Martingale inequalities for spline sequences

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Abstract

We show that D. Lépingle's $L_1(\ell_2)$ -inequality

$$\left\| \left(\sum_n \mathbb{E}[f_n | \mathcal{F}_{n-1}]^2 \right)^{1/2} \right\|_1 \leq 2 \cdot \left\| \left(\sum_n f_n^2 \right)^{1/2} \right\|_1, \quad f_n \in \mathcal{F}_n,$$

extends to the case where we substitute the conditional expectation operators with orthogonal projection operators onto spline spaces and where we can allow that f_n is contained in a suitable spline space $\mathcal{S}(\mathcal{F}_n)$. This is done provided the filtration $(\mathcal{F}_n)$ satisfies a certain regularity condition depending on the degree of smoothness of the functions contained in $\mathcal{S}(\mathcal{F}_n)$. As a by-product, we also obtain a spline version of H_1 -BMO duality under this assumption.

Keywords Martingale inequalities · Polynomial spline spaces · Orthogonal projection operators

Mathematics Subject Classification 65D07 · 60G42 · 42C10

1 Introduction

This article is part of a series of papers that extend martingale results to polynomial spline sequences of arbitrary order (see e.g. [11, 14, 16–19, 22]). In order to explain those martingale type results, we have to introduce a little bit of terminology: Let k be a positive integer, $(\mathcal{F}_n)$ an increasing sequence of σ -algebras of sets in $[0, 1]$ where each $\mathcal{F}_n$ is generated by a finite partition of $[0, 1]$ into intervals of positive length. Moreover, define the spline space

$$\mathcal{S}_k(\mathcal{F}_n) = \{f \in C^{k-2}[0, 1] : f \text{ is a polynomial of order } k \text{ on each atom of } \mathcal{F}_n\}$$

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and let $P_n^{(k)}$ be the orthogonal projection operator onto $\mathcal{S}_k(\mathcal{F}_n)$ with respect to the L_2 inner product on $[0, 1]$ with the Lebesgue measure $|\cdot|$. The space $\mathcal{S}_1(\mathcal{F}_n)$ consists of piecewise constant functions and $P_n^{(1)}$ is the conditional expectation operator with respect to the σ -algebra $\mathcal{F}_n$. Similarly to the definition of martingales, we introduce the following notion: let $(f_n)_{n \geq 0}$ be a sequence of integrable functions. We call this sequence a *k-martingale spline sequence* (adapted to $(\mathcal{F}_n)$) if, for all n ,

$$P_n^{(k)} f_{n+1} = f_n.$$

For basic facts about martingales and conditional expectations, we refer to [15].

Classical martingale theorems such as Doob's inequality or the martingale convergence theorem in fact carry over to *k*-martingale spline sequences corresponding to arbitrary filtrations $(\mathcal{F}_n)$ of the above type, just by replacing conditional expectation operators by the projection operators $P_n^{(k)}$. Indeed, we have

- (i) (Shadrin's theorem) there exists a constant C_k depending only on k such that

$$\sup_n \|P_n^{(k)} : L_1 \rightarrow L_1\| \leq C_k,$$

- (ii) (Doob's weak type inequality for splines)
there exists a constant C_k depending only on k such that for any *k*-martingale spline sequence (f_n) and any $\lambda > 0$,

$$|\{\sup_n |f_n| > \lambda\}| \leq C_k \frac{\sup_n \|f_n\|_1}{\lambda},$$

- (iii) (Doob's L_p inequality for splines)
for all $p \in (1, \infty]$ there exists a constant $C_{p,k}$ depending only on p and k such that for all *k*-martingale spline sequences (f_n) ,

$$\left\| \sup_n |f_n| \right\|_p \leq C_{p,k} \sup_n \|f_n\|_p,$$

- (iv) (Spline convergence theorem)
if (f_n) is an L_1 -bounded *k*-martingale spline sequence, then (f_n) converges almost surely to some L_1 -function,
(v) (Spline convergence theorem, L_p -version)
for $1 < p < \infty$, if (f_n) is an L_p -bounded *k*-martingale spline sequence, then (f_n) converges almost surely and in L_p .

Property (i) was proved by Shadrin in the groundbreaking paper [22]. We also refer to the paper [25] by von Golitschek, who gives a substantially shorter proof of (i). Properties (ii) and (iii) are proved in [19] and properties (iv) and (v) in [14], but see also [18], where it is shown that, in analogy to the martingale case, the validity of (iv) and (v) for all *k*-martingale spline sequences with values in a Banach space X characterize the Radon–Nikodým property of X (for background information on that material, we refer to the monographs [6,20]).

Here, we continue this line of transferring martingale results to k -martingale spline sequences and extend Lépingle's $L_1(\ell_2)$ -inequality [12], which reads

$$\left\| \left(\sum_n \mathbb{E}[f_n | \mathcal{F}_{n-1}]^2 \right)^{1/2} \right\|_1 \leq 2 \cdot \left\| \left(\sum_n f_n^2 \right)^{1/2} \right\|_1, \quad (1.1)$$

provided the sequence of (real-valued) random variables f_n is adapted to the filtration $(\mathcal{F}_n)$, i.e. each f_n is $\mathcal{F}_n$ -measurable. Different proofs of (1.1) were given by Bourgain [3, Proposition 5], Delbaen and Schachermayer [4, Lemma 1] and Müller [13, Proposition 4.1]. The spline version of inequality (1.1) is contained in Theorem 4.1.

This inequality is an L_1 extension of the following result for $1 < p < \infty$, proved by Stein [24], that holds for *arbitrary* integrable functions f_n :

$$\left\| \left(\sum_n \mathbb{E}[f_n | \mathcal{F}_{n-1}]^2 \right)^{1/2} \right\|_p \leq a_p \left\| \left(\sum_n f_n^2 \right)^{1/2} \right\|_p, \quad (1.2)$$

for some constant a_p depending only on p . This can be seen as a dual version of Doob's inequality $\|\sup_\ell |\mathbb{E}[f | \mathcal{F}_\ell]|\|_p \leq c_p \|f\|_p$ for $p > 1$, see [1]. Once we know Doob's inequality for spline projections, which is point (iii) above, the same proof as in [1] works for spline projections if we use suitable positive operators T_n instead of $P_n^{(k)}$ that also satisfy Doob's inequality and dominate the operators $P_n^{(k)}$ pointwise (cf. Sects. 3.1, 3.2).

The usage of those operators T_n is also necessary in the extension of inequality (1.1) to splines. Lépingle's proof of (1.1) rests on an idea by Herz [10] of splitting $\mathbb{E}[f_n \cdot h_n]$ (for f_n being $\mathcal{F}_n$ -measurable) by Cauchy–Schwarz after introducing the square function $S_n^2 = \sum_{\ell \leq n} f_\ell^2$:

$$(\mathbb{E}[f_n \cdot h_n])^2 \leq \mathbb{E}[f_n^2 / S_n] \cdot \mathbb{E}[S_n h_n^2] \quad (1.3)$$

and estimating both factors on the right hand side separately. A key point in estimating the second factor is that S_n is $\mathcal{F}_n$ -measurable, and therefore, $\mathbb{E}[S_n | \mathcal{F}_n] = S_n$. If we want to allow $f_n \in \mathcal{S}_k(\mathcal{F}_n)$, S_n will not be contained in $\mathcal{S}_k(\mathcal{F}_n)$ in general. Under certain conditions on the filtration $(\mathcal{F}_n)$, we will show in this article how to substitute S_n in estimate (1.3) by a function $g_n \in \mathcal{S}_k(\mathcal{F}_n)$ that enjoys similar properties to S_n and allows us to proceed (cf. Sect. 3.4, in particular Proposition 3.4 and Theorem 3.6). As a by-product, we obtain a spline version (Theorem 4.2) of C. Fefferman's theorem [7] on H^1 -BMO duality. For its martingale version, we refer to A. M. Garsia's book [8] on Martingale Inequalities.

