\mathbb{R})$.

Proposition 5.2 *For a subset G of a Riesz space E the following assertions are equivalent:*

(1) *G is consistent;*
(2) *every finite subset of G is laterally bounded;*
(3) *every finite subset of G has a $\sqsubseteq$-supremum in E.*

Proof By Corollary 3.7, (2) and (3) are equivalent. Implication (2) $\Rightarrow$ (1) is obvious. We prove implication (1) $\Rightarrow$ (2). By Corollary 3.6, it is enough to consider the case where $G \subseteq E^+$. Let $G \subseteq E^+$ be consistent and let $\{x_1, \dots, x_n\}$ be a finite subset of G, $x_i \neq x_j$ for $i \neq j$. By Proposition 3.2, $x_i \sqcup x_j = x_i \vee x_j$ for all $i, j = 1, \dots, n$. We show that, for all $i = 1, \dots, n$ one has $x_i \sqsubseteq x_1 \vee \dots \vee x_n$, which completes the proof. Fix any i. Since $x_i \perp (x_i \vee x_j) - x_i$ for all j, one has

$$x_i \perp (x_i \vee x_1 - x_i) \vee \dots \vee (x_i \vee x_n - x_i) = x_1 \vee \dots \vee x_n - x_i,$$

and hence $x_i \sqsubseteq x_1 \vee \dots \vee x_n$. □

Recall that a Riesz space E is called *laterally complete* if every disjoint family from E^+ has the supremum. The following simple observation shows that, using lateral terminology, one can give an equivalent definition for not necessarily positive disjoint families.

Proposition 5.3 *A Riesz space E is laterally complete if and only if every disjoint family from E has a lateral supremum.*

Proof By (2) of Proposition 3.4, if every disjoint family from E has a lateral supremum then E is laterally complete. Let E be laterally complete and $G \subset E$ be a disjoint family. Obviously, $G^+ = \{g^+ : g \in G\}$ and $G^- = \{g^- : g \in G\}$ are disjoint families from E^+ which have suprema $\sup G^+$ and $\sup G^-$ by the lateral completeness of E. By (2) of Proposition 3.4, there are lateral suprema $\bigsqcup G^+$ and $\bigsqcup G^-$. Finally, by Corollary 3.6, there exists the lateral supremum $\bigsqcup G = \bigsqcup G^+ - \bigsqcup G^-$. □

The following theorem characterizes laterally complete Riesz spaces in terms of consistent sets.

Theorem 5.4 *A Riesz space E is laterally complete if and only if every consistent set $A \subset E$ has a lateral supremum $\bigsqcup A \in E$.*

First we need some lemmas.

Lemma 5.5 *Let A be a consistent subset of a Riesz space E, $B \subseteq A$, $a \in A$. If there exists $\bigsqcup B$ then the two-point set $\{a, \bigsqcup B\}$ is laterally bounded.*

Proof By Corollary 3.6, with no loss of generality, we may and do assume that $A \subset E^+$. Hence, $b := \bigsqcup B = \bigvee B$. We prove that $\{a, b\}$ is laterally bounded by $e = a \vee b$. First we show that $a \sqsubseteq e$, that is, $a \perp a \vee b - a$. Since $a \vee b - a = 0 \vee (b - a) = (b - a)^+$, one has

$$|a| \wedge |a \vee b - a| = |a| \wedge \Big(\bigvee_{b \in B} b - a\Big)^+ = |a| \wedge \Big(\bigvee_{b \in B} (b - a)\Big)^+ = |a| \wedge \bigvee_{b \in B} (b - a)^+.$$

By the infinite distributivity property of any Archimedean Riesz space [26, Theorem III.5.1] we continue the chain of equalities as

$$= \bigvee_{b \in B} |a| \wedge (b - a)^+ = \bigvee_{b \in B} 0 = 0,$$

because for all $b \in B$, $a \sqsubseteq a \vee b$ and hence $a \perp a \vee b - a$, that is, $|a| \wedge (b - a)^+ = 0$.

Now we show that $b \sqsubseteq e$, that is, $b \perp b \vee a - b$. Fix any $b' \in B$. Since $a - b \leq a - b'$, one has that $(a - b)^+ \leq (a - b')^+$ and hence

$$|b'| \wedge |b \vee a - b| = |b'| \wedge (a - b)^+ \leq |b'| \wedge (a - b')^+ = 0.$$

We have proved that, for every $b' \in B$, $b' \perp b \vee a - b$. Since the orthogonal complement to any element is a band, we obtain that $b = \bigvee B \perp b \vee a - b$. □

Lemma 5.6 *Let ω_β be a cardinal, $A = \{a_\alpha : \alpha < \omega_\beta\}$ be a consistent subset of a Riesz space E. Assume that for all ordinals α, $0 < \alpha < \omega_\beta$ there exists $\bigsqcup A_\alpha$, where $A_\alpha = \{a_\xi : \xi < \alpha\}$. Set*

$$b_0 = a_0 \text{ and } b_\alpha = a_\alpha \setminus \bigsqcup A_\alpha \text{ for } 0 < \alpha < \omega_\beta. \tag{5.1}$$

If there exists $\bigsqcup \{b_\alpha : \alpha < \omega_\beta\} = e \in E$ then $\bigsqcup A$ exists and equals e.

Remark that the set-theoretical difference in (5.1) is well defined by Lemma 5.5.

Proof Set $A_{\omega_\alpha} = A$ and prove the following claim by transfinite induction.

$(*)$ *For every ordinal α, $0 < \alpha \leq \omega_\alpha$ one has $\bigsqcup A_\alpha = \bigsqcup \{b_\xi : \xi < \alpha\}$.*

For $\alpha = 1$ we have $\bigsqcup A_0 = a_0 = b_0 = \bigsqcup \{b_\xi : \xi < 1\}$ and Claim $(*)$ holds. Fix any ordinal γ, $1 < \gamma \leq \omega_\alpha$. Assume Claim $(*)$ holds for all $\alpha < \gamma$ and prove the claim for $\alpha = \gamma$.