2 Preliminaries

In this section, we collect all tools that are needed subsequently.

2.1 Properties of polynomials

We will need Remez' inequality for polynomials:

Theorem 2.1 *Let $V \subset \mathbb{R}$ be a compact interval in $\mathbb{R}$ and $E \subset V$ a measurable subset. Then, for all polynomials p of order k (i.e. degree $k - 1$) on V ,*

$$\|p\|_{L_\infty(V)} \leq \left(4 \frac{|V|}{|E|}\right)^{k-1} \|p\|_{L_\infty(E)}.$$

Applying this theorem with the set $E = \{x \in V : |p(x)| \leq 8^{-k+1} \|p\|_{L_\infty(V)}\}$ immediately yields the following corollary:

Corollary 2.2 *Let p be a polynomial of order k on a compact interval $V \subset \mathbb{R}$. Then*

$$|\{x \in V : |p(x)| \geq 8^{-k+1} \|p\|_{L_\infty(V)}\}| \geq |V|/2.$$

2.2 Properties of spline functions

For an interval σ -algebra $\mathcal{F}$ (i.e. $\mathcal{F}$ is generated by a finite collection of intervals having positive length), the space $\mathcal{S}_k(\mathcal{F})$ is spanned by a very special local basis (N_i) , the so called B-spline basis. It has the properties that each N_i is non-negative and each support of N_i consists of at most k neighboring atoms of $\mathcal{F}$. Moreover, (N_i) is a partition of unity, i.e. for all $x \in [0, 1]$, there exist at most k functions N_i so that $N_i(x) \neq 0$ and $\sum_i N_i(x) = 1$. In the following, we denote by E_i the support of the B-spline function N_i . The usual ordering of the B-splines (N_i) —which we also employ here—is such that for all i , $\inf E_i \leq \inf E_{i+1}$ and $\sup E_i \leq \sup E_{i+1}$.

We write $A(t) \lesssim B(t)$ to denote the existence of a constant C such that for all t , $A(t) \leq CB(t)$, where t denote all implicit and explicit dependencies the expression A and B might have. If the constant C additionally depends on some parameter, we will indicate this in the text. Similarly, the symbols $\gtrsim$ and $\simeq$ are used.

Another important property of B-splines is the following relation between B-spline coefficients and the L_p -norm of the corresponding B-spline expansions.

Theorem 2.3 (B-spline stability, local and global) *Let $1 \leq p \leq \infty$ and $g = \sum_j a_j N_j$. Then, for all j ,*

$$|a_j| \lesssim |J_j|^{-1/p} \|g\|_{L_p(J_j)}, \quad (2.1)$$

where J_j is an atom of $\mathcal{F}$ contained in E_j having maximal length. Additionally,

$$\|g\|_p \simeq \|(a_j |E_j|^{1/p})\|_{\ell_p}, \quad (2.2)$$

where in both (2.1) and (2.2), the implied constants depend only on the spline order k .

Observe that (2.1) implies for $g \in \mathcal{S}_k(\mathcal{F})$ and any measurable set $A \subset [0, 1]$

$$\|g\|_{L_\infty(A)} \lesssim \max_{j: |E_j \cap A| > 0} \|g\|_{L_\infty(J_j)}. \quad (2.3)$$

We will also need the following relation between the B-spline expansion of a function and its expansion using B-splines of a finer grid.

Theorem 2.4 *Let $\mathcal{G} \subset \mathcal{F}$ be two interval σ -algebras and denote by $(N_{\mathcal{G},i})_i$ the B-spline basis of the coarser space $\mathcal{S}_k(\mathcal{G})$ and by $(N_{\mathcal{F},i})_i$ the B-spline basis of the finer space $\mathcal{S}_k(\mathcal{F})$. Then, given $f = \sum_j a_j N_{\mathcal{G},j}$, we can expand f in the basis $(N_{\mathcal{F},i})_i$*

$$\sum_j a_j N_{\mathcal{G},j} = \sum_i b_i N_{\mathcal{F},i},$$

where for each i , b_i is a convex combination of the coefficients a_j with $\text{supp } N_{\mathcal{G},j} \supseteq \text{supp } N_{\mathcal{F},i}$.

For those results and more information on spline functions, in particular B-splines, we refer to [21] or [5].

2.3 Spline orthoprojectors

We now use the B-spline basis of $\mathcal{S}_k(\mathcal{F})$ and expand the orthogonal projection operator P onto $\mathcal{S}_k(\mathcal{F})$ in the form

$$Pf = \sum_{i,j} a_{ij} \left(\int_0^1 f(x) N_i(x) dx \right) \cdot N_j \quad (2.4)$$

for some coefficients (a_{ij}) . Denoting by E_{ij} the smallest interval containing both supports E_i and E_j of the B-spline functions N_i and N_j respectively, we have the following estimate for a_{ij} [19]: there exist constants C and $0 < q < 1$ depending only on k so that for each interval σ -algebra $\mathcal{F}$ and each i, j ,

$$|a_{ij}| \leq C \frac{q^{|i-j|}}{|E_{ij}|}. \quad (2.5)$$

2.4 Spline square functions

Let $(\mathcal{F}_n)$ be a sequence of increasing interval σ -algebras in $[0, 1]$ and we assume that each $\mathcal{F}_{n+1}$ is generated from $\mathcal{F}_n$ by the subdivision of exactly one atom of $\mathcal{F}_n$ into two atoms of $\mathcal{F}_{n+1}$. Let P_n be the orthogonal projection operator onto $\mathcal{S}_k(\mathcal{F}_n)$. We denote $\Delta_n f = P_n f - P_{n-1} f$ and define the spline square function

$$Sf = \left(\sum_n |\Delta_n f|^2 \right)^{1/2}.$$

We have Burkholder's inequality for the spline square function, i.e. for all $1 < p < \infty$ [16], the L_p -norm of the square function Sf is comparable to the L_p -norm of f :

$$\|Sf\|_p \simeq \|f\|_p, \quad f \in L_p \quad (2.6)$$

with constants depending only on p and k . Moreover, for $p = 1$, it is shown in [9] that

$$\|Sf\|_1 \simeq \sup_{\varepsilon \in \{-1,1\}^{\mathbb{Z}}} \left\| \sum_n \varepsilon_n \Delta_n f \right\|_1, \quad Sf \in L_1, \quad (2.7)$$

with constants depending only on k and where the proof of the $\lesssim$ -part only uses Khintchine's inequality whereas the proof of the $\gtrsim$ -part uses fine properties of the functions $\Delta_n f$.

2.5 $L_p(\ell_q)$ -spaces

For $1 \leq p, q \leq \infty$, we denote by $L_p(\ell_q)$ the space of sequences of measurable functions (f_n) on $[0, 1]$ so that the norm

$$\|(f_n)\|_{L_p(\ell_q)} = \left(\int_0^1 \left(\sum_n |f_n(t)|^q \right)^{p/q} dt \right)^{1/p}$$

is finite (with the obvious modifications if $p = \infty$ or $q = \infty$). For $1 \leq p, q < \infty$, the dual space (see [2]) of $L_p(\ell_q)$ is $L_{p'}(\ell_{q'})$ with $p' = p/(p-1)$, $q' = q/(q-1)$ and the duality pairing

$$\langle (f_n), (g_n) \rangle = \int_0^1 \sum_n f_n(t) g_n(t) dt.$$

Hölder's inequality takes the form $|\langle (f_n), (g_n) \rangle| \leq \|(f_n)\|_{L_p(\ell_q)} \|(g_n)\|_{L_{p'}(\ell_{q'})}$.