If γ is an isolated ordinal then $\gamma = \alpha + 1$ for some $\alpha < \gamma$, and by the induction assumption

$$\begin{aligned}\bigcup A_\gamma &= a_\alpha \cup \bigcup A_\alpha \\ &\overset{\text{by Prop.3.18(4)}}{=} \left(a_\alpha \setminus \bigcup A_\alpha\right) \cup \bigcup A_\alpha \\ &= b_\alpha \cup \bigcup \{b_\xi : \xi < \alpha\} = \bigcup \{b_\xi : \xi < \gamma\}.\end{aligned}$$

If γ is a limited ordinal then by the induction assumption

$$\begin{aligned}\bigcup A_\gamma &= \bigcup \{a_\xi : \xi < \gamma\} \\ &= \bigcup_{\alpha<\gamma} \bigcup \{a_\xi : \xi < \alpha\} \\ &= \bigcup_{\alpha<\gamma} \bigcup \{b_\xi : \xi < \alpha\} \\ &= \bigcup \{b_\xi : \xi < \gamma\}.\end{aligned}$$

So, Claim $(*)$ is proved. The lemma follows from the claim for $\gamma = \omega_\beta$. □

Proof of Theorem 5.4 Let E be laterally complete. Assume $A \subset E$ is consistent. By Corollary 3.6, with no loss of generality, we may and do assume that $A \subset E^+$. So, it is enough to prove the following claim.

For every ordinal β every consistent set $A \subset E^+$ of cardinality $< \aleph_\beta$ has a lateral supremum.

For $\beta = 0$ the claim follows from Proposition 5.2. Assume the claim is true for every ordinal $0 \le \beta < \gamma$ and prove the claim for $\beta = \gamma$. Let $A \subseteq E^+$ be a consistent subset and $|A| < \aleph_\gamma$. If γ is a limit ordinal then there is $\beta < \gamma$ such that $|A| < \aleph_\beta$, and the lateral boundedness of A follows from the induction assumption. Let γ be an isolated ordinal, $\gamma = \beta + 1$. Then $|A| \le \aleph_\beta$. If $|A| < \aleph_\beta$ then the lateral boundedness of A follows again from the induction assumption. So, consider the case where $|A| = \aleph_\beta$. Let $A = \{a_\alpha : \alpha < \omega_\beta\}$ be any well ordering of A. Set $A_\alpha = \{a_\xi : \xi < \alpha\}$ for all $0 < \alpha < \omega_\beta$ and observe that $|A_\alpha| < \aleph_\beta$. By the induction assumption, $\bigcup A_\alpha$ is well defined for all $\alpha < \omega_\beta$. Then we define an ω_β-sequence $(b_\alpha)_{\alpha<\omega_\beta}$ in E by setting

$$b_0 = a_0 \ \text{ and } b_\alpha = a_\alpha \setminus \bigcup A_\alpha \ \text{ for } 0 < \alpha < \omega_\beta. \tag{5.2}$$

The set-theoretical difference in (5.2) is well defined by Lemma 5.5. It is immediate that $(b_\alpha)_{\alpha<\omega_\beta}$ is a disjoint system in E^+. By the lateral completeness of E, there exists $e = \bigcup \{b_\alpha : \alpha < \omega_\beta\}$. By Lemma 5.6, $\bigcup A = e$.

Assume every consistent set in E has a lateral supremum. Let $a_i \in E^+$, $i \in I$, be a disjoint system. Obviously, $A = \{a_i : i \in I\}$ is a consistent set. By the assumption, there exists $\bigcup A \in E$. By (2) of Proposition 3.4, $\sup A = \bigcup A$, and hence, E is laterally complete. □

Theorem 5.4 has an important consequence which concerns the property of a Riesz space to have a lateral suprema of every laterally bounded set.

Definition 5.7 A Riesz space E is said to be *C-complete* if every nonempty laterally bounded subset G of E has a lateral supremum $\bigsqcup G \in E$.

Corollary 5.8 *If a Riesz space E is either Dedekind complete or laterally complete then E is C-complete.*

Proof If a Riesz space E is Dedekind complete then E is C-complete by Corollary 3.9. If a Riesz space E is laterally complete then E is C-complete by Theorem 5.4. □

Observe that the Banach lattice $C[0, 1]$ is a C-complete Riesz space which is both not Dedekind complete and not laterally complete. We do not know whether there is an atomless example of the kind.

6 Lateral fields, ideals and bands

Definition 6.1 A nonempty subset G of a Riesz space E is called:

- *finitely laterally closed* if for every laterally bounded two-point subset $\{x, y\}$ of E the condition $\{x, y\} \subseteq G$ implies $x \cup y \in G$;
- a *lateral field* if it is finitely laterally closed, and for every two-point subset $\{x, y\}$ of G the existence of xy implies $xy \in G$ and $x \setminus y \in G$;
- *laterally solid* if for each $x \in E$ and $y \in G$ the relation $x \sqsubseteq y$ implies $x \in G$;
- a *lateral ideal* if it is laterally solid and finitely laterally closed;
- *up-laterally closed* if for each subset $G_1 \subseteq G$ the existence of $f = \bigsqcup G_1$ in E implies that $f \in G$;
- *laterally closed* if for every net (x_α) in G and every $x \in E$ the condition $x_\alpha \xrightarrow{\ell} x$ implies that $x \in G$;
- a *lateral band* if it is a laterally closed lateral ideal.

It is immediate that a lateral band is a lateral ideal, and a lateral ideal is a lateral field, but the converse assertions are not true. Every laterally closed subset is up-laterally closed, but, if E contains a nontrivial laterally convergent to zero net (x_α) then the set $F_e \setminus \{0\}$ is obviously up-laterally closed but is not laterally closed. However, for laterally solid sets these two notions coincide.

Proposition 6.2 *Let G be an up-laterally closed lateral ideal of a Riesz space E. Then G is a lateral band.*

Proof Let (x_α) be a net in G, $x \in E$ and $x_\alpha \xrightarrow{\ell} x$. We claim that $x \in G$. Passing to positive and negative parts of the elements and using Corollary 3.9, we reduce the claim to the case where $x, x_\alpha \geq 0$. Assume $x_\alpha \sqsubseteq e$ for all $\alpha \geq \alpha_0$. Since $x_\alpha \xrightarrow{o} x$, by [4, Theorem 8.16]

$$x = \bigvee_{\alpha} \bigwedge_{\beta \geq \alpha} x_\beta. \tag{6.1}$$

Fix any $\alpha \geq \alpha_0$. By (2) of Proposition 3.4, $\bigwedge_{\beta \geq \alpha} x_\beta = \bigcap_{\beta \geq \alpha} x_\beta$ and hence $\bigwedge_{\beta \geq \alpha} x_\beta \sqsubseteq x_\alpha$. Since G is laterally solid, $\bigwedge_{\beta \geq \alpha} x_\beta \in G$. Since G is up-laterally solid, by (6.1) $x \in G$. □

Remark that in [11] we define a lateral band of a Riesz space E to be laterally solid up-laterally closed subset of E. Fortunately, this appears to be equivalent to the definition given above due to Proposition 6.2.

Observe that a nonempty subset $G \subseteq E$ is consistent and finitely laterally closed if and only if it is directed with respect to the lateral order.

An equivalent definition of a lateral ideal is given in the following simple statement.

Lemma 6.3 *Let E be a Riesz space and $G \subset E$. The following assertions are equivalent:*

(1) *G is a lateral ideal;*
(2) *G is laterally solid and for every $x, y \in G$ with $x \perp y$ one has $x + y \in G$.*

Proof (1) $\Rightarrow$ (2). Consider any $x, y \in G$ with $x \perp y$. Then the two-point subset $\{x, y\}$ is laterally bounded in E by $x + y$ and moreover $x + y = x \cup y \in G$.

(2) $\Rightarrow$ (1). Consider any laterally bounded two-point subset $\{x, y\}$ of E with $\{x, y\} \subseteq G$. By (4) of Proposition 3.18, $x \cup y = x \sqcup (y \setminus x)$. Since G is laterally solid, $y \setminus x \in G$ and hence $x \cup y \in G$ by (2).

□

The importance of lateral ideals for orthogonally additive operators is demonstrated in the following statement.

Proposition 6.4 *Let E, F be Riesz spaces and $T : E \to F$ a positive orthogonally additive operator. Then* $\ker T = \{x \in E : Tx = 0\}$ *is a lateral ideal in E.*

Proof Assume $x \in E$, $y \in \ker T$ and $x \sqsubseteq y$ hold. Then $0 \leq T(x) \leq T(x) + T(y - x) = T(y) = 0$, and hence $x \in \ker T$, so $\ker T$ is laterally solid. Assume $x, y \in \ker T$ and $x \perp y$. Then $T(x + y) = T(x) + T(y) = 0$ and $x + y = z \in \ker T$. □

The next statement asserts that not every lateral ideal is the kernel of a positive orthogonally additive separable valued operator.

Proposition 6.5 *Let E be a Dedekind complete Riesz space containing a strictly increasing up-laterally convergent sequence. Then there is a lateral ideal in E which cannot be equal the kernel of a positive orthogonally additive operator from E to any separable Banach lattice F.*

Proof Let $e \in E$ and let $(e_n)_{n=1}^\infty$ be a sequence with $e_n \sqsubseteq e_{n+1}$, $e_n \neq e_{n+1}$ and $e_n \xrightarrow{\ell\uparrow} e$. Set $H = \bigcup_{n=1}^\infty \mathfrak{F}_{e_n}$ and prove that H possesses the desired properties. Since $e_n \sqsubseteq e_{n+1}$, we have that $\mathfrak{F}_{e_n} \subseteq \mathfrak{F}_{e_{n+1}}$ for all n, which implies that H is a lateral ideal.

Assume, on the contrary, that $H = \ker T$, where $T : E \to F$ is a positive orthogonally additive operator to a separable Banach lattice F. Let $\mathcal{M}$ be an uncountable collection of infinite subsets of $\mathbb{N}$ such that

$$(\forall A, B \in \mathcal{M})\Big((A \neq B) \Rightarrow \big(|A \cap B| < \aleph_0\big)\Big), \tag{6.2}$$

the well known mathematical folklore. Set $e'_1 = e_1$ and $e'_{n+1} = e_{n+1} - e_n$ for every $n \in \mathbb{N}$. Then $e = \bigsqcup_{n\in\mathbb{N}} e'_n = \bigcup_{n\in\mathbb{N}} e'_n$. For any $A \in \mathcal{M}$ set $x_A = \bigcup_{n\in A} e'_n$ (the lateral supremum exists by Corollary 3.9, because the set under the supremum is laterally bounded by e). Since every $A \in \mathcal{M}$ is infinite, $x_A \notin H$ and so, $T(x_A) \neq 0$. By the separability of F, there is a nonzero condensation point $f \in F$ of the set $\{T(x_A) : A \in \mathcal{M}\}$, that is, for every $\varepsilon > 0$ one has $|\mathcal{M}_\varepsilon| > \aleph_0$, where $\mathcal{M}_\varepsilon = \{A \in \mathcal{M} : \|T(x_A) - f\| < \varepsilon\}$. Fix any $\delta > 0$ and $m \in \mathbb{N}$. Choose any $A_1, \dots, A_m \in \mathcal{M}_\delta$. Then, by (6.2), there is $n \in \mathbb{N}$ such that for the infinite sets $B_k = A_k \setminus \{1, \dots, n\}$, $k = 1, \dots, m$ one has

$$B_i \cap B_j = \emptyset \ \text{ for all } \ i, j \in \{1, \dots, m\}. \tag{6.3}$$