3 Main results

In this section, we prove our main results. Section 3.1 defines and gives properties of suitable positive operators that dominate our (non-positive) operators $P_n = P_n^{(k)}$ pointwise. In Sect. 3.2, we use those operators to give a spline version of Stein's inequality (1.2). A useful property of conditional expectations is the tower property $\mathbb{E}_{\mathcal{G}} \mathbb{E}_{\mathcal{F}} f = \mathbb{E}_{\mathcal{G}} f$ for $\mathcal{G} \subset \mathcal{F}$. In this form, it extends to the operators (P_n) , but not to the operators T from Sect. 3.1. In Sect. 3.3 we prove a version of the tower property for those operators. Section 3.4 is devoted to establishing a duality estimate using a spline square function, which is the crucial ingredient in the proofs of the spline versions of both Lépingle's inequality (1.1) and H_1 -BMO duality in Sect. 4.

3.1 The positive operators T

As above, let $\mathcal{F}$ be an interval σ -algebra on $[0, 1]$, (N_i) the B-spline basis of $\mathcal{S}_k(\mathcal{F})$, E_i the support of N_i and E_{ij} the smallest interval containing both E_i and E_j . Moreover, let q be a positive number smaller than 1. Then, we define the linear operator $T = T_{\mathcal{F}, q, k}$ by

$$Tf(x) := \sum_{i,j} \frac{q^{|i-j|}}{|E_{ij}|} \langle f, \mathbb{1}_{E_i} \rangle \mathbb{1}_{E_j}(x) = \int_0^1 K(x, t) f(t) dt,$$

where the kernel $K = K_T$ is given by

$$K(x, t) = \sum_{i,j} \frac{q^{|i-j|}}{|E_{ij}|} \mathbb{1}_{E_i}(t) \cdot \mathbb{1}_{E_j}(x).$$

We observe that the operator T is selfadjoint (w.r.t the standard inner product on L_2) and

$$k \leq K_x := \int_0^1 K(x, t) dt \leq \frac{2(k+1)}{1-q}, \quad x \in [0, 1], \quad (3.1)$$

which, in particular, implies the boundedness of the operator T on L_1 and L_∞ :

$$\|Tf\|_1 \leq \frac{2(k+1)}{1-q} \|f\|_1, \quad \|Tf\|_\infty \leq \frac{2(k+1)}{1-q} \|f\|_\infty.$$

Another very important property of T is that it is a positive operator, i.e. it maps non-negative functions to non-negative functions and that T satisfies Jensen's inequality in the form

$$\varphi(Tf(x)) \leq K_x^{-1} T(\varphi(K_x \cdot f))(x), \quad f \in L_1, x \in [0, 1], \quad (3.2)$$

for convex functions φ . This is seen by applying the classical Jensen inequality to the probability measure $K(t, x) dt / K_x$.

Let $\mathcal{M}f$ denote the Hardy–Littlewood maximal function of $f \in L_1$, i.e.

$$\mathcal{M}f(x) = \sup_{I \ni x} \frac{1}{|I|} \int_I |f(y)| dy,$$

where the supremum is taken over all subintervals of $[0, 1]$ that contain the point x . This operator is of weak type $(1, 1)$, i.e.

$$|\{\mathcal{M}f > \lambda\}| \leq C\lambda^{-1} \|f\|_1, \quad f \in L_1, \lambda > 0$$

for some constant C . Since trivially we have the estimate $\|\mathcal{M}f\|_\infty \leq \|f\|_\infty$, by Marcinkiewicz interpolation, for any $p > 1$, there exists a constant C_p depending only on p so that

$$\|\mathcal{M}f\|_p \leq C_p \|f\|_p.$$

For those assertions about $\mathcal{M}$, we refer to (for instance) [23].

The significance of T and $\mathcal{M}$ at this point is that we can use formula (2.4) and estimate (2.5) to obtain the pointwise bound

$$|Pf(x)| \leq C_1(T|f|)(x) \leq C_2\mathcal{M}(x), \quad f \in L_1, x \in [0, 1], \quad (3.3)$$

where $T = T_{\mathcal{F},q,k}$ with q given by (2.5), C_1 is a constant that depends only on k and C_2 is a constant that depends only on k and the geometric progression q . But as the parameter $q < 1$ in (2.5) depends only on k , the constant C_2 will also only depend on k .

In other words, (3.3) tells us that the positive operator T dominates the non-positive operator P pointwise, but at the same time, T is dominated by the Hardy–Littlewood maximal function $\mathcal{M}$ pointwise and independently of $\mathcal{F}$.

3.2 Stein's inequality for splines

We now use this pointwise dominating, positive operator T to prove Stein's inequality for spline projections. For this, let $(\mathcal{F}_n)$ be an interval filtration on $[0, 1]$ and P_n be the orthogonal projection operator onto the space $\mathcal{S}_k(\mathcal{F}_n)$ of splines of order k corresponding to $\mathcal{F}_n$. Working with the positive operators $T_{\mathcal{F}_n,q,k}$ instead of the non-positive operators P_n , the proof of Stein's inequality (1.2) for spline projections can be carried over from the martingale case (cf. [1,24]). For completeness, we include it here.

Theorem 3.1 *Suppose that (f_n) is a sequence of arbitrary integrable functions on $[0, 1]$. Then, for $1 \leq r \leq p < \infty$ or $1 < p \leq r \leq \infty$,*

$$\|(P_n f_n)\|_{L_p(\ell_r)} \lesssim \|(f_n)\|_{L_p(\ell_r)} \quad (3.4)$$

where the implied constant depends only on p, r and k .

Proof By (3.3), it suffices to prove this inequality for the operators $T_n = T_{\mathcal{F}_n,q,k}$ with q given by (2.5) instead of the operators P_n . First observe that for $r = p = 1$, the assertion follows from Shadrin's theorem ((i) on page 1). Inequality (3.3) and the $L_{p'}$ -boundedness of $\mathcal{M}$ for $1 < p' \leq \infty$ imply that

$$\left\| \sup_{1 \leq n \leq N} |T_n f| \right\|_{p'} \leq C_{p',k} \|f\|_{p'}, \quad f \in L_{p'} \quad (3.5)$$

with a constant $C_{p',k}$ depending on p' and k . Let $1 \leq p < \infty$ and $U_N : L_p(\ell_1^N) \rightarrow L_p$ be given by $(g_1, \dots, g_N) \mapsto \sum_{j=1}^N T_j g_j$. Inequality (3.5) implies the boundedness of

the adjoint $U_N^* : L_{p'} \rightarrow L_{p'}(\ell_\infty^N)$, $f \mapsto (T_j f)_{j=1}^N$ for $p' = p/(p-1)$ by a constant independent of N and therefore also the boundedness of U_N . Since $|T_j f| \leq T_j |f|$ by the positivity of T_j , letting $N \rightarrow \infty$ implies (3.4) for T_n instead of P_n in the case $r = 1$ and outer parameter $1 \leq p < \infty$.