Set $y_k = \bigcup\{e'_n : n \in B_k\}$ and $z_k = \bigcup\{e'_n : n \in A_k \setminus B_k\}$ for $k = 1, \dots, m$. Then $x_{A_k} = y_k \sqcup z_k$ for $k = 1, \dots, m$. Since $z_k \in H$, by the orthogonal additivity of T we get $T(y_k) = T(x_{A_k})$, and $A_k \in \mathcal{M}_\delta$ yields $\|T(y_k) - f\| < \delta$ for $k = 1, \dots, m$. By (6.3), $(y_k)_{k=1}^m$ is a disjoint sequence. Hence, taking into account that $y_1 \sqcup \dots \sqcup y_m \sqsubseteq e$, by positivity of T we obtain $T(y_1) + \dots + T(y_m) \leq T(e)$. Thus,

$$\begin{aligned} \|T(e)\| &\geq \|T(y_1) + \dots + T(y_m)\| \\ &= \|mf + T(y_1) - f + \cdots T(y_m) - f\| \\ &\geq m\|f\| - \|T(y_1) - f\| - \dots - \|T(y_m) - f\| > m\big(\|f\| - \delta\big), \end{aligned}$$

which contradicts the arbitrariness of $\delta > 0$ and $m \in \mathbb{N}$. □

The following statement shows that the separability assumption in Proposition 6.5 is essential.

Proposition 6.6 *There exist a lateral ideal H in $E = l_\infty$ such that*

(i) *H cannot be equal the kernel of a positive orthogonally additive operator acting from E to any separable Banach lattice F;*
(ii) *H is the kernel of a positive orthogonally additive operator T from E to some Banach lattice F_0.*

Proof We consider the lateral ideal

$$H = \{x = (\xi_n)_{n\in\mathbb{N}} \in \{0, 1\}^{\mathbb{N}} : (\exists m \in \mathbb{N})\, (\forall n \geq m)\, (\xi_n = 0)\}.$$

By Proposition 6.5, H cannot be equal the kernel of a positive orthogonally additive operator from E to any separable Banach lattice F.

Now let F_0 be the quotient Banach space l_∞ / c_0 endowed with the norm

$$\|\hat{x}\| = \limsup_{n\to\infty} |\xi_n|,$$

where $x = (\xi_n)_{n\in\mathbb{N}} \in l_\infty$ and $\hat{x} = \pi(x) = \{y \in l_\infty : x - y \in c_0\}$. Notice that F_0 is a Banach lattice with respect to the order

$$\hat{x} \leq \hat{y} \equiv (\exists\, u = (\xi_n)_{n\in\mathbb{N}} \in \hat{x})\, (\exists\, v = (\eta_n)_{n\in\mathbb{N}} \in \hat{y})\, (\forall\, n \in \mathbb{N})\, (\xi_n \leq \eta_n).$$

For any $A \subseteq \mathbb{N}$ set $x_A = \mathbf{1}_A$, where $\mathbf{1} = (1, 1, 1, \dots) \in l_\infty$. Moreover, for every $x = (\xi_n)_{n\in\mathbb{N}} \in l_\infty$ set $N_x = \{n \in \mathbb{N} : \xi_n \neq 1\}$ and $M_x = \{n \in \mathbb{N} : \xi_n = 1\}$. Fix any sequence $(A_n)_{n\in\mathbb{N}}$ of disjoint infinite sets $A_n \subseteq \mathbb{N}$. Consider the operator $T : l_\infty \to F_0$ defined by $T(x) = \pi(\bigsqcup_{n\in N_x} |\xi_n| \cdot x_{A_n}) + \pi(\mathbf{1}_{M_x})$. It is easy to see that T is a positive orthogonally additive operator and $\ker T = H$. □

Problem 6.7 *Are there a Riesz space E and a lateral ideal in E which is not equal to the kernel of any positive orthogonally additive operator $T : E \to F$ for any Riesz space F?*

Obviously, the intersection of lateral fields (ideals or bands) is a lateral field (resp., ideal or band). So, given any subset G of P, there exists the minimal lateral field $\mathcal{F}(G)$, the minimal lateral ideal $\mathcal{I}(G)$ and the minimal lateral band $\mathcal{B}(G)$ containing G, which equals the intersection of all lateral fields (ideals or bands, resp.) containing G. We say that $\mathcal{F}(G)$ ($\mathcal{I}(G)$ or $\mathcal{B}(G)$, resp.) is the *lateral field (ideal or band, resp.) generated by G*.

Obviously, $\mathcal{F}(G) \subseteq \mathcal{I}(G) \subseteq \mathcal{B}(G)$ for any subset $G \subseteq E$.

Proposition 6.8 *Let E be a Riesz space and $G \subseteq E$. Then the following assertions hold.*

(1) $\mathcal{F}(G) = \bigcup_{n=0}^{\infty} G_n$, *where $G_0 = G$ and for all $n = 0, 1, \dots$*

$$
\begin{aligned}
G_{n+1} = \{x \sqcup y : x, y \in G_n \text{ and } x \sqcup y \text{ exists}\} \\
\cup \{xy : x, y \in G_n \text{ and } xy \text{ exists}\} \\
\cup \{x \setminus y : x, y \in G_n \text{ and } x \setminus y \text{ exists}\}
\end{aligned}
$$

(2) $\mathcal{I}(G) = \Big\{\bigsqcup_{k=1}^{n} x_k : (\exists f_1, \dots, f_n \in G)\big(x_1 \sqsubseteq f_1\big)\&\dots\&\big(x_n \sqsubseteq f_n\big),\ n \in \mathbb{N}\Big\}$.

(3) $\mathcal{B}(G) = \Big\{\bigsqcup X : X \subseteq \bigcup_{x\in G} \mathfrak{F}_x \text{ and } \bigsqcup X \text{ exists}\Big\}$.

The proof is straightforward.

Observe that the lateral ideal generated by a single element $x \in E$ equals $\mathfrak{F}_x$.

There is a natural projection of a Dedekind complete Riesz space onto a lateral band having nice properties. The following theorem strengthens Theorem 3 of [11].