If $1 < r \leq p$, we use Jensen's inequality (3.2) and estimate (3.1) to obtain

$$\sum_{j=1}^N |T_j g_j|^r \lesssim \sum_{j=1}^N T_j (|g_j|^r)$$

and apply the result for $r = 1$ and the outer parameter p/r to get the result for $1 \leq r \leq p < \infty$. The cases $1 < p \leq r \leq \infty$ now just follow from this result using duality and the self-adjointness of T_j . $\square$

3.3 Tower property of T

Next, we will prove a substitute of the tower property $\mathbb{E}_{\mathcal{G}} \mathbb{E}_{\mathcal{F}} f = \mathbb{E}_{\mathcal{G}} f$ ($\mathcal{G} \subset \mathcal{F}$) for conditional expectations that applies to the operators T .

To formulate this result, we need a suitable notion of regularity for σ -algebras which we now describe. Let $\mathcal{F}$ be an interval σ -algebra, let (N_j) be the B-spline basis of $\mathcal{S}_k(\mathcal{F})$ and denote by E_j the support of the function N_j . The k -regularity parameter $\gamma_k(\mathcal{F})$ is defined as

$$\gamma_k(\mathcal{F}) := \max_i \max(|E_i|/|E_{i+1}|, |E_{i+1}|/|E_i|),$$

where the first maximum is taken over all i so that E_i and E_{i+1} are defined. The name k -regularity is motivated by the fact that each B-spline support E_i of order k consists of at most k (neighboring) atoms of the σ -algebra $\mathcal{F}$.

Proposition 3.2 (Tower property of T) *Let $\mathcal{G} \subset \mathcal{F}$ be two interval σ -algebras on $[0, 1]$. Let $S = T_{\mathcal{G}, \sigma, k}$ and $T = T_{\mathcal{F}, \tau, k'}$ for some $\sigma, \tau \in (0, 1)$ and some positive integers k, k' . Then, for all $q > \max(\tau, \sigma)$, there exists a constant C depending on q, k, k' so that*

$$|STf(x)| \leq C \cdot \gamma^k \cdot (T_{\mathcal{G}, q, k} |f|)(x), \quad f \in L_1, x \in [0, 1], \quad (3.6)$$

where $\gamma = \gamma_k(\mathcal{G})$ denotes the k -regularity parameter of $\mathcal{G}$.

Proof Let (F_i) be the collection of B-spline supports in $\mathcal{S}_{k'}(\mathcal{F})$ and (G_i) the collection of B-spline supports in $\mathcal{S}_k(\mathcal{G})$. Moreover, we denote by F_{ij} the smallest interval containing F_i and F_j and by G_{ij} the smallest interval containing G_i and G_j .

We show (3.6) by showing the following inequality for the kernels K_S of S and K_T of T (cf. 3.1)

$$\int_0^1 K_S(x, t) K_T(t, s) dt \leq C \gamma^k \sum_{i,j} \frac{q^{|i-j|}}{|G_{ij}|} \mathbb{1}_{G_i}(x) \mathbb{1}_{G_j}(s), \quad x, s \in [0, 1] \quad (3.7)$$

for all $q > \max(\tau, \sigma)$ and some constant C depending on q, k, k' . In order to prove this inequality, we first fix $x, s \in [0, 1]$ and choose i such that $x \in G_i$ and ℓ such that $s \in F_\ell$. Moreover, based on ℓ , we choose j so that $s \in G_j$ and $G_j \supset F_\ell$. There are at most $\max(k, k')$ choices for each of the indices i, ℓ, j and without restriction, we treat those choices separately, i.e. we only have to estimate the expression

$$\sum_{m,r} \frac{\sigma^{|m-i|} \tau^{|r-\ell|} |G_m \cap F_r|}{|G_{im}| |F_{\ell r}|}.$$

Since, for each r , there are also at most $k + k' - 1$ indices m so that $|G_m \cap F_r| > 0$ (recall that $\mathcal{G} \subset \mathcal{F}$), we choose one such index $m = m(r)$ and estimate

$$\Sigma = \sum_r \frac{\sigma^{|m(r)-i|} \tau^{|r-\ell|} |G_{m(r)} \cap F_r|}{|G_{i,m(r)}| |F_{\ell r}|}.$$

Now, observe that for any parameter choice of r in the above sum,

$$G_{i,m(r)} \cup F_{\ell r} \supseteq (G_{ij} \setminus G_j) \cup G_i$$

and therefore, since also $G_{m(r)} \cap F_r \subset G_{i,m(r)} \cap F_{\ell r}$,

$$\Sigma \leq \frac{2}{|(G_{ij} \setminus G_j) \cup G_i|} \sum_r \sigma^{|m(r)-i|} \tau^{|r-\ell|},$$

which, using the k -regularity parameter $\gamma = \gamma_k(\mathcal{G})$ of the σ -algebra $\mathcal{G}$ and denoting $\lambda = \max(\tau, \sigma)$, we estimate by

$$\begin{aligned} \Sigma &\leq \frac{2\gamma^k}{|G_{ij}|} \sum_m \lambda^{|m-i|} \sum_{r:m(r)=m} \lambda^{|r-\ell|} \lesssim \frac{\gamma^k}{|G_{ij}|} \sum_m \lambda^{|i-m|+|m-j|} \\ &\lesssim \frac{\gamma^k}{|G_{ij}|} (|i-j|+1) \lambda^{|i-j|}, \end{aligned}$$

where the implied constants depend on λ, k, k' and the estimate $\sum_{r:m(r)=m} \lambda^{|r-\ell|} \lesssim \lambda^{|m-j|}$ used the fact that, essentially, there are more atoms of $\mathcal{F}$ between F_r and F_ℓ (for r as in the sum) than atoms of $\mathcal{G}$ between G_m and G_j . Finally, we see that for any $q > \lambda$,

$$\Sigma \lesssim C \gamma^k \frac{q^{|i-j|}}{|G_{ij}|}$$

for some constant C depending on q, k, k' , and, as $x \in G_i$ and $s \in G_j$, this shows inequality (3.7). $\square$

As a corollary of Proposition 3.2, we have

Corollary 3.3 *Let (f_n) be functions in L_1 . We denote by P_n the orthogonal projection onto $\mathcal{S}_k(\mathcal{F}_n)$ and by P'_n the orthogonal projection onto $\mathcal{S}_{k'}(\mathcal{F}_n)$ for some positive integers k, k' . Moreover, let T_n be the operator $T_{\mathcal{F}_n, q, k}$ from (3.3) dominating P_n pointwise.*

Then, for any integer n and for any $1 \leq p \leq \infty$,

$$\left\| \sum_{\ell \geq n} P_n((P'_{\ell-1} f_\ell)^2) \right\|_p \lesssim \left\| \sum_{\ell \geq n} T_n((P'_{\ell-1} f_\ell)^2) \right\|_p \lesssim \gamma_k(\mathcal{F}_n)^k \cdot \left\| \sum_{\ell \geq n} f_\ell^2 \right\|_p,$$

where the implied constants only depend on k and k' .

We remark that by Jensen's inequality and the tower property, this is trivial for conditional expectations $\mathbb{E}(\cdot | \mathcal{F}_n)$ instead of the operators $P_n, T_n, P'_{\ell-1}$ even with an absolute constant on the right hand side.

Proof We denote by T_n the operator $T_{\mathcal{F}_n, q, k}$ and by T'_n the operator $T_{\mathcal{F}_n, q', k'}$, where the parameters $q, q' < 1$ are given by inequality (3.3) depending on k and k' respectively. Setting $U_n := T_{\mathcal{F}_n, \max(q, q')^{1/2}, k}$, we perform the following chain of inequalities, where we use the positivity of T_n and (3.3), Jensen's inequality for $T'_{\ell-1}$, the tower property for $T_n T'_{\ell-1}$ and the L_p -boundedness of U_n , respectively:

$$\begin{aligned} \left\| \sum_{\ell \geq n} T_n((P'_{\ell-1} f_\ell)^2) \right\|_p &\lesssim \left\| \sum_{\ell \geq n} T_n((T'_{\ell-1} f_\ell)^2) \right\|_p \\ &\lesssim \left\| \sum_{\ell \geq n} T_n(T'_{\ell-1} f_\ell^2) \right\|_p \\ &\leq \|T_n(T'_{n-1} f_n^2)\|_p + \left\| \sum_{\ell > n} T_n(T'_{\ell-1} f_\ell^2) \right\|_p \\ &\lesssim \|f_n^2\|_p + \gamma_k(\mathcal{F}_n)^k \cdot \left\| \sum_{\ell > n} U_n(f_\ell^2) \right\|_p \\ &\lesssim \gamma_k(\mathcal{F}_n)^k \cdot \left\| \sum_{\ell \geq n} f_\ell^2 \right\|_p, \end{aligned}$$

where the implied constants only depend on k and k' . □

3.4 A duality estimate using a spline square function

In order to give the desired duality estimate contained in Theorem 3.6, we need the following construction of a function $g_n \in \mathcal{S}_k(\mathcal{F}_n)$ based on a spline square function.

Proposition 3.4 *Let (f_n) be a sequence of functions with $f_n \in \mathcal{S}_k(\mathcal{F}_n)$ for all n and set*

$$X_n := \sum_{\ell \leq n} f_\ell^2.$$

Then, there exists a sequence of non-negative functions $g_n \in \mathcal{S}_k(\mathcal{F}_n)$ so that for each n ,

- (1) $g_n \leq g_{n+1}$,
- (2) $X_n^{1/2} \leq g_n$
- (3) $\mathbb{E}g_n \lesssim \mathbb{E}X_n^{1/2}$, where the implied constant depends on k and on $\sup_{m \leq n} \gamma_k(\mathcal{F}_m)$.

For the proof of this result, we need the following simple lemma.

Lemma 3.5 *Let c_1 be a positive constant and let $(A_j)_{j=1}^N$ be a sequence of atoms in $\mathcal{F}_n$. Moreover, let $\ell : \{1, \dots, N\} \rightarrow \{1, \dots, n\}$ and, for each $j \in \{1, \dots, N\}$, let B_j be a subset of an atom D_j of $\mathcal{F}_{\ell(j)}$ with*

$$|B_j| \geq c_1 \sum_{\substack{i: \ell(i) \geq \ell(j), \\ D_i \subseteq D_j}} |A_i|. \quad (3.8)$$

Then, there exists a map φ on $\{1, \dots, N\}$ so that

- (1) $|\varphi(j)| = c_1|A_j|$ for all j ,
- (2) $\varphi(j) \subseteq B_j$ for all j ,
- (3) $\varphi(i) \cap \varphi(j) = \emptyset$ for all $i \neq j$.

Proof Without restriction, we assume that the sequence (A_j) is enumerated such that $\ell(j+1) \leq \ell(j)$ for all $1 \leq j \leq N-1$. We first choose $\varphi(1)$ as an arbitrary (measurable) subset of B_1 with measure $c_1|A_1|$, which is possible by assumption (3.8). Next, we assume that for $1 \leq j \leq j_0$, we have constructed $\varphi(j)$ with the properties

- (1) $|\varphi(j)| = c_1|A_j|$,
- (2) $\varphi(j) \subseteq B_j$,
- (3) $\varphi(j) \cap \bigcup_{i < j} \varphi(i) = \emptyset$.

Based on that, we now construct $\varphi(j_0 + 1)$. Define the index sets $\Gamma = \{i : \ell(i) \geq \ell(j_0 + 1), D_i \subseteq D_{j_0+1}\}$ and $\Lambda = \{i : i \leq j_0 + 1, D_i \subseteq D_{j_0+1}\}$. Since we assumed that ℓ is decreasing, we have $\Lambda \subseteq \Gamma$ and by the nestedness of the σ -algebras $\mathcal{F}_n$, we have for $i \leq j_0 + 1$ that either $D_i \subset D_{j_0+1}$ or $|D_i \cap D_{j_0+1}| = 0$. This implies

$$\begin{aligned} \left| B_{j_0+1} \setminus \bigcup_{i \leq j_0} \varphi(i) \right| &= |B_{j_0+1}| - \left| B_{j_0+1} \cap \bigcup_{i \leq j_0} \varphi(i) \right| \\ &\geq c_1 \sum_{i \in \Gamma} |A_i| - \left| D_{j_0+1} \cap \bigcup_{i \leq j_0} \varphi(i) \right| \\ &\geq c_1 \sum_{i \in \Lambda} |A_i| - \left| \bigcup_{i \in \Lambda \setminus \{j_0+1\}} \varphi(i) \right| \\ &\geq c_1 \sum_{i \in \Lambda} |A_i| - \sum_{i \in \Lambda \setminus \{j_0+1\}} c_1|A_i| = c_1|A_{j_0+1}|. \end{aligned}$$

Therefore, we can choose $\varphi(j_0 + 1) \subseteq B_{j_0+1}$ that is disjoint to $\varphi(i)$ for any $i \leq j_0$ and $|\varphi(j_0 + 1)| = c_1|A_{j_0+1}|$ which completes the proof. $\square$

Proof of Proposition 3.4 Fix n and let $(N_{n,j})$ be the B-spline basis of $\mathcal{S}_k(\mathcal{F}_n)$. Moreover, for any j , set $E_{n,j} = \text{supp } N_{n,j}$ and $a_{n,j} := \max_{\ell \leq n} \max_{r: E_{\ell,r} \supset E_{n,j}} \|X_\ell\|_{L_\infty(E_{\ell,r})}^{1/2}$ and we define $\ell(j) \leq n$ and $r(j)$ so that $E_{\ell(j),r(j)} \supseteq E_{n,j}$ and $a_{n,j} = \|X_{\ell(j)}\|_{L_\infty(E_{\ell(j),r(j)})}^{1/2}$. Set

$$g_n := \sum_j a_{n,j} N_{n,j} \in \mathcal{S}_k(\mathcal{F}_n)$$

and it will be proved subsequently that this g_n has the desired properties.

PROPERTY (1): In order to show $g_n \leq g_{n+1}$, we use Theorem 2.4 to write

$$g_n = \sum_j a_{n,j} N_{n,j} = \sum_j \beta_{n,j} N_{n+1,j},$$

where $\beta_{n,j}$ is a convex combination of those $a_{n,r}$ with $E_{n+1,j} \subseteq E_{n,r}$, and thus

$$g_n \leq \sum_j \left(\max_{r: E_{n+1,j} \subseteq E_{n,r}} a_{n,r} \right) N_{n+1,j}.$$

By the very definition of $a_{n+1,j}$, we have

$$\max_{r: E_{n+1,j} \subseteq E_{n,r}} a_{n,r} \leq a_{n+1,j},$$

and therefore, $g_n \leq g_{n+1}$ pointwise, since the B-splines $(N_{n+1,j})_j$ are nonnegative functions.

PROPERTY (2): Now we show that $X_n^{1/2} \leq g_n$. Indeed, for any $x \in [0, 1]$,

$$g_n(x) = \sum_j a_{n,j} N_{n,j}(x) \geq \min_{j: E_{n,j} \ni x} a_{n,j} \geq \min_{j: E_{n,j} \ni x} \|X_n\|_{L_\infty(E_{n,j})}^{1/2} \geq X_n(x)^{1/2},$$

since the collection of B-splines $(N_{n,j})_j$ forms a partition of unity.