Theorem 6.9 *Let G be a lateral band of a Dedekind complete Riesz space E. Then the function $P_G : E \to E$ defined by setting for every $x \in E$*

$$P_G(x) = \bigsqcup (\mathfrak{F}_x \cap G), \tag{6.4}$$

is:

(1) *a projection of E onto G such that $P_G(x) \sqsubseteq x$ for all $x \in E$;*
(2) *a disjointness preserving operator;*
(3) *an orthogonally additive operator;*

(4) *laterally contractive, that is,* $|P_G(u) - P_G(v)| \leq |u - v|$ *for every laterally bounded two-point set* $\{u, v\} \subset E^+$*;*
(5) *up-laterally continuous;*
(6) *laterally continuous.*

Proof (1) The fact that P_G a projection of E onto G is proved in [11, Theorem 3], and the property $P_G(x) \sqsubseteq x$ for all $x \in E$ is obvious.

Properties (2) and (5) are proved in [11, Theorem 3].

(3) Fix any $x, y \in E$ with $x \perp y$ and $z \in \mathfrak{F}_{x+y} \cap G$. By Corollary 3.9 we may and do assume that $x, y \in E^+$. Then by the Riesz decomposition property [5, Theorem 1.13], there exists a decomposition $z = z_1 \sqcup z_2$, where $z_1 \in \mathfrak{F}_x \cap G$ and $z_2 \in \mathfrak{F}_y \cap G$. Thus

$$z = z_1 + z_2 \leq P_G(x) + P_G(y),$$

and passing to the lateral supremum over all $z \in \mathfrak{F}_{x+y} \cap G$ in the left hand side of the above formula we obtain $P_G(x + y) \leq P_G(x) + P_G(y)$. On the other hand, for every $z_1 \in \mathfrak{F}_x \cap G$ and $z_2 \in \mathfrak{F}_y \cap G$ the sum $z_1 + z_2$ belongs to $\mathfrak{F}_{x+y} \cap G$ and therefore

$$z_1 + z_2 = z \leq P_G(x + y).$$

Passing to the lateral supremum in the left hand side first over all $z_1 \in \mathfrak{F}_x \cap G$ and then over all $z_2 \in \mathfrak{F}_y \cap G$, we obtain $P_G(x) + P_G(y) \leq P_G(x + y)$. Finally, we have the equality $P_G(x) + P_G(y) = P_G(x + y)$.

(4) Let $\{u, v\} \subset E^+$ be any laterally bounded two-point set, and let $e \in E$ be such that $u, v \in \mathfrak{F}_e$. Since $u = (u \backslash v) \sqcup uv$, by (2), $P_G(u) = P_G(u \backslash v) + P_G(uv)$. Analogously, $P_G(v) = P_G(v \backslash u) + P_G(uv)$. Hence, $P_G(u) - P_G(v) = P_G(u \backslash v) - P_G(v \backslash u)$. Since $(u \backslash v) \perp (v \backslash u)$, by (2) one has $P_G(u \backslash v) \perp P_G(v \backslash u)$. Hence,

$$|P_G(u) - P_G(v)| = P_G(u \backslash v) + P_G(v \backslash u). \tag{6.5}$$

Since

$$\begin{aligned} |u - v| &= |(u \backslash v) + uv - (v \backslash u) - vu| \\ &= |(u \backslash v) + uv - (v \backslash u) - vu| \\ &= |(u \backslash v) - (v \backslash u)| \\ &= (u \backslash v) \sqcup (v \backslash u), \end{aligned}$$

by (2),

$$P_G|u - v| = P_G(u \backslash v) + P_G(v \backslash u). \tag{6.6}$$

Therefore, by (6.5), (6.6) and (1),

$$|P_G(u) - P_G(v)| = P_G|u - v| \leq |u - v|.$$

(6) Let (x_α) be a net in E, $x \in E$ and $x_\alpha \xrightarrow{\ell} x$. First we assume that $x_\alpha \in E^+$ for all α. Then $x \in E^+$ as well. Let α_0 be an index and $e \in E$ be such that $x_\alpha \sqsubseteq e$ for all $\alpha \geq \alpha_0$. Let (u_α) be a net in E such that $|x_\alpha - x| \leq u_\alpha \downarrow 0$. Since $P_G(x_\alpha) \sqsubseteq x_\alpha \sqsubseteq e$ for all $\alpha \geq \alpha_0$, the net $\big(P_G(x_\alpha)\big)$ is eventually laterally bounded. It remains to show that $P_G(x_\alpha) \xrightarrow{o} P_G(x)$. Fix any α. By Proposition 4.5, $x \sqsubseteq e$, and thus, $\{x_\alpha, x\}$ is a laterally bounded set. By (4), $|P_G(x_\alpha) - P_G(x)| \leq |x_\alpha - x| \leq u_\alpha \downarrow 0$. So, $P_G(x_\alpha) \xrightarrow{o} P_G(x)$.

The case where $-x_\alpha \in E^+$ for all α is considered similarly.

Consider the general case $x_\alpha \in E$ for all α. By Proposition 4.7, $x_\alpha^+ \xrightarrow{\ell} x^+$ and $x_\alpha^- \xrightarrow{\ell} x^-$. Let α_0 and $e \in E$ be such that $x_\alpha \sqsubseteq e$ for all $\alpha \geq \alpha_0$. By Proposition 3.1, $x_\alpha^+ \sqsubseteq e^+$ and $x_\alpha^- \sqsubseteq e^-$ for all $\alpha \geq \alpha_0$. Then by the above

$$\begin{aligned} P_G(x_\alpha) &= P_G(x_\alpha^+ - x_\alpha^-) = P_G(x_\alpha^+) + P_G(-x_\alpha^-) \xrightarrow{0} P_G(x^+) + P_G(-x^-) \\ &= P_G(x^+ - x^-) = P_G(x). \end{aligned}$$

By (1), $P_G(x_\alpha) \sqsubseteq x_\alpha \sqsubseteq e$ for all $\alpha \geq \alpha_0$. Thus, $P_G(x_\alpha) \xrightarrow{o} P_G(x)$. □

The following theorem and its corollaries concern relationships between consistent sets and lateral bands.