PROPERTY (3): Finally, we show $\mathbb{E} g_n \lesssim \mathbb{E} X_n^{1/2}$, where the implied constant depends only on k and on $\sup_{m \leq n} \gamma_k(\mathcal{F}_m)$. By B-spline stability (Theorem 2.3), we estimate the integral of g_n by

$$\mathbb{E} g_n \lesssim \sum_j |E_{n,j}| \cdot \|X_{\ell(j)}\|_{L_\infty(E_{\ell(j),r(j)})}^{1/2}, \quad (3.9)$$

where the implied constant only depends on k . In order to continue the estimate, we next show the inequality

$$\|X_\ell\|_{L_\infty(E_{\ell,r})} \lesssim \max_{s: |E_{\ell,r} \cap E_{\ell,s}| > 0} \|X_\ell\|_{L_\infty(J_{\ell,s})}, \quad (3.10)$$

where by $J_{\ell,s}$ we denote an atom of $\mathcal{F}_\ell$ with $J_{\ell,s} \subset E_{\ell,s}$ of maximal length and the implied constant depends only on k . Indeed, we use Theorem 2.3 in the form of (2.3) to get $(f_m \in \mathcal{S}_k(\mathcal{F}_\ell))$ for $m \leq \ell$

$$\begin{aligned} \|X_\ell\|_{L_\infty(E_{\ell,r})} &\leq \sum_{m \leq \ell} \|f_m\|_{L_\infty(E_{\ell,r})}^2 \\ &\lesssim \sum_{m \leq \ell} \sum_{s: |E_{\ell,s} \cap E_{\ell,r}| > 0} \|f_m\|_{L_\infty(J_{\ell,s})}^2 = \sum_{s: |E_{\ell,s} \cap E_{\ell,r}| > 0} \sum_{m \leq \ell} \|f_m\|_{L_\infty(J_{\ell,s})}^2. \end{aligned} \quad (3.11)$$

Now observe that for atoms I of $\mathcal{F}_\ell$, due to the equivalence of p -norms of polynomials (cf. Corollary 2.2),

$$\sum_{m \leq \ell} \|f_m\|_{L_\infty(I)}^2 \lesssim \sum_{m \leq \ell} \frac{1}{|I|} \int_I f_m^2 = \frac{1}{|I|} \int_I X_\ell \leq \|X_\ell\|_{L_\infty(I)},$$

which means that, inserting this in estimate (3.11),

$$\|X_\ell\|_{L_\infty(E_{\ell,r})} \lesssim \sum_{s: |E_{\ell,s} \cap E_{\ell,r}| > 0} \|X_\ell\|_{L_\infty(J_{\ell,s})},$$

and, since there are at most k indices s so that $|E_{\ell,s} \cap E_{\ell,r}| > 0$, we have established (3.10).

We define $K_{\ell,r}$ to be an interval $J_{\ell,s}$ with $|E_{\ell,r} \cap E_{\ell,s}| > 0$ so that

$$\max_{s: |E_{\ell,r} \cap E_{\ell,s}| > 0} \|X_\ell\|_{L_\infty(J_{\ell,s})} = \|X_\ell\|_{L_\infty(K_{\ell,r})}.$$

This means, after combining (3.9) with (3.10), we have

$$\mathbb{E} g_n \lesssim \sum_j |J_{n,j}| \cdot \|X_{\ell(j)}\|_{L_\infty(K_{\ell(j),r(j)})}^{1/2}. \quad (3.12)$$

We now apply Lemma 3.5 with the function ℓ and the choices

$$\begin{aligned} A_j &= J_{n,j}, \quad D_j = K_{\ell(j),r(j)}, \\ B_j &= \left\{ t \in D_j : X_{\ell(j)}(t) \geq 8^{-2(k-1)} \|X_{\ell(j)}\|_{L_\infty(D_j)} \right\}. \end{aligned}$$

In order to see Assumption (3.8) of Lemma 3.5, fix the index j and let i be such that $\ell(i) \geq \ell(j)$. By definition of $D_i = K_{\ell(i),r(i)}$, the smallest interval containing $J_{n,i}$ and D_i contains at most $2k - 1$ atoms of $\mathcal{F}_{\ell(i)}$ and, if $D_i \subset D_j$, the smallest interval containing $J_{n,i}$ and D_j contains at most $2k - 1$ atoms of $\mathcal{F}_{\ell(j)}$. This means that, in particular, $J_{n,i}$ is a subset of the union V of $4k$ atoms of $\mathcal{F}_{\ell(j)}$ with $D_j \subset V$. Since

each atom of $\mathcal{F}_n$ occurs at most k times in the sequence (A_j) , there exists a constant c_1 depending on k and $\sup_{u \leq \ell(j)} \gamma_k(\mathcal{F}_u) \leq \sup_{u \leq n} \gamma_k(\mathcal{F}_u)$ so that

$$|D_j| \geq c_1 \sum_{\substack{i: \ell(i) \geq \ell(j) \\ D_i \subset D_j}} |A_i|,$$

which, since $|B_j| \geq |D_j|/2$ by Corollary 2.2, shows that the assumption of Lemma 3.5 holds true and we get a function φ so that $|\varphi(j)| = c_1 |J_{n,j}|/2$, $\varphi(j) \subset B_j$, $\varphi(i) \cap \varphi(j) = \emptyset$ for all i, j . Using these properties of φ , we continue the estimate in (3.12) and write

$$\begin{aligned} \mathbb{E}g_n &\lesssim \sum_j |J_{n,j}| \cdot \|X_{\ell(j)}\|_{L_\infty(D_j)}^{1/2} \leq 8^{k-1} \cdot \sum_j \frac{|J_{n,j}|}{|\varphi(j)|} \int_{\varphi(j)} X_{\ell(j)}^{1/2} \\ &= \frac{2}{c_1} \cdot 8^{k-1} \cdot \sum_j \int_{\varphi(j)} X_{\ell(j)}^{1/2} \\ &\lesssim \sum_j \int_{\varphi(j)} X_n^{1/2} \leq \mathbb{E}X_n^{1/2}, \end{aligned}$$

with constants depending only on k and $\sup_{u \leq n} \gamma_k(\mathcal{F}_u)$. $\square$

Employing this construction of g_n , we now give the following duality estimate for spline projections (for the martingale case, see for instance [8]). The martingale version of this result is the essential estimate in the proof of both Lépingle's inequality (1.1) and the H^1 -BMO duality.

Theorem 3.6 *Let $(\mathcal{F}_n)$ be such that $\gamma := \sup_n \gamma_k(\mathcal{F}_n) < \infty$ and $(f_n)_{n \geq 1}$ a sequence of functions with $f_n \in \mathcal{S}_k(\mathcal{F}_n)$ for each n . Additionally, let $h_n \in L_1$ be arbitrary. Then, for any N ,*

$$\sum_{n \leq N} \mathbb{E}[|f_n \cdot h_n|] \lesssim \sqrt{2} \cdot \mathbb{E} \left[\left(\sum_{\ell \leq N} f_\ell^2 \right)^{1/2} \right] \cdot \sup_{n \leq N} \|P_n \left(\sum_{\ell=n}^N h_\ell^2 \right)\|_\infty^{1/2},$$

where the implied constant is the same constant that appears in (3) of Proposition 3.4 and therefore only depends on k and γ .