Theorem 6.10 *The lateral band $\mathcal{B}(G)$ in a Riesz space E generated by a consistent set G is consistent.*

Proof Let $x, y \in \mathcal{B}(G)$, say $x = \bigsqcup X$ and $y = \bigsqcup Y$, where $X, Y \subseteq \bigcup_{g \in G} \mathfrak{F}_g$. By Theorem 3.8, $x^+ = \sup X^+$ and $y^+ = \sup Y^+$, where $X^+ = \{u^+ : u \in X\}$ and $Y^+ = \{v^+ : v \in Y\}$.

Fix any $u \in X$ and $v \in Y$. Since G is consistent, the two-point subset $\{u^+, v^+\}$ is laterally bounded, and hence $u^+ \sqsubseteq u^+ \sqcup v^+ = u^+ \vee v^+$ by (2) of Proposition 3.4. Hence

$$u^+ \wedge (u^+ \vee v^+ - u^+) = 0. \tag{6.7}$$

Observe that

$$x^+ \vee v^+ - x^+ \leq u^+ \vee v^+ - u^+. \tag{6.8}$$

Indeed, $u^+ \leq x^+$ and hence

$$\begin{aligned} x^+ \vee v^+ - x^+ &= (-x^+ + x^+) \vee (-x^+ + y^+) = 0 \vee (-x^+ + y^+) \\ &\leq 0 \vee (-u^+ + y^+) = (-u^+ + u^+) \vee (-u^+ + y^+) = u^+ \vee v^+ - u^+. \end{aligned}$$

Combining (6.7) with (6.8) we obtain

$$(\forall u \in X)(\forall v \in Y) \qquad 0 \leq u^+ \wedge (x^+ \vee v^+ - x^+) \leq u^+ \wedge (u^+ \vee v^+ - u^+) = 0.$$

Hence

$$(\forall v \in Y) \qquad x^+ \wedge (x^+ \vee v^+ - x^+) = \sup_{u \in X} \big(u^+ \wedge (x^+ \vee v^+ - x^+)\big) = 0.$$

Then

$$x^+ \wedge (x^+ \vee y^+ - x^+) = \sup_{v \in Y} \big(x^+ \wedge (x^+ \vee v^+ - x^+)\big) = 0,$$

that is, $x^+ \sqsubseteq x^+ \vee y^+$. Analogously, $y^+ \sqsubseteq x^+ \vee y^+$. Therefore, $\{x^+, y^+\}$ is a laterally bounded two-point set. Analogously, the two-point set $\{x^-, y^-\}$ is laterally bounded as well. By Corollary 3.6, $\{x, y\}$ is laterally bounded. □

Corollary 6.11 *The lateral field $\mathcal{F}(G)$ and the lateral ideal $\mathcal{I}(G)$ in a Riesz space E generated by a consistent subset $G \subseteq E$ is consistent.*

Corollary 6.12 *Any maximal consistent subset of a Riesz space E is a lateral band in E.*

7 Extensions of orthogonally additive operators

The questions of the extension of orthogonally additive operators from subsets of a Riesz space to the entire space were considered in [11] and [21]. Here we review known results and prove our main result (Theorem 7.5) which gives a positive answer to a problem posed in [11].

One can consider an orthogonally additive operator defined on an arbitrary subset of a Riesz space. More precisely, let E be a Riesz space, X a vector space and $D \subseteq E$. A function $T : D \to X$ is called an *orthogonally additive operator* if for any $x, y \in D$ with $x \perp y$ and $x+y \in D$ one has $T(x+y) = T(x)+T(y)$. The very general extension problem of whether every orthogonally additive operator defined on an arbitrary subset D of a Riesz space E has an extension to an orthogonally additive operator on E, has a negative answer, even for "good lattices" E. Indeed, let E denote the Riesz space $\mathbb{R}^5$ with the usual coordinate-wise order. Denote by $e_1, \dots, e_5$ be the unit vector basis of E. Let D consists of all sums $\sum_{k \in A} e_k$ over three-point subsets $A \subset \{1, \dots, 5\}$. Since D consists of $C_5^3 = 10$ elements, the set F of all functions $T : D \to \mathbb{R}$ is a 10-dimensional vector space with respect to the coordinate-wise operations. Since D contains no orthogonal elements, every element of F is an orthogonally additive operator. On the other hand, the set of all orthogonally additive operators defined on the set $\mathfrak{F}_e$ of all fragments of $e = e_1 + \cdots + e_5$ is a 5-dimensional vector space. Hence, not every orthogonally additive operator defined on D can be extended to an orthogonally additive operator on E. Clearly, such an example exists for every Riesz space E with $\dim E \geq 5$.

In [11] and [21] the authors extend orthogonally additive operators from lateral ideals and lateral bands in a Dedekind complete Riesz space.

Theorem 7.1 [11,21] *Let E, F be Riesz spaces with F Dedekind complete, D a lateral ideal in E and $T_0 : D \to F$ a positive order bounded orthogonally additive operator. Then there exists an abstract Uryson operator $T : E \to F$ which extends T_0. Moreover, the operator $T : E \to F$ defined by the formula*

$$T(x) = \sup T_0(D \cap \mathfrak{F}_x) \text{ for all } x \in E \tag{7.1}$$

is the minimal abstract Uryson extension of T_0 in the sense that if $S : E \to F$ is another positive order bounded orthogonally additive operator extension of T_0 then $T(x) \le S(x)$ for all $x \in E$.

The first part of Theorem 7.1 is proved in [11] and the second part in [21].

The operator $T : E \to F$ defined by (7.1) is called the *minimal Uryson extension* of T_0. The following result of [21] say that the minimal Uryson extension preserves different compactness type properties (actually we combine two theorems of [21] in the following one).