Proof Let $X_n := \sum_{\ell \leq n} f_\ell^2$ and $f_\ell \equiv 0$ for $\ell > N$ and $\ell \leq 0$. By Proposition 3.4, we choose an increasing sequence (g_n) of functions with $g_0 = 0$, $g_n \in \mathcal{S}_k(\mathcal{F}_n)$ and the properties $X_n^{1/2} \leq g_n$ and $\mathbb{E}g_n \lesssim \mathbb{E}X_n^{1/2}$ where the implied constant depends only on k and γ . Then, apply Cauchy–Schwarz inequality by introducing the factor $g_n^{1/2}$ to get

$$\begin{aligned}\sum_n \mathbb{E}[|f_n \cdot h_n|] &= \sum_n \mathbb{E} \left[\left| \frac{f_n}{g_n^{1/2}} \cdot g_n^{1/2} h_n \right| \right] \\ &\leq \left[\sum_n \mathbb{E}[f_n^2/g_n] \right]^{1/2} \cdot \left[\sum_n \mathbb{E}[g_n h_n^2] \right]^{1/2}.\end{aligned}$$

We estimate each of the factors on the right hand side separately and set

$$\Sigma_1 := \sum_n \mathbb{E}[f_n^2/g_n], \quad \Sigma_2 := \sum_n \mathbb{E}[g_n h_n^2].$$

The first factor is estimated by the pointwise inequality $X_n^{1/2} \leq g_n$:

$$\begin{aligned}\Sigma_1 &= \mathbb{E} \left[\sum_n \frac{f_n^2}{g_n} \right] \leq \mathbb{E} \left[\sum_n \frac{f_n^2}{X_n^{1/2}} \right] \\ &= \mathbb{E} \left[\sum_n \frac{X_n - X_{n-1}}{X_n^{1/2}} \right] \leq 2\mathbb{E} \sum_n (X_n^{1/2} - X_{n-1}^{1/2}) = 2\mathbb{E} X_N^{1/2}.\end{aligned}$$

We continue with Σ_2 :

$$\begin{aligned}\Sigma_2 &= \mathbb{E} \left[\sum_{\ell=1}^N g_\ell h_\ell^2 \right] = \mathbb{E} \left[\sum_{\ell=1}^N \sum_{n=1}^\ell (g_n - g_{n-1}) h_\ell^2 \right] \\ &= \mathbb{E} \left[\sum_{n=1}^N (g_n - g_{n-1}) \cdot \sum_{\ell=n}^N h_\ell^2 \right] \\ &= \mathbb{E} \left[\sum_{n=1}^N P_n (g_n - g_{n-1}) \cdot \sum_{\ell=n}^N h_\ell^2 \right] \\ &= \mathbb{E} \left[\sum_{n=1}^N (g_n - g_{n-1}) \cdot P_n \left(\sum_{\ell=n}^N h_\ell^2 \right) \right] \\ &\leq \mathbb{E} \left[\sum_{n=1}^N (g_n - g_{n-1}) \right] \cdot \sup_{1 \leq n \leq N} \left\| P_n \left(\sum_{\ell=n}^N h_\ell^2 \right) \right\|_\infty,\end{aligned}$$

where the last inequality follows from $g_n \geq g_{n-1}$. Noting that by the properties of g_n , $\mathbb{E} \left[\sum_{n=1}^N (g_n - g_{n-1}) \right] = \mathbb{E} g_N \lesssim \mathbb{E} X_N^{1/2}$ and combining the two parts Σ_1 and Σ_2 , we obtain the conclusion. $\square$

4 Applications

We give two applications of Theorem 3.6, (i) D. Lépingle's inequality and (ii) an analogue of C. Fefferman's H_1 -BMO duality in the setting of splines. Once the results

from Sect. 3 are known, the proofs of the subsequent results proceed similarly to their martingale counterparts in [8, 12] by using spline properties instead of martingale properties.

4.1 Lépingle's inequality for splines

Theorem 4.1 *Let k, k' be positive integers. Let $(\mathcal{F}_n)$ be an interval filtration with $\sup_n \gamma_k(\mathcal{F}_n) < \infty$ and, for any n , $f_n \in \mathcal{S}_k(\mathcal{F}_n)$ and P'_n be the orthogonal projection operator on $\mathcal{S}_{k'}(\mathcal{F}_n)$. Then,*

$$\|(P'_{n-1}f_n)\|_{L_1(\ell_2)} = \left\| \left(\sum_n (P'_{n-1}f_n)^2 \right)^{1/2} \right\|_1 \lesssim \left\| \left(\sum_n f_n^2 \right)^{1/2} \right\|_1 = \|(f_n)\|_{L_1(\ell_2)},$$

where the implied constant depends only on k, k' and $\sup_n \gamma_k(\mathcal{F}_n)$.

We emphasize that the parameters k and k' can be different here, k being the spline order of the sequence (f_n) and k' being the spline order of the projection operators P'_{n-1} . In particular, the constant on the right hand side does not depend on the k' -regularity parameter $\sup_n \gamma_{k'}(\mathcal{F}_n)$.

Proof We first assume that $f_n = 0$ for $n > N$ and begin by using duality

$$\mathbb{E} \left[\left(\sum_n (P'_{n-1}f_n)^2 \right)^{1/2} \right] = \sup_{(H_n)} \mathbb{E} \left[\sum_n (P'_{n-1}f_n) \cdot H_n \right],$$

where sup is taken over all $(H_n) \in L_\infty(\ell_2)$ with $\|(H_n)\|_{L_\infty(\ell_2)} = 1$. By the self-adjointness of P'_{n-1} ,

$$\mathbb{E}[(P'_{n-1}f_n) \cdot H_n] = \mathbb{E}[f_n \cdot (P'_{n-1}H_n)].$$

Then we apply Theorem 3.6 for f_n and $h_n = P'_{n-1}H_n$ to obtain (denoting by P_n the orthogonal projection operator onto $\mathcal{S}_k(\mathcal{F}_n)$)

$$\sum_{n \leq N} |\mathbb{E}[f_n \cdot h_n]| \lesssim \mathbb{E} \left[\left(\sum_{\ell \leq N} f_\ell^2 \right)^{1/2} \right] \cdot \sup_{n \leq N} \left\| P_n \left(\sum_{\ell=n}^N (P'_{\ell-1}H_\ell)^2 \right) \right\|_\infty^{1/2}. \quad (4.1)$$

To estimate the right hand side, we note that for any n , by Corollary 3.3,

$$\left\| P_n \left(\sum_{\ell=n}^N (P'_{\ell-1}H_\ell)^2 \right) \right\|_\infty \lesssim \left\| \sum_{\ell=n}^N H_\ell^2 \right\|_\infty.$$

Therefore, (4.1) implies

$$\mathbb{E} \left[\left(\sum_n (P'_{n-1} f_n)^2 \right)^{1/2} \right] = \sup_{(H_n)} \mathbb{E} \left[\sum_n f_n \cdot (P'_{n-1} H_n) \right] \lesssim \mathbb{E} \left[\left(\sum_{\ell \leq N} f_\ell^2 \right)^{1/2} \right],$$

with a constant depending only on k, k' and $\sup_{n \leq N} \gamma_k(\mathcal{F}_n)$. Letting N tend to infinity, we obtain the conclusion. $\square$