Theorem 7.2 [21] *Let E be a Dedekind complete Riesz space, F an order complete Banach lattice, D a lateral band in E and $T_0 : D \to F$ an order bounded orthogonally additive operator with the minimal Uryson extension $T : E \to F$.*

(1) *If T_0 is AM-compact (or C-compact) then so is T.*
(2) *If, in addition, E is atomless and T_0 is narrow (or strictly narrow) then so is T.*

The following result is an easy consequence of Theorem 6.9.

Theorem 7.3 (A. Gumenchuk, M. Pliev, M. Popov) *Let E, F be Riesz spaces with E Dedekind complete, E_0 a lateral band of E and $T_0 : E_0 \to F$ an orthogonally additive operator. Then there is an orthogonally additive extension $T : E \to F$ of T_0. If, moreover, T_0 is positive (laterally bounded, preserves disjointness or laterally continuous) then so is T.*

A more delicate problem is to extend an orthogonally additive operator from a lateral field.

Problem 7.4 (A. Gumenchuk, M. Pliev, M. Popov, [11]) *Let F be a lateral field in a Riesz space E, X a linear space. Whether every orthogonally additive operator $T_0 : F \to X$ can be extended to an orthogonally additive operator $T : E \to X$?*

We solve this problem for Riesz spaces with the intersection property which is much less restrictive than for Dedekind complete ones, as in theorems 7.1 and 7.3. Moreover, the following theorem, which is the main result of the section, deals with arbitrary orthogonally additive operators.

Theorem 7.5 *Let D be a lateral field in a Riesz space E with the intersection property, X a vector space and $T_0 : D \to X$ an orthogonally additive operator. Then there exists an orthogonally additive extension $T : E \to X$ of T_0.*

First we need some lemmas.

Lemma 7.6 *Let F be a finite laterally bounded field in a Riesz space E. Then there is a finite disjoint sequence $(f_j)_{j=1}^m$ in F such that*

$$F = \Big\{\bigsqcup_{j\in J} f_j : \ J \subseteq \{1, \dots, m\}\Big\}.$$

Proof We say that a nonzero element $f_0 \in F$ is *elementary* if for all $f \in F$ the relation $f \sqsubseteq f_0$ yields $f = 0$ or $f = f_0$. Since F is finite, for every nonzero element $f \in F$ there is an elementary element $f_0 \in F$ with $f_0 \sqsubseteq f$. Let $(f_j)_{j=1}^m$ be the collection of all pairwise distinct elementary elements of F. Evidently, it is a disjoint sequence. Given any $f \in F$, we set $J = \{j \leq m : \ f_j \sqsubseteq f\}$. Then it is immediate that $f = \bigsqcup_{j\in J} f_j$. □

Lemma 7.7 *Let E be a Riesz space, $x \in E$, $D \subseteq \mathfrak{F}_x$ a lateral field in E, X a vector space and $T_0 : D \to X$ an orthogonally additive operator. Then there exists an orthogonally additive extension $T : \mathfrak{F}_x \to X$.*

Proof Denote by Z the linear span of D in E. At the first step, we define a linear operator $S_0 : Z \to X$ which extends T_0. Let $z = \sum\limits_{k=1}^{n} \alpha_k x_k$ be any element of $Z\setminus\{0\}$, where $n \in \mathbb{N}$, $\alpha_k \in \mathbb{R}$ and $x_k \in D$ for $k = 1, \dots, n$. Let F be the lateral field in E generated by the set $\{x_k : \ k = 1, \dots, n\}$. Since $x_k \in \mathfrak{F}_x$ for all k and $\mathfrak{F}_x$ is a Boolean algebra, F is finite. Using Lemma 7.6, one can easily show that z can be represented as $z = \sum\limits_{i=1}^{s} \beta_i y_i$ where $(y_i)_{i=1}^s$ is a disjoint sequence in D and $\beta_i \neq 0$ for all i. Then we define $S_0 : Z \to X$ by setting $S_0(0) = 0$ and

$$S_0(z) = \sum_{i=1}^{s} \beta_i T_0(y_i).$$

We show that $S_0(z)$ does not depend on the expansion of z. Indeed, let z have another expansion $z = \sum\limits_{j=1}^{t} \gamma_j z_j$, where $(z_j)_{j=1}^t$ is a disjoint sequence in D and $\gamma_j \neq 0$ for all j. Observe that if $y_i z_j \neq 0$ for given i and j then $\beta_i = \gamma_j$. Then

$$\begin{aligned}\sum_{i=1}^{s} \beta_i T_0(y_i) &= \sum_{i=1}^{s} \beta_i T_0\Big(\sum_{j=1}^{t} y_i z_j\Big) = \sum_{i=1}^{s}\sum_{j=1}^{t} \beta_i T_0(y_i z_j)\\ &= \sum_{i=1}^{s}\sum_{j=1}^{t} \gamma_j T_0(y_i z_j) = \sum_{j=1}^{t} \gamma_j T_0\Big(\sum_{i=1}^{s} y_i z_j\Big) = \sum_{j=1}^{t} \gamma_j T_0(z_j).\end{aligned}$$

Obviously, S_0 extends T_0 to Z. Using the well known technique of Hamel's bases, one can easily extend S_0 to a linear operator $S : E \to X$. Finally we set $T = S|_{\mathfrak{F}_x}$. □

Lemma 7.8 *Let D_0 be a lateral field in a Riesz space E with the intersection property, $x_0 \in E \setminus D_0$, D the lateral field in E generated by $D_0 \cup \{x_0\}$, X a vector space and $T_0 : D_0 \to X$ an orthogonally additive operator. Then there exists an orthogonally additive extension $T : D \to X$.*

Proof Let Y be the lateral field in E generated by $\mathfrak{F}_{x_0} \cap D$. We set $Y_0 = D_0 \cap Y$. Then $Y_0 \subseteq \mathfrak{F}_{x_0}$ and Y_0 is a lateral field in E. By Lemma 7.7, there exists an orthogonally additive extension $S : Y \to X$ of $S_0 = T_0|_{Y_0}$. Set $Z = \{x \setminus x_0 : x \in D_0\}$ and define an operator $R : Z \to X$ by setting

$$R(z) = T_0(x) - S(xx_0),$$

where $x \in D_0$ is such that $z = x \setminus x_0$. Show that R is well defined. Assume $z \in Z$ and $x_1, x_2 \in D_0$ satisfy $z = x_1 \setminus x_0 = x_2 \setminus x_0$. Denote $u_1 = x_1 x_0$, $u_2 = x_2 x_0$, $x = x_1 x_2$ and $u = u_1 u_2$. Observe that $x = z + u$. Taking into account that $u_i \setminus u = x_i \setminus u \in D_0$, we write

$$\begin{aligned} T_0(x_1) - S(u_1) &= T_0(x + u_1 \setminus u) - S(u_1) = T_0(x) + T_0(u_1 \setminus u) - S(u_1) \\ &= T_0(x) + S(u_1 \setminus u) - S(u_1) = T_0(x) - S(u). \end{aligned}$$

Analogously,

$$T_0(x_2) - S(u_2) = T_0(x) - S(u).$$

Now we are going to prove the following property of R.

Claim 1 For any finite disjoint sequence $(z_k)_{k=1}^n$ in Z with $z = z_1 + \cdots + z_n \in Z$ one has $R(z) = R(z_1) + \ldots + R(z_n)$.

Note that the claim for $n = 2$ logically does not imply the claim for an arbitrary n, because we do not assume here that $z_1 + z_2 \in Z$.

Let $x, x_1, \ldots, x_n \in D_0$ be such that $z = x - xx_0$ and $z_k = x_k - x_k x_0$ for all $k = 1, \ldots, n$. Setting $\tilde{x}_k = x_k x$ for each $k = 1, \ldots, n$, we obtain $z_k = \tilde{x}_k - (\tilde{x}_k x_0)$ for every $k = 1, \ldots, n$. The for $\tilde{x} = \tilde{x_1} + \cdots + \tilde{x}_{n-1}$ one has $z_1 + \cdots + z_{n-1} = \tilde{x} - (\tilde{x} x_0)$. Now the claim is reduced to the case where $n = 2$, $x_1 \sqsubseteq x$ and $x_2 = x - x_1$.

Set $u_1 = x_1 x_0$, $u_2 = x_2 x_0$ and $u = xx_0$. Then $u_1 + u_2 = u$ and

$$\begin{aligned} R(z_1) + R(z_2) &= T_0(x_1) - S(u_1) + T_0(x_2) - S(u_2) \\ &= T_0(x) - T_0(x_2) - S(u_1) + T_0(x_2) - S(u_2) = T_0(x) - S(u) = R(z), \end{aligned}$$

which completes the proof of Claim 1.

Now we need the next property of R.

Claim 2 Let $(z_k)_{k=1}^n$ and $(w_j)_{j=1}^m$ be disjoint sequences in Z with $z_1 + \cdots + z_n = w_1 + \ldots + w_m$. Then $R(z_1) + \ldots + R(z_n) = R(w_1) + \ldots + R(w_m)$.

Indeed, by item (vii) of Proposition 3.21, Z is closed under intersections. Then by Claim 1,

$$R(z_1) + \ldots + R(z_n) = \sum_{k=1}^{n} \sum_{i=1}^{m} R(z_k w_i) = \sum_{i=1}^{m} \sum_{k=1}^{n} R(z_k w_i) = R(w_1) + \ldots + R(w_m),$$

and Claim 2 is proved.

Claim 3 For every $d \in D$ there is a unique $y \in Y$ and (not necessarily unique) finite collection $z_1, \ldots, z_n \in Z$ such that $d = y \sqcup z_1 \sqcup \ldots \sqcup z_n$.

First we show that if an element $d \in E$ has a representation $d = y \sqcup z_1 \sqcup \ldots \sqcup z_n$, where $y \in Y$ and $z_i \in Z$, then $y = dx_0$. Indeed, by (v) of Proposition 3.21,

$$dx_0 = yx_0 \sqcup z_1x_0 \sqcup \ldots \sqcup z_nx_0 = yx_0, \tag{7.2}$$

because if $z_i = x_i \backslash x_0$ then by (ii) and (v) of Proposition 3.21,

$$x_0x_i = x_0\big(x_0x_i \sqcup z_i\big) = x_0x_i \sqcup x_0z_i,$$

which implies that $x_0z_i = 0$. On the other hand, $y \sqsubseteq x_0$ implies that $yx_0 = y$. Hence by (7.2), $y = dx_0$. In particular, the uniqueness of y is established.

To complete the proof of the lemma, it is enough to observe that D consists of all vectors of the form $y + z_1 + \cdots + z_n$, where $y \in Y$, $z_1, \ldots, z_n \in Z$ is a disjoint sequence, and set

$$T(y + z_1 + \cdots + z_n) = S(y) + R(z_1) + \cdots + R(z_n).$$

□

Proof of Theorem 7.5 The proof is standard using Zorn'z lemma and Lemma 7.8 which is considered as an extension by one step. □

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Affiliations

Volodymyr Mykhaylyuk[1,2] · Marat Pliev[3,6] · Mikhail Popov[4,5]

✉ Mikhail Popov
misham.popov@gmail.com

Volodymyr Mykhaylyuk
vmykhaylyuk@ukr.net

Marat Pliev
plimarat@yandex.ru

1 Jan Kochanowski University in Kielce, Kielce, Poland

2 Yurii Fedkovych Chernivtsi National University, Chernivtsi, Ukraine

3 Southern Mathematical Institute of the Russian Academy of Sciences, Str. Markusa 22, Vladikavkaz, Russia 362027

4 Institute of Exact and Technical Sciences, Pomeranian University in Słupsk, Słupsk, Poland

5 Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine

6 Department of Mathematics, North-Ossetian State University, Vladikavkaz, Russia 362025