4.2 H_1 -BMO duality for splines

We fix an interval filtration $(\mathcal{F}_n)_{n=1}^\infty$, a spline order k and the orthogonal projection operators P_n onto $\mathcal{S}_k(\mathcal{F}_n)$ and additionally, we set $P_0 = 0$. For $f \in L_1$, we introduce the notation

$$\Delta_n f := P_n f - P_{n-1} f, \quad S_n(f) := \left(\sum_{\ell=1}^n (\Delta_\ell f)^2 \right)^{1/2}, \quad S(f) = \sup_n S_n(f).$$

Observe that for $\ell < n$ and $f, g \in L_1$,

$$\mathbb{E}[\Delta_\ell f \cdot \Delta_n g] = \mathbb{E}[P_\ell(\Delta_\ell f) \cdot \Delta_n g] = \mathbb{E}[\Delta_\ell f \cdot P_\ell(\Delta_n g)] = 0. \quad (4.2)$$

Let V be the L_1 -closure of $\cup_n \mathcal{S}_k(\mathcal{F}_n)$. Then, the uniform boundedness of P_n on L_1 implies that $P_n f \rightarrow f$ in L_1 for $f \in V$. Next, set

$$H_{1,k} = H_{1,k}((\mathcal{F}_n)) = \{f \in V : \mathbb{E}(S(f)) < \infty\}$$

and equip $H_{1,k}$ with the norm $\|f\|_{H_{1,k}} = \mathbb{E}S(f)$. Define

$$\text{BMO}_k = \text{BMO}_k((\mathcal{F}_n)) = \left\{ f \in V : \sup_n \left\| \sum_{\ell \geq n} T_n((\Delta_\ell f)^2) \right\|_\infty < \infty \right\}$$

and the corresponding quasinorm

$$\|f\|_{\text{BMO}_k} = \sup_n \left\| \sum_{\ell \geq n} T_n((\Delta_\ell f)^2) \right\|_\infty^{1/2},$$

where T_n is the operator from (3.3) that dominates P_n pointwise.

Let us now assume $\sup_n \gamma_k(\mathcal{F}_n) < \infty$. In this case we identify, similarly to H_1 -BMO-duality (cf. [7,8,10]), BMO_k as the dual space of $H_{1,k}$.

First, we use the duality estimate Theorem 3.6 and (4.2) to prove, for $f \in H_{1,k}$ and $h \in \text{BMO}_k$,

$$|\mathbb{E}[(P_n f) \cdot (P_n h)]| \leq \sum_{\ell \leq n} \mathbb{E}[|\Delta_\ell f| \cdot |\Delta_\ell h|] \lesssim \|S_n(f)\|_1 \cdot \|h\|_{\text{BMO}_k}.$$

This estimate also implies that the limit $\lim_n \mathbb{E}[(P_n f) \cdot (P_n h)]$ exists and satisfies

$$|\lim_n \mathbb{E}[(P_n f) \cdot (P_n h)]| \lesssim \|f\|_{H_{1,k}} \cdot \|h\|_{\text{BMO}_k}.$$

On the other hand, similarly to the martingale case (see [8]), given a continuous linear functional L on $H_{1,k}$, we extend L norm-preservingly to a continuous linear functional Λ on $L_1(\ell_2)$, which, by Sect. 2.5, has the form

$$\Lambda(\eta) = \mathbb{E} \left[\sum_{\ell} \sigma_{\ell} \eta_{\ell} \right], \quad \eta \in L_1(\ell_2)$$

for some $\sigma \in L_{\infty}(\ell_2)$. The k -martingale spline sequence $h_n = \sum_{\ell \leq n} \Delta_{\ell} \sigma_{\ell}$ is bounded in L_2 and therefore, by the spline convergence theorem ((v) on page 2), has a limit $h \in L_2$ with $P_n h = h_n$ and which is also contained in BMO_k . Indeed, by using Corollary 3.3, we obtain $\|h\|_{\text{BMO}_k} \lesssim \|\sigma\|_{L_{\infty}(\ell_2)} = \|\Lambda\| = \|L\|$ with a constant depending only on k and $\sup_n \gamma_k(\mathcal{F}_n)$. Moreover, for $f \in H_{1,k}$, since L is continuous on $H_{1,k}$,

$$\begin{aligned} L(f) &= \lim_n L(P_n f) = \lim_n \Lambda((\Delta_1 f, \dots, \Delta_n f, 0, 0, \dots)) \\ &= \lim_n \sum_{\ell=1}^n \mathbb{E}[\sigma_{\ell} \cdot \Delta_{\ell} f] = \lim_n \mathbb{E}[(P_n f) \cdot (P_n h)]. \end{aligned}$$

This yields the following

Theorem 4.2 *If $\sup_n \gamma_k(\mathcal{F}_n) < \infty$, the linear mapping*

$$u : \text{BMO}_k \rightarrow H_{1,k}^*, \quad h \mapsto (f \mapsto \lim_n \mathbb{E}[(P_n f) \cdot (P_n h)])$$

is bijective and satisfies

$$\|u(h)\|_{H_{1,k}^*} \simeq \|h\|_{\text{BMO}_k},$$

where the implied constants depend only on k and $\sup_n \gamma_k(\mathcal{F}_n)$.

Remark 4.3 We close with a few remarks concerning the above result and we assume that $(\mathcal{F}_n)$ is a sequence of increasing interval σ -algebras with $\sup_n \gamma_k(\mathcal{F}_n) < \infty$.

- (1) By Khintchine's inequality, $\|Sf\|_1 \lesssim \sup_{\varepsilon \in \{-1,1\}^{\mathbb{Z}}} \|\sum_n \varepsilon_n \Delta_n f\|_1$. Based on the interval filtration $(\mathcal{F}_n)$, we can generate an interval filtration $(\mathcal{G}_n)$ that contains $(\mathcal{F}_n)$ as a subsequence and each $\mathcal{G}_{n+1}$ is generated from $\mathcal{G}_n$ by dividing exactly one atom of $\mathcal{G}_n$ into two atoms of $\mathcal{G}_{n+1}$. Denoting by $P_n^{\mathcal{G}}$ the orthogonal projection operator onto $\mathcal{S}_k(\mathcal{G}_n)$ and $\Delta_j^{\mathcal{G}} = P_j^{\mathcal{G}} - P_{j-1}^{\mathcal{G}}$, we can write

$$\sum_n \varepsilon_n \Delta_n f = \sum_n \varepsilon_n \sum_{j=a_n}^{a_{n+1}-1} \Delta_j^{\mathcal{G}} f$$

for some sequence (a_n) . By using inequalities (2.7) and (2.6) and writing $(S^{\mathcal{G}} f)^2 = \sum_j |\Delta_j^{\mathcal{G}} f|^2$, we obtain for $p > 1$

$$\|Sf\|_1 \lesssim \|S^{\mathcal{G}} f\|_1 \leq \|S^{\mathcal{G}} f\|_p \lesssim \|f\|_p.$$

This implies $L_p \subset H_{1,k}$ for all $p > 1$ and, by duality, $\text{BMO}_k \subset L_p$ for all $p < \infty$.

- (2) If $(\mathcal{F}_n)$ is of the form that each $\mathcal{F}_{n+1}$ is generated from $\mathcal{F}_n$ by splitting exactly one atom of $\mathcal{F}_n$ into two atoms of $\mathcal{F}_{n+1}$ and under the condition $\sup_n \gamma_{k-1}(\mathcal{F}_n) < \infty$ (which is stronger than $\sup_n \gamma_k(\mathcal{F}_n) < \infty$), it is shown in [9] that

$$\|Sf\|_1 \simeq \|f\|_{H_1},$$

where H_1 denotes the atomic Hardy space on $[0, 1]$, i.e. in this case, $H_{1,k}$ coincides with H_1 .

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